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Contents

Introduction

1 Euler – Poisson equations and integrable cases. In this introduction, we provide a brief outline of the main milestones in the development of rigid body dynamics. The equations of rigid body motion about a fixed point in various force fields were inspired by the work of Euler, d'Alembert, Clairaut, Poisson and Lagrange. The latter presented them in a fairly general and modern form in his *Analytical Mechanics*. Of particular interest was and still is the problem of the motion of a rigid body about a fixed point in a homogeneous gravitational field (i.e., of a heavy rigid body). A basis for research along these lines is the system of equations obtained by Euler and Poisson, which combine the elegance and simplicity of representation with the difficulty and even impossibility of obtaining a general solution in explicit form. Historically, integrable cases, i.e., cases in which a general solution can be obtained (in terms of quadratures, as the authors of classic works put it) were the first to be studied. Obviously, this requires that some system parameters or initial conditions be given. The most popular of them were found already by Euler (1758) and Lagrange (1788) when the general principles of dynamics were formed and developed.

A much more complicated case of integrability of the Euler – Poisson equations, which gave impetus to new research in the area of integrable systems, was found by S. V. Kovalevskaya in 1888. This result was highly regarded by the Paris Academy of Sciences, which awarded the Bordin prize to S. V. Kovalevskaya in 1888 for a memoir on the rotation of a rigid body about a fixed point. We note that the Academy of Sciences had announced a competition for research on the same topic twice before, but no one had been awarded the prize. In the spring of 1889, Kovalevskaya was awarded a prize by the Royal Swedish Academy of Sciences for her second memoir on the problem of rigid body rotation.

The integrability of the Euler and Lagrange cases is due to natural dynamical symmetries and preservation of the corresponding first integrals. S. V. Kovalevskaya found her own integrable case from nonobvious analytical considerations, using the well-developed theory of algebraic functions (elliptic functions are a particular case of these functions). This required uniqueness of the general solution on the complex plane of time, which laid the foundations for the Painlevé – Kovalevskaya test, one of the most advanced methods of integrability analysis of dynamical systems. The Kovalevskaya integral has no natural symmetry origin; its symmetries are said to be hidden, and the problem of motion description and explicit integration in this case is much more complicated.

After Kovalevskaya, no general integrable cases have been found for the Euler – Poisson equations, and it has even been proved that they do not exist. Nevertheless, the problem of obtaining various partial integrals and partial solutions, of which about two dozens have been obtained, has long been a topic of special interest. But this number does not cover even a small part of the full range of possible motions of a heavy rigid body.

2 Poincaré and Helmholtz equations. In the second half of the 19th century and in the early 20th century, integrable cases were found in rigid body dynamics for various versions of the problem of rigid body motion: motion of a body in a fluid, motion of a body containing cavities filled with fluid, gyrostats, and nonholonomic problems. The study of these problems was made possible by the development of the general formalism of rigid body dynamics, which culminated in the Poincaré equations allowing the equations of rigid body motion to be represented in terms of group variables.

Here, mention should be made of progress in the hydrodynamics of an ideal fluid and vortex theory, whose foundations were laid by H. Helmholtz.

In this way, equations for the vorticity vector, similar to dynamical equations for angular momentum, were obtained, and Poincaré studied for the first time the precession of the Earth axis, using as the Earth model a rigid body (mantle) containing cavities filled with an incompressible vortex fluid (core). Using simplified and integrable Poincaré equations, Volterra posed the model problem of gyrostats, which he applied to describe the motion of the Earth poles.

3 Integrable cases. The classical period. As mentioned above, for various forms of equations, finding cases that are fixed by restrictions on parameters and initial conditions and finding cases of explicit solvability of the problem in quadratures (i.e., integrable cases, as they are called nowadays) was considered to be of primary importance in the classical period.

Integrable cases are usually associated with the names of those who have discovered them. We have already mentioned the cases bearing the names of Euler, Lagrange and Kovalevskaya. Other names include those of the well-known Western mathematicians and mechanics researchers: G. Kirchhoff, A. Clebsch, P. Appell, F. Brun, V. Volterra; major advances are due to Russian scientists such as A. M. Lyapunov, V. A. Steklov, N. E. Zhukovsky, S. A. Chaplygin, and D. N. Goryachev. In this sense, rigid body dynamics can be regarded as an area that is the most rich in interesting, nontrivial integrable problems, which comprise a precious heritage of treasures for the study of modern dynamics.

In the classical period, great emphasis was placed not only on finding first integrals, but also on obtaining explicit solutions in various classes of functions (mainly elliptic ones). Particular progress along these lines was made by S. V. Kovalevskaya, V. Volterra, G. Halphen, F. Kötter, and their techniques are still being adopted.

4 Nonintegrability results by Poincaré and Kozlov. Qualitative analysis. In the first half of the 20th century, interest in searching for integrable cases waned somewhat. This is mainly due to the fact that many mathematicians gained insight into the results of H. Poincaré on the nonintegrability of a typical Hamiltonian dynamical system [474] (in particular, the well-known three-body problem). As a consequence, many results of the classical works lost much of their value in the eyes of mathematicians and this prompted them to develop new methods of perturbation theory: the principle of averaging, KAM-theory etc.

The basic equations of rigid body dynamics are generally nonintegrable as well, and hence they exhibit complicated, unpredictable behavior [474], which are the subject of a new area of research called *determined chaos*. A systematical treatment of the effects of nonintegrability in rigid body dynamics is given in the book by V. V. Kozlov [333]. We have gathered the main results on the nonintegrability of equations of rigid body dynamics in one of the appendices.

There is another fact which makes the book by V. V. Kozlov [333] so important. Unlike the authors of classical works who had an unnatural propensity to obtain explicit solutions telling us little about the real motion of the system, he raises the problem of qualitative analysis of integrable dynamical systems, and, using the Kovalevskaya and Goryachev – Chaplygin tops as examples, draws general conclusions on the behavior of the line of nodes and angles of proper rotation. The latest results were obtained by applying the Liouville – Arnold theorem and the Weyl theorem on uniform distribution.

5 Poisson structures and Lie algebras. In addition to the idea of a wide application of computer methods, we tried to present here in more detail the modern methods of Poisson dynamics and geometry, the theory of Lie groups and Lie algebras, which we had outlined only roughly in our previous book, *Poisson structures and Lie algebras in Hamiltonian mechanics* [95]. Rigid body dynamics plays a special role in developing these methods. As mentioned above, it is a good area for testing new mathematical theories, and its significance, particularly to the development of many branches of topology and nonlinear Poisson structures, nonholonomic geometry, the theory of symmetries and tensor invariants, can hardly be overestimated.

It may even be asserted that, just as the analysis of the three-body problem has enabled the understanding of the profound ideas of H. Poincaré about the nonintegrability of dynamical systems, the results and methods of Sophus Lie have been integrated into the general mathematical culture, as a consequence of their application to the dynamics of tops, which provide examples of mechanical realization of the most natural Lie groups and Lie algebras. Moreover, as opposed to celestial mechanics, vibration theory and vortex dynamics, rigid body dynamics contains, on the one hand, a number of nontrivial integrable cases and, on the other hand, in view of the compactness of configuration space, is more preferable for analysis of chaotic motions.

6 Topological analysis. The application of methods of topological analysis to the integration of problems of rigid body dynamics, namely, the study of rearrangements of Liouville tori when a parameter passes through critical values, was first proposed by M. P. Kharlamov [286] and developed further in the theory of topological invariants, which was created (mainly by the school of A. T. Fomenko) for the classification of integrable Hamiltonian systems with two degrees of freedom. Most of the results obtained by these techniques are presented in the book [66]. Complex methods which lead basically to the same results are presented in the book by M. Audin [25].

7 The L-A pair method. The development of the method of isospectral deformation (Lax representation, L-A pair) revived interest in integrable problems of rigid body dynamics in 1970–1990. As a result, quite a number of new integrable cases were found. As a rule, most publications of that period were concerned with multidimensional generalizations of well-known natural physical analogs. The development of this avenue of research is also due to the fact that the ideas of the theory of Lie groups and Lie algebras and ideas of analysis of general (nonlinear and degenerate) Poisson structures have found their way into rigid body dynamics. An account of the current status of research on these topics can be found in our book [87].

It turned out that many constructions of the Lie-algebraic approach and methods of qualitative analysis can be extended to nonholonomic problems of rigid body dynamics, where some new integrable systems [90] have been added for the last few decades.

8 Systems of analytical computations and first integrals. We also note that systems of analytical computations play a large role in the problem of finding high-degree integrals. One of the first integrable cases obtained by this method is the Sokolov case in the Kirchhoff equations (2001). Since then, rigid body dynamics has been enriched with a number of more complex first integrals, even those of degree 6, and the possibilities of finding them have obviously not been exhausted.

It has become standard practice to apply systems of analytical computations to stability analysis of fixed points and periodic solutions by normalization methods. This analysis is widely used in studies on rigid body dynamics, where there exist many interesting partial solutions which provide fertile ground for such activities (A. P. Markeev, [413, 414, 415, 416]).

9 The motion of a rigid body in an ideal fluid. One of the new problems addressed in this book and worthy of special note is that of a heavy rigid body moving in an ideal fluid. The first problems of the motion of a rigid body interacting with a liquid medium go back to Maxwell, Kirchhoff, Lamb, Zhukovsky and Chaplygin. However, in the general case such problems are very complicated and still remain unresolved. During a real motion one should take account of the lifting force and various drag forces due to vortex shedding and friction of the fluid with the surface of the body. Generally speaking, most interesting effects arising from interaction between the medium and the body are due to viscosity. One usually uses phenomenological models of interaction which incorporate viscosity as one of the force (as a rule, nonpotential) factors. However, even if viscosity is excluded altogether, the problem can be of great interest, since there remain factors of fluid resistance due to added masses and the inertia of fluid. In this setting, problems of a heavy rigid body falling in a fluid were considered as far back as the end of the 19th century by V. A. Steklov, S. A. Chaplygin, and D. N. Goryachev. The rapid development of aerohydrodynamics in the early 20th century and the explanation of the nature of the lifting force by Kutta, Zhukovsky and Chaplygin made it possible to introduce into these equations additional terms related to circulation. It turned out that all these formulations of the problem *keep us within the scope of Hamiltonian formalism* and can be used to explain the results of various experiments. We note that full-scale experiments with falling plates or disks are still being conducted in various hydrodynamical laboratories both in Russia and abroad, but none of the existing theories is capable of completely explaining the results of such experiments.

A more accurate description of the fall is possible beyond the scope of Hamiltonian formalism — dissipative factors need to be taken into account. However, serious research on this would have to be fairly extensive and, although there have been a number of outstanding publications on these topics, we do not discuss them in this book.

10 Physical applications. In recent decades, a few other lines of research have emerged that are concerned with the dynamics of tops. One of them arose in quantum mechanics from the analysis of systems of interacting spins with anisotropy (the Heisenberg chain or XYZ-model).

The classical model is here a basis for understanding the dynamics at the quantum level, which, in a sense, can also be integrable and chaotic. Research on quantum chaos is only beginning to emerge [560, 481], but evidently will soon develop into a separate branch of science, where a great deal of attention will be given to the quantum description of tops. This is primarily due to the

fact that the model of a top is the main one in the quantum theory of angular momentum,which is used in quantum chemistry and molecular spectroscopy.

It is also interesting to note that the integrability conditions and the integrals for the spin model which are presented in the modern literature on quantum mechanics (see, e.g., [481], 2004) are simplified results obtained in classical works (W. Frahm, F. Schottka) over a century ago. This is due to the fact that many of modern physicists who might have gone a bit too far in their abstract and intricate theories (like quantum field theory or the theory of gravitation) do not have a very good understanding of problems that have a natural origin and are related to the dynamics of a usual spinning top used as a toy.

11 Computer methods of analysis. In a sense, even in the analysis of an integrable situation for which in principle a complete classification of all solutions can be given, computers opened up a new era. Previously, integrable systems had been studied predominantly by analytical methods allowing one to obtain explicit quadratures and geometric interpretations, which in many cases looked rather artificial. A combination of the ideas of topological analysis (bifurcation diagrams), stability theory, the method of phase sections and immediate computer visualization of “particularly special solutions” can provide a better illustration of the particularity of an integrable situation and of the most distinctive features of motion, and can be useful in obtaining a number of new results even for seemingly well-understood problems (such as the Kovalevskaya and Goryachev – Chaplygin tops, and the Bobylev – Steklov solution). The thing is that these results are very difficult to spot in cumbersome analytical expressions. After the computer has identified the results, the proof of these facts can also be obtained analytically. Of special note is the fact that to date no analysis has been made of rigid body motion in absolute space.

Some interesting motions of integrable tops may be capable of giving rise to concrete ideas concerning their practical application. We recall, for example, that so far no application has been found for the Kovalevskaya top discovered over a century ago, since almost nothing has been known of its motion, in spite of the complete solution in terms of elliptic functions.

In this book we present analytical expressions for families of doubly asymptotic solutions generated by unstable periodic solutions. For integrable cases the behavior of the system is the most complicated and looks irregular even despite the existence of an additional integral (recall that in the general case the motion of an integrable system is regular). Under perturbations such solutions are the first to break down, and regions appear which are filled with "real" chaotic trajectories and are located in phase space near these solutions.

Computer analyses give ample cause for "revision" and make it possible to

grasp the real significance of analytical investigations. While some analytical results (concerning, for example, the separation of variables) turn out to be useful for the study of bifurcations and various topological problems, it is as good as useless to "develop" them further to obtain explicit quadratures (in terms of θ-functions). These results are contained, for example, in [228, 162], but they can be used only as exercises in differential equations rather than as methods of dynamical analysis.

12 Transition to chaos. The value of the results of the classical works in rigid body dynamics was questioned as far back as the 1970s by K. Magnus [401]. The epoch of faith in boundless opportunities offered by computers led to the conviction that all these results are useless and that a sufficiently powerful computer is capable of forecasting the motion on any time interval with fair accuracy. However, in view of the exponentially fast divergence of trajectories (due to instability in some regions of phase space) in typical dynamical systems, which are nonintegrable, such a computation on sufficiently large time intervals has no physical meaning, since the initial conditions for specific (applied) systems are always given with some error.

It seems that numerical methods can be relied on only in an integrable situation in which no such divergence occurs. However, it turns out that conservative systems retain many elements of integrable dynamics even in a stochastic situation. Under small perturbations of the integrable problem, nondegenerate periodic orbits persist and most of quasi-periodic motions (KAM-theory) do not decay.

As perturbations grow further, both periodic orbits and invariant tori undergo various bifurcations, which follow some general patterns. They determine changes in the whole structure of the phase flow, which combines regions with regular and chaotic behavior, and define scenarios of transition to chaos. The best-known scenarios describing the evolution of the dynamical behavior of the system from regular to chaotic behavior include the homoclinic structure (discovered by H. Poincaré) and cascades of period doubling bifurcations (discovered by M. Feigenbaum) possessing the universality property. In Hamiltonian systems, as parameters change, these two effects coexist with each other, determining the evolution of the phase portrait. Evidently, in various classes of problems we have different combinations of the two scenarios, although general patterns can be observed as well. In rigid body dynamics (as opposed, for example, to celestial mechanics) there has been almost no research along these lines (which, by the way, is impossible without high-precision computer simulations). In this book we have gathered perhaps nearly all research results in this area, which, unfortunately, are very scarce.

13 Probabilistic effects. We also note that in rigid body dynamics, even in a Hamiltonian setting, there are a number of systems which cannot be described using the standard methods of Hamiltonian chaos. These systems arise in the description of a heavy rigid body falling in a fluid. The behavior of such systems, despite their Hamiltonian nature, is similar to that of dissipative systems and is due to the fact that asymptotic behavior is different under changing initial conditions. It turns out that in the general case the dependence of asymptotic behavior on the initial conditions is very complicated (fractal), so that only a probabilistic description is possible. It has already long been standard practice to use such a description to investigate the motion of a rigid body under the action of constant and dissipative moments [442]. However, there is no relevant theory to describe the motion of the body in a fluid, and so computer methods have to be applied.

Another interesting and nonclassical problem is that of rigid body motion (in vacuum or in a fluid) with the geometry and dynamical characteristics changing with time. This problem goes back to J. Liouville and is closely related to his equations of a variable rigid body. In this book we present a number of results pertaining to the hydrodynamical problem of self-propulsion (i.e., to the possibility of reaching a given point in space by controlling the geometry and the dynamical characteristics of the body) and to the problem of adiabatic (slow and periodic) behavior of such systems. By the way, the problem of self-propulsion and controllability of motion is closely related to problems concerned with the mechanism of locomotion of fish and to the popular problem of a falling cat, which rights itself as it falls to land to its feet,irrespective of its initial orientation.

14 It is beyond the scope of this book to discuss problems of the stability of special motions and most applied and engineering problems, since a complete and adequate treatment of these would require a separate book. However, even physicists and engineers can gain from this book an understanding of the general formalism of writing the basic dynamical equations, as well as the main aspects of regular and chaotic behavior in rigid body dynamics. This book may be regarded as a reference book on these topics, but it also elucidates, as far as possible, the derivation of the main results and sometimes presents complete proofs.

We have not included sections on nonholonomic systems and multidimensional generalizations of rigid body dynamics. Many of the topics mentioned here have been treated in the books [90, 95, 87].

Of special note are nonholonomic systems describing the rolling motion of rigid bodies on each other without slipping. Traditionally, discussions of rolling

problems are included in courses on rigid body dynamics (Routh [498], MacMillan [400], Magnus [401] and others). However, this area has flourished very recently; besides, many dynamical effects in nonholonomic systems have been found to differ so much from Hamiltonian ones that they require separate research (see, e.g., [90]).

15 A unique feature of this book is a wide use of numerical experiments and methods of computer visualization coupled with analytical methods. We also present the basic historical facts and a fairly detailed list of references.

Almost all modern and classical integrable cases were verified by using the computer software package MAPLE. It turned out that some well-known results are not quite correct, and the other results were found to be considerably simplified. The computer visualization of motion and numerical integration were performed by using the software package CHAOS, developed at the Institute of Computer Science.

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The Creators of Rigid Body Dynamics

This section presents, in chronological order, brief biographies of some of the scientists whose names are mentioned throughout the book. In most cases we describe only their achievements in the development of rigid body dynamics, although the results obtained by them in other fields of mathematics and mechanics are often no less notable. These short essays can be useful for understanding the evolution of the main ideas and methods of rigid body dynamics, and we shall return to the historical aspects of its development in the course of the book.


L. Euler

Euler, Leonard (15.4.1707 – 18.9.1783) was a great mathematician and mechanician. He was born in Switzerland and spent a large part of his life in Russia (1727–1741, 1766–1783). Euler made significant contributions to almost all areas of mathematics, and it is difficult to give an overview of his output, which extends to more than 860 works. Euler's contribution to other sciences including shipbuilding, artillery, turbine theory, and material resistance is also considerable.

For the theory of rigid body dynamics, Euler elaborated the moment-of-inertia theory and obtained a formula for the velocity distribution in a rigid body. In 1750 he obtained the equations of motion in a fixed coordinate system, which turned out to be less useful for applications. In a number of works written during the period 1758–1765, Euler used the notion, invented by him, of a coordinate system fixed in the body to obtain the equations — later to be called the Euler–Poisson[1] equations — in the form in which they are still commonly studied today.

These equations are formulated by means of the so-called Euler angles. He obtained the kinematic relations for the motion of a heavy rigid body, now called the Euler relations, and found the integrability case for which the force of gravity is absent. Euler reduced this integrable case to quadratures and examined various particular solutions.

1 Probably, the contribution of Poisson reflected in their name is the systematical presentation given by him in his well-known course of mechanics.

J. L. Lagrange

Lagrange, Joseph Louis (25.1.1736 – 10.4.1813) was a great French mathematician, mechanician, and astronomer. In his famous two-volume treatise *Analytical Mechanics*, he developed a general formalism for dynamics. He presented the equations of rigid body motion in an arbitrary potential force field using a coordinate system attached to the body, the projections of angular momentum, and the direction cosines (Volume II). He also presented the integrability case, which is characterized by having axial symmetry and which was reduced by him to quadratures. Following his principle of avoidance of graphical representation, Lagrange did not provide a geometric study of the motion, and the pictures of the apex behavior, which entered almost all mechanics textbooks after him, made their first appearance in the work of Poisson (1815), who treated this problem as if it were a completely new one[2]. Lagrange simplified the solution for the Euler case and gave a direct proof of the existence of real roots for the third-order equation which determines the position of principal axes. He also developed perturbation theory, which Jacobi was to make use of later to obtain the system of "osculating" variables for studying perturbations of the Euler top.

L. Poinsot

Poinsot, Louis (3.1.1777 – 5.12.1859) was a French engineer, mechanician, and mathematician. He introduced the concepts of ellipsoid of inertia and instantaneous axis of rotation and gave a geometric interpretation of the Euler case in which there arise the curves of polhodes and herpolhodes introduced by him (1851). He presented a geometric stability analysis of the rotation of a rigid body about the principal axes of the ellipsoid of inertia. In contrast to Lagrange, Poinsot insisted on the advantage of geometric methods in mechanics as opposed to analytical methods: "In all these solutions, we see only calculations without any clear picture of the body motion" [475][3]. The ideas of Poinsot were later supported and developed by N. E. Zhukovskii and S. A. Chaplygin. Also, Poinsot used a geometric approach in studying statics (*Elements of statics*, 1803).

2 Poisson did improve the notation which previously had made it difficult to understand the treatises of D'Alembert, Euler, and Lagrange, and presented various particular cases of motion: in some textbooks, the Lagrange case is called the Lagrange–Poisson case.

3 Translated from French into English.

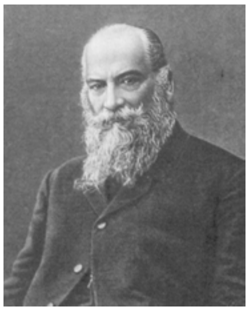
Kirchhoff, Gustav Robert (12.3.1824 – 17.10. 1887) was a German physicist and mathematician. In his *Lectures on Mathematical Physics* (1874–1894, Vols. 1–4) he laid the foundations of modern elasticity theory, hydrodynamics, optics, and electrodynamics. He showed the analogy between the Euler–Poisson equations and the equations of the elastic line bend. Developing ideas of Thomson and Tait, he was able to reduce the problem of rigid body motion in an ideal fluid to a system of ordinary differential equations. He found the integrable case, which is characterized by axial symmetry, presented its solution in elliptic functions and studied various particular motions.


G. R. Kirchhoff

Clebsch, Rudolph Friedrich Alfred (19.1. 1833 – 7.11.1872) was a German mathematician and mechanician. He founded the journal *Mathematische Annalen*, which is still one of the leading mathematical journals. He was a specialist in projective geometry and the theory of invariants of algebraic forms. He proposed a change of representation for the Kirchhoff equations, equivalent to passing from the Lagrangian description to the Hamiltonian one. Using the new equations, he was able to find a new case having an additional quadratic integral which, as became clear later, is identical to the Brun and Tisserand integrals.


R. F. A. Clebsch

Zhukovskii, Nikolai Egorovich (17.1.1847 – 17.3.1921) was a Russian mechanician, mathematician, and engineer. He was hailed by V. I. Lenin as the "father of Russian aviation". In his master's thesis (1876), he laid the foundations for the theory of the motion of a rigid body with cavities filled with an ideal incompressible fluid. For multiply connected cavities, he showed that the equations of motion of the system are equivalent to those modelling the motion of a gyrostat – a rigid body with a fly wheel. He introduced the appropriate dynamical characteristics and calculated them for cavities of various shapes. He showed the integrability of a free gyrostat, whose explicit solution was subsequently obtained by V. Volterra using elliptic functions (1899). He studied "plane" motions of a rigid body in Lobachevskii space. He put forward a geometrical interpretation of the Kovalevskaya case and his own method, making use of a certain auxiliary curvilinear coordinate system, for reducing it to quadratures. He observed that the center-of-mass motion in the Hess case is pendulum-like, and suggested an interesting geometric study for it. In connection with his studies in hydro- and aero-mechanics, he considered a number of


N. E. Zhukovskii

model statements of problems of the plane motions of plates under the action of the lift force due to circulation. In mechanics, following Poinsot, N. E. Zhukovskii held the geometrical picture of a motion to be the ideal form of solution, although his geometric interpretations of the motions of the free gyrostat and the heavy rigid body case of Kovalevskaya are neither very simple nor very natural.

S. V. Kovalevskaya

Kovalevskaya, Sof'ya Vasil'evna (15.1.1850 – 10.2.1891) was a famous mathematician. In 1874 she defended her dissertation in Göttingen and was granted a Ph.D. In 1884 she was appointed to a five year position as "Professor Extraordinarius" (Professor without Chair) at the University of Stockholm and became a member of the editorial board of the journal *Acta Mathematica*. In 1889 she was appointed Professor Ordinarius (Professorial Chair Holder) at the same university and elected a corresponding member of the Petersburg Academy of Sciences. She was the first woman in the world to become a professor of mathematics.

For the discovery of the third integrability case of Euler–Poisson equations (the first two being those of Euler and Lagrange), she was awarded the Bordin prize (1888), and for another work on rigid body rotation, she was awarded a prize by the Swedish Royal Academy of Sciences. In these works she proposed a new method, now known as the Kovalevskaya method and still widely used, which furnishes a test for integrability related to the behavior of the general solution on the complex plane of time. She then also obtained the explicit solution by quadratures using theta-functions of two variables. Up to the present day, the transformations carried out by Kovalevskaya are considered to be far from trivial and have never been improved upon.

Kovalevskaya also dealt with general problems of integrating partial differential equations (producing the so-called Cauchy–Kovalevskaya theorem), analysis of the stability of the annuli of Saturn, and of light propagation in crystals.

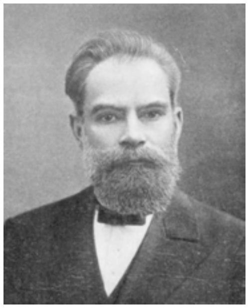
Having a literary talent and having lived through a fascinating historical period in Europe, she left us the additional inheritance of several novels and memoirs, which remain popular today.

Poincaré, Henry Jules (29.4.1854 – 17.7.1912) was a famous French mathematician, physicist, astronomer, and philosopher. In his three-volume treatise *New Methods of Celestial Mechanics*, examining the three-body problem, he initiated the qualitative study of dynamical systems and discussed obstructions to the existence of analytical first integrals for a wide class of dynamical systems. He proposed (without proving them) arguments for such obstructions appropriate to the Euler–Poisson equations. He established a new form of dynamical equations on a group, which systematized the particular results of Euler and Lagrange and turned out to be the most suitable point of view for various problems of rigid body dynamics. The Hamiltonian variant of these equations was suggested by N. G. Chetaev.

H. Poincaré

Poincaré applied the formalism developed by him to derive the equations of motion for a rigid body having cavities filled with a vortex ideal incompressible fluid. For these equations, he found the integrability case characterized by having a dynamical symmetry. Moreover, he obtained the elliptic quadrature and used it to explain various effects of the Earth's precession; he pictured the Earth as a rigid shell filled with a fluid nucleus. Also, he found explicit formulae for the frequencies of small oscillations and obtained the necessary conditions for stability.

Lyapunov, Aleksandr Mikhailovich (6.6. 1857 – 3.11.1918) was a famous Russian mathematician and mechanician, the creator of the stability theory of a dynamical system. He discovered one of the integrability cases of the Kirchhoff equations for a rigid body moving in a fluid. In a vast memoir of 1888, he showed and studied the stability of the rigid body screw motions in a fluid. He clarified the Kovalevskaya method mentioned above, demonstrating that her arguments concerning the uniqueness of solutions in the integrable cases had been correct. He then proposed a method of his own (the so-called Kovalevskaya–Lyapunov method) consisting of the introduction of a small parameter and analysis of the variational equation.



A. M. Lyapunov

V. A. Steklov

Steklov, Vladimir Andreevich (9.1.1864 – 30.5.1926) was a Russian mathematician and mechanician, a student of A. M. Lyapunov. In 1894 he defended his master's thesis *On Rigid Body Motion in a Fluid* (published in 1893), in which he gave a new integrability case of the Kirchhoff equations and proved the theorem on the impossibility of other cases for which the additional integral is quadratic.

He found the equivalence between the Clebsch integrable case of Kirchhoff's equations and the Brun problem. In 1909, he found a new integrable family for the problem of the motion of a rigid body with cavities filled with fluid (Poincaré–Zhukovskii equations). He presented two particular solutions of the Euler–Poisson equations (one of which was found simultaneously by D. K. Bobylev).

S. A. Chaplygin

Chaplygin, Sergei Alekseevich (5.4.1869 – 8.10.1942) was a Russian mathematician and mechanician, one of the founders of modern hydro-aerodynamics. He found the particular integrability case of the Euler–Poisson equations for zero area constant, generalizing the more special solution of D.G. Goryachev and also more particular solutions characterized by a system of linear invariant relations. For the Kirchhoff equations, he also found an analogous particular case of partial integrability and its generalizations, studied the screw motions, and gave a geometric interpretation of various other motions (in particular, for the Clebsch case). He derived the equations of motion of a heavy rigid body in a fluid and studied the cases of plane and axially-symmetric motions in detail.

His reputation is especially due to his work on nonholonomic mechanics. In this area he found a number of integrable problems of rigid body dynamics: the rolling of an axially-symmetric body on a plane, the Chaplygin ball, the Chaplygin sleigh, etc. Like N. E. Zhukovskii, he strove for geometric clarity in his masterly analytical calculations.

Goryachev, Dmitrii Nikanorovich (20.10. 1867–10.7.1949) was a Russian mechanician, a student of N. E. Zhukovskii. He found an integrability case and existence of particular solutions in the Euler–Poisson equations for zero area constant. He studied the equations of rigid body motion in several force fields. He found a certain general form of potential for the rigid body system, admitting integrals of third and fourth degrees. He worked on the problem of heavy rigid body motion in a fluid. His master's thesis *On Some Cases of Motions of Rectilinear Parallel Vortices* defended in 1899 became a classical work in vortex structure dynamics. He also worked on the well-known three-body problem from celestial mechanics.

D. N. Goryachev

Kozlov, Valerii Vasil'evich (was born 1.1. 1950) is a Russian mathematician and mechanician, a member of the Russian Academy of Sciences (since 2000). In a number of works collected in the book *Qualitative Analysis Methods in Rigid Body Dynamics* (MGU, 1980), he proved the nonexistence of analytical integrals of the Euler–Poisson equations and also showed the dynamical effects preventing the integrability of these equations: the separatrix splitting and the birth of a large number of nondegenerate periodic solutions. These studies "closed" the problem of Poincaré posed in his *New Methods of Celestial Mechanics* (Vol. 1), thus opening a new epoch in the theory of rigid body dynamics: now, rather than searching for particular solutions with a given algebraic structure, it is qualitative methods for the study of general solutions which are of prime interest.

V. V. Kozlov

Also, V. V. Kozlov suggested new methods for analyzing integrable systems based on the use of the geometric Liouville–Arnol'd theorem and Weyl's uniform distribution theorem. Justifying the Kovalevskaya method, V. V. Kozlov proved a number of assertions relating the branching of the general solution on the complex time plane to the nonexistence of single-valued first integrals (Painlevé–Golubev conjecture). To find periodic solutions for rigid body dynamics, he introduced the use of variational methods. V. V. Kozlov has written a large number of works related to the motion of a body in a fluid in which many classical problems were solved and also new models and methods based on qualitative analysis are presented. In recent years he has turned his attention to problems of statistical mechanics.

1 Rigid Body Equations of Motion and Their Integration

1.1 Poisson Brackets and Hamiltonian Formalism

1 Poisson manifolds

The majority of problems to be considered in this book can be written in canonical Hamiltonian form and also possess a first integral—the energy integral. However, in many cases, the equations of motion admit a more convenient algebraic form based on some different system of variables and simplifying the search for integrals and particular solutions, the analysis of stability, etc. In this algebraic form, many properties of canonical Hamiltonian systems are preserved, but also some characteristic distinctions appear; these are studied in the general theory of Poisson structures (for details, see [95, 577, 418]).

We briefly present some basic definitions and results necessary for studying problems of rigid body dynamics. Note that the development of the theory of Poisson structures was in many respects motivated by the dynamics of tops, which allows one to make abstract formulations of many theorems more evident and natural.

We refer the reader to the textbooks [18, 21, 170, 418] on differential and symplectic geometry. All results of this section can be represented in coordinate form; the reader can ignore formal mathematical terminology, which is based on simple dynamical facts, but may seem to be alien to them on first acquaintance.

Poisson brackets and their properties. The equations of dynamics can be written in standard *Hamiltonian form*

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}, \qquad H = H(\boldsymbol{q}, \boldsymbol{p}), \tag{1.1}$$

where the canonical coordinates $(\boldsymbol{q}, \boldsymbol{p})$ are defined on an even-dimensional manifold $(\boldsymbol{q}, \boldsymbol{p}) \in M^{2n}$, called the *phase space*. The function H is called the *Hamiltonian*. The number $n = \frac{\dim M}{2}$ is called the number of degrees of freedom of the Hamiltonian system (1.1).

The divergence of the vector field (1.1) vanishes, i.e., the phase flow is incompressible(*the Liouville theorem*).

If we introduce the *Poisson bracket* of two functions F and G by the formula

$$\{F, G\} = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right), \tag{1.2}$$

then (1.1) can be rewritten as

$$\dot{q}_i = \{q_i, H\}, \qquad \dot{p}_i = \{p_i, H\}. \tag{1.3}$$

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The evolution of any differentiable function $F = F(\boldsymbol{q}, \boldsymbol{p})$ is governed by the Hamiltonian law:

$$\dot{F} = \{F, H\}. \tag{1.4}$$

Equations (1.1) are not invariant under arbitrary transformations of coordinates. Moreover, the form (1.1) of the main equations of rigid body dynamics does not possess good algebraic properties but has singularities that are not related to the nature of the problem considered (see Sec. 1.3). Before discussing more convenient forms of the equations of motion, which preserve the main properties of the canonical form, we present an invariant exposition of Hamiltonian dynamics.

In the invariant construction of the Hamiltonian formalism, following P. Dirac, we start from Eqs. (1.3) and postulate the following properties of *Poisson brackets* defined for functions given on a manifold M, of arbitrary dimension with coordinates $\boldsymbol{x} = (x^1, \dots, x^n)$:

1° $\{\lambda F_1 + \mu F_2, G\} = \lambda\{F_1, G\} + \mu\{F_2, G\}$, $\quad \lambda, \mu \in \mathbb{R}$ (*bilinearity*),
2° $\{F, G\} = -\{G, F\}$ (*skew symmetry*),
3° $\{F_1 F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}$ (*the Leibniz rule*),
4° $\{\{H, F\}, G\} + \{\{G, H\}, F\} + \{\{F, G\}, H\} = 0$ (*the Jacobi identity*).

The Poisson bracket $\{\cdot, \cdot\}$ is also called a *Poisson structure* and the manifold M on which it is defined is called a *Poisson manifold*.

In this definition, we omit the *nondegeneracy* property (i.e., for any function $F(\boldsymbol{x}) \not\equiv$ const, there exists $G \not\equiv$ const, $\{F, G\} \not\equiv 0$), which obviously holds for the canonical structure (1.2). This allows one, for example, to introduce a Poisson bracket for odd-dimensional systems. In our considerations, a Poisson structure may be *degenerate* and possess *Casimir functions* $F_k(\boldsymbol{x})$ that commute with all variables x_i and, therefore, with all functions $G(\boldsymbol{x})$ on M: $\{F_k, G\} = 0$. Casimir functions are also called *central functions, Casimirs, or annihilators*.

Properties 1° – 4° allow us to express the Poisson bracket of functions F and G in explicit form in coordinates

$$\{F, G\}(\boldsymbol{x}) = \sum_{i,j} \{x^i, x^j\} \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial x^j}. \tag{1.5}$$

The basic brackets $J^{ij} = \{x^i, x^j\}$ are called *structure functions* of a Poisson manifold M with respect to a given (generally speaking, local) coordinate system $\boldsymbol{x} = (x_1, \dots, x_n)$ (see [21, 450]). They form the *structure matrix (tensor)* $\mathbf{J} = \|J^{ij}\|$ of size $n \times n$.

If

$$\mathbf{J} = \begin{pmatrix} 0 & \mathbf{E} \\ -\mathbf{E} & 0 \end{pmatrix}, \quad \mathbf{E} = \|\delta_i^j\|, \tag{1.6}$$

then we obtain the canonical Poisson bracket defined by formula (1.2).

The following conditions satisfied by the structure matrix $\mathbf{J}(\boldsymbol{x})$ can be deduced from 1° – 4°:

I. skew symmetry:

$$J^{ij}(\boldsymbol{x}) = -J^{ji}(\boldsymbol{x}), \tag{1.7}$$

II. the Jacobi identity:

$$\sum_{l=1}^{n} \left(J^{il} \frac{\partial J^{jk}}{\partial x^l} + J^{kl} \frac{\partial J^{ij}}{\partial x^l} + J^{jl} \frac{\partial J^{ki}}{\partial x^l} \right) = 0. \tag{1.8}$$

Therefore, for example, an arbitrary constant, skew-symmetric matrix $\|J^{ij}\|$ determines a Poisson structure.

The invariant object determined by the tensor $\mathbf{J}$ is a bivector (bivector field):

$$\mathbf{J}(dF, dG) = \sum J^{ij}(\boldsymbol{x}) \frac{\partial F}{\partial x^i} \wedge \frac{\partial G}{\partial x^j},$$

where dF is a covector with components $\frac{\partial F}{\partial x^i}$.

A smooth function H on the manifold defines the vector field $\boldsymbol{X}_H = \{\boldsymbol{x}, H\}$, called a Hamiltonian system. It has the following coordinate representation:

$$\dot{x}^i = X_H^i = \{x^i, H\} = \sum_j J^{ij}(x) \frac{\partial H}{\partial x^j}. \tag{1.9}$$

The function $H = H(\boldsymbol{x})$ is called the *Hamiltonian* or *Hamiltonian function*.

The commutator of vector fields and the Poisson brackets are related by

$$[\boldsymbol{X}_H, \boldsymbol{X}_F] = -\boldsymbol{X}_{\{H,F\}}.$$

It is easy to verify that any Hamiltonian vector field generates a transformation (phase flow) preserving the Poisson brackets.

A function $F(\boldsymbol{x})$ is called *first integral* if its derivative along the vector field vanishes, $\dot{F} = \boldsymbol{X}_H(F) = 0$; this condition is equivalent to the relation $\{F, H\} = 0$.

The system

$$F_1(\boldsymbol{x}) = 0, \ \ldots, \ F_k(\boldsymbol{x}) = 0 \tag{1.10}$$

determines a *system of invariant relations* (defining the *invariant manifold*) if $\{F_i, H\} = 0$ on the submanifold defined by conditions (1.10).

Nondegenerate brackets. Symplectic structures. If a Poisson bracket is nondegenerate, then we can uniquely assign to it a closed, nondegenerate 2-form. Indeed, for any smooth function F, the operation $\boldsymbol{X}_F = \{F, \cdot\}$ is a differentiation and defines a tangent vector at any point of M. Using $1^\circ - 4^\circ$, we can prove that any tangent vector at a point of the manifold can be represented in such a form.

We define the nondegenerate 2-form ω^2 by the rule

$$\omega^2(\boldsymbol{X}_G, \boldsymbol{X}_F) = \{F, G\}.$$

Axioms $1^\circ - 4^\circ$ imply that this 2-form is bilinear, skew-symmetric and closed. It is also nondegenerate. This 2-form is called a *symplectic structure* and the manifold M is a *symplectic manifold*.

In coordinates, ω^2 has the form $\sum\limits_{i,j} \omega_{ij} dx_i \wedge dx_j$, where $\|\omega_{ij}\| = \|J^{ij}\|^{-1}$; in the canonical case (1.6), $\omega^2 = \sum\limits_i dp_i \wedge dq_i$. Any symplectic structure can be locally reduced to this form; this is the *Darboux theorem* (see [450, 18, 21]). In the next section, we state this theorem in a more general form.

Symplectic foliations. Generalized Darboux theorem. If a Poisson bracket is degenerate, then the Poisson manifold (phase space) is foliated by *symplectic leaves* (*fibers*). The restriction of the Poisson structure to a symplectic leaf is nondegenerate. Usually these leaves are common level surfaces of all Casimir functions. On any leaf, the Darboux theorem holds and the equations of motion may be written in canonical form. However, the reduction to such a system is not always necessary in applied problems since this leads to the loss of algebraicity of differential equations and restrictions in the use of geometrical and topological methods.

REMARK. In rigid body dynamics, the algebraic form of the equations of motion is usually used to search for integrals, particular solutions and to perform stability analysis. This form is also preferable for numerical integration since the canonical form often gives rise to singularities related to degeneration of local variables at some points (for example, Euler angles at the poles of the Poisson sphere; see Sec. 1.3).

For problems of qualitative analysis and perturbation theory, however, the canonical form is usually preferred since the corresponding methods are most developed and algorithmized for this form.

The *rank of a Poisson structure at a point* $\boldsymbol{x} \in M$ is the rank of the structural tensor at this point (obviously, it is even). The *rank of a Poisson structure on the whole manifold* M is the maximal rank, which it has at some point $\boldsymbol{x} \in M$. For a symplectic manifold the rank of the Poisson structure is constant and maximal at each point.

We state the general *Darboux theorem* for arbitrary Poisson manifolds (see [95, 450]).

Theorem. *Let* $(M, \{\cdot\,,\cdot\})$ *be a Poisson manifold of dimension* n, *and at a point* $\boldsymbol{x} \in M$, *let the rank of the bracket* $\{\cdot\,,\cdot\}$ *be locally constant, maximal and equal to* $2r$ = *the rank of the Poisson structure. Then there exists a local system of canonical coordinates* $x_1, \dots, x_r, y_1, \dots, y_r, z_1, \dots, z_{n-2r}$, *in which the Poisson brackets have the form*

$$\{x_i, x_j\} = \{y_i, y_j\} = \{x_i, z_k\} = \{y_i, z_k\} = \{z_k, z_l\} = 0,$$
$$\{x_i, y_j\} = \delta_{ij},$$

where $1 \leqslant i, j \leqslant r, \quad 1 \leqslant k, l \leqslant n - 2r.$

In these coordinates, any symplectic leaf in some neighborhood of $\boldsymbol{x}$ is defined by the equations $z_i = c_i$ (c_i = const) and the symplectic structure on the leaf is defined by the form $\omega = \sum_i dx_i \wedge dy_i$. Note that by this theorem, a symplectic leaf $z_i = k_i$ is defined only locally, while the global Casimir functions may not exist.

Through points at which the rank of the Poisson bracket is not maximal (i.e., less than $2r$) there pass *singular symplectic leaves* (see, e.g., [95]). Systems on singular symplectic leaves often arise in mechanics (see [95, 462]).

2 The Lie – Poisson bracket

One of the most important examples of a Poisson structure is that associated with a *Lie algebra*. Let c_{ij}^k be the structure constants of a Lie algebra $\mathfrak{g}$ in a basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. The *Lie – Poisson bracket* of functions F and H defined on another (generally speaking) vector space V with coordinates $\boldsymbol{x} = (x_1, \ldots, x_n)$ and a basis $\omega^1, \ldots, \omega^n$ is defined by the formula

$$\{F, H\} = \sum_{i,j=1}^{n} J_{ij}(\boldsymbol{x}) \frac{\partial F}{\partial x_i} \frac{\partial H}{\partial x_j}, \tag{1.11}$$

where the structure tensor $J_{ij}(\boldsymbol{x}) = \sum_k c_{ij}^k x_k$ is linear in x_k. All necessary identities $1^\circ - 4^\circ$ (see Sec. 1) for the structure tensor can be obtained from the properties of the structure constants of Lie algebras:

1. $c_{ij}^k = -c_{ji}^k$,
2. $\sum_m (c_{im}^l c_{jk}^m + c_{km}^l c_{ij}^m + c_{jm}^l c_{ki}^m) = 0$.

As is known from the theory of Lie algebras, symplectic leaves of a Lie – Poisson structure are exactly orbits of the coadjoint representation of the corresponding Lie group (see [18, 21, 450]). A formal presentation and the corresponding proof can be found, e.g., in [18]. Hamilton's equations for the Lie – Poisson structure in the coordinates $\boldsymbol{x}$ are

$$\dot{x}_i = \{x_i, H\} = \sum_{k,j} c_{ij}^k x_k \frac{\partial H}{\partial x^j}. \tag{1.12}$$

Remark. Equations (1.12) can be written in a more invariant, coordinate-free form as follows:

$$\dot{\boldsymbol{x}} = \mathrm{ad}_{dH}^{\star}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathfrak{g}^{\star}, \tag{1.13}$$

where $\mathrm{ad}_{\xi}^{\star}$, $(\boldsymbol{\xi} \in \mathfrak{g})$ is the operator of the coadjoint representation of the Lie algebra $\mathfrak{g}$: $\mathrm{ad}_{\xi}^{\star} : \mathfrak{g}^{\star} \to \mathfrak{g}^{\star}$.

In rigid body dynamics, the Lie – Poisson bracket occurs very often. This is due to the fact that the configuration space of a system is, as a rule, some combination of natural Lie groups ($SO(3)$, $E(3)$, ...). However, reduction with respect to cyclic variables can give rise to nonlinear Poisson brackets (see Ch. 4, Sec. 4.1).

Next we derive the equations of rigid body motion from general dynamical principles.

1.2 Poincaré and Poincaré – Chetaev Equations

1 Poincaré Equations

The most natural and convenient form of the equations of motion of a rigid body is obtained from the general dynamical equations in *quasi-coordinates*. The Lagrangian form of these equations was obtained by *H. Poincaré* [473], and the Hamiltonian form by *N. G. Chetaev* [134]. Possible generalizations to the nonholonomic case (i.e., for nonintegrable constraints) were considered in [332, 510]. For rigid body dynamics, the Poincaré – Chetaev equations lead to Hamiltonian equations with linear structure tensor, i.e., to the Lie – Poisson structure described above (see Sec. 1.1). The Poincaré and Poincaré – Chetaev equations are derived in this section, since an account of this cannot be found in the standard literature.

Let us consider the equations of motion of a Lagrange dynamical system determined by generalized coordinates $\boldsymbol{q} = (q_1, \dots, q_n)$, with redundancy — in general, these are not independent, i.e., they satisfy $m < n$ *holonomic* constraints of the form $f_j(\boldsymbol{q}) = 0$, $j = 1, \dots, m$ and *quasi-velocities* $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$, which are expressed in terms of the generalized velocities $\dot{q}_i$ by the formulas

$$\dot{q}_i = \sum_{s=1}^{k} v_i^s(\boldsymbol{q})\omega_s, \quad i = 1, \dots, n. \tag{1.14}$$

We also assume that all holonomic constraints are taken into account, thus

$$\left(\nabla f_j, \dot{\boldsymbol{q}}\right) = \sum_{i,s} v_i^s(\boldsymbol{q})\omega_s \frac{\partial f_j}{\partial q_i} \equiv 0, \quad j = 1, \dots, m.$$

In the case where $k > n - m$ this condition requires the quasi-velocities to satisfy additional relations linear in ω_i.

The quantities ω_s are called the *Poincaré parameters* and are the components of the velocity of the system in the (generally speaking) nonholonomic basis of vector fields

$$\boldsymbol{v}^s = \sum_i v_i^s(\boldsymbol{q}) \frac{\partial}{\partial q_i}. \tag{1.15}$$

From now on it will be assumed that the vector fields form a closed system

$$[\boldsymbol{v}^i, \boldsymbol{v}^j] = \sum_{s=1}^{k} c_{ij}^s(\boldsymbol{q})\boldsymbol{v}^s, \quad i, j, s = 1, \dots, k. \tag{1.16}$$

In the case where $k \leqslant n$ this conditions is implied by integrability of the constraints [450]. If all c_{ij}^s are constant, then the fields $\boldsymbol{v}^s$ define a finite-dimensional Lie algebra. The equations of motion in the variables $(q_1, \dots, q_n, \omega_1, \dots, \omega_k)$ have the Lagrangian form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \omega_i}\right) = \sum_{r,s} c_{ri}^s \omega_r \frac{\partial L}{\partial \omega_s} + \boldsymbol{v}^i(L), \quad i = 1, \dots, k, \tag{1.17}$$

and are called the *Poincaré equations*; together with (1.14) they form a complete system of equations of motion. In (1.17), the differentiation along the vector field $\boldsymbol{v}^i$ is defined by formula (1.15).

If the Lagrange function is a homogeneous quadratic form of angular velocities (for example, the kinetic energy), then $\boldsymbol{v}_i(L) = 0$ and the system (1.17) for $\boldsymbol{\omega}$ decouples and can be integrated independently. In this case, Eqs. (1.17) are called the *Euler – Poincaré equations*.

Poincaré obtained his equations using the Hamiltonian variational principle [473]. Here we restrict our attention to the case where the number of components of the quasi-velocity $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$ coincides with the dimension of the configuration M^k-space defined by the constraints $f_j(\boldsymbol{q}) = 0, j = 1, \dots, m$, i.e., $k = n - m$. In this case Eqs. (1.17) can be obtained directly from the Euler – Lagrange equations.

Indeed, introducing on M^k the local coordinates x_i, the Euler – Lagrange equations can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \left(\frac{\partial L}{\partial x_i}\right) = 0, \quad i = 1, \dots, k. \tag{1.18}$$

By (1.14) and (1.15), the following relations hold:

$$\begin{aligned} \omega_s &= \sum_{i=1}^{k} a_s^i \dot{x}_i, \quad \dot{x}_i = \sum_{s=1}^{k} b_i^s \omega_s, \\ \boldsymbol{v}^s &= \sum_{i=1}^{k} b_i^s \frac{\partial}{\partial x_i}, \quad i,\ s = 1, \dots, k, \end{aligned} \tag{1.19}$$

where the $k \times k$ matrices $\mathbf{A} = \|a_s^i\|$, $\mathbf{B} = \|b_i^s\|$ are inverse to one another ($\mathbf{AB} = \mathbf{E}$).

Denote the Lagrange function expressed in terms of the quasi-velocities by

$$\tilde{L}(\boldsymbol{x}, \boldsymbol{\omega}) = L(\boldsymbol{x}, \dot{\boldsymbol{x}}). \tag{1.20}$$

Using (1.19) we obtain

$$\begin{aligned} \frac{\partial L}{\partial x_i} &= \frac{\partial \tilde{L}}{\partial x_i} + \sum_{k,\ s} \dot{x}_k \frac{\partial \tilde{L}}{\partial \omega_s}\frac{\partial b_s^k}{\partial x_i}, \\ \frac{\partial L}{\partial \dot{x}_i} &= \sum_{s} \frac{\partial \tilde{L}}{\partial \omega_s} b_s^i, \quad i = 1, \dots, k. \end{aligned} \tag{1.21}$$

Substitute Eqs. (1.21) into Eqs. (1.18) and multiply both sides by the matrix $\mathbf{A}$; make use of Eqs. (1.19) in the resulting system and apply the following representation of the structure coefficients in (1.16):

$$c_{sp}^r(x) = \sum_{k,\ i} a_r^k \left(b_i^s \frac{\partial b_k^p}{\partial x_i} - b_i^p \frac{\partial b_k^s}{\partial x_i} \right).$$

Collecting similar terms, we arrive at Eqs. (1.17). □

In the case where the number of quasi-velocities is greater than the dimension of the configuration space, the reasoning becomes more difficult since the matrices **A** and **B** are not square and have no inverses.

2 Poincaré – Chetaev equations

N. G. Chetaev modified the Poincaré equations (1.17) and (1.14) by applying the *Legendre transform*:

$$\begin{aligned} M_i &= \frac{\partial L}{\partial \omega_i}, \\ \sum_i \omega_i M_i - L \,|_{\boldsymbol{\omega}\to\boldsymbol{M}} &= H(\boldsymbol{M}, \boldsymbol{q}). \end{aligned} \tag{1.22}$$

The variables M_i should be seen as "*quasi-momenta*". One has $\omega_i = \partial H/\partial M_i$ and Eqs. (1.17) can be written as

$$\dot{M}_i = \sum_{rs} c_{ri}^s \frac{\partial H}{\partial M_r} M_s - \boldsymbol{v}^i(H), \quad i = 1, \ldots, k. \tag{1.23}$$

To obtain a closed system, we should combine (1.23) with Eqs. (1.14) written in the form

$$\dot{q}_i = \sum_s v_i^s(\boldsymbol{q}) \frac{\partial H}{\partial M_s}, \quad i = 1, \ldots, n. \tag{1.24}$$

The combination of Eqs. (1.23) and (1.24) forms a Hamiltonian system with (in general) a degenerate Poisson bracket defined for arbitrary functions $f(\boldsymbol{M}, \boldsymbol{q})$ and $g(\boldsymbol{M}, \boldsymbol{q})$ by the formula [134]

$$\{f, g\} = \sum_i \left(\frac{\partial g}{\partial M_i} v^i(f) - \frac{\partial f}{\partial M_i} v^i(g) \right) + \sum_{sij} c_{ij}^s \frac{\partial f}{\partial M_j} \frac{\partial g}{\partial M_i} M_s. \tag{1.25}$$

It is straightforward to verify that this bracket satisfies all the conditions $1^\circ - 4^\circ$ (see Sec. 1.1) required for it to be a Poisson structure. From relation (1.25) one can easily obtain the structure matrix J^{ij}:

$$\begin{aligned} \{M_i, M_j\} &= \sum_s c_{ij}^s(\boldsymbol{q}) M_s, \\ \{q_i, q_j\} = 0, \quad \{q_i, M_j\} &= v_i^j(\boldsymbol{q}). \end{aligned} \tag{1.26}$$

Historical comments. For dynamical equations of the form (1.23), (1.24) N. G. Chetaev [134] developed a theory of integration similar to the Hamilton – Jacobi method. In the canonical case, success in the separation of variables stems from special coordinate systems on the configuration space (e.g., elliptic or sphero-conic coordinates). However, for the algebraic form (1.23), (1.24), only trivial symmetries can be studied (for example, the symmetries that arise in the Lagrange case; see Ch. 2).

For similar reasons, Chetaev's considerations on the generalizations of the Routh theorem in the presence of a cyclic integral and reduction of order have not received further development either. For the Poincaré – Chetaev equations possessing first integrals of cyclic type, a new reduction procedure is presented in Ch. 4, Sec. 4.1, 4.2. It provides a way to obtain the equations of the reduced system in the most simple algebraic form and sometimes leads to nonlinear Poisson brackets.

3 Equations on Lie groups

As a rule, the configuration space for rigid body dynamics is a Lie group. For example, to describe the rotation of a rigid body about a fixed point the configuration space is the group $SO(3)$ and in the case of free motion of a rigid body it is the group $E(3) = SO(3) \otimes_s \mathbb{R}^3$, the semi-direct product of the rotation group $SO(3)$ and the commutative group of translations $\mathbb{R}^3$.

On a group, it is convenient to choose as a basis of the vector fields $\boldsymbol{v}^s$ (1.15) left-invariant (or right-invariant) vector fields which form the corresponding Lie algebra. In this case, the tensor c_{ij}^k is independent of the choice of coordinates and is determined by the structure constants of the Lie algebra. The brackets (1.25) define the so-called *canonical structure on the cotangent bundle* over the group [95, 466].

If the Hamiltonian H is independent of q_i, i.e., ($\boldsymbol{v}_i(H) = 0$), then the equations for quasi-momenta $M_1, \dots, M_k$ are closed and the Euler equations for the inertial motion of a rigid body can be obtained; in this case, the constants c_{ij}^s are defined by the Lie algebra $so(3)$. For arbitrary Lie algebra with structure constants c_{ij}^s, similar equations with quadratic Hamiltonian are also (as in subsection 1) called the Euler – Poincaré equations.

If the Hamiltonian H depends on coordinates but the redundant coordinates can be chosen such that all components of the left-invariant fields $v_r^s(\boldsymbol{q})$ are linear in $\boldsymbol{q}$, then the bracket (1.26) becomes the usual Lie – Poisson bracket and all geometric dependencies for redundant variables are its Casimir functions or invariant relations. This can be achieved by using the matrix realization of the Lie group taking the components of its matrices as redundant coordinates. The Lie – Poisson structure obtained in this case corresponds to the semi-direct sum $\mathfrak{g} \oplus_s \mathbb{R}^{n^2}$, where $\mathbb{R}^{n^2}$ is the space of $(n \times n)$-matrices and $\mathfrak{g}$ is the Lie algebra of the given Lie group; this structure is called the *natural canonical structure of the cotangent bundle* over the Lie group. This is the way, for example, to obtain the equations of motion of a rigid body in terms of direction cosines and angular momenta (see Sec. 1.4). The matrix realization of Lie groups is also used in the multidimensional rigid body dynamics [67, 95, 466].

The Hamilton equations on a Lie group with the natural canonical structure for problems of rigid body dynamics always possess a standard invariant measure (since in this case all groups are unimodular). This is an analog of the Liouville theorem on the solenoidality of the canonical Hamiltonian flow.

A detailed derivation of the equations of motion of a rigid body in an arbitrary potential field is considered in Sec. 1.4. More complicated equations, whose derivation is based on hydrodynamical principles, describing the motion of rigid bodies in a fluid or the motion of rigid bodies with cavities filled with fluid are considered in Ch. 3.

4 Comments

1. Thus, the Poincaré and Poincaré – Chetaev equations only represent a convenient tool for rewriting the equations of motion of a system in the Lagrangian and Hamiltonian form in an arbitrary (including redundant) system of variables. The possibility of such representation is related with the presence of a *tensor invariant*, i.e., a Poisson structure whose coordinate form depends on the choice of coordinates; moreover, for redundant variables, the Poisson structure is degenerate.

Note that the connection between the Lagrangian and Hamiltonian forms is clear for most mechanicians only when written in their canonical form. Thus, in [61], the Hamiltonian form of the dynamical equations is considered as found from some considerations that are not completely natural; in particular, with reference to [447], where the author, unfamiliar with the general formalism of dynamical equations, really "rediscovers" the Euler angles and conjugate momenta. In [61], some strange theorems are proved that state that the Lagrangian form can be obtained from the Hamiltonian form; some confusion appears since the components of the angular momentum commute equally with the components of the linear momentum and with the direction cosines, and the Kirchhoff equations can be treated as a part of momentum equations on the group $E(3)$, i.e., the Euler – Poincaré equations for $\boldsymbol{M}$ and $\boldsymbol{p}$. In the case where the potential is absent, this part can be separated from the position equations (i.e., equations for direction cosines). On the other hand, the Kirchhoff equations can be considered as Hamiltonian equations on $SO(3)$, and in this case, the components of the momentum force $\boldsymbol{p}$ must be treated as direction cosines. Note that the Steklov analogy consists of this [549] (see also Ch. 3, Sec. 3.1).

A priori, the non-Hamiltonian coordinate form of the Newton equations of the dynamics of a satellite is used in [33], where the presence of the energy integral becomes more clear.

2. There is an interesting result concerning the question of the existence of a global invariant measure for the Euler – Poincaré equations on a Lie algebra (it is assumed that the density of this measure is differentiable; see [335]). It should be emphasized that the result which follows speaks of a global measure: in general, so-called singular measures often occur; for such measures, their densities vanish or become infinite at some points (or on submanifolds of lower dimension).

Theorem. [335]**.** *The Euler – Poincaré equations possess a global invariant measure if and only if the Lie algebra* $\mathfrak{g}$ *corresponds to a unimodular group* G.

(In terms of the structure constants, the unimodularity criterion is $\sum_k c_{ik}^k = 0$).

Moreover, in the unimodular case the invariant measure is standard, i.e., it is constant on the whole Lie algebra g. This is a consequence of the following more general result: *a* system of ordinary differential equations with homogeneous right-hand sides possesses an invariant measure if and only if its phase flow preserves the standard measure.

In [335], the following incorrect conclusion was deduced from the absence of an invariant measure: "it is impossible to present the Euler – Poincaré equations in Hamiltonian form for an arbitrary Lie algebra". Actually, as was noted above, the Euler – Poincaré equations are a priori Hamiltonian equations, and on any symplectic leaf of the Lie algebra (which is an invariant manifold of the system), they can be reduced to canonical Hamiltonian form by the Darboux theorem. However, for a generic Lie algebra, symplectic leaves can have different dimensions and hence a global measure does not exist.

Let us consider the simplest example of a solvable two-dimensional Lie algebra, having the corresponding Poisson bracket $\{M_1, M_2\} = M_1$. The phase space (the plane of the variables M_1 and M_2) consists of two two-dimensional symplectic leaves, the upper and lower half-planes without the straight line $M_1 = 0$. Each point of this straight line is a zero-dimensional symplectic leaf. For an arbitrary quadratic Hamiltonian $H = \frac{1}{2}(a_{11}M_1^2 + a_{22}M_2^2 + 2a_{12}M_1M_2)$, the equations of motion have the form

$$\dot{M}_1 = M_1(a_{22}M_2 + a_{12}M_1), \quad \dot{M}_2 = -M_1(a_{11}M_1 + a_{12}M_2). \tag{1.27}$$

Since symplectic leaves are invariant, all points of the line are equilibria positions. Trajectories of the system are arcs of ellipses $H = \text{const}$; moving along them, a point approaches the line $M_1 = 0$. Obviously, the system (1.27) cannot possess an absolutely continuous global invariant measure, however, this does not contradict its Hamiltonian property. In each of the two half-planes, the system can be represented in canonical form, for example, by using the variables $q = \ln M_1$ and $p = M_2$:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

In this case, the equations and the Hamiltonian become nonalgebraic, $H = \frac{1}{2}(a_{11}e^{2q} + a_{22}p^2 + a_{12}pe^q)$ — the Hamiltonian of the simplest Toda lattice.

1.3 Various Systems of Variables in Rigid Body Dynamics

Various coordinate systems are used to describe the motion of rigid bodies. Each reveals its own advantages and disadvantages in each specific problem. For example, in the search for first integrals, study of stability, and topological analysis, variables in which the equations are polynomial (or even homogeneous) are the most convenient. For numerical simulation, it is usually more convenient to have a system of equations of lowest possible order. For qualitative, perturbative, and nonlinear normalizations problems, systems of canonical variables that reflect the specific character of the perturbed problem are preferred. We describe the most significant systems of variables used in rigid body dynamics. In practical applications, especially to gyroscopes, various combinations and modifications of these systems are also used.

1 Euler angles

Consider a rigid body rotating about a fixed point O in a potential field. The configuration space, i.e., the set of all possible positions of the body, is the Lie group $SO(3)$, and we can take three angles θ, φ, and ψ as coordinates defining the position of the body; for the standard choice these angles are called the *Euler angles* (see [16, 224]).

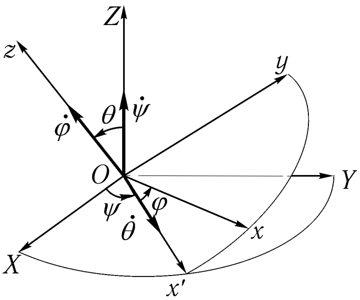

Fig. 1.1. Euler angles

At the point O, we place the vertices of two orthogonal frames: immovable $OXYZ$ and movable $Oxyz$, which is attached to the rotating body (see Fig. 1.1).

Rotation by the angle ψ (*precession angle*) about the axis OZ transfers the immovable frame into the position $Ox'y'z'$. Rotation by the angle θ (*nutation angle*) is made about the axis Ox' called the *line of nodes*. Rotation by the angle φ (*proper rotation angle*) about the axis Oz finally gives the moving frame $Oxyz$. The three rotations determined by the Euler angles θ, φ and ψ, completely define the position of the moving frame with respect to the immovable frame. The projections ω_1, ω_2, and ω_3 of the angular velocity $\boldsymbol{\omega}$ on the axes of the moving frame $Oxyz$ are expressed in terms of the Euler angles as follows:

$$\begin{aligned}
\omega_1 &= \dot\psi \sin\theta \sin\varphi + \dot\theta \cos\varphi, \\
\omega_2 &= \dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi, \\
\omega_3 &= \dot\psi \cos\theta + \dot\varphi.
\end{aligned} \tag{1.28}$$

These relations are called the *kinematic Euler formulae*. Using (1.28), one can easily obtain the Lagrange function $L = L(\varphi, \psi, \theta, \dot{\varphi}, \dot{\psi}, \dot{\theta})$ of the system (see Sec. 1.6), leading to definition of the canonical momenta (by the Legendre transform)

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}}, \quad p_\psi = \frac{\partial L}{\partial \dot{\psi}}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}}. \tag{1.29}$$

2 Euler variables. Components of momentum and the direction cosines

For this system of variables $(\boldsymbol{M}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, $\boldsymbol{M} = (M_1, M_2, M_3)$ are the components of the angular momentum in the moving frame $Oxyz$ attached to the body and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are orthogonal unit vectors of the immovable frame, written in the moving frame $Oxyz$. They are constructed as follows. The *matrix of direction cosines* (the rotation matrix) that determines the position of the body in the fixed frame

$$\mathbf{Q} = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix}, \tag{1.30}$$

is orthogonal and belongs to the group $SO(3)$.

That is,

$$\begin{aligned} (\boldsymbol{\alpha}, \boldsymbol{\alpha}) &= (\boldsymbol{\beta}, \boldsymbol{\beta}) = (\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1, \\ (\boldsymbol{\alpha}, \boldsymbol{\beta}) &= (\boldsymbol{\alpha}, \boldsymbol{\gamma}) = (\boldsymbol{\beta}, \boldsymbol{\gamma}) = 0, \end{aligned} \tag{1.31}$$

where here and in what follows, the parentheses (,) means the usual scalar product, and we find that the angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ written in the moving frame $Oxyz$ can be represented by the skew-symmetric matrix $\widetilde{\boldsymbol{\omega}} = \dot{\mathbf{Q}}\mathbf{Q}^T$, with entries $\tilde{\omega}_{ij} = \varepsilon_{ijk}\omega_k$.

Similarly, the angular velocity $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ written in the fixed frame $OXYZ$ is represented in the same fashion by the matrix $\tilde{\boldsymbol{\Omega}} = \mathbf{Q}^T\dot{\mathbf{Q}}$.

The directions of the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ in the movable and immovable frames, respectively, define conic surfaces called the *movable and immovable Poinsot axoids*. The motion of the rigid body is represented as the rolling (without sliding) of the movable axoid on the immovable axoid; each axoid touches the other along the instantaneous rotation axis. Free motions of the body (without fixed points) can be interpreted as the rolling of the movable axoid on the immovable axoid with sliding along some axis that defines the instantaneous screw motion. If we mark instantaneous values of the angular velocities on the generatrices of the axoids, then we obtain the so-called *movable* and *immovable* hodographs; in the general case, they are spatial curves.

If the angular velocity as the function of time $\boldsymbol{\omega}(t)$ is known, the orientation of the rigid body can be found by solving the Poisson equation

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \qquad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega} \tag{1.32}$$

using the relations (1.31). These equations can be reduced to the Riccati equation as follows. For the unit sphere $\boldsymbol{\gamma}^2 = 1$ we perform a stereographic projection onto the plane passing through the equator $\gamma_3 = 0$, by the formulae

$$\gamma_1 = \frac{2x}{x^2 + y^2 + 1}, \qquad \gamma_2 = \frac{2y}{x^2 + y^2 + 1}, \qquad \gamma_3 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}.$$

For the evolution of the coordinates x, y we obtain the equations

$$\dot{x} = \frac{1}{2}\omega_2 + \omega_3 y + \frac{1}{2}\omega_2(x^2 - y^2) - \omega_1 xy, \qquad \dot{y} = -\frac{1}{2}\omega_1 - \omega_3 x + \frac{1}{2}\omega_1(x^2 - y^2) - \omega_2 xy.$$

Writing in complex form, for the variable $\sigma = x + iy$ we obtain the Riccati equation (see [144])

$$\dot{\sigma} = -i\omega_3\sigma + \frac{1}{2}(\omega_2 - i\omega_1) + \frac{1}{2}(\omega_2 + i\omega_1)\sigma^2. \tag{1.33}$$

REMARK. This equation is easy to obtain if one uses a complex representation for sphero-conic coordinates of the following form

$$x + iy = \frac{\gamma_1 + i\gamma_2}{1 - \gamma_3}.$$

Angular momentum $\boldsymbol{M}$ is given by the formula

$$\boldsymbol{M} = \frac{\partial L}{\partial \boldsymbol{\omega}}, \tag{1.34}$$

where $L = L(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is the Lagrange function and ω is the angular velocity. $\boldsymbol{M}$ is related with the Euler variables φ, ψ, θ, p_φ, p_ψ, p_θ by the following relations, which are implied by the kinematic Euler equations (1.28) and (1.29):

$$\begin{aligned} M_1 &= \frac{\sin\varphi}{\sin\theta}(p_\psi - p_\varphi\cos\theta) + p_\theta\cos\varphi, \\ M_2 &= \frac{\cos\varphi}{\sin\theta}(p_\psi - p_\varphi\cos\theta) - p_\theta\sin\varphi, \\ M_3 &= p_\varphi. \end{aligned} \tag{1.35}$$

REMARK 1. Our definition in (1.34) of angular momentum differs from the physical definition used in rigid body dynamics, $\boldsymbol{M} = \sum(\boldsymbol{r}_i \times m_i\boldsymbol{v}_i)$, but they coincide if $L = T$ is kinetic energy. Distinctions appear if gyroscopic forces occur in the system; such forces lead to terms linear in generalized velocities in the Lagrangian. In this case, definition (1.34) that follows from the Chetaev transform is more convenient.

REMARK 2. The relation between the direction cosines (1.30) and the Euler angles is given by

$$\mathbf{Q} = \begin{pmatrix} \cos\varphi\cos\psi - \cos\theta\sin\psi\sin\varphi & \cos\varphi\sin\psi + \cos\theta\cos\psi\sin\varphi & \sin\varphi\sin\theta \\ -\sin\varphi\cos\psi - \cos\theta\sin\psi\cos\varphi & -\sin\varphi\sin\psi + \cos\theta\cos\psi\cos\varphi & \cos\varphi\sin\theta \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{pmatrix}.$$

3 Quaternion Rodrigues – Hamilton parameters

K. Gauss noted that the position of a rigid body can be uniquely defined by the set of quaternions $\lambda = \lambda_0 + \boldsymbol{i}\lambda_1 + \boldsymbol{j}\lambda_2 + \boldsymbol{k}\lambda_3$ with unit norm $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$. They form the group $Sp(1)$, which is the universal covering of the group $SO(3)$ ($SO(3) \approx Sp(1)/\pm 1$) [170]. A detailed discussion of these redundant coordinates called the *Rodrigues – Hamilton parameters* can be found in Whittaker's monograph [611]. We clarify the geometrical meaning of the parameters λ_s [321, 611].

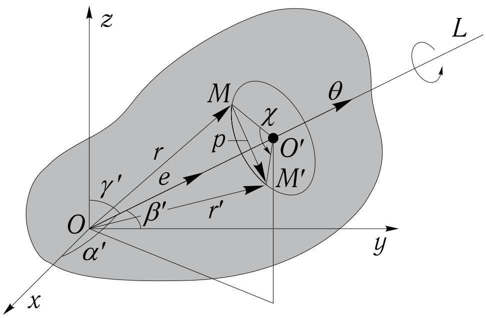

Fig. 1.2. Quaternion Rodrigues – Hamilton parameters.

It is known from kinematics that a rigid body with a fixed point O can be transferred from any position to another by a rotation through some angle χ about some axis OL attached to the body (see Fig. 1.2). The orientation of the axis OL is defined by a suitable unit vector $\boldsymbol{e}$. The position of a point M of the body is described by a vector $\overrightarrow{OM} = \boldsymbol{r}$, say. Suppose that the rotation moves the vector $\boldsymbol{r}$ to the position $\overrightarrow{OM}' = \boldsymbol{r}'$. The vector

$$\boldsymbol{p} = \overrightarrow{OM}' - \overrightarrow{OM} = \boldsymbol{r}' - \boldsymbol{r}$$

can be expressed in terms of $\boldsymbol{r}$, $\boldsymbol{e}$, and χ by the Rodrigues formula

$$\boldsymbol{p} = \frac{1}{1 + \frac{1}{4}\theta^2}\boldsymbol{\theta} \times \left(\boldsymbol{r} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{r}\right), \tag{1.36}$$

where the vector

$$\boldsymbol{\theta} = 2\tan\frac{\chi}{2}\boldsymbol{e} \tag{1.37}$$

is called the *vector of finite rotation*; its direction coincides with the direction of the unit vector $\boldsymbol{e}$ and its length is $2\tan(\chi/2)$.

Let

$$\boldsymbol{e} = \boldsymbol{i}\cos\alpha' + \boldsymbol{j}\cos\beta' + \boldsymbol{k}\cos\gamma', \tag{1.38}$$

where α', β', and γ' are the angles made by the vector $\boldsymbol{e}$ with the axes x, y, and z, respectively.

The quantities

$$\lambda_0 = \cos\frac{\chi}{2}, \qquad \lambda_1 = \cos\alpha'\sin\frac{\chi}{2},$$
$$\lambda_2 = \cos\beta'\sin\frac{\chi}{2}, \qquad \lambda_3 = \cos\gamma'\sin\frac{\chi}{2} \tag{1.39}$$

are the *Rodrigues – Hamilton parameters*. The parameter λ_0 is equal to the cosine of a half of the angle χ defining the finite rotation of the body. The parameters λ_1, λ_2, and λ_3 are proportional to the sine of $\chi/2$ multiplied by the direction cosines of the axis OL.

The relation of the Rodrigues – Hamilton parameters with the Euler angles θ, φ, ψ is

$$\lambda_0 = \cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}, \qquad \lambda_1 = \sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2},$$
$$\lambda_2 = \sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}, \qquad \lambda_3 = \cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}. \tag{1.40}$$

The direction cosines $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ are related with the quaternion parameters by the following quadratic relations that define the Cayley parametrization of the group $SO(3)$; we obtain a two-sheeted covering of $SO(3)$ by the three-dimensional sphere S^3: the same element of $SO(3)$ corresponds to two quaternions λ_i and $-\lambda_i$. In the quaternion representation the matrix of direction cosines (1.30) is

$$\mathbf{Q} = \begin{pmatrix} \lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2 & 2(\lambda_0\lambda_3+\lambda_1\lambda_2) & 2(\lambda_1\lambda_3-\lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & \lambda_0^2-\lambda_1^2+\lambda_2^2-\lambda_3^2 & 2(\lambda_0\lambda_1+\lambda_2\lambda_3) \\ 2(\lambda_0\lambda_2+\lambda_1\lambda_3) & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) & \lambda_0^2-\lambda_1^2-\lambda_2^2+\lambda_3^2 \end{pmatrix}. \tag{1.41}$$

In the index form, the entries of the matrix $\mathbf{Q} = \|Q_{ij}\|$ are expressed as

$$Q_{ij} = -2\Big(\lambda_i\lambda_j + \Big(\lambda_0^2 - \frac{1}{2}\Big)\delta_{ij} - \lambda_0\lambda_k\varepsilon_{ijk}\Big).$$

REMARK 3. The relation between the projections of the angular velocity $\boldsymbol{\omega}$ and the Rodrigues – Hamilton parameters has the form

$$\begin{aligned} \omega_1 &= 2(\lambda_0\dot\lambda_1 + \lambda_3\dot\lambda_2 - \lambda_2\dot\lambda_3 - \lambda_1\dot\lambda_0), \\ \omega_2 &= 2(-\lambda_3\dot\lambda_1 + \lambda_0\dot\lambda_2 + \lambda_1\dot\lambda_3 - \lambda_2\dot\lambda_0), \\ \omega_3 &= 2(\lambda_2\dot\lambda_1 - \lambda_1\dot\lambda_2 + \lambda_0\dot\lambda_3 - \lambda_3\dot\lambda_0). \end{aligned}$$

REMARK 4. As an alternative to the Rodrigues – Hamilton parameters, one can take complex quantities α, β, γ, and δ satisfying the condition

$$\alpha\delta - \beta\gamma = 1$$

and consider them to be the components of the complex rotation matrix

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with unit determinant. These parameters are called the *Cayley – Klein parameters*.

The relation between the Cayley – Klein and Rodrigues – Hamilton parameters is expressed by the formulae

$$\alpha = \lambda_0 + i\lambda_3, \quad \beta = -\lambda_2 + i\lambda_1, \quad \gamma = \lambda_2 + i\lambda_1, \quad \delta = \lambda_0 - i\lambda_3,$$

and the representation of the Cayley – Klein parameters in terms of the Euler angles has the form

$$\alpha = \cos\frac{\theta}{2} e^{i\frac{\psi+\varphi}{2}}, \qquad \beta = i\sin\frac{\theta}{2} e^{i\frac{\psi-\varphi}{2}},$$
$$\gamma = i\sin\frac{\theta}{2} e^{-i\frac{\psi-\varphi}{2}}, \qquad \delta = \cos\frac{\theta}{2} e^{-i\frac{\psi+\varphi}{2}}.$$

4 Andoyer variables

The Andoyer variables are the most convenient ones for perturbation theory applied to rigid body dynamics. Their geometric origin is shown in Fig. 1.3 (see also [150, 333, 95]).

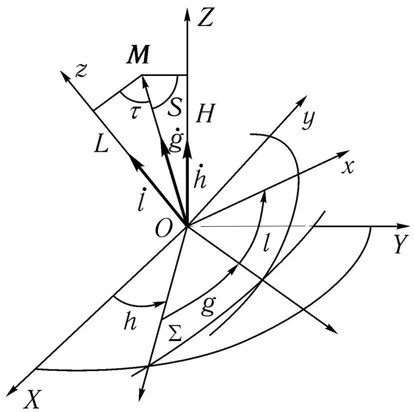

Fig. 1.3. Andoyer variables.

Here, $OXYZ$ is the immovable frame with origin at the fixed point, $Oxyz$ is the movable frame attached to the body with origin at the fixed point, and Σ is the plane passing through the fixed point and perpendicular to the angular momentum vector $\boldsymbol{M}$ (1.35) of the body. In this notation,

- L is the projection of $\boldsymbol{M}$ on the movable axis Oz;
- G is the value of $\boldsymbol{M}$;
- H is the projection of $\boldsymbol{M}$ on the fixed axis OZ;
- l is the angle between the axis Ox and the line of intersection of Σ with the plane Oxy;

g is the angle between the lines of intersection of Σ with the planes Oxy and OXY;
h is the angle between the axis OX and the line of intersection of Σ with the plane OXY.

The expressions of the components of the angular momentum in terms of the variables L, G, H, l, g, and h have the form

$$M_1 = \sqrt{G^2 - L^2}\sin l, \quad M_2 = \sqrt{G^2 - L^2}\cos l, \quad M_3 = L, \quad G^2 = \boldsymbol{M}^2, \tag{1.42}$$

i.e., L and l are cylindrical coordinates on the two-dimensional sphere in the space of momenta M_1, M_2, M_3.

The components of all direction cosines can be expressed as follows (as far as the authors know, not all of these relations are presented in the literature):

$$\begin{aligned}
\alpha_1 &= -\sin l \sin h \cos g \sin\tau \sin\zeta + \sin l \sin h \cos\tau \cos\zeta \\
&\quad - \sin l \sin g \cos h \sin\tau - \cos l \sin h \sin g \sin\zeta + \cos l \cos g \cos h, \\
\alpha_2 &= \cos l \cos g \sin h \sin\tau \sin\zeta - \cos l \sin h \cos\tau \cos\zeta \\
&\quad + \cos l \cos h \sin g \sin\tau - \sin l \sin g \sin\zeta \sin h + \sin l \cos h \cos g, \\
\alpha_3 &= \sin h \cos\tau \cos g \sin\zeta + \sin h \sin\tau \cos\zeta + \cos\tau \sin g \cos h, \\
\beta_1 &= -(\sin l \cos h \cos g \sin\tau \sin\zeta - \sin l \cos h \cos\zeta \cos\tau \\
&\quad - \sin l \sin g \sin h \sin\tau + \cos l \cos h \sin g \sin\zeta + \cos l \cos g \sin h), \\
\beta_2 &= \cos l \cos h \sin\tau \cos g \sin\zeta - \cos l \cos h \cos\zeta \cos\tau \\
&\quad - \cos l \sin g \sin h \sin\tau - \sin l \cos h \sin g \sin\zeta - \sin l \cos g \sin h, \\
\beta_3 &= -\sin h \cos\tau \sin g + \cos\tau \cos g \sin\zeta \cos h + \sin\tau \cos\zeta \cos h, \\
\gamma_1 &= (\sin\zeta \cos\tau + \sin\tau \cos\zeta \cos g) \sin l + \cos\zeta \sin g \cos l, \\
\gamma_2 &= (\sin\zeta \cos\tau + \sin\tau \cos\zeta \cos g) \cos l - \cos\zeta \sin g \sin l, \\
\gamma_3 &= \sin\zeta \sin\tau - \cos\tau \cos\zeta \cos g,
\end{aligned} \tag{1.43}$$

where $\sin\tau = \frac{L}{G}$, $\sin\zeta = \frac{H}{G}$.

Remark. The expressions of the direction cosines γ_i via the Andoyer variables can be found, for example, in [16, 333, 88]. The inverse formulae are $L = M_3$, $G = \sqrt{(\boldsymbol{M}, \boldsymbol{M})}$, $l = \arctan \frac{M_1}{M_2}$, $g = \arcsin \frac{M_2\gamma_1 - M_1\gamma_2}{\sqrt{M_1^2 + M_2^2}}$. The formulae for α_3 and β_3 can be easily obtained from geometric considerations. The other direction cosines are implied by the commutation relations (1.59) (see the next section). The relations for λ_i^2 can be obtained via the formulae

$$\lambda_0^2 = \frac{1 + \alpha_1 + \beta_2 + \gamma_3}{4}, \qquad \lambda_1^2 = \frac{1 + \alpha_1 - \beta_2 - \gamma_3}{4},$$
$$\lambda_2^2 = \frac{1 - \alpha_1 + \beta_2 - \gamma_3}{4}, \qquad \lambda_3^2 = \frac{1 - \alpha_1 - \beta_2 + \gamma_3}{4},$$

and the λ_is themselves are defined up to a sign.

5 Comments

In contrast to the system of Euler angles with their adjoint canonical momenta, the system of Andoyer variables cannot be divided into natural position and momentum parts. They prove to be very convenient for perturbation analysis applied to rigid body systems: the variables G and L are integrals of motion for both of the best-known integrable problems of rigid body dynamics, the Euler and Lagrange cases, each of which is a natural candidate for an unperturbed system. Similar systems of "osculating elements" that are not necessarily canonical were used by Poisson, Charlier, Andoyer, and Tisserand in developing the theories of lunar librations and rotational motions of planets in celestial mechanics. The French geometer P. Serret was the first (1866) to use variables similar to the Andoyer variables [528]. Another similar system of variables was considered by Rado [476]. Andoyer introduced and systematically used these variables in his famous course in celestial mechanics [8]. They are sometimes called Serret – Andoyer variables. A. Deprit had independently introduced these variables in 1962 (see [150]) to clarify the phase geometry of the Euler case (see Sec. 2.2, Ch. 2). This allowed one to realize their universal character in rigid body dynamics; they were also used in qualitative analysis in [333] (they were called *special canonical variables* there) and for numerical simulations [88]. (Note that in the first edition of this book, we called them Andoyer – Deprit variables, but now it seems to us that this is not valid.)

A systematic study of the equations of motion for a heavy gyroscope in the Rodrigues – Hamilton quaternion variables (and also in the Cayley – Klein parameters) can be found in the remarkable book of F. Klein and A. Sommerfeld *Über die Theorie des Kreisels* [305] (indeed, the main results on this problem belong to F. Klein; see also [304]). At that time, the Hamiltonian structure of these equations (as equations on a Lie algebra) was not known; however, the Rodrigues – Hamilton variables proved to be very convenient for explicit integration in elliptic functions and for the analysis of numerous particular solutions. E. Study used a system of redundant variables (of the same type as Plücker coordinates) similar to quaternions in his book *Geometrie der Dynamen* and calculated the kinetic energy of a rigid body in these coordinates.

1.4 Different Forms of Equations of Motion

1 Equations of motion of a rigid body with a fixed point

In this section, we present the equations of motion of a rigid body in the various forms they take in different variables. The choice of variables to be used depends on the questions to be answered.

Euler – Poincaré equations on the group $SO(3)$. Consider the motion of a rigid body with one fixed point, i.e., this point is fixed in space (relative to some inertial reference frame). The configuration space is the group $SO(3)$. We represent elements of this group by orthogonal matrices of direction cosines: (1.30) (see Sec. 1.3, item 2)

$$\mathbf{Q} = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \in SO(3), \tag{1.44}$$

where, as above, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ form an orthonormal basis in the immovable frame, but are represented in the moving frame attached to the body.

The angular velocity of the body $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ in the moving frame is found from the Poisson equations

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \qquad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \tag{1.45}$$

which show that the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ are constant in the absolute space. Rewriting (1.45) in matrix form, we obtain

$$\dot{\mathbf{Q}} = \widetilde{\boldsymbol{\omega}}\mathbf{Q}, \quad \widetilde{\boldsymbol{\omega}} = -\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}} = \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}, \tag{1.46}$$

where

$$\widetilde{\boldsymbol{\omega}} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

From a group-theoretic viewpoint, the projections ω_i of the angular velocity in the body correspond to the components of the velocity of the point on the group $SO(3)$ in the basis of left-invariant vector fields. Similarly, the components Ω_i of the angular velocity in the fixed frame correspond to the representation of the angular velocity in the basis of right-invariant vector fields

$$\boldsymbol{\omega} = \sum_k \omega_k \boldsymbol{\xi}_k, \qquad \boldsymbol{\xi}_k = -\sum_{ij} \varepsilon_{kij}\left(\alpha_i \frac{\partial}{\partial \alpha_j} + \beta_i \frac{\partial}{\partial \beta_j} + \gamma_i \frac{\partial}{\partial \gamma_j}\right). \tag{1.47}$$

To find the fields $\boldsymbol{\xi}_k$, we write the time derivative taking (1.46) into account:

$$\frac{df}{dt} = \mathrm{Tr}\left(\dot{\mathbf{Q}}^{\mathrm{T}} \frac{\partial f}{\partial \mathbf{Q}}\right) = \mathrm{Tr}\left((\widetilde{\boldsymbol{\omega}}\mathbf{Q})^{\mathrm{T}} \frac{\partial f}{\partial \mathbf{Q}}\right), \quad \frac{\partial f}{\partial \mathbf{Q}} = \left\| \frac{\partial f}{\partial Q_{ij}} \right\|; \tag{1.48}$$

grouping terms containing ω_i, we obtain the vector fields $\boldsymbol{\xi}_i$ (1.47).

The commutation relations for the vector fields $\boldsymbol{\xi}_k$ have the form

$$[\boldsymbol{\xi}_i, \boldsymbol{\xi}_j] = \varepsilon_{ijk}\boldsymbol{\xi}_k, \tag{1.49}$$

where ε_{ijk} are the Levi-Civita symbols.

Substituting (1.47) and (1.49) in the Euler – Poincaré equations (1.17), we obtain the equations of motion in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right) = \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega} + \frac{\partial L}{\partial \boldsymbol{\alpha}} \times \boldsymbol{\alpha} + \frac{\partial L}{\partial \boldsymbol{\beta}} \times \boldsymbol{\beta} + \frac{\partial L}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma}, \tag{1.50}$$

which, together with (1.45), form the complete system of equations of motion for a body with a fixed point. System (1.45), (1.50) was obtained by J. Lagrange in the second part of his famous treatise *Méchanique analytique* [364].

REMARK. Eqs. (1.45), (1.50) may be represented in matrix form, which can be easily generalized to the higher-dimensional case:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \widetilde{\boldsymbol{\omega}}}\right) = \left[\widetilde{\boldsymbol{\omega}}, \frac{\partial L}{\partial \widetilde{\boldsymbol{\omega}}}\right] + \frac{\partial L}{\partial \mathbf{Q}}\mathbf{Q}^{\mathrm{T}} - \left(\frac{\partial L}{\partial \mathbf{Q}}\right)^{\mathrm{T}}\mathbf{Q}, \quad \dot{\mathbf{Q}} = \widetilde{\boldsymbol{\omega}}\mathbf{Q},$$

where $\frac{\partial L}{\partial \widetilde{\boldsymbol{\omega}}} = \left\|\frac{\partial L}{\partial \widetilde{\omega}_{ij}}\right\|$, $\frac{\partial L}{\partial \mathbf{Q}} = \left\|\frac{\partial L}{\partial Q_{ij}}\right\|$, and $[\cdot\,,\cdot]$ is the usual matrix commutator.

Equations of motion in angular velocities and quaternions. In Sec. 1.3 we presented the quaternion parametrization of the group $SO(3)$ for which the vector fields (1.47) are also linear functions of the coordinates. Indeed, it is easy to show that on the unit sphere $\lambda_0^2 + \boldsymbol{\lambda}^2 = 1$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, the components of the angular velocity (1.46) and the vector fields (1.47) have the form (see [335, 321])

$$\begin{aligned}
\omega_1 &= 2(\lambda_0\dot{\lambda}_1 - \lambda_1\dot{\lambda}_0 + \lambda_3\dot{\lambda}_2 - \lambda_2\dot{\lambda}_3),\\
\omega_2 &= 2(\lambda_0\dot{\lambda}_2 - \lambda_2\dot{\lambda}_0 + \lambda_1\dot{\lambda}_3 - \lambda_3\dot{\lambda}_1),\\
\omega_3 &= 2(\lambda_0\dot{\lambda}_3 - \lambda_3\dot{\lambda}_0 + \lambda_2\dot{\lambda}_1 - \lambda_1\dot{\lambda}_2),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\xi}_1 &= \frac{1}{2}\left(\lambda_0\frac{\partial}{\partial\lambda_1} - \lambda_1\frac{\partial}{\partial\lambda_0} + \lambda_3\frac{\partial}{\partial\lambda_2} - \lambda_2\frac{\partial}{\partial\lambda_3}\right),\\
\boldsymbol{\xi}_2 &= \frac{1}{2}\left(\lambda_0\frac{\partial}{\partial\lambda_2} - \lambda_2\frac{\partial}{\partial\lambda_0} + \lambda_1\frac{\partial}{\partial\lambda_3} - \lambda_3\frac{\partial}{\partial\lambda_1}\right),\\
\boldsymbol{\xi}_3 &= \frac{1}{2}\left(\lambda_0\frac{\partial}{\partial\lambda_3} - \lambda_3\frac{\partial}{\partial\lambda_0} + \lambda_2\frac{\partial}{\partial\lambda_1} - \lambda_1\frac{\partial}{\partial\lambda_2}\right).
\end{aligned} \tag{1.51}$$

The commutation relations for the fields $\boldsymbol{\xi}_k$ also have the form (1.49).

The Poincaré equations (1.17), subject to (1.51), become

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial\boldsymbol{\omega}}\right) &= \frac{\partial L}{\partial\boldsymbol{\omega}}\times\boldsymbol{\omega} + \frac{1}{2}\lambda_0\frac{\partial L}{\partial\boldsymbol{\lambda}} - \frac{1}{2}\boldsymbol{\lambda}\frac{\partial L}{\partial\lambda_0} + \frac{1}{2}\frac{\partial L}{\partial\boldsymbol{\lambda}}\times\boldsymbol{\lambda},\\
\dot{\lambda}_0 = -\frac{1}{2}(\boldsymbol{\omega}, \boldsymbol{\lambda}), &\qquad \dot{\boldsymbol{\lambda}} = \frac{1}{2}\lambda_0\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\lambda}\times\boldsymbol{\omega}.
\end{aligned} \tag{1.52}$$

Kinetic energy of a rigid body with a fixed point in vector and matrix forms can be represented as follows:

$$T = \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) = -\frac{1}{2}\,\mathrm{Tr}(\widetilde{\boldsymbol{\omega}}\mathbf{J}\widetilde{\boldsymbol{\omega}}). \tag{1.53}$$

Here $\mathbf{I} = \|I_{ij}\|$ is the *tensor of inertia* of the rigid body with respect to the fixed point of the body; its components are defined by the formula

$$I_{ij} = \int_\tau (\boldsymbol{y}^2\delta_{ij} - y_i y_j)\rho(\boldsymbol{y})\,d^3\boldsymbol{y}, \tag{1.54}$$

where integration extends over all points $\boldsymbol{y}$ of the body τ and $\rho(\boldsymbol{y})$ is the density of the body at the point $\boldsymbol{y}$.

The tensor $\mathbf{J} = \|J_{ij}\|$ is also called the tensor of inertia, but it is defined by the formula

$$J_{ij} = \int_\tau y_i y_j \rho(\boldsymbol{y})\, d^3\boldsymbol{y}; \tag{1.55}$$

this tensor is usually used for higher-dimensional generalizations.

The relation between $\mathbf{I}$ and $\mathbf{J}$ is

$$\mathbf{J} = \frac{1}{2}(\mathrm{Tr}\,\mathbf{I})\mathbf{E} - \mathbf{I}, \qquad \mathbf{I} = (\mathrm{Tr}\,\mathbf{J})\mathbf{E} - \mathbf{J}. \tag{1.56}$$

In the reference frame attached to the body, the tensors $\mathbf{I}$ and $\mathbf{J}$ are represented by constant symmetric matrices (in the fixed frame, they are represented by coordinate-dependent matrices); since they commute ($\mathbf{IJ} = \mathbf{JI}$), they can be simultaneously diagonalized. The corresponding coordinate system in the body is said to be *principal* and its axes are called the *principal axes* (*of inertia*). As is well known, the principal moments of inertia I_1, I_2, I_3 of any body obey the inequality of the triangle $I_i + I_j \geqslant I_k$. The equality takes place for the case of an infinitely thin plate.

2 Hamiltonian form of equations of motion for different systems of variables

Equations of motion in algebraic form. Equations (1.45), (1.50) can be represented in Hamiltonian form by using the Legendre transform

$$\boldsymbol{M} = \frac{\partial L}{\partial \boldsymbol{\omega}}, \quad H = \big[(\boldsymbol{M}, \boldsymbol{\omega}) - L\big]\big|_{\boldsymbol{\omega}\to\boldsymbol{M}}. \tag{1.57}$$

For a natural system with kinetic energy (1.53) and potential energy $U(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ we have

$$\boldsymbol{M} = \mathbf{I}\boldsymbol{\omega}, \quad H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + U(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}), \tag{1.58}$$

where $\mathbf{A} = \mathbf{I}^{-1}$, $\boldsymbol{M}$ are the components of the angular momentum referred to the moving axes, and $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ are the components of the direction cosines.

The general formulae (1.26) and (1.49) imply that the Poisson bracket is defined by the algebra $so(3) \oplus_s (\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3)$, which is the semi-direct sum of the algebra of rotations and three copies of the algebra of translations

$$\begin{gathered} \{M_i, M_j\} = -\varepsilon_{ijk} M_k, \qquad \{M_i, \alpha_j\} = -\varepsilon_{ijk}\alpha_k, \\ \{M_i, \beta_j\} = -\varepsilon_{ijk}\beta_k, \qquad \{M_i, \gamma_j\} = -\varepsilon_{ijk}\gamma_k, \\ \{\alpha_i, \alpha_j\} = \{\beta_i, \beta_j\} = \{\gamma_i, \gamma_j\} = \{\alpha_i, \beta_j\} = \{\alpha_i, \gamma_j\} = \{\beta_i, \gamma_j\} = 0. \end{gathered} \tag{1.59}$$

The Hamiltonian equations are

$$\begin{gathered} \dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{\alpha} \times \frac{\partial H}{\partial \boldsymbol{\alpha}} + \boldsymbol{\beta} \times \frac{\partial H}{\partial \boldsymbol{\beta}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \\ \dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \frac{\partial H}{\partial \boldsymbol{M}}, \qquad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \frac{\partial H}{\partial \boldsymbol{M}}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{M}}, \\ H = \frac{1}{2}(\boldsymbol{M}, \mathbf{A}\boldsymbol{M}) + U(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}). \end{gathered} \tag{1.60}$$

The equations of motion of a rigid body in a generalized potential field (e.g., a magnetic field) can also be represented in the form of (1.60). In this case, the Hamiltonian H contains terms linear in $\boldsymbol{M}$ (see below).

The Poisson bracket (1.59) is degenerate and possesses six Casimir functions:

$$\begin{aligned} f_1 &= (\boldsymbol{\alpha}, \boldsymbol{\alpha}), \quad & f_2 &= (\boldsymbol{\beta}, \boldsymbol{\beta}), \quad & f_3 &= (\boldsymbol{\gamma}, \boldsymbol{\gamma}), \\ f_4 &= (\boldsymbol{\alpha}, \boldsymbol{\beta}), \quad & f_5 &= (\boldsymbol{\alpha}, \boldsymbol{\gamma}), \quad & f_6 &= (\boldsymbol{\beta}, \boldsymbol{\gamma}). \end{aligned} \tag{1.61}$$

A generic symplectic leaf has dimension equal to six and is diffeomorphic to the cotangent bundle of the rotational group $SO(3)$. Hence there is no loss of generality in taking the representative example amongst these, for which $f_1 = f_2 = f_3 = 1$ and $f_4 = f_5 = f_6 = 0$, since the system (1.60) has three degrees of freedom.

In the immovable coordinate system, the position and velocity of a rigid body can be characterized by the projections on the fixed axes of the orthonormal basis vectors attached to the body, which can be expressed in terms of the rows of the matrix $\mathbf{Q}$, and the projections of the angular momentum vector on the same axes:

$$\begin{gathered} \boldsymbol{e}_1 = (\alpha_1, \beta_1, \gamma_1), \quad \boldsymbol{e}_2 = (\alpha_2, \beta_2, \gamma_2), \quad \boldsymbol{e}_3 = (\alpha_3, \beta_3, \gamma_3), \\ N_1 = (\boldsymbol{M}, \boldsymbol{\alpha}),\ N_2 = (\boldsymbol{M}, \boldsymbol{\beta}),\ N_3 = (\boldsymbol{M}, \boldsymbol{\gamma}). \end{gathered} \tag{1.62}$$

It is easy to show that the variables $\boldsymbol{N}$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ also form a Lie – Poisson structure, which differs from (1.59) only by sign:

$$\begin{gathered} \{N_i, N_j\} = \varepsilon_{ijk} N_k, \\ \{N_i, e_{1j}\} = \varepsilon_{ijk} e_{1k}, \quad \{N_i, e_{2j}\} = \varepsilon_{ijk} e_{2k}, \quad \{N_i, e_{3k}\} = \varepsilon_{ijk} e_{3k}, \\ \{e_{ki}, e_{lj}\} = 0. \end{gathered} \tag{1.63}$$

For example, the spherical pendulum in a potential field is conveniently described in terms of the variables $\boldsymbol{N}$ and $\boldsymbol{e}_k$. The Hamiltonian has the form

$$H = \frac{1}{2ml^2} \boldsymbol{N}^2 + U(\boldsymbol{e}_3). \tag{1.64}$$

and the relation $(\boldsymbol{N}, \boldsymbol{e}_3) = 0$ holds (i.e., motion is on the zero orbit of $e(3)$). Here $\boldsymbol{e}_3$ is the unit vector connecting the fixed point and the center of gravity, $\boldsymbol{N} = ml^2 \boldsymbol{\omega}$, $\boldsymbol{\omega} = \boldsymbol{e}_3 \times \dot{\boldsymbol{e}}_3$ is the angular velocity, and l is the length of the pendulum. In other words, the spherical pendulum can be represented as the spherical top on the zero orbit of the algebra $e(3)$.

The generators $\boldsymbol{e}_i$ and $\boldsymbol{N}$ in (1.62) are also convenient for the description of reduction for systems admitting the Lagrange integral $F = M_3 = \text{const}$ as an integral of motion (see Sec. 4.1, 4.2, Ch. 4).

Quaternion representation of the equations of motion. In practice, the redundancy in Eqs. (1.60) is very inconvenient; for example, it may be expected, as often happens, that in numerical integration of these equations, the orthonormality relations (1.61) do not survive for a long time. This disadvantage is absent in the quaternion representation of the equations of motion presented by the authors in [92, 95]. The matrix of direction cosines in the quaternion representation has the form (1.41) and the corresponding commutation relations are

$$\begin{aligned} \{M_i, M_j\} &= -\varepsilon_{ijk} M_k, & \{M_i, \lambda_0\} &= \frac{1}{2}\lambda_i, \\ \{M_i, \lambda_j\} &= -\frac{1}{2}(\varepsilon_{ijk}\lambda_k + \delta_{ij}\lambda_0), & \{\lambda_\mu, \lambda_\nu\} &= 0. \end{aligned} \tag{1.65}$$

The corresponding Lie algebra is the semi-direct sum of the algebra of rotations $so(3)$ and the algebra of translations $\mathbb{R}^4$: $l(7) \approx so(3) \oplus_s \mathbb{R}^4$.

Bracket (1.65) is degenerate. It admits just one Casimir function

$$F(\lambda) = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2. \tag{1.66}$$

A generic symplectic leaf is diffeomorphic to the cotangent bundle T^*S^3 of the three-dimensional sphere and has dimension 6. The equations of motion have the form

$$\begin{aligned} \dot{\boldsymbol{M}} &= \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \frac{1}{2}\boldsymbol{\lambda} \times \frac{\partial H}{\partial \boldsymbol{\lambda}} + \frac{1}{2}\frac{\partial H}{\partial \lambda_0}\boldsymbol{\lambda} - \frac{1}{2}\lambda_0 \frac{\partial H}{\partial \boldsymbol{\lambda}}, \\ \dot{\lambda}_0 &= -\frac{1}{2}\left(\boldsymbol{\lambda}, \frac{\partial H}{\partial \boldsymbol{M}}\right), \quad \dot{\boldsymbol{\lambda}} = \frac{1}{2}\boldsymbol{\lambda} \times \frac{\partial H}{\partial \boldsymbol{M}} + \frac{1}{2}\lambda_0 \frac{\partial H}{\partial \boldsymbol{M}}. \end{aligned} \tag{1.67}$$

For integrability, two additional involutive integrals are needed.

REMARK. For a physical rigid body system, the Hamiltonian H is a single-valued function on $T^*SO(3)$. Since the covering of $SO(3)$ by quaternions (1.41) is twofold, the Hamiltonian function depends only on quadratic combinations $\lambda_i \lambda_j$. Nevertheless, systems having Hamiltonians with arbitrary dependence on quaternion components do occur in different branches of mechanics, for example, in celestial mechanics on a curved space, the Leggett system, quantum mechanics of spins (see Chs. 3, 4). Probably, the most value of the quaternionic representation (1.67) is in quantum mechanics, where some phenomena are essentially related to spin.

Canonical equations in Euler angles and Andoyer variables. In the Euler angles (θ, φ, ψ) and corresponding canonical momenta $p_\theta, p_\varphi, p_\psi$, the equations of motion have the usual Hamiltonian form

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}, \quad \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \quad \boldsymbol{q} = (\theta, \varphi, \psi), \quad \boldsymbol{p} = (p_\theta, p_\varphi, p_\psi), \tag{1.68}$$

obtained from the Lagrangian formalism in the variables $(\theta, \varphi, \psi, \dot{\theta}, \dot{\varphi}, \dot{\psi})$ by using the Legendre transform

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}, \qquad H(\boldsymbol{p}, \boldsymbol{q}) = \big[(\boldsymbol{p}, \dot{\boldsymbol{q}}) - L\big]\Big|_{\dot{\boldsymbol{q}},\boldsymbol{q} \to \boldsymbol{p},\boldsymbol{q}}.$$

Here L is the Lagrange function, which for natural systems has the form $L = T - U(\theta, \varphi, \psi)$. The kinetic energy of the body does not depend on the variable ψ and has the form

$$T = \frac{1}{2}(\mathbf{A}\boldsymbol{M},\ \boldsymbol{M}) = \frac{1}{2}\Big[a_1\Big(\frac{\sin\varphi}{\sin\theta}(p_\psi - p_\varphi\cos\theta) + p_\theta\cos\varphi\Big)^2 + a_2\Big(\frac{\cos\varphi}{\sin\theta}(p_\psi - p_\varphi\cos\theta) - p_\theta\sin\varphi\Big)^2 + a_3 p_\varphi^2\Big]. \tag{1.69}$$

The motion of a rigid body in a potential field is defined by a natural system, and the Hamiltonian has the form

$$H = T + U(\theta,\ \varphi,\ \psi). \tag{1.70}$$

If the potential energy is also independent of ψ $\left(\text{i.e. } \frac{\partial U}{\partial\psi} = 0\right)$, which corresponds to invariance of the force field with respect to rotations about the vertical axis in the immovable frame, then the variable ψ is cyclic and the generalized momentum $p_\psi = (\boldsymbol{M}, \boldsymbol{\gamma})$ is conserved. Upon Routhian reduction with respect to the precession angle ψ, we obtain a system describing the motion of a point on the sphere $\boldsymbol{\gamma}^2 = 1$ (where $\gamma_1 = \sin\theta\sin\varphi$, $\gamma_2 = \sin\theta\cos\varphi$, and $\gamma_3 = \cos\theta$), which is called the *Poisson sphere*. If $p_\psi \neq 0$, then the Hamiltonian contains terms linear in velocity (so-called gyroscopic terms) that cannot be eliminated by coordinate transforms and correspond to motion in a generalized potential field. The impossibility of eliminating gyroscopic terms is related to the global phenomenon of a "monopole", whose value is the integral of the gyroscopic 2-form over the Poisson sphere (see [447]). The "monopole" problem was raised for the first time by P. Dirac in connection with the quantization of the motion of a particle on a sphere. If $p_\psi = 0$, then the system discussed is natural, i.e., there are no gyroscopic terms.

If the dynamical symmetry condition $a_1 = a_2$ holds, the kinetic energy (1.69) becomes simpler, being also independent of the angle φ:

$$T = \frac{1}{2}\left(a_1\left(p_\theta^2 + \frac{(p_\psi - p_\varphi\cos\theta)^2}{\sin^2\theta}\right) + a_3 p_\varphi^2\right). \tag{1.71}$$

If in addition the potential U is also independent of φ (i.e., $\frac{\partial U}{\partial\psi} = \frac{\partial U}{\partial\varphi} = 0$), in other words, $U = U(\theta) = U(\gamma_3)$, then there exists one more cyclic integral – the Lagrange integral $p_\varphi = M_3 = c_2 = \text{const}$, which corresponds to invariance of the system under rotations about the axis of dynamical symmetry. The (one-degree-of-freedom) system obtained after reduction is integrable (for details, see Sec. 2.3 Ch. 2). If $p_\psi = c_1 \neq 0$ but $p_\varphi = c_2 = 0$, then the equations describe the motion of the *spherical pendulum*.

In terms of Andoyer variables, the equations of motion also have the form (1.68), where $\boldsymbol{q} = (l, g, h)$ and $\boldsymbol{p} = (L, G, H)$. Since in the variables (L, G, H, l, g, h) purely position coordinates cannot be singled out, i.e., variables on the base and on the leaves of the cotangent bundle are mixed, the potential U depends in general on all variables, $U = U(L, G, H, l, g, h)$.

The kinetic energy T has the form

$$T = \frac{1}{2}\left[(G^2 - L^2)(a_1\sin^2 l + a_2\cos^2 l) + a_3 L^2\right]. \tag{1.72}$$

It is easy to obtain the following facts:

1) if $\frac{\partial U}{\partial h} = 0$, then $H = p_\psi = (\boldsymbol{M}, \boldsymbol{\gamma})$ is preserved,
2) if $a_1 = a_2$ and $\frac{\partial U}{\partial l} = 0$, then $L = M_3$ is preserved.

$H = p_\psi = (\boldsymbol{M}, \boldsymbol{\gamma})$ is called the *area integral.*
$L = M_3$ is called the *Lagrange integral.*

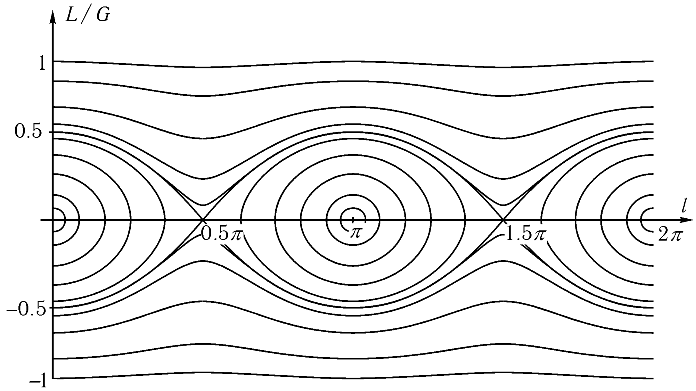

Fig. 1.4. The phase portrait of the Euler problem. Stable fixed points and the straight lines $|L| = G$ correspond to stable permanent rotations about the major and minor axes, unstable points correspond to rotations about the middle axis, separatrices are formed by double-asymptotic trajectories that connect unstable permanent rotations.

A feature of the representation of the kinetic energy in the form (1.72) is its independence of g. This allows one to integrate the Euler problem—the motion of a free top, for which $U \equiv 0$ (see Sec. 2.1, Ch. 2). The corresponding cyclic integral is $G = \text{const}$, which represents the angular momentum $G^2 = \boldsymbol{M}^2$. This makes the Andoyer variables useful for geometric interpretation and analysis of perturbation cases. The phase portrait for the Euler case on the cylindrical model of the sphere is shown in Fig. 1.4. A perturbation (such as a gravitational field) gives rise to chaotic motions on the phase portrait, in the neighborhood of separatrices joining unstable uniform rotations (Fig. 1.5). Next we describe methods to visualize of the phase flow in more detail.

3 Poincaré section and chaotic motions

The *Poincaré map* (Poincaré section, phase section) is useful for visualizing chaotic motions of a two-degree-of-freedom system. It turns the phase flow into a discrete, two-dimensional mapping.

We describe the method for construction of this mapping in the concrete case of rigid body dynamics. It is usually convenient to use the Andoyer variables , the section

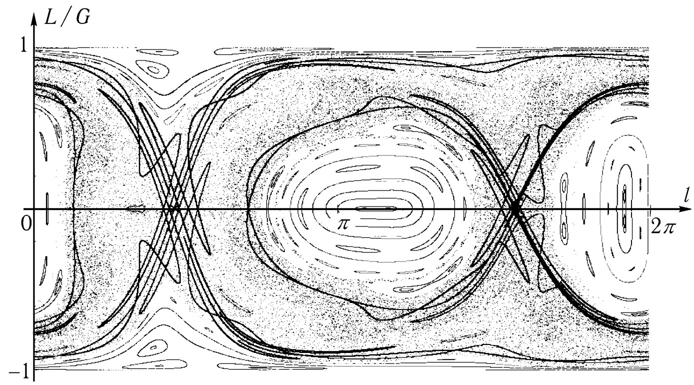

Fig. 1.5. The phase portrait (section formed by the intersection with the plane $g = \frac{\pi}{2}$) for the Euler – Poisson equation, fixing $h = 1.5$, $c = 1$, with parameters of the body: $\mathbf{I} = \mathrm{diag}(1.5; 1.2)$, $\boldsymbol{r} = (0.5,\ 0,\ 0)$. We see doubling of the period on the trajectory branching from permanent rotations near the points $(\pi, 0)$ and $(2\pi, 0)$ in Fig. 1.4 and splitting of the separatrices of unstable periodic orbits emanating from the points $(0.5\pi, 0)$ and $(1.5\pi, 0)$ in Fig. 1.4.

plane initially introduced in [205], and sometimes other section planes that clarify different aspects of the motion. In some cases, the Euler variables $(p_\theta, p_\varphi, \theta, \varphi)$ and other types of section may be useful [107, 175].

First, we fix an energy level $\mathscr{H}(L, G, H, l, g) = E = \mathrm{const}$. If we assume that the field is axially symmetric, then the variable h is cyclic and does not participate in the Hamiltonian, so the corresponding conjugate variable H, which is the constant of areas, can be considered as a parameter. Hence, on the energy level, we have a three-dimensional phase flow. Choose a section plane of the form $g = g_0 \bmod 2\pi$, $g_0 = \mathrm{const}$ (in the sequel, we sometimes use instead $l = l_0 \bmod 2\pi$, $l_0 = \mathrm{const}$) and consider the sequence of intersections of some trajectory with this plane, $\ldots, x_{n-1}, x_n, x_{n+1}, \ldots$, with the proviso that the crossings are all in the same direction, i.e., $\operatorname{sgn} \dot{g}(x_n) = \operatorname{sgn} \dot{g}(x_{n+1})$ (Fig. 1.6).

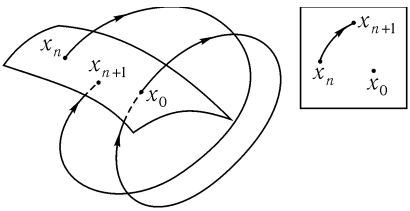

Fig. 1.6.

REMARK. The latter condition arises from the requirement that periodic orbits intersecting the plane $g = g_0$ at, generally speaking, two points, are to be fixed points of the map: $x_n = x_0$, $n = 1, \ldots$ (see Fig. 1.6).

For each point x_n, the Poincaré map puts into correspondence its iteration x_{n+1} belonging to the same trajectory. In general, this mapping is defined locally near some periodic solution since under the action of the phase flow, a point might leave the section plane and not return to it. However, this mapping is very helpful and illustrates different effects related with returning trajectories. It is also sometimes called the *first-return mapping*.

If we consider the Poincaré mapping globally, we have to identify domains on the section plane in which this mapping is well defined. Such domains are called the *domains of possible motions*. Usually they are defined by solutions of the energy equation $\mathscr{H}(\boldsymbol{p}, \boldsymbol{q}) = E$, $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^4$, $q = q_0 = \text{const}$ (in our case, $(\boldsymbol{p}, \boldsymbol{q}) = (L, G, l, g)$, $q_0 = g_0$). If the energy level surface is compact, then the Poincaré recurrence theorem is valid, and the point crosses the chosen plane infinitely many times. On the boundary of the domain of possible motions, the trajectory is tangent to the section plane, i.e., the transversality of the intersection is violated. Global Poincaré mappings are still little studied.

In the study of rigid body dynamics, we choose coordinates $(l \bmod 2\pi, L/G)$ on the section plane for reasons of compactness since $|L/G| \leqslant 1$ (see [205, 88]). Iterations of the mapping are found by numerical integration of the equations of motion in the variables $(\boldsymbol{M}, \boldsymbol{\gamma})$ and subsequent transformation to the section plane in the variables (L, G, l, g) by the formulae (1.42), (1.43):

$$
\begin{aligned}
&M_1 = \sqrt{G^2 - L^2}\sin l, \quad M_2 = \sqrt{G^2 - L^2}\cos l, \quad M_3 = L, \\
\gamma_1 &= \left(\frac{H}{G}\sqrt{1-\left(\frac{L}{G}\right)^2} + \frac{L}{G}\sqrt{1-\left(\frac{H}{G}\right)^2}\cos g\right)\sin l + \sqrt{1-\left(\frac{H}{G}\right)^2}\sin g \cos l, \\
\gamma_2 &= \left(\frac{H}{G}\sqrt{1-\left(\frac{L}{G}\right)^2} + \frac{L}{G}\sqrt{1-\left(\frac{H}{G}\right)^2}\cos g\right)\cos l - \sqrt{1-\left(\frac{H}{G}\right)^2}\sin g \sin l, \\
\gamma_3 &= \left(\frac{H}{G}\right)\left(\frac{L}{G}\right) - \sqrt{1-\left(\frac{L}{G}\right)^2}\sqrt{1-\left(\frac{H}{G}\right)^2}\cos g.
\end{aligned}
\tag{1.73}
$$

This is related to the problem of attaining sufficient accuracy of numeric integration and reduction of computation time. We also note that in the recent versions of the software we also use the quaternion equations in the variables $(\boldsymbol{M}, \lambda)$, which allow one to achieve high accuracy and also to find the absolute motion of the rigid body necessary for visualization of trajectories of different points of the body.

For integrable systems, consecutive iterations of the mapping lie on invariant curves consisting of periodic or quasi-periodic motions (see Sec. 1.7) and defined by the constant value of an additional integral (Fig. 1.4); in the general (nonintegrable) situation, a trajectory can fill, in a chaotic fashion, whole domains in the phase space (of the level surface $H = h$, Fig. 1.5).

The Poincaré mapping first appeared in the theory of nonintegrability and deterministic chaos where it continues to play an important role. It is also useful for the study of integrable cases because it allows one to represent visually the relative positions of different particular solutions in the phase space, among them some outstanding solutions, which are of great importance (see Ch. 2).

For the Euler case, the picture of the Poincaré mapping is well known (cf. Fig. 1.4). By the way, introducing the variables L, G, H, l, g, and h in [150], Deprit (1967) considered the possibility of visual interpretation of solutions to the Euler problem as their main merit; the visual interpretation can compete with the "geometric" Poinsot interpretation (Sec. 2.2, Ch. 2). A more detailed description of the Andoyer variables and their properties was presented by Deprit and Elipe in [152] (1993), where these variables are discussed in connection with the problem of full reduction and its applications to different problems of celestial mechanics. In the sequel, we use the Poincaré section as described above for the study of both integrable and nonintegrable cases.

1.5 Equations of Motion of a Rigid Body in Euclidean Space

1 Lagrangian formalism and Poincaré equations on the group $E(3)$

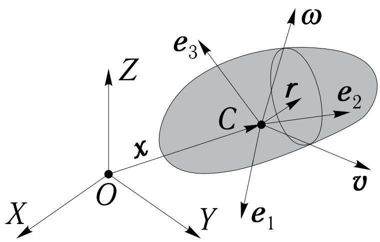

Fig. 1.7. A free rigid body.

Consider a rigid body moving in Euclidean space $\mathbb{R}^3$; its configuration space coincides with the group $E(3)$. In matrix form, elements of the group can be represented as

$$\mathbf{S} = \left(\begin{array}{c|c} \mathbf{Q}^{\mathrm{T}} & \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \\ \hline 0 & 1 \end{array}\right) \in E(3),$$

where $\mathbf{Q} \in SO(3)$ is the matrix of direction cosines (1.30) and $\boldsymbol{x}$ is the position vector of some fixed point C of the body represented in the fixed frame (see Fig. 1.7).

We may write the equations of motion for the projections of the angular velocity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ as well as the absolute velocity of the center of mass $\boldsymbol{v} = (v_1, v_2, v_3)$ on the axes attached to the body. Similarly to (1.46), we write the following obvious geometric relations:

$$\dot{\mathbf{Q}} = \widetilde{\boldsymbol{\omega}}\mathbf{Q}, \qquad \boldsymbol{v} = \mathbf{Q}\dot{\boldsymbol{x}}. \tag{1.74}$$

Now we find the corresponding basic left-invariant fields on the group $E(3)$. For this, consider the time derivative owing to Eqs. (1.74):

$$\frac{df}{dt} = \operatorname{Tr}\left(\dot{\mathbf{Q}}^T \frac{\partial f}{\partial \mathbf{Q}}\right) + \left(\frac{\partial f}{\partial \boldsymbol{x}}, \dot{\boldsymbol{x}}\right) = \operatorname{Tr}\left((\widetilde{\boldsymbol{\omega}}\mathbf{Q})^T \frac{\partial f}{\partial \mathbf{Q}}\right) + \left(\mathbf{Q}\frac{\partial f}{\partial \boldsymbol{x}}, \boldsymbol{v}\right),$$
$$\frac{\partial f}{\partial \mathbf{Q}} = \left\| \frac{\partial f}{\partial Q_{ij}} \right\|; \quad \frac{\partial f}{\partial \boldsymbol{x}} = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right).$$

Grouping terms with ω_i and v_i, we obtain

$$\begin{aligned} \boldsymbol{\omega} &= \omega_k \boldsymbol{\xi}_k, & \boldsymbol{\xi}_k &= -\sum_{ij} \varepsilon_{kij}\left(\alpha_i \frac{\partial}{\partial \alpha_j} + \beta_i \frac{\partial}{\partial \beta_j} + \gamma_i \frac{\partial}{\partial \gamma_j}\right), \\ \boldsymbol{v} &= v_i \boldsymbol{\zeta}_i, & \boldsymbol{\zeta}_i &= \alpha_i \frac{\partial}{\partial x_1} + \beta_i \frac{\partial}{\partial x_2} + \gamma_i \frac{\partial}{\partial x_3}. \end{aligned} \tag{1.75}$$

The commutators of the basic fields $\boldsymbol{\xi}_i$ and $\boldsymbol{\zeta}_j$ have the form

$$[\boldsymbol{\xi}_i, \boldsymbol{\xi}_j] = \varepsilon_{ijk}\boldsymbol{\xi}_k, \qquad [\boldsymbol{\xi}_i, \boldsymbol{\zeta}_j] = \varepsilon_{ijk}\boldsymbol{\zeta}_k, \qquad [\boldsymbol{\zeta}_i, \boldsymbol{\zeta}_j] = 0. \tag{1.76}$$

Using (1.75) and (1.76), we obtain the Poincaré equations of motion (1.17) for the dynamics of a free rigid body

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right) &= \frac{\partial L}{\partial \boldsymbol{\omega}} \times \boldsymbol{\omega} + \frac{\partial L}{\partial \boldsymbol{v}} \times \boldsymbol{v} + \frac{\partial L}{\partial \boldsymbol{\alpha}} \times \boldsymbol{\alpha} + \frac{\partial L}{\partial \boldsymbol{\beta}} \times \boldsymbol{\beta} + \frac{\partial L}{\partial \boldsymbol{\gamma}} \times \boldsymbol{\gamma}, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) &= \frac{\partial L}{\partial \boldsymbol{v}} \times \boldsymbol{\omega} + \frac{\partial L}{\partial x_1}\boldsymbol{\alpha} + \frac{\partial L}{\partial x_2}\boldsymbol{\beta} + \frac{\partial L}{\partial x_3}\boldsymbol{\gamma}, \\ \dot{\boldsymbol{\alpha}} &= \boldsymbol{\alpha} \times \boldsymbol{\omega}, \qquad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \qquad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \\ \dot{x}_1 &= (\boldsymbol{\alpha}, \boldsymbol{v}), \qquad \dot{x}_2 = (\boldsymbol{\beta}, \boldsymbol{v}), \qquad \dot{x}_3 = (\boldsymbol{\gamma}, \boldsymbol{v}). \end{aligned} \tag{1.77}$$

2 Kinetic energy of a rigid body in $\mathbb{R}^3$

The position vector of a point of a rigid body in the immovable coordinate system may be written in the form $\boldsymbol{q} = \mathbf{Q}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{x}$, where $\boldsymbol{y}$ is the time-independent position vector of the point in the body reference frame. Differentiating with respect to time, $\dot{\boldsymbol{q}} = \dot{\mathbf{Q}}^{\mathrm{T}}\boldsymbol{y} + \dot{\boldsymbol{x}}$, and integrating with respect to $\boldsymbol{y}$ over the body, we obtain the kinetic energy in vector and matrix form:

$$\begin{aligned} T &= \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) + m(\boldsymbol{v}, \boldsymbol{r} \times \boldsymbol{\omega}) + \frac{1}{2}m\boldsymbol{v}^2 \\ &= -\frac{1}{2}\operatorname{Tr}(\widetilde{\boldsymbol{\omega}}\mathbf{J}\widetilde{\boldsymbol{\omega}}) + m(\boldsymbol{v}, \widetilde{\boldsymbol{\omega}}\boldsymbol{r}) + \frac{1}{2}m\boldsymbol{v}^2, \end{aligned} \tag{1.78}$$

where $m = \int_\tau \rho(\boldsymbol{y})\, d^3\boldsymbol{y}$ is the total mass of the body and $\boldsymbol{r} = \frac{1}{m}\int \boldsymbol{y}\rho(\boldsymbol{y})\, d^3y$ is the position vector of the center of mass in the coordinate system attached to the body, $\rho(\boldsymbol{y})$ is the mass density of the body, and $\mathbf{I}$ and $\mathbf{J}$ are defined by relations (1.54) and (1.55).