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Groups, Difference Sets, and the Monster

Proceedings of a Special Research Quarter at The Ohio State University, Spring 1993

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Preface

These Proceedings are an outgrowth of a special research quarter held at the Ohio State University in Spring 1993 and supported by the O.S.U. Mathematical Research Institute and the National Security Agency.

The focus during the quarter was primarily on the following topics: finite groups from a geometric view point; abelian and non-abelian difference sets; the Monster and related topics in number theory and physics; and computational group theory.

A variety of additional group theoretic topics were discussed. The papers in this volume reflect primarily the first three main topics of the quarter. All of the topics presented represent areas of intense and active research. Most of the authors have included in their articles many stimulating questions for future research, in addition to new ideas and theories.

It is a pleasure to acknowledge the efforts of a number of people who helped make the special quarter successful. The Ohio State University Mathematics Department staff—particularly Marilyn Radcliff and Denise Witcher—provided the organized support which made the conference run smoothly. Several of the papers were retyped by Terry England. Professor S. K. Wong was a great support in arranging for the housing and other needs of the participants. Finally, the computational group theory week was expertly organized by Professor Akos Seress.

Groups and Geometry. The first week of the conference focused on groups and geometries. New light was shed on such classical topics as spreads, ovoids and generalized quadrangles by Glauberman, Shult and others. In recent years many old and new geometrical themes have been viewed from the perspective of diagram geometries as pioneered by Tits, Buekenhout, Ronan and Smith. This perspective informs the work of Baumeister, Shult and Stroth. In particular Stroth clearly formulated the challenge to develop a 'theory of sporadic geometries'. With roots in the work of Tits, Brown and Quillen, the subject of group actions on simplicial complexes has been an extremely active area. It is represented here by Smith's article on block complexes.

Certain investigations arise in contexts not normally considered geometric, but acquire a sometimes unexpected geometric flavor. Thus the work of Stroth on the 'uniqueness case' is central to the classification of the finite simple groups, but the resolution he outlines involves the geometric theory of amalgams. Likewise the work of Frohardt and Magaard addresses a conjecture of Guralnick and Thompson arising in connection with the inverse problem of Galois theory. However, central to their work is a geometric analysis of the fixed point subgeometries of elements acting on buildings. Similarly, the point of departure of Kantor's discussion is Thompson's study of the Lie algebra E_8 in connection with the simple group Th $(= F_3)$. However he illuminates relations to such geometric object as parallelisms and symplectic spreads.

Perhaps least geometric is Dowd's paper on the 1-cohomology of certain classical matrix groups, though the groups in questions are certainly of geometric significance. Besides completely solving the problem for three families of groups, he presents new techniques for addressing this difficult and important class of problems.

Besides those who have contributed papers to this volume, many other mathematicians joined in the formal and informal discussions of the conference. We thank all of them for their enthusiastic participation. The speakers and their topics were the following:

Michael Aschbacher, Cal Tech, *Foundations of the sporadic groups*;

Ulrich Meierfrankenfeld, Michigan State U., *A construction of* J_4;

Gernot Stroth, U. of Halle, *The uniqueness theorem*;

Charles Thomas, U. of Cambridge, *Cohomology of finite simple groups*;

Michael Dowd, U. of Florida, *On the cohomology of the groups* $SL(3, 3^n)$ *and* $SU(3, 3^n)$;

Jonathan Hall, Michigan State U., *Locally finite simple groups*;

Stephen D. Smith, U. Illinois at Chicago, *Groups and complexes revisited*;

Ernest Shult, Kansas State U., *M-systems and the BLT property*;

Andrew Mathas, U. Notre Dame, *Left cell representations and generic degrees*;

George Glauberman, U. of Chicago, *Outer automorphisms of* Sym(6) *and* $Sp(4, 2^n)$;

Gernot Stroth, U. of Halle, *Some sporadic geometries*;

Michael Abramson, Bowling Green State U., *Affine blueprints*;

J. J. Seidel, Tech. U. Eindhoven, *Signed graphs, root lattices and Coxeter groups*;

Barbara Baumeister, Freie U. Berlin, *Flag-transitive rank* 3 *geometries which are locally complete graphs*;

Daniel Frohardt, Wayne State U., *Applications of* n*-gons*;

Thomas Weigel, U. of Freiburg, *Primitive linear* p'*-groups and the distribution of* p*-singular elements*;

Kay Magaard, Wayne State U., *Fixed point ratios for exceptional groups*;

Robert Liebler, Colorado State U., *Antipodal distance transitive covers of complete bipartite graphs*;

Norbert Knarr, Tech. U. of Braunschweig, *Construction of translation planes*;

Chat Yin Ho, U. of Florida, *Involutions of a finite group and an application to collineation groups*;

William Kantor, U. of Oregon, *Orthogonal decompositions of Lie algebras*.

Ron Solomon

Difference Sets. The subject of *difference sets* is in the midst of a renaissance of unprecedented scope. Besides discovering fascinating properties of the designs and codes arising from difference sets, researchers all around the world, in ever increasing numbers, are establishing new existence criteria by extending the traditional character-theoretic and cyclotomic techniques which have long been applied to abelian groups to such great effect by Marshall Hall, Jr. among many others, and by developing entirely new techniques, some of which exploit the representation theory of nonabelian groups in an essential way. And, *mirabile dictu*, the most spectacular of these new results are *positive*—the construction of difference sets in groups where many had hitherto believed that they could not exist. These giant strides in our understanding are well illustrated by the recent work in the area of *Hadamard* difference sets—those that have parameters $(v, k, \lambda) = (4N^2, 2N^2 - N, N^2 - N)$, in which case the (± 1)-incidence matrix of the translate design is a Hadamard matrix.

Beginning with Turyn's thesis in 1965 and at a rate rapidly increasing in recent years, much progress was made on the fundamental problem of determining which groups could support such a Hadamard difference set. But until 1992 *all* Hadamard groups known had order $4N^2$, where N was of the form $2^a 3^b$, and, indeed, some researchers opined that no other orders were possible. Then in the spring of 1992 Ted Shorter, a young computer scientist in the Office of Mathematical Research of the National Security Agency succeeded in constructing a difference set in the nonabelian group $(\mathbb{Z}_5 \times \mathbb{Z}_5) \rtimes_2 \mathbb{Z}_4$ of order $100 = 4 \cdot 5^2$. Shorter carried out an attack which had been outlined by Ken Smith, who, himself, had found all possible F_{20} homomorphic images of such a putative difference set and had proposed searching for the four $\mathbb{Z}_5 \times \mathbb{Z}_5$ coset pieces by taking into account all these homomorphism constraints. This result was all the more exciting because Bob McFarland had shown that no Hadamard difference set could exist in any *abelian* group of order $4p^2$, for $p > 3$ a prime. The order barrier having been breached, it was shortly demolished by Ming-yuan Xia of China who constructed Hadamard difference sets with $N = p^2$, for *all* primes $p \equiv 3 \pmod 4$.

Thus it was in such a propitious atmosphere that researchers gathered in Columbus during the period 17–19 May 1993 to take stock of these startling new developments and to discuss promising new directions for future research. The speakers and their topics were the following:

James A. Davis, U. Richmond, *Nonexistence of abelian Menon difference sets using perfect binary arrays*;

Xiaohong Wu, Ohio State U., *Construction of difference sets*;

Shuhong Gao, U. Waterloo (Canada), *On nonabelian difference sets*;

Richard J. Turyn, Newton, Massachusetts, *Backtrack with lookahead*;

D. B. Meisner, London (England), *Menon difference sets*;

John F. Dillon, National Security Agency, *An update on Hadamard difference sets of both kinds*;

K. T. Arasu, Wright State U., *Updating Lander's table*;

Joel E. Iiams, Colorado State U., *Hadamard difference sets in a group G of order* $4p^2$, *where G has the Frobenius group of order* $4p$ *as homomorphic image*;

Ming-yuan Xia, Huazhong Normal U. (People's Republic of China), *Williamson matrices and difference sets*;

Stefan Loewe, TU Braunschweig (Germany), *Multipliers of partial addition sets*;

Alexander Pott, U. Augsburg (Germany), *Quasiregular collineation groups of projective planes*;

Bernhard Schmidt, U. Augsburg (Germany), *Nonexistence of some difference sets*;

Vladimir D. Tonchev, Michigan Technological U., *Designs with the symmetric difference property and their groups*;

A. R. Calderbank, AT&T Bell Laboratories, *The linearity of some notorious families of nonlinear binary codes*;

Kenneth W. Smith, Central Michigan U., *Difference sets in 2-groups of large exponent*;

Harriet Pollatsek, Mount Holyoke C., *On difference sets in groups of order* $4n$, $n = p^4$ *or* p^2q^2, p *and* q *odd primes*;

Warwick de Launey, DSTO (Australia), *Some cocyclic Hadamard matrices and their relative difference sets*;

Sonja Radas, U. Florida, $PSL(3, q)$ *as a totally irregular collineation group*;

Chat Ho, U. Florida, *Planar Singer groups*;

Qing Xiang, Ohio State U., *Some number theoretic results on multipliers*;

W. K. Chan, Ohio State U., *Nonexistence results on Menon difference sets.*

In addition to papers presented at the workshop, these Proceedings also include two excellent survey articles—one by Jim Davis and Jonathan Jedwab on Hadamard difference sets, and the other by Alex Pott on *relative difference sets*, a construct which is extremely useful in the construction and analysis of difference sets, but which is also of great interest and a source of wonderful problems and beautiful results in its own right.

Not all talks are included in these Proceedings. For example, at the time of the difference set workshop Rob Calderbank was visiting at Ohio State to discuss the then recently discovered phenomenon of the $\mathbb{Z}_4$-linearity of certain well-known nonlinear binary codes. Since the most famous class of such codes—the *Kerdock* codes—correspond to difference sets in the the elementary abelian 2-group $\mathbb{Z}_2^{2m}$, this topic fit in perfectly with the theme of the workshop which Rob therefore graced with an exposition. His paper, jointly authored with Hammons, Kumar, Sloane and Solé, has recently appeared in the *IEEE Transactions on Information Theory*.

The papers included in these Proceedings are split evenly between abelian and nonabelian groups, with the two survey papers treating both. The paper by Arasu, Davis, Jedwab, Ma and McFarland lowers Turyn's exponent bound for certain parameters (v, k, λ) including $(96, 20, 4)$ and thus completes the classification of abelian groups of order 96 which can support such a difference set. Xia's paper outlines his dramatic breakthrough on the Hadamard group order problem; and Xiang studies abelian groups which can support Paley-type *partial difference sets*, another variation on the difference set theme. Some of the latest ideas on exploiting the representation theory of nonabelian groups in the study of difference sets are presented in the paper of Iiams, Liebler and Smith and the paper of Harriet Pollatsek; in particular, much of the spirit of the Smith–Shorter construction is captured in these papers. Finally, Meisner extends to nonabelian semi-direct products an important composition theorem of Turyn.

The difference set period of concentration got off to a wonderful start on Sunday evening when all visitors were treated to an exquisite reception at the beautiful home of Dijen Ray-Chaudhuri and his wife; we are most grateful for their generous hospitality. The last night saw the difference set contingent enjoying a veritable symposium at the local brew-pub where all participating graduate students were guests of honor. The pleasant surroundings of the Ohio State campus and the well-planned accommodations made possible easy interaction among all participants. We thank those who prepared papers for this volume and the referees who worked so hard to make it a very special one. We are indebted to Mrs. Terry England, who TEXed so beautifully most of the contributions. Qing Xiang also helped with the typing and cheerfully made himself available for the local transportation of visitors. But, most of all, we thank all who attended — that geographically diverse yet intellectually focused cadre whose presence and spirited participation so well exemplified the present vitality of this subject of difference sets.

K. T. Arasu, John F. Dillon, Surinder Sehgal

The Monster. "When Ernest Rutherford dismissed nuclear energy as *moonshine* in 1933, Leo Szilard took it as a personal challenge. Nine years later, under a Chicago grandstand, Enrico Fermi demonstrated the first self-sustaining pile." (quoted from an article written by Albert Wattenberg, Physics Today, January 1993).

On the first day of the conference I asked John Conway and Simon Norton if they had in mind the *moonshine* of Rutherford when they titled their epoch-making paper, written in 1979, '*Monstrous Moonshine*'. Conway said that wasn't the reason. The words *monster* and *moonshine* are now deeply embedded in the mathematical psyche. Our *monster* is beautiful, awesome, sometime even fearful.

In one of M. Koike's survey articles written in Japanese, he writes: "In 1979, Conway and Norton published a paper with a strange title, *Monstrous Moonshine*, in *Bull. London Math. Soc. Journal*. It is a poetic title. A reason why they called in this way the phonomena they discovered may be - it can not be phrased using the present mathematical language, but it is a solid mathematical fact, and some facts are hidden now before it all becomes clear (*sunshine*) -".

The *moonshine* has not yet turned into *sunshine*, far from it actually. Our progress, however, is slow and steady. The year 1993, when the conference was held, is the 20th year since the Monster first appeared in the world. The Monster is now an adult at least physically if not mentally. The account of its discovery, Griess' construction of it, the

McKay–Thompson observation-conjecture, the construction of the *moonshine* module, etc. are presented very well in a book written by Frenkel, Lepowsky and Meurman [Vertex Operator Algebras and the Monster, Academic Press, Inc. (1988)].

The conference was stimulating for all participants. What follows in **Part 3** is the collection of papers submitted by the speakers of the conference. Most of them are original research works, a few, however, are expository. The speakers and their topics were the following:

George Glauberman, Univ. of Chicago, *Y-diagram generators for the twisted* $E_6 = \{^2E_6(2)\}$;

Noriko Yui, Queen's Univ., *Singular values of the Thompson series*;

John Mckay, Concordia Univ., *A Hauptmodul for all seasons*;

Geoffrey Mason, Univ. of Calif., Santa Cruz, *Modular invariance and the boson-fermion correspondence*;

John H. Conway, Princeton Univ., Colloquium Talk, *Understanding* $\Gamma_0(N)$ *and similar groups*;

Shogo Aoyama, Leuven, Belgium, *The Virasoro invariant anti-bracket formalism in the string theory*;

Michael P. Tuite, Univ. College Galway and Dublin Inst. for Advanced Studies, *Monstrous moonshine and the uniqueness of the Moonshine module*;

Paul S. Montague, Univ. of Cambridge, *A third order twisted construction of the Monster conformal field theory*;

Hiromichi Yamada, Hitotsubashi Univ. Japan, *A generalization of the Kac–Moody algebras*;

Simon P. Norton, Univ. of Cambridge, *Non-monstrous moonshine*;

John H. Conway, Princeton Univ., *The 'square root of the Monster construction'*;

Masahiko Miyamoto, Ehime Univ. Japan, *Deep hole isotropic elements and* 21*-node systems on the Monster module*;

Robert L. Griess, Jr., Univ. of Michigan, *Codes, loops, and p-locals*;

Charles R. Ferenbaugh, Yale Univ., *Lattices and generalized Hecke operators*;

Chongying Dong, Univ. of Calif., Santa Cruz, *Representations of vertex operator algebras*;

Yves Martin, Univ. of Calif., Santa Cruz, *On multiplicative eta-quotients*;

Alex Ryba, Marquette Univ., *A natural invariant algebra for the Harada-Norton group*.

Koichiro Harada

Contents

Part 3 The Monster

PART I

GROUPS AND GEOMETRY

On flag-transitive $c.c^*$-geometries

Barbara Baumeister

Introduction

In this paper we continue the classification of the groups G, which act flag-transitively on a geometry Γ belonging to

$$\begin{array}{ccccc} 1 & \subset & 2 & \supset & 3 \\ \circ & \text{———} & \circ & \text{———} & \circ \\ 1 & & n-2 & & 1 \end{array}.$$

In the diagram the integer below a node of type i is one less than the number of maximal flags, which contain a fixed flag of cotype $\{i\}$, $i \in \{1, 2, 3\}$. A geometry with the diagram $\overset{1}{\circ}\overset{\subset}{\text{—}}\overset{2}{\circ}\overset{\supset}{\text{—}}\overset{3}{\circ}$ is also called a $c.c^*$-geometry and we call the objects of type 1 points, those of type 2 lines and those of type 3 circles.

For notation and definitions concerning geometries see **[Bue1]**. For the convenience of the reader we recall the definition of a $c.c^*$-geometry.

A geometry Γ consisting of points, lines and circles belongs to $\begin{array}{ccccc} 1 & \subset & 2 & \supset & 3 \\ \circ & \text{—} & \circ & \text{—} & \circ \\ 1 & & n-2 & & 1 \end{array}$ if

(1) for every point P, the residue Γ_P of P is the complete graph K_n on n vertices, where the circles and the lines in Γ_P are the vertices and the edges respectively;
(2) for every line L the residue Γ_L of L is a generalized 2-gon consisting of two points and two circles;
(3) for every circle C, the residue Γ_C of C is the complete graph K_n, where the points and the lines in Γ_C are the vertices and the edges respectively.

Furthermore for X an element of Γ, we denote by G_X the stabilizer of X in G and by K_X the kernel of the action of G_X on the residue Γ_X of X in Γ.

In **[Ba2]** we gave two examples with $n = 15$ and $G_P \cong A_7$, which admit $2M_{22}$ and M_{22} as flag-transitive automorphism group, respectively. Moreover we determined all flag-transitive $c.c^*$-geometries with $n = 15$.

It is known that for each point P the group G_P is a doubly-transitive permutation group of degree n **[Ba2]**, **[GM]**. On the other hand for each doubly-transitive permutation group L of degree n there exists a $c.c^*$-geometry, the two-coloured hypercube $H(n)$, with automorphism group G, such that the stabilizer of a point is isomorphic to L, see for instance **[Wi]**, **[Ba2]**.

Now assume that G_P has no regular normal subgroup. We are going to show that Γ is covered by the two-coloured hypercube or $G_P \cong A_7$ or G_P is a group of Lie-type of rank 1 (Theorem A). Furthermore we determine all flag-transitive $c.c^*$-geometries with $n \le 20$ (Theorem B).

Finally we give all known examples (Section 2). There the geometries appearing in Theorem A or B are described in more detail.

Grams and Meixner **[GM]** independently studied some of the geometries assuming $n \leq 12$.

Theorem A. *Suppose that G acts flag-transitively on a $c.c^*$-geometry Γ, and that G_P/K_P has no normal elementary abelian subgroup. Then G_P and G_C are isomorphic. If furthermore G_P/K_P is not isomorphic to a group of Lie-type of rank 1, then one of the following holds:*

(1) *G is isomorphic to a factor group of $2^{n-1}\colon G_P$, where G_P is a doubly-transitive permutation group of degree n and the universal 2-cover of Γ is the two-coloured hypercube $H(n)$.*

(2) *$G \cong 2M_{22}$ or M_{22}, $G_P \cong G_C \cong A_7$ and $G_L \cong S_4 \times Z_2$.*

Theorem B. *Suppose that G acts flag-transitively on a $c.c^*$-geometry Γ, that G_P/K_P has no normal elementary abelian subgroup and that each point is incident to n circles, $n \leq 20$. Then G is isomorphic to a factor group of $\widetilde{G}$, where $\widetilde{G}$ is one of the following.*

(1) *$\widetilde{G} \cong 2^{n-1}\colon G_P$, and G_P is a doubly-transitive permutation group of degree n.*

(2) *$\widetilde{G} \cong M_{12}$, $G_P \cong L_2(11)$ and $G_L \cong D_{12} \times Z_2$, $n = 11$.*

(3) *$L_2(q) \leq G_P \leq \mathrm{Aut}(L_2(q))$ and $n = q + 1$.*
   (i) *$q = 4$, $\widetilde{G} \cong L_2(11)$, $G_P \cong A_5$ and $G_L \cong D_{12}$.*
   (ii) *$q = 5$, $\widetilde{G} \cong 3A_6$ or $3S_6$ and $G_L \cong D_8$ or D_{16} respectively.*
   (iii) *$q = 9$, $2L_3(4) \leq \widetilde{G} \leq 2L_3(4)\langle f, g\rangle$, f a field and g a graph automorphism and G_L an extension of $D_8 * Z_4$.*
   (iv) *$q = 11$, $\widetilde{G} \cong M_{12}$ or $\mathrm{Aut}(M_{12})$ and $G_L \cong D_{20}$ or $D_{20}2$.*

(4) *$\widetilde{G} \cong U_3(3)$, $G_P \cong L_3(2)$ and $G_L \cong (Z_4 \times Z_2)\colon Z_2$, $n = 7$.*

(5) *$\widetilde{G} \cong 2M_{22}$, $G_P \cong A_7$ and $G_L \cong S_4 \times Z_2$, $n = 15$.*

In particular the examples (26) *and* (32) *from* **[Bue4]**, *which are listed in* (2) *and in* (3)(iii), *are simply connected.*

Some words about the proof of Theorems A and B. We use the method of generators and relations, see for instance **[Yo]**. By **[As]** we can identify Γ with the group geometry $\Gamma(G, (G_P, G_L, G_C))$ for $\{P, L, C\}$ a flag of Γ. Now the strategy is to determine the amalgam of G_P, G_L and G_C and its completion $\widetilde{G}$. Then we obtain G as a factor group of $\widetilde{G}$.

We show, that if K and L are two doubly-transitive permutation groups of the same order and the same degree and if K is almost simple, then $\mathrm{Soc}(K) \cong \mathrm{Soc}(L)$ holds (Section 3). Supposing that G_P/K_P is almost simple we derive from this $G_P \cong G_C$ as permutation groups on the circles in Γ_P or the points in Γ_C, respectively. Looking at the generators and relations of the two-coloured hypercube, **[Ba2]**, we give a sufficient condition on G_P that Γ is covered by the hypercube. Using this condition we obtain Theorem A.

Now suppose $n \leq 20$ (Section 5). By our condition on G_P we only need to consider the doubly-transitive groups G_P, whose stabilizer of 2 points has a nontrivial center, and the two exceptional cases $G_P \cong L_2(11)$, A_7 of degree $11, 15$, respectively. Using coset enumeration we obtain $|\widetilde{G}|$. We complete the determination of $\widetilde{G}$ by examining the examples from section 2. The enumeration was done with the algebra system CAYLEY.

We exclude the case G_P/K_P being an affine group, since for some of them, e.g. the Frobenius groups, it is not clear how to glue G_P and G_C together. Moreover most of them fail the sufficient condition on G_P to be covered by the hypercube, see also **[Ba1]**. For the moment a classification of these amalgams seems to be out of range.

Notation. We write G^* for $G \setminus \{1\}$, G a group.

Examples

In this section we give examples of groups G acting flag-transitively on $c.c^*$-geometries Γ. Only the examples in (6) do not appear in the statement of Theorem B. There G_P/K_P has a normal elementary abelian subgroup.

Remark. Let N be a normal subgroup of G, which acts semiregularly on the points, lines and circles of Γ. Moreover suppose for $n \in N^*$ and for X an element of Γ that the residues Γ_X and Γ_{X^n} have an empty intersection. Then, as usual for group geometries, we get a new $c.c^*$-geometry identifying, respectively, points, lines and circles iff they are in the same orbit of N. The obtained quotient is covered by Γ and G/N acts flag-transitively on it.

(1) Semibiplanes.

Each semibiplane induces a $c.c^*$-geometry Γ. A semibiplane S is a rank 2 geometry satisfying:

(i) any two points are incident with 0 or 2 common blocks;

(ii) any two blocks are incident with 0 or 2 common points (see for example **[Wi]**).

As points and circles of Γ we take, respectively, the points and blocks of S and as lines the quadruples (P_1, P_2, B_1, B_2), where the two different points P_1, P_2 are incident with the two different blocks (circles) B_1, B_2.

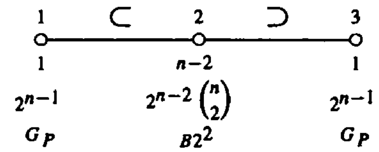
If G is a flag-transitive automorphism group of S, such that the stabilizer G_P of a point P acts transitively on the points of S at distance 1 from P, then G acts flag-transitively on Γ too.

Conversely if Γ is a $c.c^*$-geometry for which the Intersection Property **[Bue2]** holds, then the truncation of Γ to points and circles (blocks) is a semibiplane.

All of the following examples are semibiplanes except for the example admitting S_6 as automorphism group and except for some nontrivial quotients of the two-coloured hypercube and also of the third example of (6).

(2) The two-coloured hypercube.

$2^{n-1}G_P$



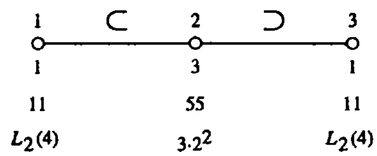
G_P an arbitrary doubly-transitive permutation group on $\Omega = \{1, \ldots, n\}$ and $B = \mathrm{stab}_{G_P}(\{1\}, \{2\})$. The point-circle incidence graph of this geometry Γ is an n-dimensional cube. Hence the points and the circles are the vertices and the lines are the rectangles of the n-dimensional cube. This geometry appears in **[Wi]** and we can construct Γ also in the following way:

Take an n-dimensional $GF(2)$-vector space $V = \langle e_1, \ldots, e_n \rangle$ and let G_P act on V by permuting the index of the basis $\{e_1, \ldots, e_n\}$. We can identify the n-dimensional cube with V, so that the points are the elements in $U := \langle e_1 + e_i, i \in \Omega \rangle$ and the circles are those in $V \setminus U$. Then $G = U{:}G_P \cong 2^{n-1}G_P$ acts flag-transitively on Γ, see also **[Ba2]**.

The geometry Γ is simply connected and G is the completion of the amalgam of G_P, G_L and G_C, see for instance **[Ba2]**.

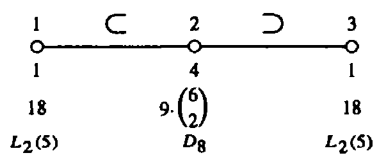
(3) Examples with $E(G_P) \cong L_2(q)$.

$L_2(11)$



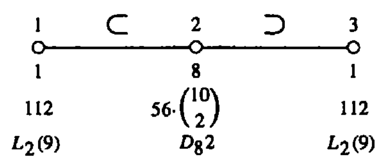
In this case G_P and G_C are not conjugated in G and any two points as well as any two circles are at distance at most 2. This geometry can be found in **[Bue3]**.

$3A_6$



The stabilizer G_P and G_C are not conjugated in G and we have $G_L = \langle N_{G_P}(i), N_{G_C}(i) \rangle$, i an arbitrary involution of $G_P \cap G_C \cong D_{10}$. In the quotient each point is incident to each circle. These geometries are due to **[JvT]**. By Theorem B we obtain $\mathrm{Aut}(\Gamma) \cong 3S_6$.

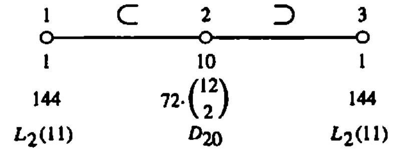
$2L_3(4)$



The quotient $\Gamma(L_3(4), (L_2(9), D_{16}, L_2(9)))$ can be constructed in the Steiner-system $\mathcal{S} = S(3, 6, 22)$ on the set $\Delta = \{\alpha_1, \ldots, \alpha_{22}\}$. The points and the circles are the hexads of $\mathcal{S}$, which do not contain α_1. A point P is incident to a circle C iff their intersection is empty. Then two different points P_1 and P_2 are simultaneously incident with 0 circles iff $| P_1 \cap P_2 |= 0$ and with 2 circles iff $| P_1 \cap P_2 |= 2$.

Let f be a field and g a graph automorphism of $L_3(4)$. By Theorem B the full automorphism group of Γ is $2L_3(4)\langle f, g\rangle$.

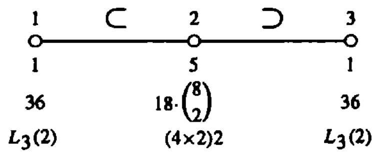
M_{12}



This geometry was found by **[Leo]** and a construction is given in **[BCN**, p. 371**]**. Take the Steiner-system $\mathcal{S} = S(5, 8, 24)$ and two complementary dodecads D_1 and D_2. Then $\mathrm{stab}_{M_{24}}(D_1) \cong M_{12}$. Define a graph Δ with vertex set $D_1 \times D_2$, where two pairs (d_1, d_2), (e_1, e_2) are nonadjacent either if $d_1 = e_1$ or $d_2 = e_2$ or if there is an octad B in $\mathcal{S}$ with $B \cap D_1 = \{d_1, e_1\}$ and $\{d_2, e_2\} \subset B \cap D_2$. Then Δ has exactly 144 12-cliques. The points are the vertices of Δ and the circles the 12-cliques. Thus the stabilizer of a point is contained in a maximal subgroup of G which is isomorphic to M_{11} and the stabilizer of a circle is a maximal subgroup in M_{12}. In fact, $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(M_{12})$.

(4) Example with $G_P \cong L_n(q)$, $n > 2$.

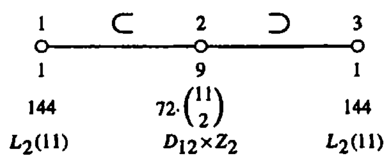
$U_3(3)$



The group $G \cong U_3(3)$ has a rank 4 representation on 36 points over $H \cong L_3(2)$ with orbitals of lenghts $1, 21, 7, 7$. Define a graph Δ, whose vertices are the conjugates of H in G and where two vertices are adjacent iff they intersect in a subgroup isomorphic to D_8. Then G has two orbits of 7-cliques, each of length 36. The group $\mathrm{Aut}(G)$, also acting on Δ, interchanges these two orbits. The points of Γ are the vertices and the circles are the 7-cliques in one of these two orbits. This example is due to **[Neu]**, see also **[Ch]**.

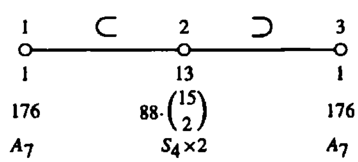
(5) Examples with exceptional doubly-transitive action of G_P.

M_{12}



In this geometry, which was found by Buekenhout **[Bue3]**, the stabilizer of a point and the stabilizer of a circle are conjugated maximal subgroups in M_{12}.

$2M_{22}$

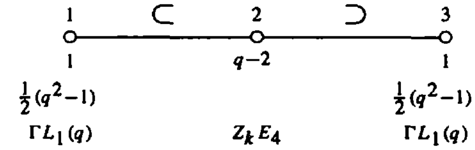


The quotient with flag-transitive automorphism group M_{22} can be constructed in the Steiner-system $S(5, 8, 24)$ on the set $\Omega = \{\alpha_1, \ldots, \alpha_{24}\}$, **[Ba2]**. The points are the octads which contain α_1, but not α_{24} and the circles are the octads, which contain α_{24}

but not α_1. The lines are two-coloured sextets $\{L_1, L_2, L_3\}$ $\{L_4, L_5, L_6\}$, such that $\alpha_1 \in L_1$ and $\alpha_{24} \in L_6$. A point P is incident to a circle C iff their intersection is empty. Moreover the geometry $\widetilde{\Gamma} = \Gamma(L_3(4), (L_2(9), D_{16}, L_2(9)))$ can be found in Γ. The points and the circles of $\widetilde{\Gamma}$ are the points of Γ, which contain also α_2.

(6) Examples with affine G_P.

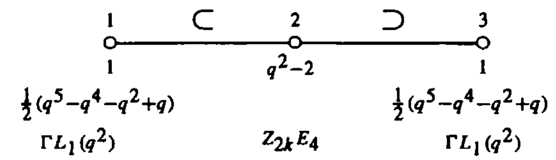
$\Gamma L_2(q)/Z(SL_2(q))$



, $q = p^k$ odd.

Take the projective plane $\mathcal{P} = PG(2, q)$ and an homology α of $\mathcal{P}$ of order 2. Then the points of Γ are the unordered pairs (P, P^α), P a point of $\mathcal{P}$, which is not fixed by α, and the circles are the unordered pairs (l, l^α), l a line of $\mathcal{P}$, which is not fixed by α. A point (P, P^α) is incident to a circle (l, l^α) iff P is on l or l^α. This geometry was discovered by Hughes **[Hug]** and Γ admits the quotients G/Z, $Z(GL_2(q)) \geq Z \geq Z(SL_2(q))$. Also $H \leq G$, $H \cong GL_2(q)/Z(SL_2(q))$, acts flag-transitively on Γ where $H \cap G_P \cong \mathrm{Frob}(q(q-1))$.

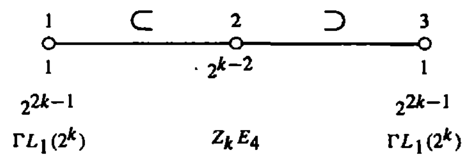
$\Gamma L_3(q)$



, $q = p^k$ odd.

This geometry can be found in **[Ba1]** and the quotient $G/Z(GL_3(q))$ is described in **[Hug]**: Take the projective plane $\mathcal{P} = PG(2, q^2)$ and a Baer involution of $\mathcal{P}$. Then the points, the circles and the incidence of the quotient are defined as in the previous example. Also $H \leq G$, $H \cong GL_3(q)$ acts flag-transitively on Γ.

$2^{(k-1)+2k} Z_{(2^k-1)} Z_k$



This is the third example of **[Hug]**. Take the projective plane $\mathcal{P} = PG(2, q)$, $q = 2^k$ and an elation of $\mathcal{P}$. Then the points, the circles and the incidence are defined as in the previous examples.

On doubly-transitive permutation groups

In this section we prove special facts about doubly-transitive permutation groups, that will be needed in the proof of Theorem A. Let G be a doubly-transitive permutation group on a set Ω. Denote by G_0 and G_{00} the stabilizer of one point and of two points of Ω, respectively. By **[Ca]** either $G \cong p^m G_0$, $p^m = |\Omega|$, or G is an almost simple group listed in **[Ca, p. 8]**. Since we are only interested in the latter, we suppose in this section $E(G) \neq 1$. For a doubly-transitive permutation group, the subgroup $E(G)$ of G is the nonabelian socle of G.

About $N_G(G_{00})$.

Lemma 3.1. *If G is not isomorphic to $L_d(2), d > 2$, $L_2(8){:}3$ or A_7 of degree $2^d - 1, 28, 15$ respectively, then $N_G(G_{00})/G_{00} \cong Z_2$.*

Proof. As $N_G(G_{00})$ acts doubly-transitive on the fixed points of G_{00} in Ω, **[HuI**, II. 1.13], we have $N_G(G_{00})/G_{00} \cong Z_2$, G_{00} fixes exactly two points of Ω and this holds $N_{G_0}(G_{00}) = G_{00}$. By **[Ca]** $E(G)$ operates doubly-transitive on Ω, too. If $N_{G_0}(G_{00}) > G_{00}$, then G_{00} and $E(G)_{00}$ fixes more than two points, hence in this case $N_{E(G)_0}(E(G)_{00}) > E(G)_{00}$. Thus we may assume $E(G) = G$.

We inspect the list of **[Ca]**. If G acts 3-transitively on Ω or if G_0 is a Frobenius group, then the assertion follows. Hence the Lemma is proved for $G \cong A_n$, $U_3(q)$, $Sz(q)$, $R(q)$, M_{11} of degree 11 or 12, M_{12}, M_{22}, M_{23} and M_{24}.

If $G \cong L_k(r)$, $r > 2$, then it is easy to see, that G_{00} fixes exactly two points and if $G \cong Sp_{2d}(2)$, $d > 2$, of degree $2^{2d-1} + 2^{d-1}$ or $2^{2d-1} - 2^{d-1}$, then $G_0 \cong O^+_{2d}(2), O^-_{2d}(2)$ and $G_{00} \cong 2^{2(d-1)}O^+_{2(d-1)}(2)$, $2^{2(d-1)}O^-_{2(d-1)}(2)$ respectively. Hence in this case G_{00} is a maximal parabolic subgroup in G_0.

The assertion holds also for the remaining groups, since if $G \cong L_2(11), HS$ or C_3 of degree $11, 176, 276$, then $G_{00} \cong S_3, A_6 2, U_4(3)2$ is a maximal subgroup of $G_0 \cong A_5, U_3(5){:}2, \mathrm{Aut}(McL)$ respectively.

Extensions of automorphisms of G_0.

Lemma 3.2. *If G is not isomorphic to $L_2(11)$, M_{11}, M_{22}, M_{23}, $L_3(4)$ of degree 11, 11, 22, 23, 21 respectively, then each automorphism of G_0, which leaves G_{00} invariant, can be extended to an automorphism of G.*

Proof. If $E(G) \cong Sp_{2d}(2), A_n, M_{12}, M_{24}, C_3, A_7, M_{11}, HS$ of degree $2^{2d-1} + 2^{d-1}$ or $2^{2d-1} - 2^{d-1}, n, 12, 24, 276, 15, 12, 176$ respectively, then $[G_{00}, \alpha] \leq G_{00}$ yields $\alpha \in \mathrm{Inn}(G_0)$. Hence we may assume $E(G) \cong L_d(q), Sz(q), R(q), U_3(q)$ or $L_2(8) \cong R(3)'$.

Now suppose there exists an automorphism $\alpha \in \mathrm{Aut}(G_0)$, which can not be extended to an automorphism of G. Without loss of generality we may assume $o(\alpha) = r^s$, r a prime.

In the proof we distinguish three cases $E(G) \cong L_d(q)$, $E(G) \cong Sz(q), R(q)$, $U_3(q)$ and $E(G) \cong L_2(8)$. Set $q = p^n$, p a prime.

Case 1. $E(G) \cong L_d(q)$ of degree $(q^d - 1)/(q - 1)$, $(d, q) \neq (3, 4)$.
We have $G = E(G).F$, where F are diagonal and field automorphism, and $G_0 = Q{:}(H{:}D).F$, $Q = O_p(G_0) \cong E_{q^{d-1}}$ is a natural module for $H \cong SL_{d-1}(q)$ and $D \cong Z_{(q-1)/(d,q-1)}$ induces a diagonal automorphism of order $(q-1,d-1)$ on H. Furthermore $G_{00} = N_{G_0}(Q_1) = Q_1{:}\mathrm{stab}_H(Q_1).D.F$, $Q_1 := Q \cap G_{00} \cong E_{q^{d-2}}$. We have

(1) $C_{G\cdot A}(Q) = Q$, for $G \cdot A \leq \mathrm{Aut}(G)$ with $G \cdot A \cong P\Gamma L_d(q)$.

(2) Without loss of generality $[Q, \alpha] = 1$ and $[G_0, \alpha] \leq Q$:

Without loss of generality we have $[Q_1, A] \leq Q_1$ and by (1) there is an homomorphism ψ from $G_0 A$ into $\mathrm{Aut}(Q)$ with kernel Q. As $[Q, \alpha] \leq Q$, α induces also an automorphism $\widetilde{\alpha}$ on Q. Hence $\widetilde{\alpha} \in N_{\mathrm{Aut}(Q)}(H^{\psi})$ and $[Q_1, \widetilde{\alpha}] \leq Q_1$. Thus there exists an element $\beta \in G_{00}A$ with $[Q, \alpha\beta] = 1$ and $[Q_1, \beta] \leq Q_1$. With the Three-Subgroup Lemma we get $[G_0, \alpha\beta] \leq C_{\mathrm{Aut}(G)}(Q) = Q$ by (1) and then $[G_{00}, \alpha\beta] \leq N_{G_0}(Q_1) = G_{00}$. As $G = E(G)G_0$ due to Frattini, we also get $\beta \in \mathrm{Aut}(G)$. Hence we may suppose $[Q, \alpha] = 1$ and $[G_0, \alpha] \leq Q$.

(3) $d > 2$:
Suppose $d = 2$. Then we have $H = 1$, $G_0 = Q{:}\,D.F$, $Q{:}\,D$ a Frobenius group of order $q(q-1)/(q-1, 2)$, and $G_{00} = D.F$. As D char G_{00} we obtain $[D, \alpha] \leq Q \cap D = 1$. Thus we get $[G_0, \alpha] \leq C_{G_0}(Q{:}\,D) = C_Q(D) = 1$ in contradiction to our assumption, that α can not be extended.

(4) $o(\alpha) = p$:
Suppose $r \neq p$ and set $W := \langle\alpha\rangle \times Q$. By (2) we have $W \trianglelefteq G_0\langle\alpha\rangle$. Hence $\langle\alpha\rangle$ char W, so $[H, \alpha] \leq Q \cap \langle\alpha\rangle = 1$ in contradiction to our general assumption. Hence $r = p$. Since α induces an automorphism of order p on QH, we get $o(\alpha) = p$ with (1).

(5) $C_W(H) = 1$:
Suppose $1 \neq g \in C_W(H)$. Then $g = x\alpha^i$ for some $x \in Q, i \in \{1, \ldots, p\}$ and $[QH, x\alpha^i] = 1$. We get $[G_0, x\alpha] \leq C_{G_0}(Q{:}\,H) = 1$. Hence α^i and also α can be extended to an automorphism of G, contradiction.

(6) There is no counterexample:
By (5) H and H^{α} are not conjugated in QH, thus either $d = 3$ and $q = 2^n$, $n > 1$ or $(d, q) = (4, 2)$ **[JP]**. As $(d, q) \neq (4, 2)$ by **[Ba2]** and as we suppose $(d, q) \neq (3, 4)$, we have $D \neq 1$. From $W = [W, D] \times C_W(D) = Q \times C_W(D)$ we obtain $C_W(D) = \langle w\alpha\rangle$, $w \in Q$. Hence , as $H \leq C_{G_0}(D)$, we get $[H, w\alpha] \leq Q \cap C_{G_0}(D) = 1$. This yields the contradiction $[G_0, w\alpha] = 1$.

Case 2. $E(G) \cong Sz(q)$, $q = 2^{2m+1} > 2$, $R(q)$, $q = 3^{2m+1} > 3$, $U_3(q)$, $q > 2$, of degree $q^2 + 1, q^3 + 1, q^3 + 1$ respectively.
We have $E(G)_0 = Q{:}\,H$, $Q = O_p(G_0)$, and $E(G)_{00} = H$.

If $E(G) \cong Sz(q)$, then $|Q| = q^2$, $Z(Q) = Q' = \phi(Q) \cong E_q$, $Q/\phi(Q) \cong E_q$ and $H \cong Z_{q-1}$. H acts transitively on $\phi(Q)^*$ and also on $(Q/\phi(Q))^*$, **[Hu III, XI.** 3.1**]**.

If $E(G) \cong R(q)$ then $|Q| = q^3$, $Q' = \phi(Q) \cong E_{q^2}$, $Z(Q) \cong E_q$ and $H \cong Z_{q-1}$. H acts transitively on $(Q/\phi(Q))^*$, $(\phi(Q)/Z(Q))^*$ and on $Z(Q)^*$, **[Hu III, XI.** 13.2**]**.

If $E(G) \cong U_3(q)$, then $|Q| = q^3$, $Z(Q) = Q' = \phi(Q) \cong E_q$, $Q/\phi(Q) \cong E_{q^2}$ and $H \cong Z_{(q^2-1)/d}$, $d = (q + 1, 3)$. Furthermore $Q/\phi(Q)H$ is a Frobenius group and H acts transitively on $\phi(Q)^*$, **[Hu I,** II. 10.12**]**.

(1) Without loss of generality $G = E(G)$:
If $\alpha \in \mathrm{Aut}(G_0)$, then $[E(G)_0, \alpha] \leq E(G)_0$ and $[E(G)_{00}, \alpha] \leq E(G)_{00}$. Suppose there exists an element $\beta \in \mathrm{Aut}(E(G))$ with $[E(G)_0, \alpha\beta] = 1$. Then $[G_0, \alpha\beta] \leq C_Q(H) = 1$. Due to Frattini we get $G = E(G)G_0$ and $\beta \in \mathrm{Aut}(G)$.

(2) Without loss of generality $[Q/\phi(Q), \alpha] = 1$:
We get due to **[HuI,** II 7.3**]** $[Q/\phi(Q), \alpha\beta] = 1$ for an element $\beta \in H \cdot A$, where $\mathrm{Aut}(G) = E(G)A$. As $\alpha\beta$ normalizes G_0 and G_{00}, we may assume $[Q/\phi(Q), \alpha] = 1$.

Hence we obtain also

(3) $[H, \alpha] = 1$ and due to Burnside

(4) $o(\alpha) = p^s$, q being a power of p.

(5) If $G \cong Sz(q), U_3(q)$, then $[\phi(Q), \alpha] = 1$ and if $G \cong R(q)$, then $[Z(Q), \alpha] = 1 = [\phi(Q)/Z(Q), \alpha]$:
Suppose $G \cong Sz(q)$ or $U_3(q)$. Then $C_{\phi(Q)}(\alpha) \neq 1$ by (4). As H acts on $C_{\phi(Q)}(\alpha)$ by (3), we get $C_{\phi(Q)}(\alpha) = \phi(Q)$.

The same argument holds in the case $G \cong R(q)$.

(6) $G \cong U_3(q)$:
First suppose $G \cong Sz(q)$. As $Z(Q) = \phi(Q)$ we can define an H-module homomorphism ψ between $Q/\phi(Q)$ and $\phi(Q)$ given by $(q\phi(Q))^{\psi} = [q, \alpha]$. Since H acts transitively on $Q/\phi(Q)$, we have $C_Q(\alpha) = \phi(Q)$, thus ψ is an H-module isomorphism.

By **[Hu III**, XI. 3.1**]** there exist monomorphisms $\beta_1: (Q/\phi(Q))H \to A\Gamma L_1(q)$ and $\beta_2: \phi(Q)H \to A\Gamma L_1(q)$ with $a^{h^{\beta_1}} = \lambda a$ and $a^{h^{\beta_2}} = \lambda^{2^{m+1}+1}a$ for some $\lambda \in GF(q)^*$ and for all $a \in GF(q)$, $\langle h \rangle = H$.

Hence there exists an $A \in N_{\mathrm{Aut}(\phi(Q)^{\beta_2})}(\langle h^{\beta_2} \rangle)$, such that $h^{\beta_1 A} = h^{\beta_2}$. This is impossible, since $N_{\mathrm{Aut}(\phi(Q)^{\beta_2})}(\langle h^{\beta_2} \rangle) = H^{\beta_2}\,\mathrm{Aut}(GF(q))$.

If $G \cong R(q)$, then $Z(Q)$ and $\phi(Q)/Z(Q)$ are isomorphic H-modules, in contradiction to **[Hu III**, XI. 13.2**]** and **[HuI**, II. 7.3**]**.

(7) There exists no counterexample:
By (6) we have $G \cong U_3(q)$. Again there is an H-module homomorphism ψ between Q and $\phi(Q)$ given by $q^{\psi} = [q, \alpha]$. Since $Q/\phi(Q) \cong E_{q^2}$ and $\phi(Q) \cong E_q$, we get $C_Q(\alpha)/\phi(Q) \neq 1$. As H acts on $C_Q(\alpha)$, we obtain $[Q, \alpha] = 1$. Thus $\alpha = 1$ in contradiction to our assumption.

Case 3. $E(G) \cong L_2(8)$ of degree 28.
We have $G \cong L_2(8)3$, $G_0 = \langle c, d \rangle$, $\langle d \rangle \cong Z_6$ induces the full automorphism group on $\langle c \rangle \cong Z_9$ and $G_{00} = \langle d^3 \rangle$. As $[G_{00}, \alpha] \leq G_{00}$ we get $[d^3, \alpha] = 1$ and $[\langle c \rangle, \alpha] \leq \langle c \rangle$. Thus we may suppose $[\langle c \rangle, \alpha] = 1$. Then $[\langle d \rangle, \alpha] \leq C_{\langle d \rangle}(\langle c \rangle) = 1$, the contradiction $[G_0, \alpha] = 1$ follows.

Lemma 3.3. *Let* $G \cong M_{11}, M_{22}, M_{23}, L_3(4)$ *of degree* $11, 22, 23, 21$ *respectively and let* α *be an automorphism of* G_0 *with* $[\alpha, G_{00}] \leq G_{00}$, *such that there exists an involution* $a \in N_G(G_{00}) \setminus G_{00}$ *with* $(aa^{\alpha})^2 \in \mathrm{Inn}(G_{00})$. *Then* α *can be extended to an automorphism of* G.

Proof. First we examine $E(G) \cong M_{22}$ and M_{23}.

Let H be isomorphic to M_{24} acting 5-transitively on $\Omega = \{1, \dots, 24\}$ and set $H_0 := \mathrm{stab}_H(\{1\})$, $H_{00} := \mathrm{stab}_H(\{1\}, \{2\})$, $H_{000} := \mathrm{stab}_H(\{1\}, \{2\}, \{3\})$ and $H_{(4)} := \mathrm{stab}_H(\{1\}, \dots, \{4\})$.

If $E(G) \cong M_{23}$, then we have $G = E(G)$, $H_0 \cong G$, $H_{00} \cong G_0$, $G_{00} \cong H_{000}$ and $N_H(H_{00}) \cong \mathrm{Aut}(M_{22})$. Identifying H_{000} and G_{00} there exists an element $\widetilde{a} \in N_{H_0}(G_{00}) \setminus G_{00}$ and an $\widetilde{\alpha} \in N_H(H_{00}) \cap N_H(G_{00})$, such that $[a^{-1}\widetilde{a}, G_{00}] = 1$ and $[\alpha^{-1}\widetilde{\alpha}, G_0] = 1$. Hence $\widetilde{a}$ has the orbits $\{1\}, \{2, 3\}$ on $\{1, 2, 3\}$ and $\widetilde{\alpha}$ fixes 3. Since $(aa^{\alpha})^2 \in G_{00}$, we obtain $\widetilde{\alpha} \in G_{00}$. Thus $\alpha \in \mathrm{Inn}(G_{00})$ and the assertion follows.

$E(G) \cong M_{22}$. If $G \cong \mathrm{Aut}(M_{22})$, then $G_0 \cong L_3(4)2, G_{00} \cong 2^4L_2(4)2$ and $N_{\mathrm{Aut}(G_0)}(G_{00}) = G_{00}$. Thus $\alpha \in \mathrm{Inn}(G_0)$.

If $G \cong M_{22} \cong H_{00}$, then $G_0 \cong H_{000}$ and $G_{00} \cong H_{(4)}$. Furthermore $N_{\mathrm{Aut}(G_0)}$ (G_{00}) is isomorphic to a subgroup of $N_H(H_{000})$. Identifying $H_{(4)}$ and G_{00} we get $[\alpha^{-1}\widetilde{\alpha}, G_{00}] = 1$ for an $\widetilde{\alpha} \in N_H(H_{000})$, which yields the assertion.

$E(G) \cong M_{11}$. Hence $G \cong M_{11}$. Embedding G in M_{12} we obtain in the same manner as above, that α can be extended.

$E(G) \cong L_3(4)$. Then $E(G)_0 = Q{:}H$, $Q \cong 2^4$ a natural module for $H \cong SL_2(4)$, $E(G)_{00} = (Q_1 \times Q_2)\langle e \rangle$, $Q_1 \cong Q_2 \cong 2^2$, $o(e) = 3$ and $\langle e \rangle$ acts transitively on Q_i^*, $i = 1, 2$. Furthermore $Q_1 = Q \cap E(G)_{00}$. Assume that the assertion is false. With the proof of Lemma 3.2 we get without loss of generality $[Q, \alpha] = 1 = [QH/Q, \alpha]$ and α induces an automorphism of order 2 on G_0. Let $Q_1 = \langle x_1, x_2 \rangle$, $x_2 := x_1^e$ and $Q_2 = \langle x_3, x_4 \rangle$, $x_4 := x_3^e$. Then we may assume $x_1^\alpha = x_1, x_2^\alpha = x_2$ and $x_3^\alpha = x_3x_1, x_4^\alpha = x_4x_2$. Moreover we have $[a, e] = 1$ and we may suppose $Q_1^a = Q_2$. Hence $x_1^a = x_3, x_3x_4$ or x_4. In each case we get $[Q_i, (aa^\alpha)^2] \not\leq Q_i$, $i = 1, 2$ in contradiction to $(aa^\alpha)^2 \in \mathrm{Inn}(G_{00})$.

About $Z(G_{00})$.

Lemma 3.4. *We have $Z(G_{00}) = 1$ or $E(G)$ is isomorphic to a member of $\Lambda := \{L_2(q), L_3(2), U_3(q), Sz(q), R(q), L_2(8)\}$ of degree $q+1, 7, q^3+1, q^2+1, q^3+1, 28$ respectively.*

Proof. Assume $E(G)$ is not isomorphic as a permutation group to any member of Λ. If $E(G) \cong A_n, L_2(11), A_7, M_{11}$ of degree 11 or 12, $M_{12}, M_{23}, M_{24}, HS, C_3$, then $G_{00} = E(G)_{00} \cong A_{n-2}, S_3, A_4, E_9{:}Q_8, A_5, M_{10}, L_3(4), M_{22}, \mathrm{Aut}(A_6)$, $\mathrm{Aut}(U_4(3))$ respectively. In these cases $Z(G_{00}) = 1$ holds. Suppose $E(G) \cong Sp_{2d}(2)$ of degree $2^{2d-1} + 2^{d-1}$ or $2^{2d-1} - 2^{d-1}$. Then $E(G)_{00} = G_{00} = Q{:}H$, $Q \cong E_{2^{2(d-1)}}$ and $H \cong O^+_{2(d-1)}(2)$ or $O^-_{2(d-1)}(2)$ respectively. Since Q is a natural module for H, we get $Z(G_{00}) = 1$.

Thus we only have to consider $E(G) \cong L_d(q)$, $d > 2$ and $(d, q) \neq (3, 2)$. We use the terminology introduced in Lemma 3.2 and set $Q_2 := O_p(N_H(Q_1))$. It is not difficult to see $C_{G_{00}}(Q_1 \times Q_2) = Q_1 \times Q_2$. As $C_{Q_1 \times Q_2}(N_H(Q_1)) = C_{Q_1}(N_H(Q_1)) \times C_{Q_2}(N_H(Q_1)) = 1$ we obtain again $Z(G_{00}) = 1$.

Two doubly-transitive groups of the same degree and the same order. Let H be a further doubly-transitive group of the same degree n and the same order as G. If G and H have normal p and r subgroups respectively, p, r primes, then $n = p^m = r^s$ for some $r, s \leq 1$. Thus we get $r = p$ and $\mathrm{Soc}(G) = O_p(G) \cong p^m \cong O_p(H) = \mathrm{Soc}(H)$. We want to show $\mathrm{Soc}(G) \cong \mathrm{Soc}(H)$ as permutation groups provided G is almost simple.

Lemma 3.5. *The group H is almost simple.*

Proof. Suppose $O_p(H) \neq 1$ for some prime p. Then $O_p(H) \cong p^m = n$, $m \in \mathbb{N}$. As $n \neq 28$ the group $E(G)$ acts doubly-transitively on the cosets of $E(G)_0$ by **[Ca]**, hence $|E(G){:} E(G)_0| = n$.

Due to **[Gu]** and **[Ca]** we get $E(G) \cong A_n, L_d(q), L_2(11), M_{11}, M_{23}$ of degree $n, (q^d-1)/(q-1), 11, 11, 23$ respectively. If $G \cong L_2(11), M_{11}$ or M_{23}, then it follows $H_0 \leq \Gamma L_1(p)$, thus $|G| = |H|$ does not hold.

Hence $E(G) \cong A_n$ or $L_d(q)$. As $|\Gamma L_1(p)| = p-1$ we have $m > 1$.
First suppose $E(G) \cong A_n$ and let $T \in \mathrm{Syl}_p(H)$ and $S \in \mathrm{Syl}_p(E(G))$. Then we have $|S| = p^{p^{m-1}+\dots+p+1}$. As $H \leq O_p(H)GL_m(p)$, we get $|T| \mid p^{m+m(m-1)/2}$. Since $n \geq 5$ it follows $m = 1$, a contradiction.

Thus $E(G) \cong L_d(q)$ and $n = (q^d-1)/(q-1) = p^m$. In **[Li, appendix 1]** all affine doubly-transitive permutation groups are determined. There are 4 infinite classes and some exceptional cases. If H is one of the exceptional affine groups, then we obtain $n = p^m = 11^2$, $E(G) \cong L_5(3)$ and $H_0 \leq SL_2(3)2 \times Z_5$ or $SL_2(5) \times 5$. But this yields a contradiction to $|G| = |H|$. Thus H belongs to an infinite class. So $H_0 \leq \Gamma L_1(p^m)$ or H_0 contains a normal subgroup isomorphic to $SL_a(t), Sp_{2a}(t), G_2(t)'$, $n = t^a$ or t^{2a} respectively.

Let $S \in \mathrm{Syl}_p(G)$ and suppose $q = r^s$, r a prime. As $r^{sd}-1 \neq 63$, due to Zsigmondy **[Zsig]** p does not divide $r^i - 1$ for $i < sd$. Then $(p, s) = 1$ and $|S| = p^m$. If H_0 is not contained in $\Gamma L_1(p^m)$, then $|T| > p^m$ for $T \in \mathrm{Syl}_p(H)$, which contradicts to our assumption. So $H_0 \leq \Gamma L_1(p^m)$. Again we obtain a contradiction to $|G| = |H|$.

Lemma 3.6. *Suppose $E(G) \cong L_d(q)$ and $E(H) \cong L_k(s)$. Then $(q^d-1)/(q-1) = (s^k-1)/(s-1)$ and $|G| = |H|$ iff $q = s$ and $d = k$.*

Proof. Artin showed, if $|E(G)| = |E(H)|$, then $E(G) \cong E(H)$ **[Art]**. We will use his argumentation modified for our problem. We have

$$M := |G| = \frac{m}{e} q^{\frac{d(d-1)}{2}}(q^2-1)\dots(q^d-1) =$$

$$|H| = \frac{m_1}{e_1} s^{\frac{k(k-1)}{2}}(s^2-1)\dots(s^k-1),$$

where $q = p^r$, $m \mid r$, $e \mid (d, q-1)$, $s = p_1^{r_1}$, $m_1 \mid r_1$, $e_1 \mid (k, s-1)$. Furthermore

$$M < s^{k^2},$$

as

$$M \leq s^{k(k-1)/2+(k+1)k/2-1}(1-\frac{1}{s^2}) \cdot \dots \cdot (1-\frac{1}{s^k})m_1 \leq s^{k^2-1}m_1 < s^{k^2}.$$

If $p = p_1$, then $s = q$ and $d = k$ hold. Hence we may assume $p \neq p_1$. As $s(s^{k-2} + .. + s + 1) = q(q^{d-2} + \dots + q + 1)$, we get

$$s \mid q^{d-2} + \dots + q + 1.$$

If $d = 2$, then $q + 1 = s^{k-1} + \dots + s + 1$, hence $q = s$ and $d = k$.

Moreover we may suppose $d = 3$ or $d = 4$:
Assume $d, k > 4$. We may choose notations, such that $p_1 \neq 2$, and let f be the smallest positive number, such that $p_1 \mid (q^f - 1)$. As $s^{k(k-1)/2} \mid (q^{d-1}-1) \cdot \dots \cdot (q-1)m$ we

get

$$s^{k(k-1)/2} < (q^f - 1)_{p_1}^{\left[\frac{d-1}{f}\right]} p_1^{\left[\frac{d-1}{fp_1}\right]+\left[\frac{d-1}{fp_1^2}\right]+\cdots} m$$
$$< q^{d-1} p_1^{\frac{\left[\frac{(d-1)}{f}\right]}{p_1-1}} m \leq q^{d-1} 3^{\frac{d-1}{2}} m < q^d 3^{\frac{d}{2}},$$

see [**Art**, p. 362]. Since $k \geq 5$ we have $\frac{2k}{k-1} \leq \frac{5}{2}$. Hence, as $|E(G)| < s^{k^2}$, we obtain $|E(G)| < q^{\frac{5d}{2}} 3^{\frac{5d}{4}}$, which is impossible [**Art**, p. 362–363].

Suppose $d = 3$. Then $(q^d - 1)/(q - 1) = (s^k - 1)/(s - 1)$ yields $q \geq 11$ and $s \mid q + 1$. As $|G| = \frac{m}{e} q^3(q^3 - 1)(q^2 - 1)$ we get $s^{k(k-1)/2} \mid 4(q + 1)m$. Since $k \geq 3$, we have $2k/(k - 1) \geq 3$. Thus

$$\frac{m}{3} q^8 (1 - 1/q)^2 \leq |G| < s^{k^2} \leq 4^3 (q + 1)^3 m^3.$$

Hence

$$q^8 < 4^4 q^3 \frac{(1 + \frac{1}{q})^3}{(1 - \frac{1}{q})^2} m^2 \leq 4^4 q^5 (\frac{14}{13})^3 (\frac{13}{12})^2 \leq 4^4 q^5 \frac{14^3}{12^3}$$

and $q < 4^{\frac{4}{3}} \frac{7}{6} < \frac{7^2}{6} < 9$ in contradiction to $q \geq 11$.

A similar argumentation yields also a contradiction for $d = 4$.

Proposition 3.7. *If G and H are two doubly-transitive permutation groups of the same order and the same degree, then* $\mathrm{Soc}(G) \cong \mathrm{Soc}(H)$.

Proof. We shall look at the list of the almost simple doubly-transitive permutation groups [**Ca**, page 8], and show one by one $E(G) \cong E(H)$. Let q be a prime power p^r.

$E(G) \cong A_n$ of degree n. Then, as H is a permutation group of degree n, the group H is isomorphic to a subgroup of S_n, hence $E(G) \cong E(H)$.

$E(G) \cong L_d(q)$ of degree $(q^d - 1)/(q - 1)$. Due to [**Ca**] and Lemma 3.6 we get $E(G) \cong E(H)$ or $E(H) \cong Sp_{2k}(2)$. Hence suppose $E(H) \cong Sp_{2k}(2)$ of degree $2^{2k-1} + 2^{k-1}$ or $2^{2k-1} - 2^{k-1}$, $k \geq 3$. Then $|G| = |H| = 2^{k^2}(2^2 - 1) \cdot \ldots \cdot (2^{2k} - 1)$ and q is odd.

If $d = 2$, then 2^{k^2} divides $|\Gamma L_2(q)| = (q + 1)q(q - 1)r$, but a Sylow 2-subgroup of G has order at most $2^{k-1} 2r < 2^k q < 2^k 2^{2k} \leq 2^{k^2}$.

Thus $d > 2$ and, as G and H have the same degree, $k > 3$. Then we obtain $q^{d(d-1)/2} \leq 2^{2k} 3^{k/2}$ with the same argumentation as in 3.6 and, as $|G| < q^{d^2}$ and $2d/(d - 1) \leq 3$,

$$2^{k^2+k(k+1)} (1 - \frac{1}{2})^k < 2^{k^2+k(k+1)} \prod_{i=1}^{k} (1 - \frac{1}{2^i})(1 + \frac{1}{2^i}) = |H| < q^{d^2} < 2^{6k} 3^{3k/2}.$$

Now $2^{2k} < 2^6 3^{3/2}$, so that $2^k < 2^3 3^{3/4} \leq 8 \cdot 2 = 16$, which yields the contradiction $k = 3$.

$E(G) \cong U_3(q)$ of degree q^3+1. Then $|G| = \frac{1}{(q+1,3)^\varepsilon}(q^3+1)q^3(q^2-1)2m$, $m \mid r$ and $\varepsilon \in \{0,1\}$.

If $E(H) \cong Sz(s)$ of degree s^2+1, $s = 2^b$, $b \geq 3$, then $q^3 = s^2$ and $q = 2^{2b/3}$. As $|H|$ divides $s^2(s^2+1)(s-1)b$ we obtain

$$\frac{1}{3}(q^3+1)q^3(q^2-1) \mid s^2(s^2+1)(s-1)b,$$

hence $(2^{4b/3}-1) \mid 3b(2^b-1)$. Since $a \neq 6$, $a := 4b/3$, there exists a prime t, such that $t \mid (2^a-1)$, but $t \nmid (2^i-1)$, $i < a$ **[Zsig]**. Hence $t \equiv 1 \bmod a$ and $t \neq 3$, as $a > 2$, which is a contradiction to $t \mid 3b(2^b-1)$.

It follows $E(H) \cong E(G)$ or $E(H) \cong Sp_{2k}(2)$ due to the list of **[Ca]**.

Suppose $H = E(H) \cong Sp_{2k}(2)$ of degree $2^{2k-1}+2^{k-1}$ or $2^{2k-1}-2^{k-1}$. Then $q^3+1 = 2^{k-1}(2^k+1)$ or $2^{k-1}(2^k-1)$, so that q is odd and $q^3+1 \equiv 2^{k-1} \bmod 2^k$. Hence $q+1 \equiv 2^{k-1} \bmod 2^k$, $q-1 \equiv 2 \bmod 4$ and r is odd. As $|S| = 2^{k^2}$, $S \in \mathrm{Syl}_2(H)$, we have $2^{k^2} \mid (q^3+1)(q^2-1)2$, thus we get the false statement $2^{k^2} \mid 2^{2k}$.

If $E(G) \cong R(q)$ of degree q^3+1, $q = 3^{2a+1}$, then we only have to check the case $E(H) \cong Sp_{2k}(2)$. Since $3^{2b+1}+1 \equiv 4 \bmod 8$ for all $b \in N$, we get $k = 3$. But then G and H do not have the same degree.

For the remaining groups the assertion follows immediately.

Corollary 3.8. *If G and H are two doubly-transitive permutation groups of the same order with isomorphic point stabilizers and if G is almost simple, then G and H are isomorphic as permutation groups.*

Proof. Because of Proposition 3.7 it follows $E(G) \cong E(H)$. Without loss of generality we may assume $H \leq \mathrm{Aut}(E(G))$.

If $\mathrm{Aut}(E(G))/E(G)$ is cyclic, then we get $G = H$ because of $|G| = |H|$. Thus due to the list of **[Ca]** we only have to consider $E(G_P) \cong L_d(q)$ and $U_3(q)$ with $(d, q-1) \neq 1$ and $3|q+1$, respectively. Now, as $G_0 \cong H_0$, we obtain $G \cong H$.

An inspection of the doubly-transitive almost simple permutation groups yields, that G and H are isomorphic as permutation groups, too (see also **[Ba2, (3.4)]**).

Proof of Theorem A

From now on we suppose that the group G acts flag-transitively on a $c.c^*$-geometry Γ. Let P, L, C be a flag, P a point, L a line and C a circle. In **[Ba2]** we showed

Lemma 4.1. $K_P = K_C = 1$.

and

Lemma 4.2. *If a group G acts flag-transitively on a geometry Γ belonging to the diagram* $\underset{1}{\overset{1}{\circ}}\overset{c}{\text{——}}\underset{n-2}{\overset{2}{\circ}}\overset{c^*}{\text{——}}\underset{1}{\overset{3}{\circ}}$, *$n \geq 1$, there are pairwise distinct subgroups $G_1, G_2, G_3 \leq G$, satisfying the following conditions:*

(1) G_i *acts doubly-transitively on* $\{(G_1 \cap G_3)g, g \in G_i\}$, $i \in \{1,3\}$;
(2) $B \trianglelefteq G_2$, $G_2/B \cong E_4$, $(G_2 \cap G_i)/B \cong Z_2$ *and* $G_i = \langle a_i, G_1 \cap G_3\rangle$, $a_i \in (G_2 \cap G_i) \setminus B$, $i \in \{1,3\}$, *and* $B := G_1 \cap G_2 \cap G_3$;
(3) $(G_1 \cap G_3) \cap (G_1 \cap G_3)^{a_i} = B$;
(4) $G = \langle G_1, G_3\rangle$.

By Lemma 4.1 and 4.2 the stabilizers G_P and G_C are doubly-transitive permutation groups of the same order, which have also isomorphic point stabilizers. From now on we suppose, that G_P has no regular normal subgroup. Thus Corollary 3.8 yields

Proposition 4.3. *The groups G_P and G_C are isomorphic as permutation groups.*

We can choose the isomorphism between G_P and G_C in a favourable way.

Lemma 4.4. *If $G_P \ncong L_2(11)$ of degree 11, then there exists an isomorphism $\phi: G_P \to G_C$, such that $\phi_{|G_P \cap G_C} = id$. If furthermore $G_P \ncong A_7, L_2(8){:}3$ of degree 15, 28 respectively, then also $(G_P \cap G_L)^\phi = G_C \cap G_L$ holds.*

Proof. By Propositon 4.3 and Lemma 3.1, 3.2 and 3.3 there exists an isomorphism $\phi: G_P \to G_C$ with $\phi_{|G_P \cap G_C} = \mathrm{id}$. We show $(G_P \cap G_L)^\phi = (G_C \cap G_L)$. By Lemma 3.1 we only have to consider $G_P \cong L_d(2)$, $d > 2$.

Let $G_P \cong L_d(2)$. Then $N_{G_P}(B) = BU$, $U \cong S_3$ and BU acts as S_3 on $\{Q_1^U\}$ with kernel B, $Q_1 := O_p(G_P \cap G_C) \cap B$. Let $G_P \cap G_L = \langle B, a_1\rangle$ and $G_C \cap G_L = \langle B, a_3\rangle$. Then the order of $a_3^{-1}a_1^\phi B$ is 1 or 3. We get, as $[(a_3^{-1}a_1)^{-1}(a_3^{-1}a_1^\phi), B] = 1$ and $(a_3^{-1}a_1)^2 \in B$, that $[(a_3^{-1}a_1^\phi)^2 h, B] = 1$ for some $h \in B$. Hence $(a_3^{-1}a_1^\phi)^2$ acts trivially on $\{Q_1^U\}$, which implies $(a_3^{-1}a_1^\phi)^2 \in B$ and $a_3 = a_1^\phi$.

Remark. Suppose, that Γ is simply connected. Then G is the completion of the amalgam of G_P, G_C and G_L, [**Pa**, p. 234–236]. Let $U \leq G_P$ be a point stabilizer in the doubly-transitive action of G_P and let ϕ be an isomorphism between G_P and G_C. If $G_P \ncong L_2(11)$ of degree 11, then, by Lemma 4.4, G_P and G_C are amalgamated in the following way:

$$u = u^\phi \text{ for all } u \in U.$$

But if $G_P \cong L_2(11)$ of degree 11, then G_P and G_C can be amalgamated in a twisted way:

$$u = u^{\phi\psi} \text{ for all } u \in U,$$

where ψ is an automorphism of $U^\phi \cong A_5$, which can not be extended to one of G_C. This situation happens in $\Gamma(M_{12}, (L_2(11), D_{12} \times Z_2, L_2(11)))$. In this case there exists no isomorphism between G_P and G_C, which is the identity on $G_P \cap G_C$.

If $G_P \cong A_7$ or $L_2(8){:}3$ of degree 15, 28 respectively, then $N_{G_P}(B) > G_P \cap G_L$. For the group geometry $\Gamma(M_{22}, (A_7, S_4 \times Z_2, A_7))$ there exists actually no isomorphism $\phi: G_P \to G_C$ with $\phi_{|G_P \cap G_C} = \mathrm{id}$ and $(G_P \cap G_L)^\phi = G_C \cap G_L$.

Now suppose that $G_P \not\cong L_2(11)$, A_7, $L_2(8){:}3$ of degree $11, 15, 28$ respectively and that Γ is simply connected. Then by Lemma 4.3 we can give a presentation of G, which depends on the stabilizers G_P and G_L.

Let $B = \langle X_{123} \mid R_{123} \rangle$, X_{123} being generators and R_{123} relations, $G_P \cap G_C = \langle X_{13} \mid R_{13} \rangle$, where $X_{123} \subset X_{13}$ and $R_{123} \subset R_{13}$, $G_P \cap G_L = \langle \{a_1\} \cup X_{123} \mid R_{21} \rangle$. Then we get $G_L = \langle \{a_1, a_3\} \cup X_{123} \mid R_{21} \cup R_{12}^{a_3}, (a_1^{-1}a_3)^2 = b \rangle$ for some $b \in Z(B)$ and $G_C = \langle \{a_3\} \cup X_{13} \mid R_1^{a_3} \rangle$, where $a_3 \in G_L \cap G_C$ and where $R_{i,j}^{a_3}$ is the set of relations which we obtain from the relations $R_{i,j}$ by replacing a_1 by a_3, $i, j \in \{1, 2, 3\}$. Hence

$$G = \langle \{a_1, a_3\} \cup X_3 \mid R_1 \cup R_1^{a_3}, (a_1^{-1}a_3)^2 = b \rangle .$$

The following has been proved in **[Ba2]**.

Lemma 4.5. *The geometry Γ is the two-coloured hypercube with point stabilizer G_P iff there exist $a_1 \in (G_P \cap G_L) \setminus B$ and $a_3 \in (G_C \cap G_L) \setminus B$, such that $G = \langle \{a_1, a_3\} \cup X_{13} \mid R_1 \cup R_1^{a_3}, (a_1^{-1}a_3)^2 = 1 \rangle$.*

Thus we obtain:
If $G_P \not\cong L_2(11), A_7, L_2(8){:}3$ of degree $11, 15, 28$ and if $Z(B) = 1$,
then Γ is the hypercube with point-stabilizer G_P.

Now Lemma 3.4 yields

Proposition 4.6. *If G_P has no normal elementary abelian subgroup, then Γ is covered by the two-coloured hypercube with point stabilizer G_P or $E(G_P) \cong U$, $U \in \{L_2(q), L_3(2), U_3(q), Sz(q), R(q), L_2(11), A_7, L_2(8)\}$ of degree $q+1, 7, q^3+1, q^2+1, q^3+1, 11, 15, 28$ respectively.*

Proof of Theorem A. Suppose that Γ is simply connected. By assumption G_P is neither an affine group nor isomorphic to a group of Lie-type of rank 1. As $L_3(2) \cong L_2(7)$ we have by Proposition 4.6 $G_P \cong A_7$ of degree 15 or Γ is the hypercube with point stabilizer G_P. In **[Ba2]** it was shown that, if $|G_P{:}G_P \cap G_C| = 15$, then Γ is the hypercube with point stabilizer G_P or $\Gamma \cong \Gamma(2M_{22}, (A_7, S_4 \times Z_2, A_7))$. Hence Theorem A holds.

Proof of Theorem B

Suppose Γ is simply connected and $|G_P{:}G_P \cap G_C| \leq 20$. Let $\Lambda := \{L_2(q), L_3(2), L_2(11), L_2(8)\}$ of degree $q+1, 7, 11, 28$, respectively. If $E(G_P) \notin \Lambda$, then either G is a factor group of $2^{n-1}G_P$ and Γ is covered by the two-coloured hypercube or $G \cong 2M_{22}$ or M_{22} by Proposition 4.6 and by **[Ba2]**. Hence we have only to examine $E(G_P) \in \Lambda$. We distinguish the three cases $E(G_P) \cong L_2(q), L_3(2), L_2(11)$ of degree $q+1, 7$ or 11 respectively.

Case 1. $E(G_P) \cong L_2(q)$, $4 \leq q \leq 19$, of degree $n = q+1$.
Due to **[HP]** $L_2(q)$ is generated by a_1, c, d and the following relations always hold:

$$1 = d^{(q-1)/t} = c^p = (a_1c)^3 = d^{a_1}d = a_1^2,$$

where $q = p^n$. In $\Gamma L_2(q)$ we have furthermore

$$e^2 = d, e^{a_1} = e^{-1}, 1 = f^n = [a_1, f] = [c, f], e^f = e^p,$$

e a diagonal and f a field automorphism. Adding for each q respectively the following relations we get a presentation $\langle Y_{13} \cup \{a_1\} \mid R\rangle$, $Y_{13} = \{c, d, e, f\}$, for $\Gamma L_2(q)$:

$$\begin{array}{ll}
q = 4 & 1 = (cd)^3 = (a_1dc)^5 \\
q = 5 & 1 = c^dc = (a_1dc)^5 = c^ec^2 \\
q = 7 & 1 = c^dc^3 = c^ec^5 \\
q = 8 & 1 = (c^dc)^2 = (cd)^7 = (a_1cd)^7 \\
q = 9 & 1 = [c^d, c] = (d^2c)^2 = (a_1cd)^5 = c^ec^dc^{-1} \\
q = 11 & 1 = c^dc^2 = c^ec^3 \\
q = 13 & 1 = c^dc^9 = c^ec^{11} \\
q = 16 & 1 = d^{-4}cd^3cdc^{-1} \\
q = 17 & 1 = c^dc^8 = d^{-1}c^5a_1c^{-1}a_1c^2a_1c^6a_1 = c^ec^3 \\
q = 19 & 1 = c^dc^{-4} = c^ec^{-2}.
\end{array}$$

As $G_P \leq \Gamma L_2(q)$ this provides us with a presentation $\langle X_{13} \cup \{a_1\} \mid R_1\rangle$ of G_P, $\{c, d\} \subseteq X_{13} \subseteq Y_{13}$. Then due to Lemma 4.4 G is isomorphic to $G(i)$ for some $i \in \{1, \ldots, (q-1)/t\}$,

$$G(i) := \langle X_{13} \cup \{a_1, a_3\} \mid R_1 \cup R_1^{a_3}, (a_1^{-1}a_3)^2 = d^i\rangle$$

If $i = 0$, then Lemma 4.5 yields case (1) of Theorem B.

For $i \geq 1$ application of coset enumeration yields $|G(i)| = 1$ or

$$\begin{array}{ll}
q = 4 & |G(1)| = |G(2)| = 11 \\
q = 5 & |G(1)| = 18 \\
q = 9 & |G(2)| = 112 \\
q = 11 & |G(1)| = |G(4)| = 144.
\end{array}$$

It is not difficult to see, that if $q = 4, 11$, then $G(1) \cong G(2)$ and $G(1) \cong G(4)$ respectively. Since we know flag-transitive $c.c^*$-geometries with point stabilizer isomorphic as permutation group to G_P, Γ is the universal 2-cover of one of these and (3) of Theorem B holds.

Case 2. $E(G_P) \cong L_3(2)$ of degree 7.

Because of the doubly-transitive action we have $G_P \cong L_3(2)$. Choosing

$$w_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we have the following relations for G_P

$$R_1 = \{1 = w_1^2 = w_2^2 = [w_1, w_2] = s^2 = [w_1, s] = w_2^s w_1 w_2 = r^3 = w_1^r w_2 =$$

$$w_2^r w_1 w_2 = r^s r = a_1^2 = [s, a_1] = w_1^a w_1 s = (w_2 a_1)^3 = (a_1 r)^3\}.$$

Let a_3 be chosen as in the previous case. As $(a_1 a_3)^2 \in B$ and B is elementary abelian of order 4, we get $(a_1 a_3)^4 = 1$ and $[(a_1 a_3)^2, a_1] = 1$, hence $(a_1 a_3)^2 = s^i$ for some $i \in \{0, 1\}$. Thus G is isomorphic to

$$G(i) = \langle X_{13} \cup \{a_1, a_3\} \mid R_1 \cup R_1^{a_3}, (a_1^{-1} a_3) = s^i \rangle$$

for some $i \in \{0, 1\}$.

If $i = 0$, again Lemma 4.5 yields (1) of Theorem B.

Since we have the example (4) in section 2, we get $G(1)/N \cong U_3(3)$ for a normal subgroup N of $G(1)$. By coset enumeration follows $|G(1): G_P| = 36$, so (4) holds.

Case 3. $E(G_P) \cong L_2(11)$ of degree $n = 11$.

Then the stabilizer of a circle in $E(G_P)$ is isomorphic to A_5. In $L_2(11)$ there are two conjugacy classes of subgroups isomorphic to A_5, which are interchanged by $\mathrm{Aut}(L_2(11))$. Thus $G_P = E(G_P)$. In Case 1 we got a presentation of $G_P = G(1)$ for $q = 4$. Hence replacing there a_1 by b and a_3 by a_1, we obtain the relations

$$R_1 = \{1 = b^2 = d^3 = (bd)^2 = c^2 = (cd)^3 = (bc)^3 = (bdc)^5 =$$

$$a_1^2 = (a_1 c)^3 = (a_1 d)^2 = (a_1 dc)^5 = (ba_1)^2 d^2\},$$

where $B = \langle b, d \rangle$ and $G_P \cap G_C = \langle b, d, c \rangle$.

Unfortunately for $G_P \cong L_2(11)$ of degree 11 Lemma 4.4 does not hold. As for X a point or a circle $G_X \cap G_L = N_{G_X}(B)$, where $B \cong S_3$, Lemma 3.1 and Proposition 4.3 yields that there is an automorphism $\psi: G_P \to G_C$ with $(G_P \cap G_C)^\psi = G_P \cap G_C$ and $(G_P \cap G_L)^\psi = G_C \cap G_L$. Hence we can choose a_3 as a_1^ψ. If there is an automorphism $\varphi \in \mathrm{Aut}(G_C)$ with $[G_P \cap G_C, \psi\varphi] = 1$, then, as $Z(B) = 1$, case (1) of Theorem B holds by Lemma 4.5.

Now suppose, that ψ induces an automorphism on $G_P \cap G_C$, which can not be extended to one of G_C. Without loss of generality we may assume

$$d = (123),\ b = (12)(45),\ c = (12)(34) \text{ and } \psi = (12).$$

Hence $[a_1 a_3 d, B] = 1$ holds and, as $(a_1 a_3)^2 \in B$, we obtain $(a_1 a_3 d)^2 \in Z(B)$. Thus G is isomorphic to

$$\langle a_1, a_3, b, d, c \mid R_1, 1 = a_3^2 = (a_3 c)^3 = (a_3 d)^2 = (a_3 dc)^5 = (ba_3)^2 d =$$

$$(a_1 a_3)^2 d^{-1} \rangle,$$

R_1 the relations of G_P. Due to example (5) section 2 we have $G/N \cong M_{12}$ for a normal subgroup N of G. Now by coset enumeration $|G| = |M_{12}|$, hence $N = 1$. Thus Theorem B is proved.

Remark. If $G_P \cong L_2(8)3$ of degree 28, then we do not get new interesting examples. In this case the geometry Γ is covered by the hypercube and G is a factor group of $2^{27} G_P$.

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On the 1-cohomology of the groups $SL_4(2^n)$, $SU_4(2^n)$, and $\mathrm{Spin}_7(2^n)$

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Abstract. We compute the first cohomology groups with values in the simple modules for the algebraic groups $A_3(\bar{\mathbb{F}}_2)$, $B_3(\bar{\mathbb{F}}_2)$, and related finite groups.

Introduction

In this note, we compute the cohomology groups $H^1(G, M)$, where G is either the finite group $SL_4(2^n)$, the finite group $SU_4(2^n)$, the finite group $\mathrm{Spin}_7(2^n)$, the algebraic group $A_3(\bar{\mathbb{F}}_2)$, or the algebraic group $B_3(\bar{\mathbb{F}}_2)$, and M is a simple module. The bulk of the argument for the finite groups involves the reduction of the problem to a reasonable finite number of cases where the cohomology might be nonzero. We show that the 1-cohomology groups vanish in a large number of cases by using a generalization of Alperin's induction step [1] obtained from the long exact sequence in cohomology. We then handle the remaining cases by using information about cohomology over the algebraic group; with a suitable bound on n, we may use the relationship between rational and generic cohomology, as documented by Cline, Parshall, Scott, and van der Kallen ([3]), and Andersen ([2]). In the course of many of the arguments, we need to show that certain hom groups are zero; thus we need to develop a lot of information about which simple modules appear as composition factors of certain tensor products of simple modules. An important tool in this type of analysis will be the concept of module "mass", which was first introduced in the papers of Sin ([6], [7], [8]). In the case of the B_3-type groups, we are able to take advantage of a very simple consequence of the special isogeny that exists between the algebraic groups of type B_l and C_l.

§1. Notation and preliminaries

We fix an algebraic closure F of $\mathbb{F}_2$, and regard finite extensions of $\mathbb{F}_2$ as subfields of F. For $n \in \mathbb{N}$, we denote by G the simply connected semisimple algebraic group of type A_3 or B_3 over F, and by $G(n)$ either the finite group $SL_4(2^n)$, the finite group $\mathrm{Spin}_7(2^n)$, or the finite group $SU_4(2^n)$. The latter is by definition the subgroup of $SL_4(2^{2n})$ preserving the hermitian form on $\mathbb{F}_{2^{2n}}^4$ represented in the standard basis by the identity matrix. Thus, $G(n)$ can always be regarded as the subgroup of fixed points under an appropriate endomorphism of G. Let T be a maximal torus of G, and for dominant weights $\mu \in X^+(T)$, with respect to a fixed choice of Borel subgroup containing T, let $L(\mu)$ denote the unique (up to isomorphism) simple module for G with highest weight μ. For a module M, over G or $G(n)$, we denote by M^* its dual (contragredient). We denote by M_i the i^{th} Frobenius twist of M. The set of (isomorphism classes of) simple modules for $SL_4(2^n)$ (resp. $SU_4(2^n)$, $\mathrm{Spin}_7(2^n)$) is comprised of the restriction to $SL_4(2^n)$ (resp. $SU_4(2^n)$, $\mathrm{Spin}_7(2^n)$) of the 2^n - restricted modules for G. By Steinberg's tensor product theorem, this will be exactly the restriction to $G(n)$ of the set of modules of the form $L(\mu_0) \otimes L(\mu_1)_1 \otimes \cdots \otimes L(\mu_n)_n \cong L(\mu_0) \otimes L(2\mu_1) \otimes \cdots \otimes L(2^n\mu_n) \cong L(\mu_0 + 2\mu_1 + \cdots + 2^n\mu_n)$, as $\mu_0, \ldots, \mu_n$ range over the restricted weights (i.e. those integral weights λ for which $0 \leq \langle \lambda, \alpha_i^\vee \rangle < 2$ for each simple root α_i). We note however, that $M_{i+n} \cong M_i$ if $G(n) = SL_4(2^n)$, or $\mathrm{Spin}_7(2^n)$, while $M_{i+n} \cong M^*$ if $G(n) = SU_4(2^n)$.

We label the modules corresponding to the restricted weights as follows. Let $\lambda_1, \lambda_2, \lambda_3$ denote the standard fundamental dominant weights for a root system of type A_3 (resp. B_3).

Table 1.1

symbol	A_3	B_3	dim	mass(A_3)	mass(B_3)
Θ	$L(\lambda_1)$		4	3	
Λ	$L(\lambda_2)$	$L(\lambda_1)$	6	4	3
Γ	$L(\lambda_2 + \lambda_3)$		20	7	
Ψ	$L(\lambda_1 + \lambda_3)$	$L(\lambda_2)$	14	6	5
σ		$L(\lambda_3)$	8		3
Σ	$L(\lambda_1 + \lambda_2 + \lambda_3)$	$L(\lambda_1 + \lambda_2)$	64	10	8
S		$L(\lambda_1 + \lambda_2 + \lambda_3)$	512		11

Remark. The notation of Table 1.1 has been chosen to be compatible with the restriction map from B_3 to A_3, but for didactic reasons, we will hereafter refer to the module $L(\rho) = L(\lambda_1 + \lambda_2 + \lambda_3)$ in A_3 as "S", instead of "Σ".

For convenience in notation, we shall often denote tensor products of distinct Frobenius twists by juxtaposition (and the subscript zero shall be supressed). For example, the module $\Lambda_0 \otimes \Theta_1^* \otimes \Gamma_2$, for simply connected A_3, will be denoted by $\Lambda\Theta_1^*\Gamma_2$. Because of the special isogeny which exists between the simply connected algebraic groups of type B_3 and C_3, it turns out that for $G = B_3$, we have $L(\mu) \otimes L(\lambda_3) \cong L(\mu + \lambda_3)$, for $\mu \in \{\lambda_1, \lambda_2, \lambda_1 + \lambda_2\}$. (See **[11]**.) Thus, we refer to $L(\lambda_1 + \lambda_3)$ as $\Lambda\sigma$, etc. The fact that S, the first Steinberg module, is actually a tensor product of two smaller modules will be of great help in some of the induction step arguments. Later on, we will need to use variables to represent some indeterminate restricted module isomorphism types; we will use the capital greek letters A, Π, Υ, Ξ, and Ω.

For a finite set I of natural numbers, we let $V_I = \bigotimes_{i \in I} V_i$. The collection of simple $FG(n)$-modules then consists of the set of all (isomorphism classes of) modules of the form

$$\Theta_I \otimes \Theta_J^* \otimes \Lambda_K \otimes \Psi_L \otimes \Gamma_M \otimes \Gamma_P^* \otimes S_R,$$

if $G = A_3$, or the set of modules of the form

$$\Lambda_I \otimes \Psi_J \otimes \Sigma_K \otimes \sigma_L \otimes (\Lambda\sigma)_M \otimes (\Psi\sigma)_P \otimes S_R,$$

if $G = B_3$, where $I, J, \ldots, R$ are disjoint subsets of $N = \{0, 1, \ldots, n-1\}$. It is well-known that the module S_N is projective; it is the Steinberg module for $G(n)$. The group of field automorphisms $\mathrm{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2)$ acts on the set of isomorphism classes of simple $FG(n)$-modules by acting on the set of ordered 7-tuples of disjoint subsets of N. The automorphism $\gamma \mapsto \gamma^{2^i}$ acts by adding i to each element of N and then taking the remainder modulo n, if $G(n) = SL_4(2^n)$, or $\mathrm{Spin}_7(2^n)$. If $G(n) = SU_4(2^n)$, this is followed by the transpositions (I, J) and (M, P). Thus, the main result of the paper can be stated as follows:

Theorem. A) (A_3 version) *For $n > 8$, if $I, J, \ldots, R$ is an ordered 7-tuple of disjoint subsets of $N = \{0, 1, \ldots, n-1\}$, then*

$$H^1(G(n), \Theta_I \otimes \Theta_J^* \otimes \Lambda_K \otimes \Psi_L \otimes \Gamma_M \otimes \Gamma_P^* \otimes S_R) \cong F$$

for $(I, \ldots, R)$ Galois conjugate to $(\emptyset, \emptyset, \emptyset, \{0\}, \emptyset, \emptyset, \emptyset)$, $(\emptyset, \emptyset, \{1\}, \{0\}, \emptyset, \emptyset, \emptyset)$, $(\{1\}, \emptyset, \{0\}, \emptyset, \emptyset, \emptyset, \emptyset)$ or $(\emptyset, \{1\}, \{0\}, \emptyset, \emptyset, \emptyset, \emptyset)$, and is zero otherwise.

B) (B_3 version) *For $n > 6$, if $I, J, \ldots, R$ is an ordered 7-tuple of disjoint subsets of $N = \{0, 1, \ldots, n-1\}$, then*

$$H^1(G(n), \Lambda_I \otimes \Psi_J \otimes \Sigma_K \otimes \sigma_L \otimes (\Lambda\sigma)_M \otimes (\Psi\sigma)_P \otimes S_R) \cong F$$

for $(I, \ldots, R)$ Galois conjugate to $(\{0\}, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$, $(\{1\}, \{0\}, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$, or $(\{0\}, \emptyset, \emptyset, \{1\}, \emptyset, \emptyset, \emptyset)$, and is zero otherwise.

The same results hold for G (in both cases) if $I, J, \ldots, R$ are allowed to be disjoint finite sets of nonnegative integers and conjugation is by $\mathbb{Z}$.

The result for G follows from the result for $G(n)$ because of Theorem 7.1 of **[3]**, which asserts that the restriction map

$$\mathrm{Ext}^1_G(L(\mu), L(\nu)) \longrightarrow \mathrm{Ext}^1_{FG(n)}(L(\mu), L(\nu))$$

is injective if μ and ν are 2^n-restricted, and that it is an isomorphism if n is larger than a bound which depends on μ and ν.

Most of our results will hinge on whether or not particular simple modules appear as composition factors of certain tensor products of simple modules; the main tool for this type of analysis will be the concept of module "mass", as first introduced in the papers of Sin (**[6]**, **[7]**, **[8]**).

We must first define "mass" for modules over the algebraic group. In the following lemmas, we let G be an arbitrary semisimple, simply connected, algebraic group over an algebraically closed field of characteristic p. Let T be a maximal torus of G, and fix a choice of Borel subgroup containing T. Let $\mathbb{Z}\Phi$ denote the root lattice, Δ a (fixed) base of simple roots corresponding to the choice of Borel subgroup, $\mathbb{Z}^+\Delta$ the set of nonnegative integral linear combinations of positive roots, $X(T)$ the weight lattice, $X(T)^+$ the set of dominant weights, and let $X_1(T)$ be the set of p-restricted weights

$= \{\nu \in X(T)\colon\ 0 \leq \langle \nu, \alpha_i^\vee \rangle < p \ \forall\, \alpha_i \in \Delta\}$. Let $\mathbf{E} = \mathbf{E}_{\mathbb{R}} = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Fix some $f \in \mathbf{E}^*$ such that $f(\alpha_i) > 0$ for all $\alpha_i \in \Delta$ (e.g., we may take $f = (t\rho^\vee, \cdot)$, where $\rho^\vee = 1/2 \sum_{\alpha \in \Phi^+} \alpha^\vee$, and where t is the torsion coefficient of $X(T)/\mathbb{Z}\Phi$; this will ensure that m will take values in $\mathbb{Z}^+$. Note also that this choice will make the definition of mass invariant under duality.) Define the (p-restricted) "mass" of a module, $m(V) \in \mathbb{R}$, for G-modules V as follows:

i) For $\lambda = \sum_{i=0}^r p^i \lambda_i \in X(T)^+$ (where $\lambda_i \in X_1(T)$ for all i), we let $m(\lambda) = \sum_{i=0}^r f(\lambda_i)$.

ii) Define $m(V) = \sup\{m(\lambda)\colon L(\lambda)$ is a composition factor of $V\}$. (In particular, we have $m(L(\lambda)) = m(\lambda)$.)

In the notation established above, if we define the mass function by taking $f = (2\rho^\vee, \cdot)$ for A_3 and $f = (\rho^\vee, \cdot)$ for B_3, we have the masses for the simple restricted modules for A_3 and B_3 as listed in Table 1.1.

Lemma 1.1. *Let $\lambda, \lambda' \in X(T)^+$, with $\lambda = \sum_{i=0}^r p^i \lambda_i$, $\lambda' = \sum_{i=0}^r p^i \lambda_i'$ (where $\lambda_i, \lambda_i' \in X_1(T)$ for all i). Then $m(L(\lambda) \otimes L(\lambda')) \leq m(\lambda) + m(\lambda')$ with equality if and only if $\lambda_i + \lambda_i' \in X_1(T)$ for all i, in which case $L(\lambda + \lambda')$ is the unique composition factor of $L(\lambda) \otimes L(\lambda')$ of greatest mass.*

Proof. **Case 1.** λ, λ' both p-restricted.

Suppose $V = L(\nu)$ is a composition factor of $L(\lambda) \otimes L(\lambda')$. Then $\nu \preceq \lambda + \lambda'$ in the $\mathbb{Z}^+\Delta$ (usual) partial order. If $\nu = \sum p^i \nu_i$ ($\nu_i \in X_1(T)$), then we have $m(\nu) = \sum f(\nu_i) \leq \sum p^i f(\nu_i) = f(\nu) \leq f(\lambda + \lambda') = f(\lambda) + f(\lambda') = m(\lambda) + m(\lambda')$ with equality if and only if $\nu = \nu_0 \in X_1(T)$ and $\nu = \lambda + \lambda'$. Thus $m(L(\lambda) \otimes L(\lambda')) \leq m(\lambda) + m(\lambda')$ with equality if and only if $\nu = \lambda + \lambda' \in X_1(T)$.

Case 2. $\{\lambda, \lambda'\} \not\subseteq X_1(T)$.

We induct on the quantity $m(\lambda) + m(\lambda')$. Write $\lambda = \lambda_0 + p\bar{\lambda}$, $\lambda' = \lambda_0' + p\bar{\lambda}'$. Since mass is preserved under Frobenius twisting, we may assume that $\lambda_0 + \lambda_0' \neq 0$. Also, we have $\bar{\lambda} + \bar{\lambda}' \neq 0$ by assumption. Now,

$$m(L(\lambda) \otimes L(\lambda')) = m(L(\lambda_0) \otimes L(\lambda_0') \otimes L(p\bar{\lambda}) \otimes L(p\bar{\lambda}')) = m(L(\nu) \otimes L(\nu'))$$

for some composition factors $L(\nu), L(\nu')$ of $L(\lambda_0) \otimes L(\lambda_0')$, $L(p\bar{\lambda}) \otimes L(p\bar{\lambda}')$ respectively. By induction then, $m(\nu) \leq m(\lambda_0) + m(\lambda_0')$ and $m(\nu') \leq m(p\bar{\lambda}) + m(p\bar{\lambda}')$. If equality holds in both, we would have that $\lambda_i + \lambda_i' \in X_1(T)$ for all i, that $L(\nu) = L(\lambda_0 + \lambda_0')$ and $L(\nu') = L(p\bar{\lambda} + p\bar{\lambda}')$ are the unique composition factors of greatest mass of $L(\lambda_0) \otimes L(\lambda_0')$ and $L(p\bar{\lambda}) \otimes L(p\bar{\lambda}')$, respectively, and thus that $L(\lambda + \lambda') = L(\nu) \otimes L(\nu')$ is the unique composition factor of $L(\lambda) \otimes L(\lambda')$ of greatest mass $m(\lambda + \lambda') = m(\lambda) + m(\lambda')$. Otherwise, $m(\nu) + m(\nu') < m(\lambda) + m(\lambda')$, so that the induction hypothesis could be applied to $L(\nu) \otimes L(\nu')$ conclude that $m(L(\lambda) \otimes L(\lambda')) = m(L(\nu) \otimes L(\nu')) \leq m(\nu) + m(\nu') < m(\lambda) + m(\lambda')$. □

Corollary 1.2. *If $\lambda, \lambda' \in X_1(T)$ and $L(\nu)$ is a composition factor of $L(\lambda) \otimes L(\lambda')$ with $\nu \notin X_1(T)$, then $m(L(\nu)) \leq m(\lambda) + m(\lambda') - (p-1) \cdot \{\min_{\mu \in X_1(T) \setminus \{0\}} (m(\mu))\}$.*

Proof. Suppose $\nu = \sum_{i=0}^{r} p^i \nu_i$ is the p-adic expansion of ν. We rewrite the inequality from the proof of Case 1: $m(\lambda) + m(\lambda') - m(\nu) = f(\lambda + \lambda') - \sum f(\nu_i) \geq f(\nu) - \sum f(\nu_i) = \sum p^i f(\nu_i) - \sum f(\nu_i) = \sum (p^i - 1) m(\nu_i) \geq (p-1) m(\nu_k)$ for some $1 \leq k \leq r$ and $\nu_k \neq 0$, by assumption on ν. □

We may also define, for any $k \in \mathbb{N}$, the p^k-restricted mass, by letting $m_{p^k}(\lambda) = \sum_{i=0}^{r} p^{r(i,k)} f(\lambda_i)$, (and extending to nonsimple modules as in (ii) above,) where $r(i,k)$ is the least non-negative residue of $i \mod k$. It is an easy exercise to check that statements analogous to Lemma 1.1 and Corollary 1.2 hold for p^k-restricted mass. There is a natural way of extending the definition of mass to $G(n)$-modules by representing the simple modules as restrictions to $G(n)$ of G-modules with p^n-restricted highest weight; it can then be shown that the p^k-restricted mass of a G-module is $\geq$ to the p^k-restricted mass (as $G(n)$-module) of its restriction to $G(n)$. The following is a refinement of Lemma 2.3 of **[10]** which works for the finite groups $G(n)$.

Lemma 1.3. *Let* $\lambda = \sum_{i=0}^{n-1} p^i \lambda_i = \lambda^0 + p\bar{\lambda}$, $\mu = \sum_{i=0}^{n-1} p^i \mu_i = \mu^0 + p\bar{\mu}$, $\nu = \sum_{i=1}^{n-1} p^i \nu_i = p\bar{\nu}$, *where* $\lambda_0 \neq \mu_0$, *and* $m(\lambda_i) \geq m(\mu_i)$ *for all* $i = 1, \ldots, n-1$. *If* $L(\lambda)$ *is a composition factor of*

$$L(\nu) \otimes L(\mu) = L(p\bar{\nu}) \otimes L(\mu),$$

as $G(n)$*-modules, then* $(p^n - 1) \cdot \theta \leq (m(\mu_0) - m(\lambda_0)) + \sum_{i=1}^{n-1} p^i m(\nu_i)$, *where* $\theta = \{\min_{\beta \in X_1(T) \setminus \{0\}} (m(\beta))\}$.

Proof. Since $\lambda_0 \neq \mu_0$, $L(\lambda)$ cannot be a composition factor (as G-module) of

$$L(\nu) \otimes L(\mu) \cong L(\mu_0) \otimes [L(p\bar{\nu}) \otimes L(p\bar{\mu})],$$

by Steinberg's tensor product theorem. Therefore, we must have that $L(\lambda)$ is a composition factor of $\mathrm{res}_{G(n)}(L(\omega))$, for some non-$p^n$-restricted weight ω such that $L(\omega)$ is a G-composition factor of $L(\nu) \otimes L(\mu)$. This implies that

$$\begin{aligned}
m_{p^n}(L(\lambda)) &= \sum_{i=0}^{n-1} p^i m(\lambda_i) \leq m_{p^n}(L(\omega)) \\
&\leq m_{p^n}(L(\nu)) + m_{p^n}(L(\mu)) - (p^n - 1) \cdot \theta \\
&= \sum_{i=1}^{n-1} p^i m(\nu_i) + \sum_{i=0}^{n-1} p^i m(\mu_i) - (p^n - 1) \cdot \theta \\
&\leq \sum_{i=1}^{n-1} p^i m(\nu_i) + (m(\mu_0) - m(\lambda_0)) + \sum_{i=0}^{n-1} p^i m(\lambda_i) - (p^n - 1) \cdot \theta
\end{aligned}$$

whence

$$(p^n - 1) \cdot \theta \leq (m(\mu_0) - m(\lambda_0)) + \sum_{i=1}^{n-1} p^i m(\nu_i). \qquad \square$$

§2. Tensor products of simple modules

Lemma 2.1. *The composition factors which appear in each tensor product of pairs of restricted simple modules are as indicated in Tables 2.1 and 2.2. (We have omitted those tensor products in B_3 which are immediately obtainable from given ones, e.g., $\Lambda\sigma \otimes \Psi \cong \sigma \otimes (\Lambda \otimes \Psi)$.)*

Remark. In many cases, we are only interested in which modules appear as composition factors in a product of restricted modules, and not in their multiplicities as composition factors.

Proof. The weight multiplicities of the restricted simple modules can be completely determined from those of the Weyl modules using the Jantzen Sum Formula. The tensor products are then computed by calculating the weight orbits under the Weyl group and then multiplying the appropriate formal characters. □

Table 2.1 (Type A_3).

product	composition factors
$\Theta \otimes \Theta$	$2\Lambda, \Theta_1$
$\Theta \otimes \Theta^*$	$2F, \Psi$
$\Theta \otimes \Lambda$	Θ^*, Γ^*
$\Theta \otimes \Psi$	$2\Gamma, \Theta^*\Theta_1$
$\Theta \otimes \Gamma$	$2\Lambda, S, \Theta_1^*$
$\Theta \otimes \Gamma^*$	$2F, 3\Psi, 2\Lambda_1, \Lambda\Theta_1$
$\Lambda \otimes \Lambda$	$2F, 2\Psi, \Lambda_1$
$\Lambda \otimes \Psi$	$2\Lambda, S, \Theta_1, \Theta_1^*$
$\Lambda \otimes \Gamma$	$\Theta^*, 3\Gamma^*, 2\Theta\Theta_1^*, \Theta^*\Lambda_1$
$\Psi \otimes \Psi$	$6F, 4\Psi, 4\Lambda_1, 2\Lambda\Theta_1, 2\Lambda\Theta_1^*, \Psi_1$

Table 2.1 (Type A_3 cont.).

product	composition factors
$\Psi \otimes \Gamma$	$2\Theta, 4\Gamma, 3\Theta^*\Theta_1, \Theta^*\Theta_1^*, 2\Theta\Lambda_1, \Gamma^*\Theta_1^*$
$\Gamma \otimes \Gamma$	$6\Lambda, 2S, 4\Theta_1, 4\Theta_1^*, 2\Psi\Theta_1^*, 2\Lambda\Lambda_1, \Gamma_1$
$\Gamma \otimes \Gamma^*$	$10F, 7\Psi, 6\Lambda_1, 3\Lambda\Theta_1, 3\Lambda\Theta_1^*, 2\Psi_1, \Psi\Lambda_1$
$S \otimes \Theta$	$4\Gamma^*, 3\Theta\Theta_1^*, 2\Theta^*\Lambda_1, \Gamma\Theta_1$
$S \otimes \Lambda$	$8F, 6\Psi, 6\Lambda_1, 3\Lambda\Theta_1, 3\Lambda\Theta_1^*, 2\Psi_1, \Psi\Lambda_1$
$S \otimes \Psi$	$10\Lambda, 2S, 8\Theta_1, 8\Theta_1^*, 3\Psi\Theta_1,$
	$3\Psi\Theta_1^*, 4\Lambda\Lambda_1, 2\Gamma_1, 2\Gamma_1^*, \Lambda\Psi_1$
$S \otimes \Gamma$	$8\Theta^*, 8\Gamma^*, 4\Theta\Theta_1, 6\Theta\Theta_1^*, 4\Theta^*\Lambda_1,$
	$2\Gamma\Theta_1, 3\Gamma\Theta_1^*, 2\Theta^*\Psi_1, 2\Gamma^*\Lambda_1, \Theta\Gamma_1$
$S \otimes S$	$40F, 20\Psi, 32\Lambda_1, 14\Lambda\Theta_1, 14\Lambda\Theta_1^*, 16\Psi_1,$
	$4\Theta_2, 4\Theta_2^*, 8\Psi\Lambda_1, 4\Lambda_2, 2S\Theta_1, 2S\Theta_1^*, 2\Lambda\Gamma_1, 2\Lambda\Gamma_1^*, 2\Psi\Psi_1, S_1$

Table 2.2 (Type B_3).

product	composition factors
$\Lambda \otimes \Lambda$	$2F, 2\Psi, \Lambda_1$
$\Lambda \otimes \Psi$	$2\Lambda, \Sigma, \sigma_1$
$\Psi \otimes \Psi$	$6F, 4\Psi, 4\Lambda_1, 2\Lambda\sigma_1, \Psi_1$
$\Sigma \otimes \Lambda$	$8F, 6\Psi, 6\Lambda_1, 3\Lambda\sigma_1, 2\Psi_1, \Psi\Lambda_1$
$\Sigma \otimes \Psi$	$10\Lambda, 2\Sigma, 6\sigma_1, 3\Psi\sigma_1, 4\Lambda\Lambda_1, 2(\Lambda\sigma)_1, \Lambda\Psi_1$
$\Sigma \otimes \Sigma$	$40F, 20\Psi, 32\Lambda_1, 12\Lambda\sigma_1, 16\Psi_1, 4\sigma_2,$ $8\Psi\Lambda_1, 4\Lambda_2, 2\Sigma\sigma_1, 2\Lambda(\Lambda\sigma)_1, 2\Psi\Psi_1, \Sigma_1$
$\sigma \otimes \sigma$	$4F, 4\Lambda, 2\Psi, \sigma_1$
$\sigma \otimes \Lambda\sigma$	$8F, 8\Lambda, 8\Psi, 2\Sigma, 4\Lambda_1, 2\sigma_1, \Lambda\sigma_1$
$\sigma \otimes \Psi\sigma$	$12F, 8\Lambda, 12\Psi, 4\Sigma, 8\Lambda_1, 4\sigma_1, 4\Lambda\sigma_1, 2\Psi_1, \Psi\sigma_1$
$\sigma \otimes S$	$32F, 20\Lambda, 24\Psi, 8\Sigma, 24\Lambda_1, 12\sigma_1, 8\Psi_1,$ $4(\Lambda\sigma)_1, 8\Lambda\Lambda_1, 12\Lambda\sigma_1, 4\Psi\Lambda_1, 6\Psi\sigma_1, 2\Lambda\Psi_1, \Sigma\sigma_1$
$\Lambda\sigma \otimes \Lambda\sigma$	$32F, 24\Lambda, 28\Psi, 10\sigma_1, 8\Sigma, 20\Lambda_1,$ $8\Lambda\sigma_1, 4\Psi_1, 2\Psi\sigma_1, 4\Lambda\Lambda_1, 2\Psi\Lambda_1, (\Lambda\sigma)_1$
$\Lambda\sigma \otimes \Psi\sigma$	$52F, 36\Lambda, 40\Psi, 12\Sigma, 36\Lambda_1, 20\sigma_1, 10\Psi_1,$ $18\Lambda\sigma_1, 4\Psi\Lambda_1, 8\Psi\sigma_1, 8\Lambda\Lambda_1, 4(\Lambda\sigma)_1, 2\Lambda\Psi_1, \sigma_2, \Sigma\sigma_1$

Table 2.2 (Type B_3 cont.).

product	composition factors
$\Psi\sigma \otimes \Psi\sigma$	$88F, 64\Lambda, 60\Psi, 38\sigma_1, 16\Sigma, 64\Lambda_1, 20\Psi_1, 32\Lambda\sigma_1, 20\Psi\sigma_1,$ $24\Lambda\Lambda_1, 8\Psi\Lambda_1, 12(\Lambda\sigma)_1, 8\Lambda\Psi_1, 4\Sigma\sigma_1, 4\sigma_2, 2\Psi\Psi_1, 2\Lambda\sigma_2, (\Psi\sigma)_1$
$S \otimes \Lambda\sigma$	$144F, 92\Lambda, 88\Psi, 56\sigma_1, 24\Sigma, 112\Lambda_1, 48\Psi_1,$ $48\Lambda\sigma_1, 30\Psi\sigma_1, 44\Lambda\Lambda_1, 24\Psi\Lambda_1, 22(\Lambda\sigma)_1, 6\Sigma\sigma_1, 4\Sigma\Lambda_1,$ $8\sigma_2, 14\Lambda\Psi_1, 3\Lambda\sigma_2, 4\Psi\Psi_1, 2(\Psi\sigma)_1, 8\Lambda_2, 4\Lambda(\Lambda\sigma)_1, 2\Sigma_1, \Psi(\Lambda\sigma)_1$
$S \otimes \Psi\sigma$	$264F, 168\Lambda, 140\Psi, 36\Sigma, 216\Lambda_1, 104\sigma_1, 108\Psi_1, 82\Lambda\sigma_1,$ $52\Psi\Lambda_1, 36\Lambda\Psi_1, 60\Psi\sigma_1, 96\Lambda\Lambda_1, 48(\Lambda\sigma)_1, 14\Sigma\sigma_1, 26\sigma_2,$ $8(\Psi\sigma)_1, 14\Psi\Psi_1, 12\Lambda\sigma_2, 3\Psi\sigma_2, 24\Lambda_2, 8\Sigma_1, 8\Sigma\Lambda_1, 2\Sigma\Psi_1,$ $12\Lambda(\Lambda\sigma)_1, 4\Psi(\Lambda\sigma)_1, 2\Lambda_1\sigma_2, \Lambda(\Psi\sigma)_1$
$S \otimes S$	$840F, 480\Lambda, 368\Psi, 736\Lambda_1, 296\sigma_1, 88\Sigma, 352\Lambda\Lambda_1, 216\Lambda\sigma_1,$ $464\Psi_1, 208\Psi\Lambda_1, 164\Psi\sigma_1, 144\Lambda_2, 176(\Lambda\sigma)_1, 120\sigma_2, 168\Lambda\Psi_1, 40\Sigma\sigma_1,$ $40\Sigma\Lambda_1, 80\Psi\Psi_1, 56\Lambda\sigma_2, 64\Lambda(\Lambda\sigma)_1, 24\Lambda\Lambda_2, 56(\Psi\sigma)_1, 60\Sigma_1, 20\Psi\sigma_2,$ $32\Psi(\Lambda\sigma)_1, 8\Psi\Lambda_2, 20\Lambda_1\sigma_2, 4\sigma_1\sigma_2, 12\Sigma\Psi_1, 12\sigma_1\Lambda_2, 12\Lambda(\Psi\sigma)_1, 8\Lambda\Sigma_1,$ $4\Psi_2, 2\Sigma\sigma_2, 4\Sigma(\Lambda\sigma)_1, 2\Lambda\Lambda_1\sigma_2, 2\Psi\Sigma_1, 2\Psi(\Psi\sigma)_1, S_1$

Lemma 2.2. A) (A_3 version) *Let I, J, K, L, M, P, and R be disjoint subsets of $N = \{0, 1, \ldots, n-1\}$, and let $i \in N$. Then $A_i \otimes (\Theta_I \otimes \Theta^*_J \otimes \Lambda_K \otimes \Psi_L \otimes \Gamma_M \otimes \Gamma^*_P \otimes S_R)$ contains no composition factor of the form S_T with $|T| > |R| + 1$, where A denotes any of Θ, Θ^*, Λ, Ψ, Γ, Γ^*, S.*

B) (B_3 version) *Let* I, J, K, L, M, P, *and* R *be disjoint subsets of* $N = \{0, 1, \ldots, n-1\}$, *and let* $i \in N$. *Then* $A_i \otimes (\Lambda_I \otimes \Psi_J \otimes \Sigma_K \otimes \sigma_L \otimes (\Lambda\sigma)_M \otimes (\Psi\sigma)_P \otimes S_R)$ *contains no composition factor of the form* $\Lambda_{I'} \otimes \Psi_{J'} \otimes \Sigma_{K'} \otimes \sigma_{L'} \otimes (\Lambda\sigma)_{M'} \otimes (\Psi\sigma)_{P'} \otimes S_{R'}$ *with* $|R'| > |R| + 1$, *where* A *denotes any of* $\Lambda, \Psi, \Sigma, \sigma, \Lambda\sigma, \Psi\sigma$. *Furthermore, if* $i \in N \setminus (I \cup J \cup K \cup L \cup M \cup P \cup R)$, *then* $(S_i \otimes S_i) \otimes (\Lambda_I \otimes \Psi_J \otimes \Sigma_K \otimes \sigma_L \otimes (\Lambda\sigma)_M \otimes (\Psi\sigma)_P \otimes S_R)$ *contains no composition factor of the form* S_T *with* $|T| > |R| + 2$.

Proof. We induct on the quantity $m(A_i) + m(\Lambda_I \otimes \cdots \otimes S_R)$, if $G = B_3$. The argument for A_3 is similar. If $i \notin I \cup \cdots \cup R$, there is nothing to prove, so we analyze the filtration of $A_i \otimes (\Lambda_I \otimes \cdots \otimes S_R) = (A_i \otimes \Upsilon_i) \otimes (\Lambda_I \otimes \cdots \otimes \Upsilon_{U \setminus \{i\}} \otimes \cdots \otimes S_R)$, resulting from a composition series of $A_i \otimes \Upsilon_i$, for each choice of Υ corresponding to the various possibilities $i \in I, i \in J$, etc., and the various possible choices for A. Inspection of Table 2.2 shows that the resulting filtration factors are of one of the following 3 forms:

i) irreducible of the form $\Lambda_{I'} \otimes \Psi_{J'} \otimes \Sigma_{K'} \otimes \sigma_{L'} \otimes (\Lambda\sigma)_{M'} \otimes (\Psi\sigma)_{P'} \otimes S_{R'}$ with $|R'| \leq |R| + 1$,

ii) of the form $A'_j \otimes (\Lambda_{I'} \otimes \cdots \otimes S_{R'})$ with $j \in \{i+1, i+2\}$, $R' \subseteq R$, and

$$m(A'_j) + m(\Lambda_{I'} \otimes \cdots \otimes S_{R'}) < m(A_i) + m(\Lambda_I \otimes \cdots \otimes S_R),$$

by Corollary 1.2, or

iii) of the form $A'_{i+1} \otimes A''_{i+2} \otimes (\Lambda_{I'} \otimes \cdots \otimes S_{R'})$ with $|R'| = |R| - 1$ and

$$m(A'_{i+1}) + m(A''_{i+2}) + m(\Lambda_{I'} \otimes \cdots \otimes S_{R'}) < m(A_i) + m(\Lambda_I \otimes \cdots \otimes S_R),$$

by Corollary 1.2.

Thus, we may apply the induction hypothesis (twice if necessary).

Finally, to prove the last assertion, we examine the composition factors of $S_i \otimes S_i$. (cf. Table 2.2) We observe that we may apply the first assertion of the theorem at most twice in succession to terms of the form $A_k \otimes (\Lambda_I \otimes \Psi_J \otimes \Sigma_K \otimes \sigma_L \otimes (\Lambda\sigma)_M \otimes (\Psi\sigma)_P \otimes S_R)$ (where $k = i+1$ or $i+2$) to obtain the result. □

We will need information about the structure of the module $A_i \otimes S_R$. A restriction on which composition factors can appear in the head and socle is obtained by determining the decomposition into (projective) indecomposables of $A_i \otimes S_N$. In the following, $P(M)$ denotes the projective cover of M.

Lemma 2.3A. (A_3 version)

a) $\Theta_i \otimes S_N \cong P(\Gamma_i^* \otimes S_{N \setminus \{i\}})$

b) $\Gamma_i \otimes S_N \cong P(\Theta_i^* \otimes S_{N \setminus \{i\}}) \oplus 2P(\Gamma_i^* \otimes S_{N \setminus \{i\}})$

c) $\Lambda_i \otimes S_N \cong P(\Psi_i \otimes S_{N \setminus \{i\}})$

d) $\Psi_i \otimes S_N \cong P(\Lambda_i \otimes S_{N \setminus \{i\}}) \oplus 2S_N$

e) $S_i \otimes S_N \cong P(S_{N \setminus \{i\}}) \oplus 2P(\Psi_i \otimes S_{N \setminus \{i\}}) \oplus 2P(\Gamma_{i+1} \otimes S_{N \setminus \{i+1\}}) \oplus 2P(\Gamma_{i+1}^* \otimes S_{N \setminus \{i+1\}})$

Proof. a) $\dim_F(\mathrm{Hom}_{FG(n)}(\Theta_i \otimes S_N, \Theta_I \otimes \cdots \Gamma_P^* \otimes S_R)) = \dim_F(\mathrm{Hom}_{FG(n)}(S_N, \Theta_i^* \otimes (\Theta_I \otimes \cdots \otimes \Gamma_P^* \otimes S_R)) =$ multiplicity of S_N as a composition factor of $\Theta_i^* \otimes (\Theta_I \otimes \cdots \otimes \Gamma_P^* \otimes S_R)$ since S_N is simple and projective. However, $m(\Theta_i^* \otimes (\Theta_I \otimes \cdots \otimes S_R)) \leq$

$m(\Theta_i^*) + m(\Theta_I \otimes \cdots \otimes S_R) = 3 + 3|I| + 3|J| + 4|K| + 6|L| + 7|M| + 7|P| + 10|R|$, which is $< 10n$, unless $|R| \geq n-1$, and, if $|R| = n-1$, either $|M| = 1$ or $|P| = 1$. This reduces us to three cases:

i) $i \in R$.

$(\Theta_i^* \otimes S_i) \otimes (\Gamma_M \otimes \Gamma_P^* \otimes S_{R\setminus\{i\}}) \approx (4\Gamma_i + 3\Theta_i^*\Theta_{i+1} + 2\Theta_i\Lambda_{i+1} + \Gamma_i\Theta_{i+1}^*) \otimes (\Gamma_M \otimes \Gamma_P^* \otimes S_{R\setminus\{i\}})$, and a mass argument applies to all filtration factors except $\Gamma_i\Theta_{i+1}^* \otimes (\Gamma_M \otimes \Gamma_P^* \otimes S_{N\setminus\{i\}})$. However, expanding $\Theta_{i+1}^* \otimes A_{i+1}$ (for some A), if necessary, results in a reduction of mass estimate by at least 3 (cf. Cor. 1.2), together with simple filtration factors that are clearly not isomorphic to S_N. Thus the multiplicity of S_N is zero in this situation.

ii) $M = \{i\}$ (and $R = N \setminus \{i\}$).

$m((\Theta_i^* \otimes \Gamma_i) \otimes S_{N\setminus\{i\}}) \leq m(\Theta_i^* \otimes \Gamma_i) + m(S_{N\setminus\{i\}}) = 7 + 10(n-1) < m(S_N)$.

iii) $P = \{i\}$ (and $R = N \setminus \{i\}$).

$(\Theta_i^* \otimes \Gamma_i^*) \otimes S_{N\setminus\{i\}} \approx (2\Lambda_i + \Theta_{i+1} + S_i) \otimes S_{N\setminus\{i\}}$. Thus (by applying the mass argument to the filtration factor $\Theta_{i+1} \otimes S_{N\setminus\{i\}}$) the multiplicity of S_N is shown to be equal to one in this case.

b) $\mathrm{Hom}_{FG(n)}(\Gamma_i \otimes S_N, \Theta_I \otimes \cdots \otimes S_R)) \cong \mathrm{Hom}_{FG(n)}(S_N, \Gamma_i^* \otimes (\Theta_I \otimes \cdots \otimes S_R))$. Inspection of Table 2.1 in light of the type of mass considerations used in part (a) shows that the multiplicity of S_N is nonzero only when $R = N \setminus \{i\}$; in that situation, the multiplicity is one when $i \in J$, two when $i \in P$, and zero otherwise. This is immediate except if $i \in L, M$ or R; but in those cases the mass argument still goes through for all filtration factors except those of the form $\Gamma_i\Lambda_{i+1} \otimes (\Theta_{I'} \otimes \cdots \otimes S_{R'})$ (or similar). However, further expansion (if indeed $i+1 \in I' \cup \cdots \cup R'$) reduces the mass estimate by at least 3 (i.e., resulting in irreducible filtration factors not isomorphic to S_N, or filtration factors with mass less than $m(S_N)$.)

c) $\mathrm{Hom}_{FG(n)}(\Lambda_i \otimes S_N, \Theta_I \otimes \cdots \otimes S_R)) \cong \mathrm{Hom}_{FG(n)}(S_N, \Lambda_i \otimes (\Theta_I \otimes \cdots \otimes S_R))$. We argue as in (b). (The case $i \in R$ is handled by further expansion of the filtration factor $\Psi_i\Lambda_{i+1} \otimes (\Theta_I \otimes \cdots \otimes S_{R\setminus\{i\}})$.)

d) $\mathrm{Hom}_{FG(n)}(\Psi_i \otimes S_N, \Theta_I \otimes \cdots \otimes S_R)) \cong \mathrm{Hom}_{FG(n)}(S_N, \Psi_i \otimes (\Theta_I \otimes \cdots \otimes S_R))$. The argument of (b) goes through immediately to give the stated result.

e) $\mathrm{Hom}_{FG(n)}(S_i \otimes S_N, \Theta_I \otimes \cdots \otimes S_R) \cong \mathrm{Hom}_{FG(n)}(S_N, S_i \otimes (\Theta_I \otimes \cdots \otimes S_R))$. An argument similar to that of (b) goes through except if $i \in R$; but in that situation we make the observation that with further expansion of the filtration factor $S_i\Theta_{i+1} \otimes (\Theta_I \otimes \cdots \otimes S_{R\setminus\{i\}})$, a nonzero result is obtained only when $i+1 \in P$ and $R = N \setminus \{i+1\}$ (similarly for $S_i\Theta_{i+1}^* \otimes (\Theta_I \otimes \cdots \otimes S_{R\setminus\{i\}})$). □

Lemma 2.3B. (B_3 version)

a) $\Lambda_i \otimes S_N \cong P((\Psi\sigma)_i \otimes S_{N\setminus\{i\}})$

b) $\Psi_i \otimes S_N \cong P((\Lambda\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 2S_N$

c) $\Sigma_i \otimes S_N \cong P(\sigma_i \otimes S_{N\setminus\{i\}}) \oplus 2P((\Psi\sigma)_i \otimes S_{N\setminus\{i\}}) \oplus 2P((\Sigma_{i+1} \otimes S_{N\setminus\{i+1\}})$

d) $\sigma_i \otimes S_N \cong P(\Sigma_i \otimes S_{N\setminus\{i\}})$

e) $(\Lambda\sigma)_i \otimes S_N \cong P(\Psi_i \otimes S_{N\setminus\{i\}})$

f) $(\Psi\sigma)_i \otimes S_N \cong P(\Lambda_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Sigma_i \otimes S_{N\setminus\{i\}})$

g) $S_i \otimes S_N \cong P(S_{N\setminus\{i\}}) \oplus 2P(\Psi_i \otimes S_{N\setminus\{i\}}) \oplus 2P(\Sigma_i \otimes \Sigma_{i+1} \otimes S_{N\setminus\{i,i+1\}})$

Proof. The arguments are similar to those of version A. □

Corollary 2.4A. *Let $T \subseteq N = \{0, \ldots, n-1\}$ and let $i \in T$. Let A denote any of the symbols $\Theta, \ldots, \Gamma^*$. Then $\mathrm{Hd}_{FG(n)}(A_i \otimes S_T)$ has no constituent of the form $\Theta_I \otimes \cdots \otimes \Gamma^*_P \otimes S_R$ with $|I \cup \cdots \cup P| > 1$.*

Corollary 2.4B. *Let $T \subseteq N = \{0, \ldots, n-1\}$ and let $i \in T$. Let A denote any of the symbols $\Lambda, \Psi, \Sigma, \sigma, \Lambda\sigma, \Psi\sigma$. Then $\mathrm{Hd}_{FG(n)}(A_i \otimes S_T)$ has no constituent of the form $\Lambda_I \otimes \cdots \otimes (\Psi\sigma)_P \otimes S_R$ with $|I \cup \cdots \cup P| > 1$.*

Proof. We prove the claim if $G = B_3$ and $A = \Sigma$; the other cases are simpler. We observe that $\Sigma_i \otimes S_i$ already has 5 simple summands in its head: in fact, $\mathrm{Hd}_{FG(n)}(\Sigma_i \otimes S_i) = \sigma_i \oplus 2(\Psi\sigma)_i \oplus 2S_i\sigma_{i+1}$. For example, $\mathrm{Hom}_{FG(n)}(\Sigma_i \otimes S_i, S_i\sigma_{i+1}) \cong 2F$, since $\Sigma \otimes S$ is the restriction to $FG(n)$ of a module M over the algebraic group that has $L(\rho+2\lambda_3)$ as a composition factor with multiplicity 2, while $\mathrm{Ext}^1_G(L(\rho+2\lambda_3), L(\nu)) = 0$ for all composition factors $L(\nu)$ of M. This follows from the Lyndon-Hochschild-Serre spectral sequence for the infinitesimal subgroup G_1, since $L(\rho)$ is injective for G_1, whereas all of the other composition factors of M are of the form $L(\nu_0 + 2\tilde{\nu})$ with $\nu_0 \neq \rho$.

Thus, $\mathrm{Hd}_{FG(n)}(\Sigma_i \otimes S_T)$ must consist only of the five summands $(\sigma_i \otimes S_{T\setminus\{i\}}) \oplus 2((\Psi\sigma)_i \otimes S_{T\setminus\{i\}}) \oplus 2(\sigma_{i+1} \otimes S_T)$ if $i+1 \notin T$, or of the five summands $(\sigma_i \otimes S_{T\setminus\{i\}}) \oplus 2((\Psi\sigma)_i \otimes S_{T\setminus\{i\}}) \oplus 2(\Sigma_{i+1} \otimes S_{T\setminus\{i+1\}})$ if $i+1 \in T$, since there are only five summands in the head of $\Sigma_i \otimes S_N$. □

§3. Reduction of the problem

We show that the 1-cohomology groups vanish in a large number of cases. The following lemma is a generalization of Alperin's induction step ([**1**]) that is used frequently in the papers of Sin ([**6**], [**7**], [**8**]). It follows easily from the long exact sequence of cohomology for $FG(n)$.

Lemma 3.1. *Let D be any $FG(n)$-module, let A, B be simple $FG(n)$-modules, and let E be any simple quotient of $B \otimes D$. Let X(A,B) denote the (unique up to isomorphism) $FG(n)$-module with head isomorphic to A, and radical isomorphic to a direct sum of $d = \dim_F(\mathrm{Ext}^1_{FG(n)}(A, B))$ copies of B. Then surjectivity of the natural map*

$$\mathrm{Hom}_{FG(n)}(A \otimes D, E) \longrightarrow \mathrm{Hom}_{FG(n)}(X(A, B) \otimes D, E)$$

implies that $\dim_F(\mathrm{Ext}^1_{FG(n)}(A, B)) \leq \dim_F(\mathrm{Ext}^1_{FG(n)}(A \otimes D, E))$. □

In our applications, we will prove surjectivity by showing that

$$\mathrm{Hom}_{FG(n)}(X(A, B) \otimes D, E) = 0\,.$$

In most cases we can simply check that A is not a composition factor of $D^* \otimes E$.

Lemma 3.2. *Let I, J be subsets of $N = \{0, 1, \dots, n-1\}$ with $I \neq J$. If $G = A_3$, suppose furthermore that either*
i) $|I \Delta J| > 1$, *or*
ii) $|I \Delta J| = 1$, *and* $I \cup J = (I \cap J) \cup \{l\}$ *where* $l - 1 \in I \cap J$.
Then $\mathrm{Ext}^1_{FG(n)}(S_I, S_J) = 0$.

Proof. A) (for A_3.) We may assume $J \subsetneq I$ (as S is self-dual). Let $k \in N \setminus I$. We prove that $\dim_F(\mathrm{Ext}^1_{FG(n)}(S_I, S_J)) \leq \dim_F(\mathrm{Ext}^1_{FG(n)}(S_{I\cup\{k\}}, S_{J\cup\{k\}}))$ using Lemma 3.1; it suffices to show that $\mathrm{Hom}_{FG(n)}(X(S_I, S_J) \otimes S_k, S_J \otimes S_k)$ ($\cong \mathrm{Hom}_{FG(n)}(X(S_I, S_J), (S_k \otimes S_k) \otimes S_J)) \cong 0$. (The result then follows by downward induction on $|I|$, as S_N is projective.)

The composition factors of $S_k \otimes S_k$ are: $\{F, \Psi_k, \Lambda_{k+1}, \Lambda_k\Theta_{k+1}, \Lambda_k\Theta^*_{k+1}, \Psi_{k+1}, \Theta_{k+2}, \Theta^*_{k+2}, \Psi_k\Lambda_{k+1}, \Lambda_{k+2}, S_k\Theta_{k+1}, S_k\Theta^*_{k+1}, \Lambda_k\Gamma_{k+1}, \Lambda_k\Gamma^*_{k+1}, \Psi_{\{k,k+1\}}, S_{k+1}\}$. Since $m(S) = 10$, we need only consider those composition factors of mass ≥ 10 (i.e., $\Psi_k\Lambda_{k+1}$, $S_k\Theta_{k+1}$, $\Lambda_k\Gamma_{k+1}$, $\Psi_{\{k,k+1\}}$, S_{k+1}, and their duals), in order to show by a mass argument that S_I is not a composition factor of $(S_k \otimes S_k) \otimes S_J$.

i) $\Psi_k\Lambda_{k+1} \otimes S_J$, if not irreducible, can be written as $(S_{k+1} \otimes \Lambda_{k+1}) \otimes \Psi_k \otimes S_{J\setminus\{k+1\}}$, and thus has mass $\leq m(S_{k+1} \otimes \Lambda_{k+1}) + m(\Psi_k) + m(S_{J\setminus\{k+1\}}) \leq 10 + 6 + 10(|J| - 1) < m(S_I)$ (see Table 2.1).

ii) $S_k\Theta_{k+1} \otimes S_J$, if not irreducible, can be written as $(S_{k+1} \otimes \Theta_{k+1}) \otimes S_{(J\cup\{k\})\setminus\{k+1\}}$. Each filtration factor resulting from a composition factor, A, of $S_{k+1} \otimes \Theta_{k+1}$ has mass $\leq m(A) + m(S_{(J\cup\{k\})\setminus\{k+1\}}) \leq 7 + 10|J| < m(S_I)$, except possibly if $A = \Gamma_{k+1}\Theta_{k+2}$ (see Table 2.1). However, the resulting filtration factor $\Gamma_{k+1}\Theta_{k+2} \otimes (S_{(J\cup\{k\})\setminus\{k+1\}})$ is either irreducible (and not isomorphic to S_I) or can be rewritten as $(S_{k+2} \otimes \Theta_{k+2}) \otimes \Gamma_{k+1} \otimes S_{(J\cup\{k\})\setminus\{k+1,k+2\}}$, which has mass at most

$$m(S_{k+2} \otimes \Theta_{k+2}) + m(\Gamma_{k+1}) + m(S_{(J\cup\{k\})\setminus\{k+1,k+2\}}) \leq 10 + 7 + 10(|J| - 1) < m(S_I).$$

iii) $\Lambda_k\Gamma_{k+1} \otimes S_J$, if not irreducible, can be written as $(S_{k+1} \otimes \Gamma_{k+1}) \otimes \Lambda_k \otimes S_{J\setminus\{k+1\}}$, and thus has mass $\leq m(S_{k+1} \otimes \Gamma_{k+1}) + m(\Lambda_k) + m(S_{J\setminus\{k+1\}}) \leq 11 + 4 + 10(|J| - 1) < m(S_I)$ (see Table 2.1).

iv) $\Psi_{\{k,k+1\}} \otimes S_J$, if not irreducible, can be written as $(S_{k+1} \otimes \Psi_{k+1}) \otimes \Psi_k \otimes S_{J\setminus\{k+1\}}$ and thus has mass $\leq m(S_{k+1} \otimes \Psi_{k+1}) + m(\Psi_k) + m(S_{J\setminus\{k+1\}}) \leq 10 + 6 + 10(|J| - 1) < m(S_I)$.

v) $S_{k+1} \otimes S_J$, if irreducible, can be isomorphic to S_I only if $I \setminus J = \{k+1\}$ which is impossible if the hypothesis of the theorem is satisfied (since $k \notin I$). Otherwise we have $m(S_{k+1} \otimes S_J) \leq m(S_{k+1} \otimes S_{k+1}) + m(S_{J\setminus\{k+1\}}) \leq 13 + 10(|J| - 1) < m(S_I)$ (see Table 1).

B) (for B_3.) We proceed by showing that

$$\begin{aligned}\dim_F(\mathrm{Ext}^1_{FG(n)}(S_I, S_J)) &\leq \dim_F(\mathrm{Ext}^1_{FG(n)}(S_I \otimes \sigma_k, S_J \otimes \sigma_k)) \\ &\leq \dim_F(\mathrm{Ext}^1_{FG(n)}(S_{I\cup\{k\}}, S_{J\cup\{k\}}))\end{aligned}$$

for arbitrary $k \in N \setminus I$, using $S_k \cong \sigma_k \otimes \Sigma_k$. The first inequality follows from Lemma 3.1, if we can show that

$$\mathrm{Hom}_{FG(n)}(X(S_I, S_J) \otimes \sigma_k, S_J \otimes \sigma_k) \; (\cong \mathrm{Hom}_{FG(n)}(X(S_I, S_J), (\sigma_k \otimes \sigma_k) \otimes S_J)) \cong 0 .$$

This is immediate, as $m((\sigma_k \otimes \sigma_k) \otimes S_J) \le m(\sigma_k \otimes \sigma_k) + m(S_J) = 5 + 11|J| < m(S_I)$. The second inequality will follow from $\mathrm{Hom}_{FG(n)}(X(S_I \otimes \sigma_k, S_J \otimes \sigma_k) \otimes \Sigma_k, (S_J \otimes \sigma_k) \otimes \Sigma_k) \cong \mathrm{Hom}_{FG(n)}(X(S_I \otimes \sigma_k, S_J \otimes \sigma_k), (\Sigma_k \otimes \Sigma_k) \otimes (S_J \otimes \sigma_k)) \cong 0$. Here we observe that all of the composition factors of $\Sigma_k \otimes \Sigma_k$ have mass less than 11, except for $\Sigma_k \sigma_{k+1}$. Now, $(\Sigma_k \sigma_{k+1}) \otimes (S_J \otimes \sigma_k) \cong S_{J \cup \{k\}} \otimes \sigma_{k+1}$, if irreducible, cannot be isomorphic to $S_I \otimes \sigma_k$ (as $n > 1$). On the other hand, $m(S_{(J\cup\{k\})\setminus\{k+1\}} \otimes (S_{k+1} \otimes \sigma_{k+1})) = m(S_{(J\cup\{k\})\setminus\{k+1\}} \otimes (\Sigma_{k+1} \otimes \sigma_{k+1}) \otimes \sigma_{k+1})) \le m(S_{(J\cup\{k\})\setminus\{k+1\}}) + m(\Sigma_{k+1}) + m(\sigma_{k+1} \otimes \sigma_{k+1}) = 11|J| + 8 + 5 \le 11|I| + 2 < m(S_I \otimes \sigma_k)$. The result follows (as in A) by the obvious downward induction. □

Lemma 3.3. *Given disjoint subsets I, J, K, L, M, P, and R, with $I \cup \cdots \cup R \subsetneq T$ for some subset $T \subseteq N = \{0, 1, \ldots, n-1\}$, with at least one of $I, J, \ldots, P$ nonempty, then*

$$\mathrm{Ext}^1_{FG(n)}(S_T, \Theta_I \otimes \cdots \otimes \Gamma^*_P \otimes S_R) = 0 ,$$

if $G = A_3$, and

$$\mathrm{Ext}^1_{FG(n)}(S_T, \Lambda_I \otimes \cdots \otimes (\Psi\sigma)_P \otimes S_R) = 0 ,$$

if $G = B_3$.

Proof. A) (for A_3.) By the usual argument, it will suffice to show that S_T is not a composition factor of $(S_k \otimes S_k) \otimes \Theta_I \otimes \cdots \otimes \Gamma^*_P \otimes S_R$, for $k \in N \setminus T$. This follows immediately by mass argument if I, J, K, or L is nonempty or if $|M \cup P| > 1$, for then $m((S_k \otimes S_k) \otimes \Theta_I \otimes \cdots \otimes \Gamma^*_P \otimes S_R) \le m(S_k \otimes S_k) + m(\Theta_I \otimes \cdots \otimes \Gamma^*_P \otimes S_R) \le 13 + [10(|T|-2) + 6] = 10|T| - 1$. Otherwise, the only composition factors of $S_k \otimes S_k$ of concern are $S_k \Theta_{k+1}$ (and its dual); then $S_k \Theta_{k+1} \otimes (\Theta_I \otimes \cdots \otimes \Gamma^*_P \otimes S_R)$, if not irreducible, can be written in one of the following 3 forms:

i) $(S_{k+1} \otimes \Theta_{k+1}) \otimes S_k \otimes \Gamma_M \otimes \Gamma^*_P \otimes S_{R\setminus\{k+1\}}$, which has mass $< m(S_T)$, by the "equality only if" assertion of Lemma 1.1,

ii) $(\Gamma^*_{k+1} \otimes \Theta_{k+1}) \otimes S_k \otimes S_R$, which has mass $< m(S_T)$, again by Lemma 1.1,
or,

iii) $(\Gamma_{k+1} \otimes \Theta_{k+1}) \otimes S_k \otimes S_R$. However, all of the composition factors of $\Gamma_{k+1} \otimes \Theta_{k+1}$ have mass < 10, except S_{k+1}, but $S_{k+1} \otimes S_k \otimes S_R$ is irreducible, $\not\cong S_T$ (as $k \notin T$).

B) (for B_3.) We show that

$$\begin{aligned}
&\dim_F(\mathrm{Ext}^1_{FG(n)}(S_T, \Lambda_I \otimes \cdots \otimes (\Psi\sigma)_P \otimes S_R)) \\
&\le \dim_F(\mathrm{Ext}^1_{FG(n)}(S_T \otimes \Sigma_k, \Lambda_I \otimes \cdots \otimes (\Psi\sigma)_P \otimes S_R \otimes \Sigma_k)) \\
&\le \dim_F(\mathrm{Ext}^1_{FG(n)}(S_{T\cup\{k\}}, \Lambda_I \otimes \cdots \otimes (\Psi\sigma)_P \otimes S_{R\cup\{k\}}))
\end{aligned}$$

for $k \in N \setminus T$. The first inequality will follow from Lemma 3.1 if we can show that S_T is not a composition factor of $(\Sigma_k \otimes \Sigma_k) \otimes \Lambda_I \otimes \cdots \otimes (\Psi\sigma)_P \otimes S_R$. The mass argument