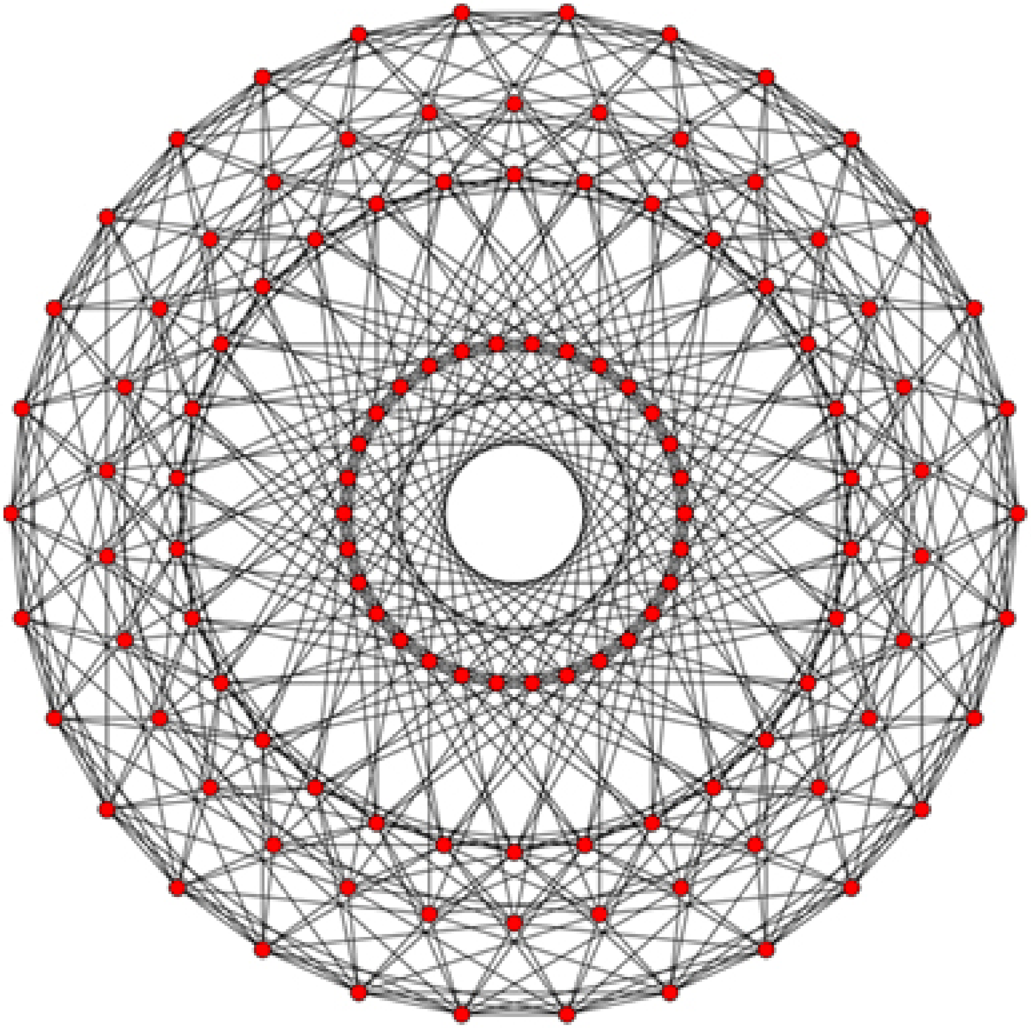
# Geometry of Alternation and Polychoron Objects



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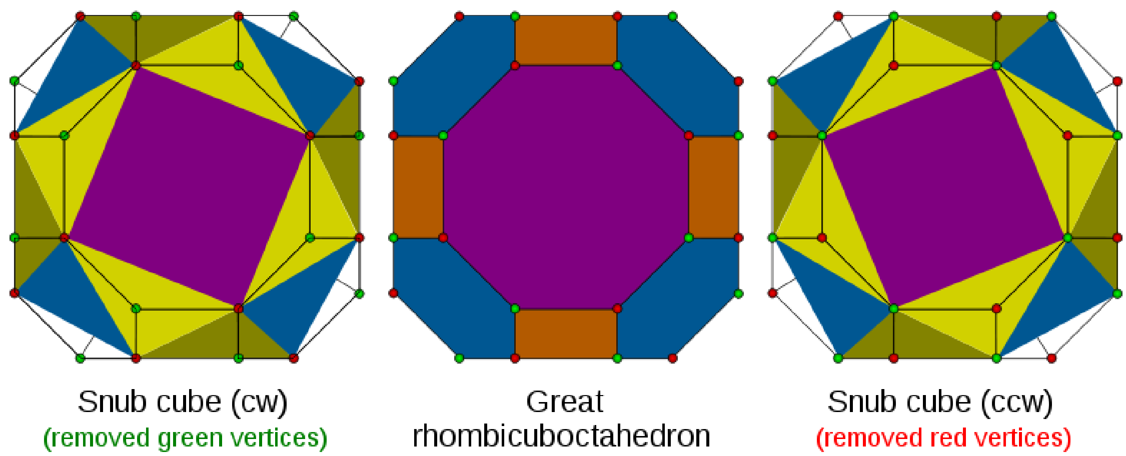
## Chapter 1

# Introduction to Alternation Geometry




Two snub cubes from great rhombicuboctahedron

See that red and green dots are placed at alternate vertices. A snub cube is generated from deleting either set of vertices, one resulting in clockwise gyrated squares, and other counterclockwise.

In geometry, an **alternation** (also called *partial truncation*, *snub* or *snubification*) is an operation on a polyhedron or tiling that birectifies alternate vertices. Only even-sided polyhedra can be alternated, for example the zonohedra. Every *2n*-sided face becomes *n*-sided. Square faces disappear into new edges.

An *alternation* of a regular polyhedron or tiling is sometimes labeled by the regular form, prefixed by an h, standing for *half*. For example h{4,3} is an alternated cube (creating a tetrahedron), and h{4,4} is an alternated square tiling (still a square tiling).

## Snub

A *snub* is a related operation. It is an *alternation* applied to an omnitruncated regular polyhedron. An omnitruncated regular polyhedron or tiling always has even-sided faces and so can always be alternated.

For instance the *snub cube* is created in two steps. First it is omnitruncated, creating the great rhombicuboctahedron. Secondly that polyhedron is alternated into a snub cube. You can see from the picture on the right that there are two ways to alternate the vertices, and they are mirror images of each other, creating two chiral forms.

Another example is the uniform antiprisms. A uniform *n*-gonal antiprism can be constructed as an alternation of a *2n*-gonal prism, and the snub of an *n*-edge hosohedron. In the case of prisms both alternated forms are identical.

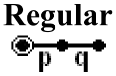
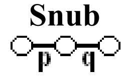
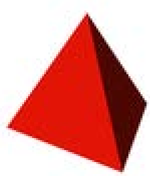
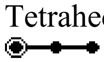
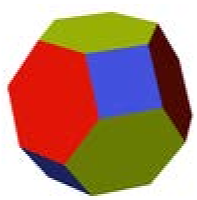
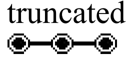
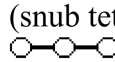
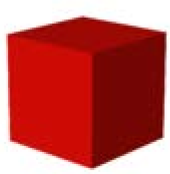
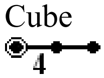
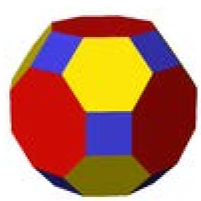
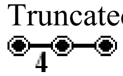
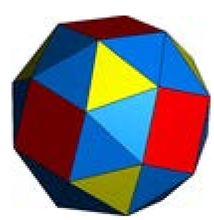
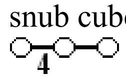
Non-uniform zonohedra can also be alternated. For instance, the Rhombic tri-acontahedron can be snubbed into either an icosahedron or a dodecahedron depending on which vertices are removed.

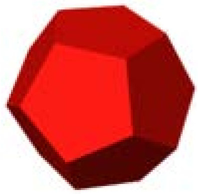
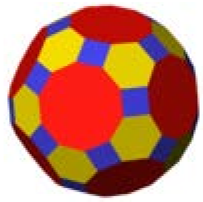
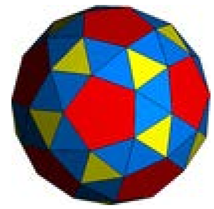
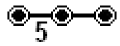
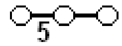
# Examples

## Platonic solid generators

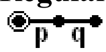
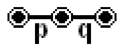
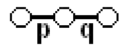
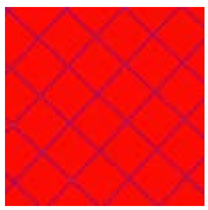
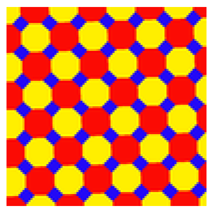
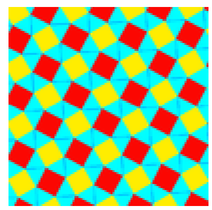
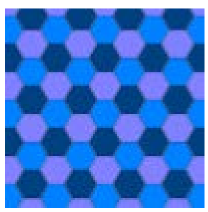
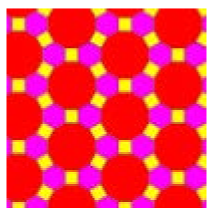
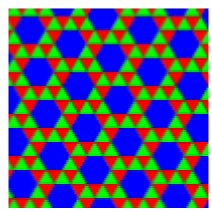
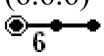
Three forms: regular → omnitruncated → snub.

The Coxeter-Dynkin diagrams are given as well. The omnitruncation actives all of the mirrors (ringed). The alternation is shown as rings with *holes*.

| **Symmetry (p q 2)** | **Regular**   | **Omnitruncated**   | **Snub**   |
| --- | --- | --- | --- |
| Tetrahedral (3 3 2) |  Tetrahedron  |  truncated octahedron  |  icosahedron (snub tetrahedron)  |
| Octahedral (4 3 2) |  Cube   |  Truncated cuboctahedron   |  snub cube   |

| | | | |
|---|---|---|---|
| Icosahedral (5 3 2) |  |  |  |
| | Dodecahedron  | Truncated icosidodecahedron  | snub dodecahedron  |

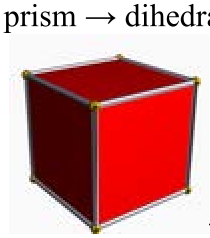
## Regular tiling generators

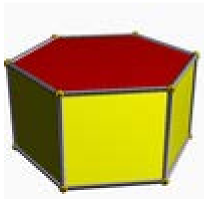
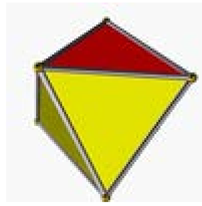
| **Symmetry (p q 2)** | **Regular**  | **Omnitruncated**  | **Snub**  |
|---|---|---|---|
| Square (4 4 2) |  |  |  |
| | (4.4.4.4)  | (4.8.8)  | (3.3.4.3.4)  |
| Hexagonal (6 3 2) |  |  |  |
| | (6.6.6)  | (3.4.6.4)  | 3.3.3.3.6  |

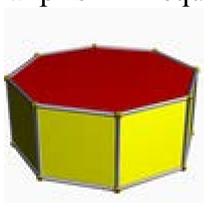
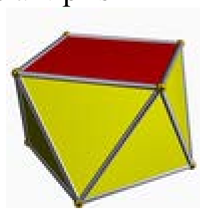
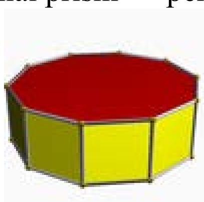
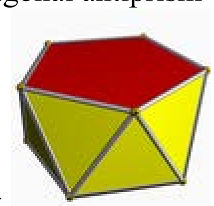
## Uniform prism generators (dihedral symmetry)

*Alternate truncations* can be applied to prisms. (A *square antiprism* may be called a *snubbed 4-edge hosohedron*, as well as an *alternated octagonal prism*.)

Two steps: 2*n*-gonal prisms → *n*-gonal antiprism.

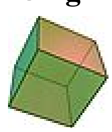
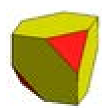
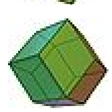
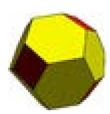
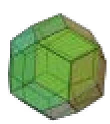
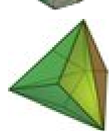
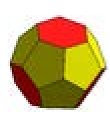
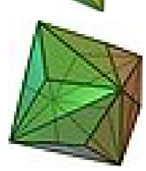
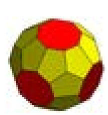
- square prism → dihedral antiprism
  -  → 
- hexagonal prism → triangular antiprism

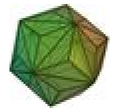
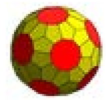
-  → 

- octagonal prism → square antiprism
  -  → 
- decagonal prism → pentagonal antiprism
  -  → 
- ....

# Alternate truncations

A similar operation can truncate alternate vertices, rather than just removing them. Below is a set of polyhedra that can be generated from the duals of Catalan solids. These have two types of vertices which can be alternately truncated. Truncating the "higher order" vertices produces these forms:

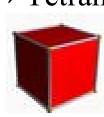
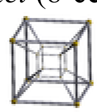
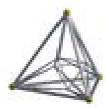
| Name | Original | Truncation | Truncated name |
|---|---|---|---|
| Cube<br>Dual of rectified tetrahedron |  |  | Alternate truncated cube |
| Rhombic dodecahedron<br>Dual of cuboctahedron |  |  | Truncated rhombic dodecahedron |
| Rhombic triacontahedron<br>Dual of icosidodecahedron |  |  | Truncated rhombic triacontahedron |
| Triakis tetrahedron<br>Dual of truncated tetrahedron |  |  | Truncated triakis tetrahedron |
| Triakis octahedron<br>Dual of truncated cube |  |  | Truncated triakis octahedron |

| | | | |
|---|---|---|---|
| Triakis icosahedron<br>Dual of truncated dodecahedron |  |  | Truncated triakis icosahedron |

# Higher dimensions

This *alternation* operation applies to higher dimensional polytopes and honeycombs as well, however in general most forms won't have uniform solution. The voids created by the deleted vertices will not in general create uniform facets.

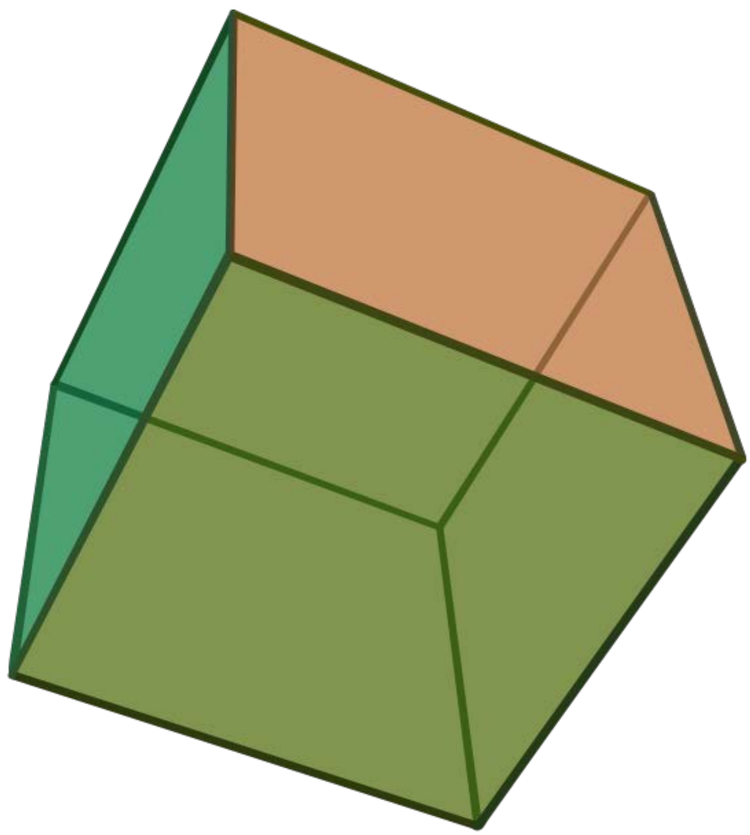
Examples:

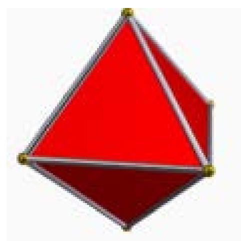
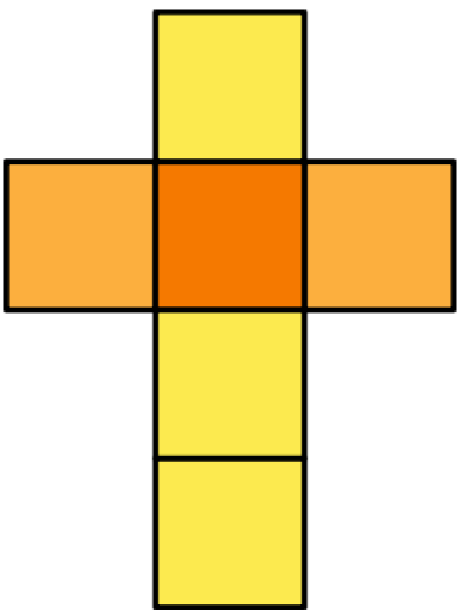
- Honeycombs
  1. An alternated cubic honeycomb is the tetrahedral-octahedral honeycomb.
  2. An alternated hexagonal prismatic honeycomb is the gyrated alternated cubic honeycomb.
- Polychora
  1. An alternated truncated 24-cell is the snub 24-cell.
- A hypercube can always be alternated into a uniform demihypercube.
  1. Cube → Tetrahedron (regular)
     -  → 
  2. *Tesseract* (8-cell) → 16-cell (regular)
     -  → 
  3. Penteract → demipenteract (semiregular)
  4. Hexeract → demihexeract (uniform)

## Chapter 2

# Cube and Tetrahedron

## Cube

| Regular Hexahedron | |
|---|---|
|  | |
| Type | Platonic solid |
| Elements | $F = 6$, $E = 12$<br>$V = 8$ ($\chi = 2$) |
| Faces by sides | 6{4} |

| | |
|---|---|
| Schläfli symbol | {4,3} |
| Wythoff symbol | 3 \| 2 4 |
| Coxeter-Dynkin |  |
| Symmetry | $O_h$, [4,3], (*432) |
| References | $U_{06}$, $C_{18}$, $W_3$ |
| Properties | Regular convex zonohedron |
| Dihedral angle | 90° |
|  4.4.4 (Vertex figure) |  Octahedron (dual polyhedron) |
|  Net | |

In geometry, a **cube** is a three-dimensional solid object bounded by six square faces, facets or sides, with three meeting at each vertex. The cube can also be called a **regular hexahedron** and is one of the five Platonic solids. It is a special kind of square prism, of rectangular parallelepiped and of trigonal trapezohedron. The cube is dual to the octahedron. It has cubical symmetry (also called octahedral symmetry).

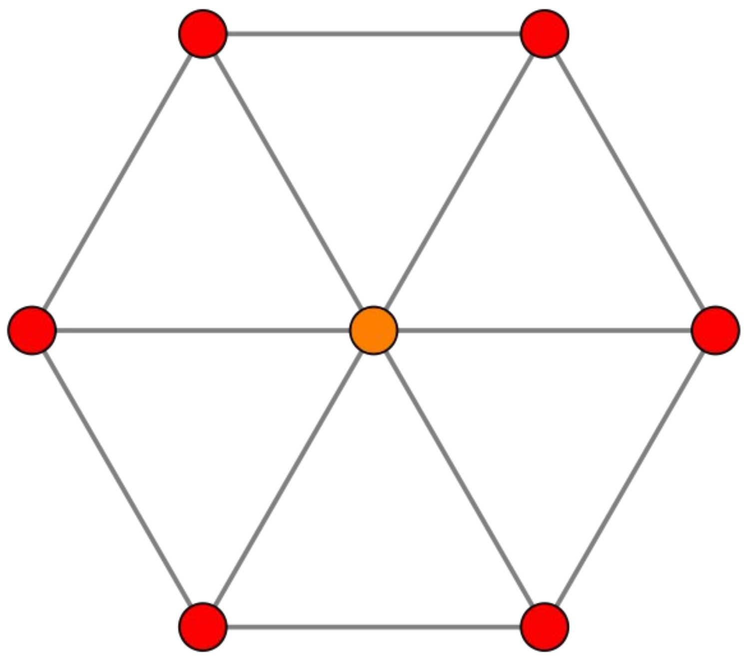
A cube is the three-dimensional case of the more general concept of a hypercube.

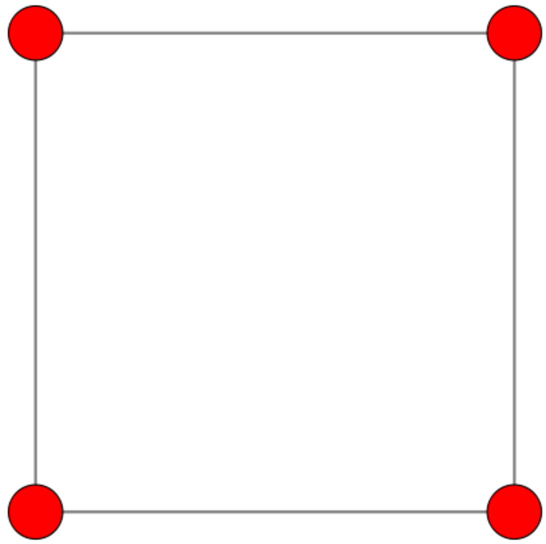
It has 11 nets. If one were to colour the cube so that no two adjacent faces had the same colour, one would need 3 colours.

If the original cube has edge length 1, its dual octahedron has edge length $\sqrt{2}$.

# Cartesian coordinates

**Orthographic projections**





For a cube centered at the origin, with edges parallel to the axes and with an edge length of 2, the Cartesian coordinates of the vertices are

$$(\pm1, \pm1, \pm1)$$

while the interior consists of all points $(x_0, x_1, x_2)$ with $-1 < x_i < 1$.

# Formulae

For a cube of edge length $a$,

| | |
|---|---|
| surface area | $6a^2$ |
| volume | $a^3$ |
| face diagonal | $\sqrt{2}a$ |
| space diagonal | $\sqrt{3}a$ |
| radius of circumscribed sphere | $\frac{\sqrt{3}}{2}a$ |
| radius of sphere tangent to edges | $\frac{a}{\sqrt{2}}$ |
| radius of inscribed sphere | $\frac{a}{2}$ |

angles between faces $\frac{\pi}{2}$

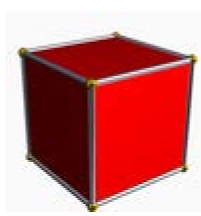
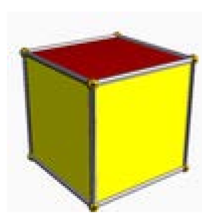
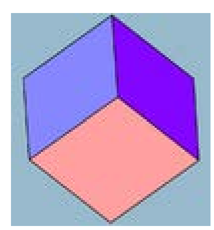
As the volume of a cube is the third power of its sides $a \times a \times a$, third powers are called *cubes*, by analogy with squares and second powers.

A cube has the largest volume among cuboids (rectangular boxes) with a given surface area. Also, a cube has the largest volume among cuboids with the same total linear size (length + width + height).

## Uniform colorings and symmetry

The cube has 3 uniform colorings, named by the colors of the square faces around each vertex: 111, 112, 123.

The cube has 3 classes of symmetry, which can be represented by vertex-transitive coloring the faces. The highest octahedral symmetry $O_h$ has all the faces the same color. The dihedral symmetry $D_{4h}$ comes from the cube being a prism, with all four sides being the same color. The lowest symmetry $D_{2h}$ is also a prismatic symmetry, with sides alternating colors, so there are three colors, paired by opposite sides. Each symmetry form has a different Wythoff symbol.

| **Name** | **Regular hexahedron** | **Square prism** | **Cuboid** | **Trigonal trapezohedron** |
|---|---|---|---|---|
| **Coxeter-Dynkin** |  |  |  | |
| **Schläfli symbol** | {4,3} | {4}x{} | {}x{}x{} | |
| **Wythoff symbol** | 3 \| 4 2 | 4 2 \| 2 | \| 2 2 2 | |
| **Symmetry** | $O_h$ (*432) | $D_{4h}$ (*422) | $D_{2h}$ (*222) | $D_{3d}$ (2*3) |
| **Symmetry order** | 24 | 16 | 8 | 12 |
| **Image (uniform coloring)** |  (111) |  (112) |  (123) |  |

## Geometric relations

These familiar six-sided dice are cube-shaped

The cube is unique among the Platonic solids for being able to tile Euclidean space regularly. It is also unique among the Platonic solids in having faces with an even number of sides and, consequently, it is the only member of that group that is a zonohedron (every face has point symmetry).

The cube can be cut into 6 identical square pyramids. If these square pyramids are then attached to the faces of a second cube, a rhombic dodecahedron is obtained.

## Other dimensions

The analogue of a cube in four-dimensional Euclidean space has a special name—a tesseract or (rarely) hypercube.

The analogue of the cube in *n*-dimensional Euclidean space is called a hypercube or ***n*-dimensional cube** or simply ***n*-cube**. It is also called a *measure polytope*.

There are analogues of the cube in lower dimensions too: a point in dimension 0, a segment in one dimension and a square in two dimensions.

## Related polyhedra

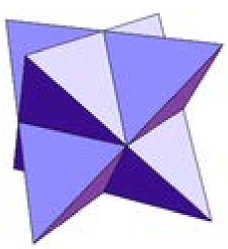
The vertices of a cube can be grouped into two groups of four, each forming a regular tetrahedron; more generally this is referred to as a demicube. These two together form a regular compound, the stella octangula. The intersection of the two forms a regular octahedron. The symmetries of a regular tetrahedron correspond to those of a cube which map each tetrahedron to itself; the other symmetries of the cube map the two to each other.

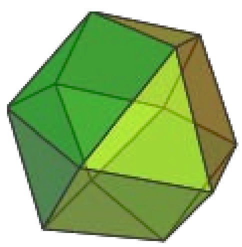
One such regular tetrahedron has a volume of ⅓ of that of the cube. The remaining space consists of four equal irregular tetrahedra with a volume of 1/6 of that of the cube, each.

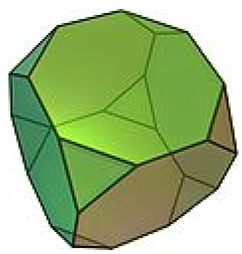
The rectified cube is the cuboctahedron. If smaller corners are cut off we get a polyhedron with 6 octagonal faces and 8 triangular ones. In particular we can get regular octagons (truncated cube). The rhombicuboctahedron is obtained by cutting off both corners and edges to the correct amount.

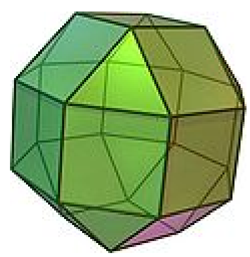
A cube can be inscribed in a dodecahedron so that each vertex of the cube is a vertex of the dodecahedron and each edge is a diagonal of one of the dodecahedron's faces; taking all such cubes gives rise to the regular compound of five cubes.

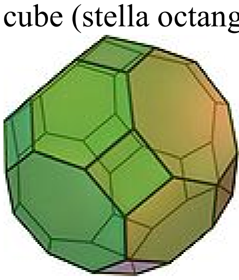
If two opposite corners of a cube are truncated at the depth of the 3 vertices directly connected to them, an irregular octahedron is obtained. Eight of these irregular octahedra can be attached to the triangular faces of a regular octahedron to obtain the cuboctahedron.

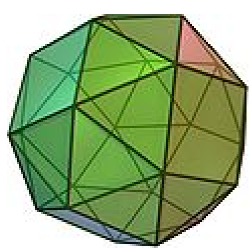

Two tetrahedra in the cube (stella octangula)

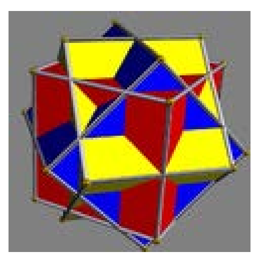

The rectified cube (cuboctahedron)

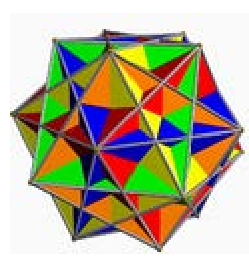

Truncated cube

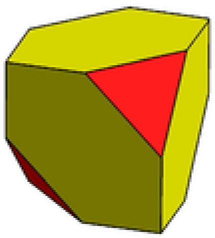

Cantellated cube (rhombicuboctahedron)


Omnitruncated cube (truncated cuboctahedron)


Snub cube

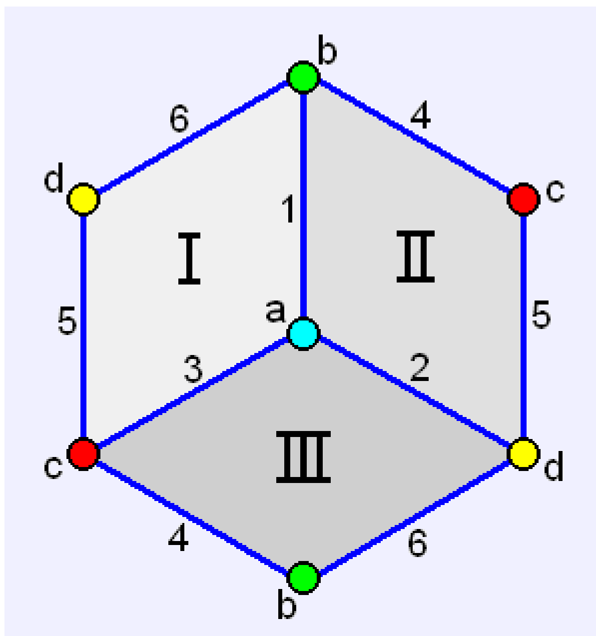

Compound of three cubes


Compound of five cubes



An alternately
truncated cube

All but the last of the figures shown have the same symmetries as the cube.




The hemicube is the 2-to-1 quotient of the cube

The quotient of the cube by the antipodal map yields a projective polyhedron, the hemicube.

The cube is a special case in various classes of general polyhedra:

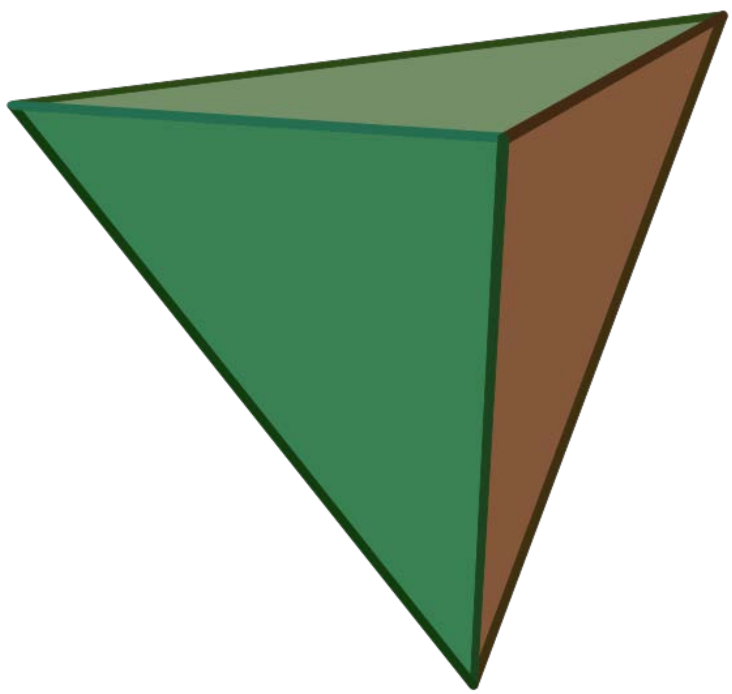
| Name | Equal edge-lengths? | Equal angles? | Right angles? |
|---|---|---|---|
| Cube | Yes | Yes | Yes |
| Rhombohedron | Yes | Yes | No |
| Cuboid | No | Yes | Yes |
| Parallelepiped | No | Yes | No |
| quadrilaterally faced hexahedron | No | No | No |

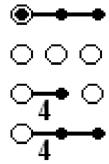
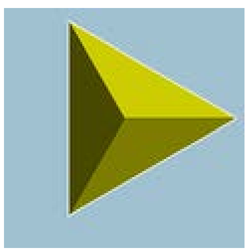
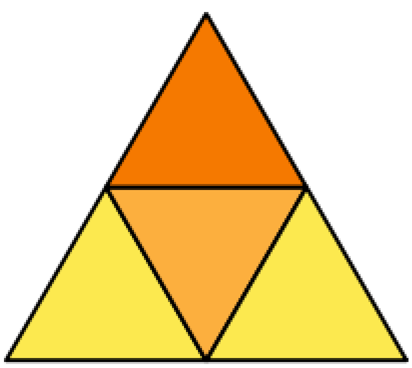
## Combinatorial cubes

A different kind of cube is the **cube graph**, which is the graph of vertices and edges of the geometrical cube. It is a special case of the hypercube graph.

An extension is the 3-dimensional *k*-ary Hamming graph, which for $k = 2$ is the cube graph. Graphs of this sort occur in the theory of parallel processing in computers.

# Tetrahedron

| Regular Tetrahedron | |
|---|---|
|  | |
| Type | Platonic solid |
| Elements | $F = 4, E = 6$<br>$V = 4$ ($\chi = 2$) |
| Faces by sides | 4{3} |
| Schläfli symbol | {3,3} and s{2,2} |
| Wythoff symbol | 3 \| 2 3 |

| | |
|---|---|
| | \| 2 2 2 |
| Coxeter-Dynkin | <br> |
| Symmetry | $T_d$<br>or (*332) |
| References | $U_{01}$, $C_{15}$, $W_1$ |
| Properties | Regular convex deltahedron |
| Dihedral angle | 70.528779° = arccos(1/3) |
| <br>3.3.3<br>(Vertex figure) | <br>Self-dual<br>(dual polyhedron) |
| <br>Net | |

In geometry, a **tetrahedron** (plural: **tetrahedra**) is a polyhedron composed of four triangular faces, three of which meet at each vertex. A **regular tetrahedron** is one in which the four triangles are regular, or "equilateral", and is one of the Platonic solids. The tetrahedron is the only convex polyhedron that has four faces.

The tetrahedron is the three-dimensional case of the more general concept of a simplex.