Strong Stable Markov Chains

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N.V. Kartashov

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VSP BV
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The Netherlands

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CONTENTS

PREFACE

This monograph presents a new approach to the investigation of ergodicity and stability problems for homogeneous Markov chains with a discrete-time and with values in a measurable space. The main purpose of this book is to highlight various methods for the explicit evaluation of estimates for convergence rates in ergodic theorems and in stability theorems for wide classes of chains. These methods are based on the classical perturbation theory of linear operators in Banach spaces and give us new results even for finite chains.

Let $X = (X_t, t \geq 0)$ be a homogeneous discrete-time Markov chain with values in a measurable state space $(E, \mathcal{E})$. From the analytical point of view, any chain can be considered as its transition kernel P on $(E, \mathcal{E})$. This kernel defines one-step ahead probabilities. A linear operator defined in the usual way on the Banach space $m\mathcal{E}$ of finite signed measures on $\mathcal{E}$ corresponds to the kernel P.

Using this relationship, the theory of quasicompact chains was developed over the last four decades by means of the linear operator theory, [Doeblin (1940), Yosida and Kakutani (1941), Bogolyubov and Krein (1946), Krein and Rootman (1948), Hille and Phillips (1957); see also Neveu (1964), Chapter 5; Revuz (1975), Chapter 6]. Quasicompact chains are often also called uniformly ergodic, uniformly recurrent, strong positive recurrent, or uniformly mixing chains. Uniformly ergodic chains possess some important properties and are widely used in applications: see Doob (1953), Nagaev (1961), Davydov (1973), Brunel and Revuz (1974), Zubkov (1979), Kalashnikov (1981), Anisimov (1988), Korolyuk and Turbin (1976), (1978), (1982). In particular, the uniform recurrence of a chain can be easily checked by a simple criterion for the quasicompactness, such as the uniform mixing condition or the Doeblin condition, that is expressed in terms of its transition kernel P. This fact makes simpler the checking of corresponding conditions and also allows us to obtain bounds for the convergence in the form of explicit inequalities. Such bounds are uniform on wide classes of Markov chains — because an "ergodicity index" of a quasicompact chain can be effectively expressed in terms of its kernel P.

Often we do not know precisely the transition probabilities and only estimates for some moment functionals of a chain are available. For this reason we must use only uniform limit theorems that are valid in certain neighborhoods of the transition operator. Therefore, uniform bounds obtained below are of a certain interest, and especially in statistics.

Ergodic properties of the quasicompactness follow from the simple structure of the spectrum of the linear operator P. It is clear that this spectrum depends not only on the analytical definition of the kernel P but also on properties of the Banach space $m\mathcal{E}$. We show that a restriction of the domain of the operator P from $m\mathcal{E}$ to some P-invariant subspace $\mathfrak{M} \subset m\mathcal{E}$ with a norm $\|\cdot\|$ can essentially improve the spectral characteristics of the operator P. Often this approach allows us to obtain ergodic properties of a chain that is not quasicompact from ones of a quasicompact chain. The investigation of the quasicompactness in this extended sense is our first goal.

Another purpose of the monograph is to introduce and investigate the stability property of a Markov chain. Each chain can be considered as a map of its transition operator to a set of finite-time distributions, ergodic distributions and other characteristics, that are calculated by one-step transition probabilities. The chain with the operator P is said to be strong stable if this map is continuous at the point P under suitable choice of metric. In other words, small perturbations of a transition operator of a strong stable Markov chain induce only small changes of finite-time and ergodic distributions. This notion is closely related to the stability theory of stochastic systems developed over the last decades by Zolotarev (1976), Borovkov (1977), and Kalashnikov (1983).

The first part of the monograph develops the theory of uniform ergodic chains with respect to a given norm. This theory is based on the approach described above and contains all basic results of the theory of quasicompact chains including period, cyclic subclasses and Doeblin's criteria.

Under certain assumptions on the norm, the equivalence between uniform ergodicity and strong stability is established. We give explicit uniform bounds for the convergence, introduce periodicity classes, and investigate an asymptotics of transition probabilities. Asymptotic expansions with explicitly evaluated correction terms in ergodicity and stability theorems are also obtained including uniform estimates of a remainder.

In the second part, we remove the condition on the uniform ergodicity. Actually, the unique condition on the initial non-perturbed chain X is the existence of a unique finite invariant measure or the finiteness of any invariant measure. Instead of the ergodicity condition, we assume that the perturbation D of the transition operator P must be subordinated to the generalized potential of the chain, that is, to an unbounded operator with its own domain. We show that this condition, without additional assumptions, is necessary and sufficient for stability of the ergodic distribution. Particularly, if X is uniformly ergodic then this condition can be reduced to the boundedness of the perturbation D in a chosen space $\mathfrak{M}$ of measures. The main part of the induced "stability" norm of a perturbation is expressed in terms of a nonnegative operator generated by means of hitting times. We also obtain explicit estimates for the stability as well as asymptotic expansions in stability theorems.

Based on the analytical methods proposed by Korolyuk and Turbin (1976-1982), we consider the problem of the states consolidation of a Markov chain as a special case of the stability problem. Explicit inequalities, asymptotic expansions, and uniform consolidation theorems are obtained. Exponential asymptotics of "rare" Markov moments on chains are also investigated and corresponding explicit bounds are given.

The main results of the first two chapters are illustrated for the problem of the stability and the ergodicity of a waiting times sequence in a one-channel queuing system under small perturbations of interarrival and service times of a general form.

For instance, the linear correction term of a distribution of the steady-state waiting time under such perturbations is evaluated.

Some new inequalities are given in Chapter 9 for the distance in the uniform metric of the distribution function of a sum of a random geometric number of independent identically distributed nonnegative random variables and the corresponding exponential function. The right-hand sides of those estimates have the first order of smallness with respect to the small parameter of a geometric distribution and contain also the second moment of the terms. The question of the best possible constants is considered.

Estimates of the convergence rate in the classical Renyi theorem are also given.

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We introduce here basic assumptions and list properties of the domain of definition $\mathfrak{M}$ of a linear operator generated by a transition kernel P. Further we give a characterization of the uniform ergodicity of a chain in terms of the generalized potential and the resolvent of the kernel P. Then we establish the equivalence with respect to the same norm between the uniform ergodicity and the strong stability of the invariant distribution of a chain.

1.1. Norms in spaces of measures, functions, and kernels

Let $(E, \mathcal{E})$ be a measurable space. Denote by $m\mathcal{E}$, $f\mathcal{E}$, and $b\mathcal{E}$ the spaces of finite signed measures on $\mathcal{E}$, measurable functions on $\mathcal{E}$, and measurable bounded functions on $\mathcal{E}$, respectively. Let $m\mathcal{E}^+$, $f\mathcal{E}^+$, and $b\mathcal{E}^+$ be the cones of nonnegative elements in these spaces.

For all transition kernels $Q(x, A)$, $x \in E$, $A \in \mathcal{E}$, and for all measurable functions $f(x)$, $x \in E$, we define linear mappings $Q\colon f\mathcal{E} \to f\mathcal{E}$, $Q\colon m\mathcal{E} \to m\mathcal{E}$, $f\colon m\mathcal{E} \to \mathbb{R}$ as follows

$$\begin{aligned} Qg(x) &= \int Q(x, dy)g(y), \\ \mu Q(A) &= \int \mu(dx)Q(x, A), \qquad (1.1) \\ \mu f &= \int \mu(dx)f(x) \end{aligned}$$

provided that these integrals are well defined.

Here and in the sequel the integral, the sum, and the supremum signs without specifications on a domain are considered on a whole domain of definition of corresponding variables.

Let the symbol $f \circ \mu$ stand for the direct product $f(x)\mu(A)$, $x \in E$, $A \in \mathcal{E}$. As usual, $|\mu|$ means the full variation of a measure μ.

Assume that a Banach space $\mathfrak{M}$ is given in $m\mathcal{E}$ with a norm $\|\cdot\|$ such that

(M1) $|\mu(E)| \leq k\|\mu\|$ for all $\mu \in \mathfrak{M}$ and some k.

For the subspace $\mathfrak{M}$, we introduce the dual space $\mathfrak{N}$ of functions on $f\mathcal{E}$ with the finite norm

$$\|f\| = \sup\{|\mu f|, \|\mu\| \leq 1\}. \qquad (1.2)$$

Such functions are supposed to coincide up to the equivalence: $f = g$ if and only if $\mu f = \mu g$ for all $\mu \in \mathfrak{M}$. We also introduce the space $\mathfrak{B}$ of transition kernels Q such that $\mathfrak{M}Q \subset \mathfrak{M}$ and the following operator norm is finite

$$\|Q\| = \sup\{\|\mu Q\|, \|\mu\| \leq 1\}. \qquad (1.3)$$

It follows from condition (M1) that the function $\mathbb{I} \in b\mathcal{E}$, that is equal to the unity identically, belongs to $\mathfrak{N}$ and $\|\mathbb{I}\| \leq K$. Note that the operator norms introduced in

(1.3) for nonequivalent initial norms on $m\mathcal{E}$ are in the general case not comparable, even in the case where one of them is subordinated with respect to another.

In what follows we treat a norm of a function or of an operator as one induced by (1.2) or (1.3). Transition kernels and corresponding linear operators are denoted by the same symbols.

For two kernels P and Q, we define their product PQ as the kernel

$$PQ(x, A) = \int P(x, dy)Q(y, A).$$

We denote by P^t the t-times product of P by itself. By the definition we put $P^0 = I$ where I is the unit operator in $\mathfrak{B}$, that is, a kernel of the form $\mathbb{I}_{x \in A}$.

It follows from definitions (1.2) and (1.3) that [see Dunford and Schvartz (1953) or Rudin (1973)]

$$\|\mu Q\| \leq \|\mu \cdot \|Q\|, \qquad \|Qf\| \leq \|Q\| \cdot \|f\|,$$
$$|\mu f| \leq \|\mu\| \cdot \|f\|, \qquad \|Q\| = \sup\{\|Qf\|, \|f\| \leq 1\}.$$

We assume throughout that condition (M1) is valid as well as usual properties of a norm in a Banach space are satisfied. In addition, we use the following consistency condition on the norm and the order structure in $m\mathcal{E}$

(M2) $\|\mu_1\| \leq \|\mu_1 + \mu_2\|$ for $\mu_i \in \mathfrak{M}^+$;

(M3) $\mathfrak{M}$ is a substructure in $m\mathcal{E}$, that is, $\|\mu_1\|, |\mu| \in \mathfrak{M}$ for $\mu \in \mathfrak{M}$ and $\|\mu_1\| \leq \|\mu_1 - \mu_2\|$ for $\mu_i \in \mathfrak{M}^+$, $\mu_1 \perp \mu_2$.

Here and in the sequel $\mathfrak{M}^+$ and $\mathfrak{N}^+$ are the cones of nonnegative elements in the spaces $\mathfrak{M}$ and $\mathfrak{N}$ generating the orders in $\mathfrak{M}$, $\mathfrak{N}$, and $\mathcal{B}$. By definition, $\mu \leq \nu$ if and only if $\nu - \mu \in \mathfrak{M}^+$, $f \leq g$ if and only if $g - f \in \mathfrak{N}^+$, and $P \leq Q$ if and only if $Q - P \in \mathcal{B}^+ = \{T \in \mathcal{B}: \mathfrak{M}^+ T \subset \mathfrak{M}^+\}$.

We also use the condition

(M3′) $\|\mu_1 - \mu_2\| = \|\mu_1 + \mu_2\|$ for $\mu_i \in \mathfrak{M}^+$, $\mu_1 \perp \mu_2$.

The last condition means that the measures μ and $|\mu|$ have the same norm in $\mathfrak{M}$. Analogously, we use the condition for the dual cone

(M2′) $\|f_1\| \leq \|f_1 + f_2\|$ for $f_i \in \mathfrak{N}^+$.

This condition evidently follows from (M1)–(M3).

Under (M2) and (M3), condition (M1) yields

(M1′) $|\mu|(E) \leq 2k\|\mu\|$ for all $\mu \in \mathfrak{M}$ and some k.

Note that the constant k may be used here instead of $2k$ if (M3′) is valid.

Conditions (M1)–(M3) and (M1′)–(M3′) are satisfied, in particular, for the following spaces $(\mathfrak{M}, \|\cdot\|)$. Let a measurable function v on E be such that $\inf v > 0$. Define, for all $\mu \in m\mathcal{E}$, a weighted variation norm

$$\|\mu\|_v = \int v(x)\, |\mu|(dx), \tag{1.4}$$

and set $\mathfrak{M}_v = \{\mu \in m\mathcal{E}: \|\mu\|_v < \infty\}$. Corresponding norms in $\mathfrak{N}_v$ and $\mathcal{B}_v$ are of the form [see Goldenstein, Gohberg, and Markus (1957) or Krein at al. (1972)]

$$\|f\|_v = \sup\left\{\frac{|f(x)|}{v(x)}, x \in E\right\},$$
$$\|Q\|_v = \sup\left\{\int |Q|(x,dy)\frac{v(y)}{v(x)}, x \in E\right\}. \tag{1.5}$$

Let $q \geq 1$ and let φ be a probability measure on $\mathcal{E}$. Put for $\mu \ll \varphi$

$$\|\mu\|_{q\varphi} = \left(\int \left|\frac{d\mu}{d\varphi}\right|^q d\varphi\right)^{1/q} \tag{1.6}$$

and $\mathfrak{M}_{q\varphi} = \{\mu \in m\mathcal{E}: \mu \ll \varphi, \|\mu\|_{q\varphi} < \infty\}$. Then the corresponding space $\mathfrak{N}_{q\varphi}$ coincides with $L^p(\varphi)$ where $p^{-1} + q^{-1} = 1$.

Due to Krein et al. (1972), chapter 8, from (M2) follows that the cone $\mathfrak{M}^+$ is normal and from (M3) follows that this cone is reproducing in $\mathfrak{M}$.

1.2. Uniformly ergodic chains

Let $X = (X_t)$ be a homogeneous Markov chain with values in a measurable space $(E, \mathcal{E})$ and with a discrete time $t = 0, 1, \ldots$ that is given by its transition kernel $P(x, A)$, $x \in E$, $A \in \mathcal{E}$.

Denote by $P^t(x, A)$ the transition probabilities over t steps. The kernel P^t is a t-fold power of P. Define also Cesaro averages $P^{(t)} = \sum_{s<t} P^s/t$, where $P^0 = P^{(0)} = I$ and I is the unit operator in $\mathfrak{M}$.

Introduce the following condition

(P) $\mathfrak{M}P \subset \mathfrak{M}$ and $\|P\| < \infty$.

Under condition (P), $P^{(t)}$ are linear bounded operators on $\mathfrak{M}$ and, therefore, the following definition is correct.

DEFINITION 1.1. A chain X is called uniformly ergodic with respect to a given norm $\|\cdot\|$ if there exists a stochastic kernel Π such that $P^{(t)} \to \Pi$ as $t \to \infty$ in the induced operator norm (1.3).

REMARK 1.1. A uniformly ergodic with respect to a norm chain may not have this property with respect to another one, even in the case where one of these measures is subordinated to the another.

DEFINITION 1.2. A stochastic kernel Π is called a stationary projector of a kernel P on $(E, \mathcal{E})$ if

$$\Pi^2 = \Pi = P\Pi = \Pi P \tag{1.7}$$

and $\mu = \mu\Pi$ provided that $\mu = \mu P$, $\mu \in \mathfrak{M}$.

THEOREM 1.1. *Let a chain X be uniformly ergodic with respect to a norm $\|\cdot\|$. Then the limit kernel Π in Definition 1.1 is the stationary projector of the kernel P.*

PROOF. It is easily seen that $\left(1 + t^{-1}\right) P^{(t+1)} - t^{-1}I = PP^{(t)} = P^{(t)}P$. Passing to the limit as $t \to \infty$ we conclude that $\Pi = P\Pi = \Pi P$. Therefore $\Pi^2 = \lim \Pi P^{(t)} =$

$\lim \Pi = \Pi$. If $\mu = \mu P$ then $\mu = \mu P^(t) \to \mu\Pi$ as $t \to \infty$ which is what had to be proved. □

REMARK 1.2. If a kernel P has a unique invariant measure π then P has a unique stationary projector Π of the form $\Pi = \mathbb{I} \circ \pi$, that is, $\Pi(x, A) = \pi(A)$ for all $x \in E$, $A \in \mathcal{E}$. Indeed, since $\mu\Pi = \mu\Pi P$, $\mu\Pi \in \{a\pi\}$ for all μ, that is, $\Pi = f \circ \pi$ for some $f \in \mathfrak{N}$. Since $\Pi\mathbb{I} = \mathbb{I}$ by the definition, $f = \mathbb{I}$ and $\Pi = \mathbb{I} \circ \pi$. If in addition $\pi \in \mathfrak{M}$ then $\Pi \in \mathcal{B}$.

REMARK 1.3. Following Korolyuk and Turbin (1978) we assume that in the space E there exists a partition $E = \bigcup_{u \in \mathfrak{U}} E_u$, $E_u \in \mathcal{E}$, such that, for some $\mathfrak{U}_1 \subset \mathfrak{U}$, the sets E_{u_1} are minimal ergodic classes for the chain X and the class $E_0 = \bigcup_{u \in \mathfrak{U} \setminus \mathfrak{U}_1} E_u$ contains zero states, that is,

$$\lim_{t \to \infty} \sup_{x \in E_0} \mathsf{P}\left(X_t \in E \setminus E_0 / X_0 = x\right) = 1$$

and, for all $u \in \mathfrak{U}_1$, $v \in \mathfrak{U} \setminus \mathfrak{U}_1$, the function

$$q(x, u) = \mathsf{P}\Big(\bigcup_{n \geq 1} \bigcap_{t \geq n} \{X_t \in E_u\} / X_0 = x\Big)$$

does not depend on x for $x \in E_v$.

Then, as is shown in Korolyuk and Turbin (1978), Chapter 6, the chain X has a stationary projector Π of the form:

$$\Pi(x, A) = \begin{cases} \pi_u(A), & \text{for } x \in E_u,\ u \in \mathfrak{U}_1, \\ \sum_{v \in \mathfrak{U}_1} q(x, v)\pi_v(A), & \text{for } x \in E_u,\ u \in \mathfrak{U} \setminus \mathfrak{U}_1, \end{cases}$$

where π_u is the ergodic distribution of the restriction of the chain X on E_u, $u \in \mathfrak{U}_1$.

1.3. Generalized potential operator

The property of the uniform ergodicity is closely related to the following notion.

DEFINITION 1.3. Let a kernel P have the stationary projector $\Pi \in \mathcal{B}$. A measure $\mu \in \mathfrak{M}$ is said to be the value $\mu = \nu R$ of a generalized potential R of the chain X if μ is a solution of the system

$$\mu(I - P) = \nu(I - \Pi), \qquad \mu\Pi = 0. \tag{1.8}$$

THEOREM 1.2. *A generalized potential R is well defined on a subspace*

$$\mathfrak{M}_0 = \mathfrak{M}(I - P + \Pi) \tag{1.9}$$

and it is a linear isomorphism between spaces $\mathfrak{M}_0(I - \Pi)$ and $\mathfrak{M}(I - \Pi)$ such that $(\mathfrak{M}_0\Pi)R = (\mathfrak{M}_0 R)\Pi = \{0\}$.

PROOF. To examine if Definition 1.3 is correct, it is sufficient to check that by Definition 1.2 from $\mu(I - P) = 0$, $\mu\Pi = 0$, follows that $\mu = \mu\Pi = 0$.

Let $\nu \in \mathfrak{M}_0$, that is, $\nu = \mu_1(I - P + \Pi)$, $\mu_1 \in \mathfrak{M}$. Then the measure $\mu = \mu_1(I - \Pi)$ meets (1.8), since $\mu\Pi = 0$ and $\mu_1(I - \Pi)(I - P) = \mu_1(I - P) = \mu_1(I - \Pi)(I - P + \Pi)$ according to (1.7) and $\mu_1(I - \Pi) \in \mathfrak{M}$ because of $\Pi \in \mathcal{B}$. So, for $\nu \in \mathfrak{M}_0$ the system (1.8) has a unique solution $\mu \in \mathfrak{M}$. The second assertion of the theorem follows from the equality $\mu = \mu_1(I - \Pi)$ obtained before. The equalities $\Pi R = R\Pi = 0$ are immediately derived from (1.8). If $\mu = \nu\Pi R$ then $\mu(I - P) = 0 = \mu\Pi$ and $\mu = 0$. Furthermore, if $\mu = \nu R\Pi$ then $\mu = \mu\Pi = 0$. The theorem is proved. □

THEOREM 1.3. *Assume that conditions* (M1)–(M3) *and* (P) *are satisfied. A chain X is uniformly ergodic with respect to a norm $\|\cdot\|$ if and only if it has a stationary projector $\Pi \in \mathcal{B}$ and one of the following equivalent (provided that $\Pi \in \mathcal{B}$) conditions holds true*:

(R) *the extension of the operator R on $\mathfrak{M}$ and $\|R\| \le r$ is bounded,*
(R1) *the operator $I - P + \Pi: \mathfrak{M} \to \mathfrak{M}$ has the bounded inverse operator;*
(R2) $\mathfrak{M}_0 = \mathfrak{M}$;
(R3) $\|\mu(I - \Pi)\| \le r\|\mu(I - P)\|$ *for all* $\mu \in \mathfrak{M}$.

PROOF. The equivalence of (R1) and (R2) directly follows from the definition of $\mathfrak{M}_0$: if the operator $S = (I - P + \Pi)^{-1}$ is bounded then $\mathfrak{M} = \mathfrak{M}S(I - P + \Pi) \subset \mathfrak{M}_0 \subset \mathfrak{M}$. On the other hand, if $\mathfrak{M}_0 = \mathfrak{M}$ then S is well defined on the whole $\mathfrak{M}$ and is closed. Indeed, $\mu(I - P + \Pi) = 0$ implies $\mu\Pi = 0$, $\mu = \mu P = \mu\Pi = 0$. Condition $\mathfrak{M}_0 = \mathfrak{M}$ yields that the equation $\mu(I - P + \Pi) = \nu$ has a solution for all $\nu \in \mathfrak{M}$ and, finally, from $\mu_n(I - P + \Pi) = \nu_n$, $\mu_n \to \mu$, $\nu_n \to \nu_0$, follows that $\mu_0(I - P + \Pi) = \nu_0$, so the operator S is closed. Therefore, under condition (R2) the operator S is bounded via the theorem on a closed graph, [see Kato (1966), Theorem 3.5.20].

The equivalence of conditions (R) and (R3) follows from Definition 1.3. Indeed, if $R \in \mathcal{B}$, $\|R\| \le r$, and $\mu = \nu R$, then $\mu\Pi = 0$ and

$$\|\mu(I - \Pi)\| = \|\mu\| = \|\nu R\| = \|\nu(I - \Pi)R\| \le r\|\nu(I - \Pi)\| = r\|\mu(I - P)\|.$$

Therefore, by Theorem 1.2, (R3) holds for $\mu \in \mathfrak{M}_0 R$ where $\mathfrak{M}_0 R = \mathfrak{M}(I - \Pi)$. It remains to observe that $\mu(I - P) = \mu(I - \Pi)(I - P)$ depends only on the measure $\mu(I - P)$. On the other hand, if condition (R3) is fulfilled, then the operator $(I - P)^{-1}$ is bounded on the Banach space $\mathfrak{M}^0 = \{\mu \in \mathfrak{M}: \mu\Pi = 0\}$ that is the closure of its domain, [see Kato (1966), Chapter 3.2 and Chapter 4.5]. Therefore, (1.8) immediately implies (R).

If condition (R1) holds true, then the system (1.8), that is equivalent to the following one $\mu(I - P + \Pi) = \nu(I - \Pi)$, $\mu\Pi = 0$, has a solution $\mu = \nu(I - \Pi)(I - P + \Pi)^{-1}$, $\|\mu\| \le c\|\nu\|$, that is, $R \in \mathcal{B}$. For $R \in \mathcal{B}$, we conclude from (1.8) that the measure $\mu = \nu R + \nu\Pi$ is a solution of the equation $\mu(I - P + \Pi) = \nu$ and $\|\mu\| \le c\|\nu\|$, that is, condition (R1) is fulfilled.

So, conditions (R)–(R3) are equivalent.

Assume that the chain X is uniformly ergodic. Then from the operator convergence $P^{(t)} \to \Pi$ we conclude that $\Pi \in \mathcal{B}$ and by Theorem 1.1 that Π is a stationary projector. For $n \ge 1$ consider the bounded operator $Q_n = I + n^{-1}\sum_{t=1}^{n}\left(P^{(t)} - \Pi\right)$. Taking into account (1.7), it is easy to check that

$$(I - P + \Pi)Q_n = Q_n(I - P + \Pi) = I - P^{(n)} + \Pi.$$

Choose n so that $\|P^{(n)} - \Pi\| < 1$. Then the operator $I - P^{(n)} + \Pi$ has the bounded inverse and commute with Q_n and $I - P + \Pi$. Dividing both sides of the equation obtained by its right-hand side, we arrive at the conclusion that the operator $(I - P + \Pi)^{-1} = Q_n\left(I - P^{(n)} + \Pi\right)$ is well defined and bounded. Therefore condition (R) is fulfilled as well as all other conditions of the theorem. □

We shall prove in Theorem 3.1 that a chain X is uniformly ergodic if $\Pi \in \mathcal{B}$ and $R \in \mathcal{B}$ and give the corresponding estimates for the convergence rate in Definition 1.1.

REMARK 1.4. The assumptions of Theorem 1.3 are necessary for the uniform ergodicity of a chain X even without conditions (M2) and (M3), since these are not used in the proof of the necessity. On the other hand, the sufficiency of these assumptions is essentially based on properties (M2) and (M3). This is illustrated in Example 1.1 below.

EXAMPLE 1.1. $E = [0,1)$, $\mathcal{E} = \mathcal{B}[0,1)$, and the space $\mathfrak{M}$ consists of finite measures such that the norm $\|\mu\| = \sum_{n\geq 0} |\int x^n \, \mu(dx)|$ is finite. Let the chain X be given by its transition kernel P defined as $Pf(x) = xf(0) + (1-x)f(x)$ for $f \in b\mathcal{B}$. For the chain X and the norm $\|\cdot\|$, conditions (M1) and (P) are satisfied and X has a unique invariant measure $\pi \in \mathfrak{M}$ that is concentrated at zero. According to Definition 1.3, one can easily derive from (1.8) that

$$\mu R(A) = \int_A x^{-1} \, \mu(dx) - \pi(A) \int_E x^{-1} \, \mu(dx)$$

for all measures μ with a support separated from zero. Note that this set is dense in $\mathfrak{M}$. By

$$\|\mu R\| = |\mu R(E)| + \sum_{n\geq 1} \left| \int x^n x^{-1} \, \mu(dx) \right| = \|\mu\|$$

we conclude that the generalized potential R is bounded, that is, condition (R) of Theorem 1.3 is satisfied.

However, the chain X is not uniformly ergodic with respect to the introduced norm $\|\cdot\|$. Indeed, the function $f(x) = x$ belongs to the dual space $\mathfrak{N}$ with a norm $\|f\| = \sup_{n,x} |f^{(n)}(x)|/n!$. The value of the corresponding Cesaro average is on f equal to $P^{(t)}f(x) = \left(1-(1-x)^t\right)/t$ and does not converge to $\Pi f(x) = 0$ in the norm $\mathfrak{N}$ which contradicts the uniform ergodicity.

It is clear that this contradiction appears because the norm is not monotone on $\mathfrak{M}$.

The following property of a potential shows that, for uniformly ergodic chains, it coincides analytically with the sum of an operator series.

THEOREM 1.4. *Let a chain X have the stationary projector $\Pi \in \mathcal{B}$, satisfies the condition* (P), *and $\|P^t\| = o(t)$ as $t \to \infty$. Then for all*

$$\mu \in \mathfrak{M}_{00} = \mathfrak{M}_0(I - P + \Pi) = \mathfrak{M}(I - P + \Pi)^2$$

the value μR coincides with the Cesaro sum of the series

$$\mu R = \sum_{t\geq 0}{}^{(c)} \mu \left(P^t - \Pi\right). \tag{1.10}$$

In particular, for uniformly ergodic chains, equation (1.10) *is valid for all $\mu \in \mathfrak{M}$.*

PROOF. It is sufficient to check that the series in (1.10) converges for $\mu \in \mathfrak{M}_{00}$ and its sum satisfies Definition 1.3.