Thomas Bedürftig and Roman Murawski

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Authors
Prof. Dr. Thomas Bedürftig
Leibniz Universität Hannover
Institut für Didaktik der Mathematik und Physik
Welfengarten 1
30167 Hannover
Germany
beduerftig@idmp.uni-hannover.de

Prof. Dr. Roman Murawski
Adam Mickiewicz Universität
Fakultät für Mathematik und Informatik
ul. Umultowska 87
61614 Poznań
Poland
rmur@amu.edu.pl

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With thanks to our wives Michaela and Hania
for patience and sympathy
in the time of our work on this book.

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Preface

A human being must philosophize
whether he wants it to or not
– he is not free here.[1]

Ludwig Feuerbach

The present book is an introduction to the philosophy of mathematics. It represents a special approach to it – one that does not look philosophically at mathematics from the outside but rather attempts to ask philosophical questions from the standpoint of a mathematical (research and teaching) practice concerning fundamental concepts, constructions and methods. It looks for answers both in mathematics and in the philosophy (of mathematics) from their beginnings till today. We think that without the historical component the philosophy of mathematics would be empty. Moreover, the historical approach is important for we should consider (even current) mathematics as being in the process of permanent development.

Philosophy of mathematics is a field between philosophy and mathematics. This is in fact a difficult position. Those being outside mathematics will ask: what at all has mathematics – the firmly stated and infallible system of numbers, formulas and methods – in common with philosophy. On the other hand philosophy – to which the philosophy of mathematics belongs – and philosophers themselves have and had always problems with mathematics, the discipline forming in a sense a paradigm of the scientific thinking and developed to such a degree that it sometimes appears to be completely unmanageable and unclear. Finally, the subject of the philosophy of mathematics is just mathematics – and mathematicians are generally not inclined to consider it in a philosophical way. This is in some sense a rational attitude because all that mathematicians are doing seems to be far away from any philosophy.

We will try to show that the situation is different. Mathematics is a vivid scientific discipline improving itself and altering permanently inside a changeful history. Its basic concepts, methods and principles are traditionally important subjects of philosophy. Furthermore, if we look into the foundations of mathematics we realize that mathematical practice, teaching and learning of mathematics are close to philosophy.

The fundamentals on which mathematics is built permit and even need to be reflected. To do this mathematically is the task of *the foundations of mathematics*, set theory and logic. However, no one is dispensed by that from further reflection because the foundations of mathematics extend to the everyday base of numbers, to mathe-

1 Der Mensch muß philosophieren, ob er mag wollen oder nicht, hierin ist er nicht frei.

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matical methods, to teaching and learning mathematics as well as to mathematical speaking and thinking. Many questions asked in the foundations of mathematics have philosophical origin and – vice versa – various answers given and results obtained there have philosophical meaning. In this new context new philosophical questions arise. The reflection on questions coming from mathematical practice, from the foundations of mathematics and from the philosophy leads to a deeper consciousness in mathematical research practice as well as in teaching and learning mathematics.

The authors of the present book are mathematicians interested in the philosophy and with a certain philosophical background. Our aim is to introduce mathematicians and teachers of mathematics as well as students of mathematics into the philosophy of mathematics. Our introduction is suitable also for professional philosophers as well as for students of philosophy, just because it approaches philosophy from the side of mathematics. The knowledge of mathematics needed to understand the book is elementary. Parts of the book where this is not sufficient are printed in small size. The same is also done when we speak about very special advanced results of mathematics and mathematical logic or about special items in philosophy. Those fragments of the book can be omitted by first reading without any serious loss. The text printed in normal size (almost all of the book!) can be in principle understood by any layman with school mathematics background.

The starting point as well as a reference point of our text are all the time real numbers. In Chapter 1 the way to them is sketched and some mathematical and philosophical problems and questions connected with them are indicated. Those questions lead to the foundations of mathematics and to the philosophy of mathematics. Chapter 2 provides a survey of various views, positions and tendencies that were formulated and developed from the very beginning of the philosophical reflection on mathematics till today. They form a sort of a background for considerations in Chapter 3 where questions and problems formulated in Chapter 1 are discussed. Chapter 2 can be also treated as an independent compendium of views and trends in the philosophy of mathematics. One can read the Introduction, Chapter 1 and Chapter 3 together and from time to time – when there appears such a demand – consult Chapter 2.

Chapter 4 is devoted to the universally accepted and applied background of the mathematical way of writing and speaking, i.e., to set theory. The usage of set-theoretical language influences the mathematical thinking. The aim of this chapter is to draw interest to that usually unconscious fundament of mathematical speaking and thinking. We shall consider two different approaches to sets and indicate some problems connected with them. Chapter 5 is devoted to the axiomatic method and to the second fundament of mathematics, namely to logic. A short survey of fundamental logical concepts as well as some information on the development of logic are provided. Mathematical logic has adopted many previously philosophical questions about mathematics and obtained results of philosophical meaning and significance.

In Chapter 6 we go out in search of infinitesimals, the infinitely small quantities – we spoke about them already in Section 3.3. They can be potentially found in school mathematics – it seems that thinking in terms of infinitesimals is not alien for pupils, however, it disappeared from mathematics. In further parts of Chapter 6 we consider elements of nonstandard analysis and discuss philosophical and mathematical problems and consequences connected with this way of thinking. In Chapter 7 we look back at what has been said in previous chapters. We attempt to characterize shortly the philosophy of mathematics introduced in the book and to state what is it for. We conclude thoughtfully. The Appendix contains short biographies of selected philosophers and mathematicians. At the end of the book one finds indices of symbols, names and concepts.

For simplicity we use in the book the masculine form (he/him/his). Whenever it appears it should be understood inclusively (and not exclusively) as referring both to her and him.

We would like to thank the editors of De Gruyter Verlag Berlin, in particular Nadja Schedensack and Apostolos Damialis for their always patient and helpful cooperation.

Hanover and Poznań
in April 2018

Thomas Bedürftig
Roman Murawski

Introduction

I am not a religious man,
but it's almost like being in touch with god
when you're thinking about mathematics.

Paul Halmos

The mathematical path of a human being begins early. First mathematics arises from contention and accord with the reality. Numbers – connected with counting and becoming natural numbers in mathematics – receive from that point their meaning. The same holds in a similar way also for negative numbers, fractions and rational numbers that arise from the contact with the everyday magnitudes by abstraction.

The situation with reals – considered in Chapter 1 at the very beginning of our book – is different. Here one has to do with an essentially new situation. To obtain reals starting from natural and rational numbers one uses in mathematics a completely new way. It is detached from ties with concrete applications. The old, simple abstraction from everyday and physical magnitudes does not work here any longer. It is just the conflict with magnitudes that causes the *theoretical* construction of the reals or the postulation of their desired properties in an axiomatic way. Some geometrical and theoretical needs challenge mathematics and demand appropriate concepts and methods that not so long ago were very new and revolutionary. Here arose and arise mathematical as well as philosophical questions. These questions point to various directions in the philosophy and history of mathematics. We indicate some of them in this introduction and mention some problems.

The reals are – and this is the common opinion today – a safe mathematical wealth. One has forgotten the heavy discussions – that still a hundred years ago affected the heads and hearts even consciences of mathematicians – or they are now treated as closed. But the problems did not disappear. One suspends them in the foundations of mathematics, omits problems and pragmatically runs over to the order of business that begins with $\mathbb{R}$. In teaching mathematics – where the main aim is to introduce as quickly as possible concepts and methods – one starts from the reals, possibly avoiding the conceptual and methodological questions and throwing away at this crucial point the possibility presenting an interesting material and deeper insight into the foundational elements of mathematics.

In Chapter 1 we consider a way to the reals that uncovers very concretely and in an elementary way problems that still today occupy the foundations as well as the philosophy of mathematics. After having presented in Chapter 2 various conceptions developed in the philosophy of mathematics, in Chapter 3 we will analyze the problems

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that lead to a better understanding of the conflicts that caused quite a deep-rooted excitement at the time of Kronecker, Frege, Cantor and Dedekind.

Those conflicts are partially or completely unnoticed in teaching at universities and at schools. In the middle of teaching mathematics at German colleges the reals are usually introduced in such a way that hides the meaning of this step. Even for a well-disposed reader this step possibly had been unnoticed during his own time at school. Before considering the problems in details in Chapter 1, now we would like to indicate shortly some important points in teaching mathematics both at schools and universities. They concern some simple issues that usually are overlooked.[2]

Teaching mathematics at universities begins usually with real numbers. When it is done properly they are introduced in an axiomatic way, i.e., by listing their properties without questioning them. This can proceed as follows.

In a handbook for teachers about number domains [258, p. 159], in the part dealing with the introduction of reals, one finds the following: At the beginning of the chapter about the reals it is stated – after a remark on the diagonal of the unit square – a remark on the number line (see below) and an indirect proof of the irrationality:

> "Hence the number $\sqrt{2}$ well known to us does not belong to the set of rational numbers."

This reflects the usual practice of introducing reals in the process of teaching mathematics and is a bit surprising in a handbook. Where from and why is the number $\sqrt{2}$ known to us? Just before that the rational numbers were introduced. Where does the number $\sqrt{2}$ belong? It is suggested that it belongs to a special type of numbers, namely to reals that are already here in a way. Where – this will be seen soon.

Such an attitude in "introducing" reals is in some cases in fact the background of teaching mathematics. How should teachers who themselves were trained and taught in such a way be able to convey pupils what in fact takes place here, at what conceptual threshold they stand? A chance to understand – at least partially – the problem and the peculiarity of solving it will be squandered. It will remain unexplained what $\sqrt{2}$ as a *number* actually *is* and why one can *calculate* with the term $\sqrt{2}$ that has not appeared so far.

An important point appears afterwards in teaching: the procedure of approximating, e.g., to $\sqrt{2}$ – whatever $\sqrt{2}$ is. Then follows a phrase like:

> "To the finite and periodic decimal fractions one adds the infinite non-periodic decimal fractions. All together are the reals." ([258, p. 189].)

2 Notice that we refer here only to the standard theory of reals. Other interpretations are considered for example in the book [91] where also historical explanations can be found – we recommend it for further study of reals.

What are infinite non-periodic decimal fractions? This is in fact only a negative description, the denial of a great problem. It concerns the *handling of the infinite*. The issue the mathematicians of the 19th century struggled heavily and in fact undecidedly for in university and school teaching will be a precondition in a subordinate clause. Still today the infinite – being a base for various further questions – is an important problem of the foundations of mathematics. The complex of problems connected with the infinite will appear throughout the whole book.

Behind such a procedure there is the so-called number line – it is not distinguished whether one speaks here about points or numbers. It is in fact a customary and justified exercise in the process of teaching mathematics to *illustrate* numbers as points on a number line. But is it justified to *identify* numbers and points? The chapter about "introduction to the reals" of the above-mentioned handbook begins with the following explanation:

> "Hence the reals will be at the beginning explained by the totality of **all** points of the number line and seen as given." [Bold in the original.]

This is useful – and murderous. The whole complex of problems of the continuum in all facings is settled at one blow. But what is in fact *the* number line? What is "the totality of all points"? How is it constructed? Does one set the points or can one take them? What type of object is a point? Is the continuum of a line exhausted by points, hence a set or "the totality" of points? If this is the case: points are initially not numbers. Can points be declared to be numbers? What numbers are they? How does one calculate with points?

The problem of the completeness of the set of the real numbers $\mathbb{R}$ will be removed by this explanation. As a set of all points of a line, the real numbers seemingly have no gaps. In fact, today one proceeds reversely. The real numbers $\mathbb{R}$ are constructed or introduced axiomatically and then one declares copies of $\mathbb{R}$ to be lines. Hence points do not become numbers, but vice versa – numbers become points.

The aim of set-theoretical constructions of reals, whatever they are, is always the completeness. Here arises ultimately and clearly the question about the connection of the constructed domain of numbers and the geometrical line. Is there a *difference* between the *set* of reals and possible conditions in the *continuum* of the geometrical line? This problem is not recognizable any more when $\mathbb{R}$ is represented as the set of all points of a line or lines are treated as copies of $\mathbb{R}$. The quite possible difference refers to the so-called nonstandard analysis. In the background appears the old idea of *infinitely small* quantities. We come to this in Chapter 3 and in Chapter 6.

Which questions referring to philosophy do appear here? We have indicated them. They are philosophical fundamental questions: it is the old question about the infinity that concerns the construction of infinite sets as well as the acceptance of infinitely small quantities, the infinitesimals. It is the question about mathematical concepts such as number or magnitude. It is the problem of the classical continuum and of the contemporary conception of the continuum as a set of elements and the identification of points and numbers. It is the question about axioms and the axiomatic method. And it is generally the question about the relation of mathematics and its concepts with the reality. What is the status of mathematical concepts? What are numbers? What is their source?

Such questions – that are relevant for the teaching of mathematics – will be made more precise in Chapter 1 on the way to $\mathbb{R}$. In Chapter 2 we shall face them again and again in the extensive outline of the history of philosophy of mathematics. There we present numerous mathematicians and philosophers as well as positions in the philosophy of mathematics from which answers can be expected.

In Chapter 3 we consider the fundamental questions in the background of the history of philosophy and mathematics. It is the question what numbers actually are. It is the question about the concept of the infinity which appears in various positions presented in Chapter 2, about the concept of a magnitude, the continuum and the infinite small. Magnitudes and the infinitesimals disappeared from pure mathematics when the classical continuum has been replaced by $\mathbb{R}$. Only names of magnitudes remained here and bare denotations of the infinitesimals.

Set theory and logic form the mathematical domain of the foundations of mathematics. They are the source of the today's way of speaking and explaining notions of a proof, a consequence, a theory. A new method of securing and presenting the mathematical knowledge provides the renewed axiomatic method. Set theory, which is in fact a theory of the infinite, together with logic as well as axiomatics will be presented in Chapters 4 and 5 – their development will be described and their problems and results will be shown and interpreted. New foundational problems naturally arise where new foundations are.

Above we have emphasized some *problems* which are hidden behind the reals. One should necessarily notice which mathematical *possibilities* they opened and what progress they produced. The step from the classical continuum into the continuum of reals made in the second half of the 19th century was in fact revolutionary. Only the reals and the set theory in their background made possible to grasp mathematically properties of the continuum hidden in the visual continuum. To them belong the fundamental concepts of completeness, continuity or dimension. Eventually, the concepts of limit, differential and integral – being used for a long time but unsecured – could be made precise. All this was possible only thanks to abstracting from some problems mentioned above – all the time against a massive resistance, e.g., of intuitionism. Today we are in another position. The foundations being then new have been for a long time proved of value. We can maintain distance and realize today on what ground we stay – without disputes concerning the foundations and problems which are still here. We can admire the mathematical achievements and attempts to overcome the problems and at the same time enter upon the mathematical venture that lies behind them.

The indicated aspects will be considered in the following chapters. But first in Chapter 1 we come to some problems connected with the foundations of $\mathbb{R}$ that in the everyday practice seem to be forgotten a bit. Chapter 1 is short, concrete, elementary and sketchy. The indicated steps on the way to the reals are known of course. However, the abstracting from any previous knowledge, the special attention paid to every single step as well as the inexorable distinguishing between geometrical and arithmetical level and the clear naming of problems can seem to be a bit unfamiliar in the beginning.

1 On the way to the reals

Between the intuitional idea and the mathematical formulation which should describe the scientifically substantial elements of our intuition in precise terms, there will always remain a gap.[1]

Richard Courant

In this chapter we will try to trace the mathematical base of the reals and to present briefly their setting and mathematical foundations, in order to formulate problems of a philosophical, methodological and mathematical nature. We state, once again, as already indicated in the Introduction, the ignorance of those problems as well as the pragmatism in putting reals as the universal base of mathematics. One pragmatically carries on doing mathematics. The reals seem to be always there. They became quasi "natural" numbers for mathematicians. In this chapter, first of all, we want to call attention to questions lying in the background of the reals by observing carefully the way they are introduced and indicating details and problems appearing there. For the sake of clarity we choose a short and pointed formulation of the problems.

The way to the real numbers $\mathbb{R}$ begins – as almost everything in mathematics does – with (genuine) natural numbers $\mathbb{N}$. We choose in this chapter a subsequent starting point: the rational numbers $\mathbb{Q}$. It is the starting point of someone learning mathematics who does not know anything about the reals. So we put ourselves consciously in the position of a student learning mathematics – it is similar to a position of the Pythagoreans 2500 years ago. *All we have and know are rational numbers and nothing else*. It is a requirement in this chapter to abandon really completely all our previous knowledge. About the way from natural to rational numbers we write briefly in Chapter 3.

1.1 Irrationality

What is irrationality? Consider the standard example. One looks for a number whose square is 2. It is called $\sqrt{2}$. The following result is stated everywhere.

Theorem. *$\sqrt{2}$ is irrational.*

1 Zwischen der intuitiven Idee und der mathematischen Formulierung, welche die wissenschaftlich wichtigen Elemente unserer Intuition in präzisen Ausdrücken beschreiben soll, wird immer eine Lücke bleiben.

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A standard indirect proof of this is then given. For completeness we also shall give a proof here, and we do this by the oldest way of arguing, which one finds in *Elements* by Euclid [109, Book X, Section 115a].

Proof. Suppose that $\sqrt{2}$ is rational, say $\sqrt{2} = \frac{m}{n}$, where m and n have no common divisor (i.e., they are relatively prime). Hence $2 = \frac{m^2}{n^2}$ and $2 \cdot n^2 = m^2$. So m^2 and consequently also m are even, e.g., $m = 2 \cdot a$. So n should be odd because by assumption m and n have no common divisor. On the other hand since $4 \cdot a^2 = m^2 = 2 \cdot n^2$, one gets $n^2 = 2 \cdot a^2$, so n^2 is even, and hence n is also even. But this is a contradiction! □

What does this theorem actually say? The suggestion, made even to experts, is: $\sqrt{2}$ is a number of another type, just an irrational number.

So what is our situation? There is nothing other than rational numbers. Hence "irrational" can mean only

$\sqrt{2}$ is not rational.

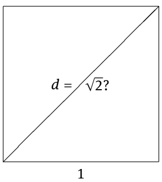
But "not rational" simply means, in the absence of other numbers, the following result.

Theorem. *$\sqrt{2}$ is not a number.*

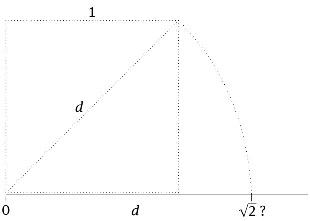
This means that there is no number whose square is 2. So what is $\sqrt{2}$?

$\sqrt{2}$ is a term without any meaning.

One can however write this term to dramatize the question how it can have a sense. But at first it has no sense. So one tries to give $\sqrt{2}$ a different sense. Can it not be seen in the next standard example?



We argue thus: from the theorem of Pythagoras it follows that the square with the side being a diagonal of the unit square has a surface area equal to 2, hence the diagonal d has length $\sqrt{2}$. If we put d on the *number line* then $\sqrt{2}$ can be seen there as the length of d:



This is the next *suggestion* which misleads an insider and lecturer and seduces a learner to accept $\sqrt{2}$ as a number without further ado.

What error have we committed? We naively assume that every point on a line on which we have visualized rational numbers as points represents a number. For numbers lie there densely. But what numbers do we have? Rational numbers! And just to such numbers belongs a point on a number line. Since $\sqrt{2}$ is not rational, we have found no *number* $\sqrt{2}$ on the number line but only a *point* to which no number corresponds. This means that there is no number corresponding to the length of the diagonal d. So:

$\sqrt{2}$ is not a number. There is no measure of the length of d.

Hence, one infers the following result.

Theorem. *The diagonal d in the unit square is not measurable.*

This is an astonishing situation. The diagonal d has a length and our experience says that there is no problem about measuring lengths. So it must be accepted that our experiences deceive us. There are magnitudes to which – by a given unit – we cannot assign a number. We are unable to measure them. In fact, the unit and a magnitude can be *incommensurable*. The domain of magnitudes cannot be grasped by our numbers.

Let us turn back to the beginning where something goes wrong. One cannot start out on the way to the reals by stating "$\sqrt{2}$ is irrational". In fact, this pretends that $\sqrt{2}$ is a number. One should start in the following way.

Theorem. *There is no number whose square is 2. There is no measure for the diagonal of the unit square.*

To operate from the very beginning with the symbol $\sqrt{2}$ is essentially problematic because this suggests "number" and pretends that the solution of our problems is available. The profound difficulty of the problems and the fact that there is a long way to go to get to the reals is usually hidden. Teachers and lecturers are shy to admit complete failure: the arithmetic developed so far does not work. The fascinating and productive question is overlooked: what can be done mathematically?

Behind such seemingly only methodological and didactic problems there are fundamental philosophical questions.

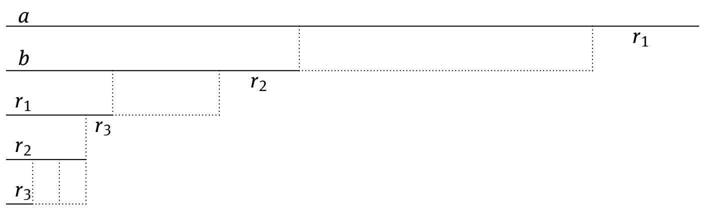
1.2 Incommensurability

Of what type the problems are can be seen in the best way when one looks back in the history of mathematics. The discovery made by us was also made by the Pythagoreans probably about 450 BC – in another, most probably more direct way. Records are unclear at this point and sources are poor. One of the reconstructions of historians in connection with this says that the phenomenon of incommensurability was discovered on the example of a regular pentagon, the emblem of Pythagoreans as well as their symbol of the cosmos.[2]

The task was to determine the proportion of the side and the diagonal in a regular pentagon. The Pythagoreans developed a procedure to *determine the proportion* of segments that could be applied everywhere – their “reciprocal subtraction” – and that became the well-known *Euclidean algorithm* in the domain of natural numbers. This was an important procedure in the history of mathematics – it will play a role again in Chapters 2 and 3. Since it is rarely applied in a geometrical form we present it briefly and apply it to the regular pentagon.

Let a and b be two segments. The Greeks did not have standardized units of measure for segments with the help of which a and b could be measured, corresponding to fixed numbers, and in such a way their proportion determined. They proceeded as follows:

Remove from a the smaller segment b as many times as possible. One gets the remainder r_1. Next remove from b twice the remainder r_1. The remainder r_2 remains, etc.



$$a = 2 \cdot b + r_1$$
$$b = 2 \cdot r_1 + r_2$$
$$r_1 = r_2 + r_3$$
$$r_2 = 3 \cdot r_3$$

In our example r_3 is contained in r_2 three times. Hence $r_1 = 4 \cdot r_3$, $b = 11 \cdot r_3$, $a = 26 \cdot r_3$ and the proportion of a to b is $26 : 11$. Now r_3 is a common measure for a and b. This was the aim of the reciprocal substraction: to determine the common measure for two given segments.

Let us transfer this procedure to the case of a regular pentagon shown in the picture below. We would like to determine the proportion of the diagonal d to the segment a. By symmetries in the regular pentagon the reciprocal substraction proceeds as follows:

$$\begin{array}{ccccc} d & = & a & + & d_1 \\ d_1 & = & a_1 & + & d_2 \\ d_2 & = & a_2 & + & d_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \qquad \begin{array}{ccccc} a & = & d_1 & + & a_1 \\ a_1 & = & d_2 & + & a_2 \\ a_2 & = & d_3 & + & a_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

What can one see? What did the ancient Greeks see?

2 Cf. the paper by H.-G. Bigalke [36]. Other authors – e.g., H. Boehme – reject the idea of this version (see [37, 39]) and establish an arithmetic-algebraic and indirect reasoning similar to the one indicated above.

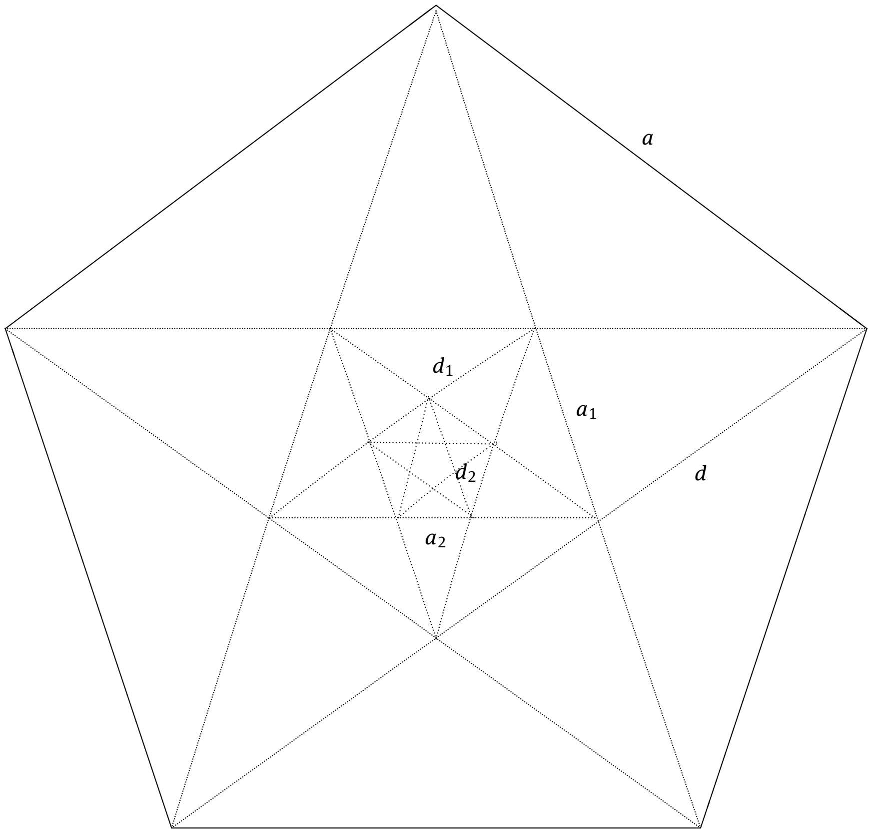


Fig. 1.1. The beginning of an infinite sequence of regular pentagons.

The procedure does not stop. It continues in a sequence of pentagons that will become smaller and smaller and that disappear at infinity. That is:

> There is no common measure for the diagonal and the segment of a regular pentagon.

The corollary is the following.

Theorem. *Diagonals and sides in a regular pentagon are incommensurable. There is no number expressing the proportion of the diagonal and a side.*

A "visible" and direct proof of the incommensurability such as the one just given is something other than an indirect reasoning for the "irrationality" of $\sqrt{2}$ given above. It provides a different experience for a learner that makes the incommensurability understandable and – much later – justifies the concept of irrationality. Also in the case of $\sqrt{2}$ there is possibly an in*sight* – aesthetically not so convincing – into the incommensurability of the diagonal and the side of a square.

In the above picture one can additionally see that, since $d_1 = d - a$, the inner and outer proportions of d and a coincide: $d : a = a : (d - a)$. Hence the proportion of d and a is the *golden ratio*. So:

There is no number expressing the proportion of the golden ratio.

The discovery of the incommensurability ca. 2450 years ago shocked the ancient Greeks. Why?

Numbers provided the foundation of mathematics and – this is noteworthy – the *metaphysics* of the Pythagoreans. For them numbers had real power that affected the material world from the higher world of numbers and formed the real things as well as their relations according to relations between numbers. This was a deep philosophical conviction of the Pythagoreans: *all is number*. And now under their eyes this conviction and their philosophy broke down. They had discovered line-segments whose relation could not be represented by natural numbers and in this way the central principle of their philosophy had been negated. Their philosophy broke in mathematics and metaphysics which before were a oneness.

Legends entwine the discovery of the incommensurability that in the fraternity of Pythagoreans was guarded as a secret and revealed by Hippasos of Metapont – in fact with dramatic consequences for the betrayer. He was cursed by Pythagoras. During his escape on the sea he was engulfed by waves stirred up by a storm.

Plato declared a hundred years later in clear words that it was a shame to know nothing about incommensurability:

> "And it seemed to me as if it were impossible for humans but rather for pigs, and I felt ashamed not only for myself but also for all Hellenes." ([276, Nomoi, Band 7, 819 d–e], translation by the authors.)

Similarly, for people beginning in mathematics today, the world breaks apart, just as it did for the Pythagoreans and for Plato. His mathematics, which had held good up to that point, now fails. But this experience can be made by someone learning mathematics only if he is given such an opportunity. Here belongs an arithmetical admission of complete failure as above. It cannot be made when one gives him $\sqrt{2}$ and the irrational numbers and proceeds as if they have always been there and everything could be measured. He needs this experience to be able to recognize the big step into a new and theoretical mathematics and to experience how mathematics invents "theoretical numbers" and *theoretically* solves and overcomes the problem of incommensurability.

Tutors who are lecturing are probably not always sensitive enough to all this. Recall the handbook mentioned in the Introduction [258]. In fact, one willingly forgets what we undertook above, namely really to take the point of view of a learner and to abstract from any previous knowledge. The reals are so close and so comfortable. Everything is measurable by them. And it is in fact hard to follow every step on the way to the reals. But the insight into the "abyss" of incommensurability and the next steps to $\mathbb{R}$ are important. One thing is clear: at the moment when the reals are there the incommensurability disappears. The irrationality remains.

Today – as it was back then – in connection with this certain philosophical questions arise. Numbers and number theory that for Pythagoreans were the *base* of philosophy detached themselves from the philosophy and now faced it. They became *subjects* of philosophical reflection. In Plato's philosophy ideas appeared in the place of numbers. What was then the status of numbers? Along with the mathematical problem there arose also a philosophical question concerning numbers:

– What are numbers?

First of all the simplest numbers, natural numbers are meant here – to them all other numbers including the later reals can be traced back. Just this question along with the question about reals led in the 19th century to the rise of the new mathematical discipline – the foundations of mathematics. It should explain *mathematically* what numbers are. We shall write about such attempts and about philosophical positions relating to them in Chapters 2 and 3.

The next question will be:

– What are reals?

This question is connected not only with mathematical problems but – as will be seen soon – also with philosophical ones. What in fact are those objects we have to do with every day? General questions from the philosophy of mathematics can arise:

– What is the status of mathematical concepts?
– What is the relation of mathematics and its concepts to reality?

The philosophical program of the Pythagoreans did not fail then in practice. The limits of numbers were indicated by geometrical magnitudes. The mathematical answer of the Greeks was plausible: they developed a theory about magnitudes that removed the theory about numbers and that so to say got priority now. This theory about magnitudes was, similarly to Greek geometry, axiomatic, i.e., it was founded on – not always explicitly given – basic statements describing the relations and the uses of magnitudes.

In today's axiomatic approach one excludes the question what in fact mathematical objects are. What in a mathematical and philosophical sense therefore remains is the question:

– What are, or better, what *were* magnitudes?

Today the old magnitudes are expatriated from pure mathematics. They have been replaced by numbers, by reals.

1.3 Calculating with $\sqrt{2}$?

Above the following question was formulated: what can be mathematically done with arithmetically empty hands? For example: how should one calculate with $\sqrt{2}$ when we do not know what $\sqrt{2}$ is and what its relation to the rational numbers is? The terms

$\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt{3}$, $\sqrt{5}$ etc. are all without meaning. Then how should we understand

$$3 \cdot \sqrt{2}, \quad 2 + 3 \cdot \sqrt{2}, \quad \sqrt{2} \cdot \sqrt{3}$$

etc.? Even $\sqrt{2} \cdot \sqrt{2}$ is a term without meaning.

Nevertheless, one calculates without any doubt as if the reals were given: one adjoins, as is usually said, $\sqrt{2}$ to $\mathbb{Q}$ and proceeds in such a way as if the set of all terms $a + b \cdot \sqrt{2}$, that is,

$$\mathbb{Q}(\sqrt{2}) = \{a + b \cdot \sqrt{2} : a, b \in \mathbb{Q}\},$$

were a subset of $\mathbb{R}$ – where $\mathbb{R}$ and calculations on it are tacitly assumed. But this is not correct as it stands. At least some words about calculating with $\sqrt{2}$ and other terms as above should be necessarily said.

One should be clear in learning and teaching and at least the following should be said: *We proceed* as if one could calculate with $\sqrt{2}$, and calculate formally with $\sqrt{2}$ in such a way *as if* its square were equal 2. We grasp, e.g., $3 \cdot \sqrt{2}$ and $2 + 3 \cdot \sqrt{2}$ as *formal* terms but calculate with them as usual, i.e., as if the calculation rules working in $\mathbb{Q}$ held also for those formal terms.

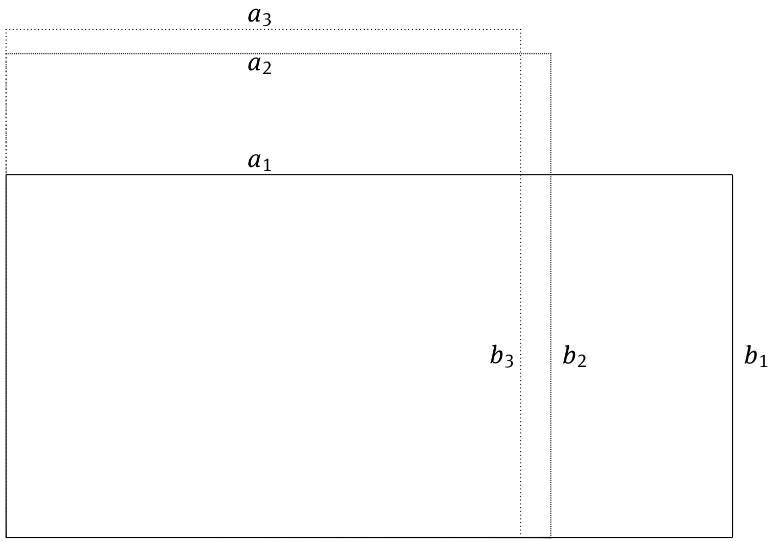
It should be made clear that something new and *theoretical* takes place here, that one deals here with a *formal expansion*, *formal* adjunction of $\sqrt{2}$ to $\mathbb{Q}$. The elementary algebraic procedure of formal adjunction should be presented early in academic lecturing – and this can be done without mentioning $\sqrt{2}$.

1.4 Procedure of approximating, nesting of intervals and completeness

To repeat once again: we have *nothing except the rational numbers* $\mathbb{Q}$. We cannot speak about the *number* $\sqrt{2}$ even when we calculate with $\sqrt{2}$.

Even when one cannot measure d, the diagonal in a unit square, one can approximate its length by rational numbers. Many procedures of approximating have been invented for this case as well as for many others. We choose an old procedure that was known already in the antiquity and which is named after Heron of Alexandria (about 100 AD).

We want to approximate a square over d having a surface area 2 by rectangles whose surface area is 2 and which have rational side lengths. We begin quite simply and very imprecisely with the rectangle R_1 whose sides have lengths $a_1 = 2$ and $b_1 = 1$. As a next approximation we take the rectangle R_2 whose side a_2 is the arithmetical mean of a_1 and b_1, that is, $a_2 = \frac{a_1+b_1}{2}$. Then there should be $b_2 = \frac{2}{a_2}$. We proceed with a_3 and b_3, respectively, etc. The following picture shows the sequence of the rectangles.

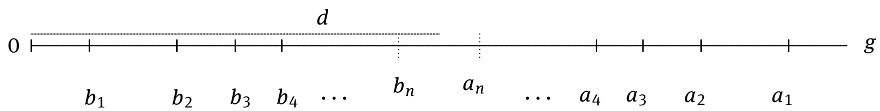


The sides in the rectangle R_n have the lengths

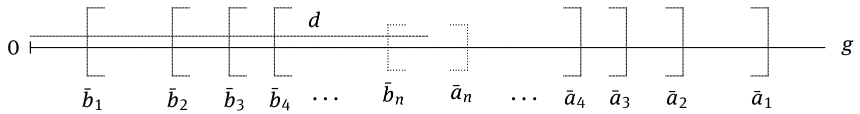
$$a_n = \frac{a_{n-1} + b_{n-1}}{2} \quad \text{and} \quad b_n = \frac{2}{a_n}.$$

All measures for the sides a_i, b_i of the rectangle R_i are rational numbers. The values a_i decrease and the values b_i increase, the differences $a_i - b_i$ approach 0.

Assume now that the sequence of the sides a_i, b_i is put on a line g. The sides a_i, b_i begin with the point 0, hence are represented as *points* on g denoted by strokes. We stress that we are talking about sides as *geometrical* intervals and their lengths. The sequence of the sides looks like this:

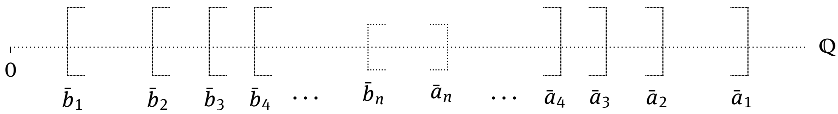


One sees a nesting of intervals that are always bounded by end points b_i and a_i. This is shown more clearly in the next picture:



Here a_i and b_i are *end points of sides of rectangles* with rational lengths. The dots "..." indicate that the procedure goes on and always leads to new sides a_i, b_i that with every step move closer to the point for d. The interval $[b_n, a_n]$ is a position with probably large n in the never-ending sequence of intervals $[b_i, a_i]$ all of them including the end point d. This is in fact a *geometrical setting*.

Besides – and this should be seen completely separately – there is the arithmetical setting in the domain $\mathbb{Q}$ of rational numbers. We imagine all rational numbers presented as points on a line in the following way: in the next picture we want to see *only* "rational points", i.e., the "number strokes" of rational numbers and the intervals of rational numbers that give the lengths a_i, b_i of sides of the rectangles. The analogy "point – number" is so commonly used that the rational measures of the sides a_i, b_i are simply marked by a_i, b_i. We should be strict and distinguish between numbers and points. So we denote the measures by $\overline{a}_i, \overline{b}_i$. Note that there is no equivalent for d since there is no rational number for the length of d.



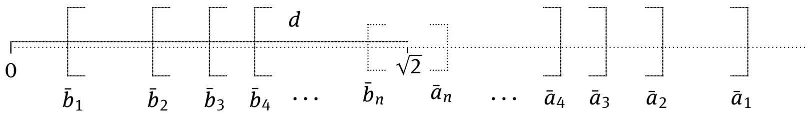
Though the arithmetical positions seem to be similar to the geometrical ones, they are in fact different. Above in the geometrical representation the end point of d is located in all intervals $[b_i, a_i]$. Here in $\mathbb{Q}$ there is no number corresponding to d:

- The intersection of all intervals $[b_i, a_i]$ is empty in $\mathbb{Q}$.

What is to be done? One *postulates* an appropriate number.

- *Claim:* There is exactly one number belonging to all intervals $[b_i, a_i]$.

It is called $\sqrt{2}$. It fills the gap in $\mathbb{Q}$ and measures d.

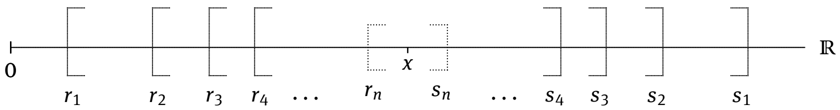


Another question is: What is this entity $\sqrt{2}$ that has turned up here. We will come to this in a moment. The procedure described above exemplifies the decisive step to the reals. We have closed a *gap* in $\mathbb{Q}$ by $\sqrt{2}$ just by a claim.

Now we claim to get the whole set of reals that should be everywhere *without gaps* and *complete*.

Claim: Let $\mathbb{R}$ be a set. One calculates with elements of $\mathbb{R}$ in a way similar to the way in which one calculates with the rational numbers. Also the order of elements should equal the order of elements of rational numbers with the following additional property of completeness.

Axiom (Axiom of completeness). Let $[r_n, s_n]$ be a nesting of intervals in $\mathbb{R}$. Then there is exactly one number $x \in \mathbb{R}$ that belongs to all intervals $[r_i, s_i]$.



"Axiom" is the Greek word for "claim", a claim that is treated as "fair", "suitable" or plausible and the statement of which should be "obvious". The conflict with intuitively clear geometrical magnitudes has provoked the axiomatic claim of completeness. It claims that from now on geometrical magnitudes can be measured and represented by reals. Just this was the intention of the claim.

Here however most essential questions about the axiomatic method arises.

– Where do axioms come from, how are they chosen, how are they justified?

And quite fundamentally:

– Is the axiomatic method suitable for grasping real or perceptual phenomena?

It is said that axioms are obvious statements. Note that the obviousness of the claim of completeness was borrowed from the perceptual completeness of the geometrical line and did *not* come out of the domain of the rational numbers itself. It is just a geometrical obviousness that is added to arithmetic – one can even say "forced into arithmetic". In the case of classical arithmetic the fact that the range up to $\mathbb{Q}$ is incomplete is simply obvious. By the axiomatic setting of $\mathbb{R}$ together with the completeness axiom a new arithmetical level, an axiomatic one, i.e., a *theoretical* one, is attained.

There still lacks a construction of the reals that would say what the claimed real numbers really are. We come to this in the next section.

Now the situation has been crucially changed: to every possible geometrical length marked by a point on a line there corresponds – this is assured by the completeness axiom – a real number. And vice versa: all those real numbers can be again *illustrated* on a geometric line as "number strokes", as points. This is a big step – thanks to the mutual correspondence of points and numbers one can identify numbers and lengths. Or better: lengths and magnitudes in general can be *replaced* by real numbers.

The consequence of this is: the number *line* becomes a line of *numbers*. A geometric line will become a *copy* of $\mathbb{R}$. We have briefly seen what gradually happened in 19th century mathematics. A linear geometrical continuum has been replaced by an arithmetical continuum. Since that time $\mathbb{R}$ *is* the continuum. The continuum became a set. Translated back to geometry: the linear continuum became a set of points.

Now when one looks at that outcome the following *question* can be asked: is this reversible correspondence between real numbers and points of the geometric line really suitable with respect to the perceptible line? Is it at all possible to understand and to grasp the geometrical "continuum" of a line by points located on it? Or is it a matter of quite alien things in the case of points and lines between which there only exists an *external relation*, the relation of incidence: points are located on lines, lines range through points?

– Can the phenomenon of continuity be grasped by points?
– Is the geometric continuum a set of points?

If these questions are answered affirmatively then one should for example be able to build a line as a set of points. However, if one wants to build a set of points, one should know what a point is. Nobody wishes to know this anymore since one has the new axiomatic approach. Everybody smiles at the "definition" by old Euclid and writes in quotation marks: "A point is that which has no part". We will anew ask this question, and we must do this:

What is a point?

The situation yielded by the construction of the reals given in the next section seems to clarify and solve the problem of building a set of points. We can put a system of coordinates on a line as real numbers when we have them, i.e., to depict reals on a line. But is "every point" caught in this way? What are "all points" of a line? The answer supposes yet again the assumption that a line is a completely built and given set of points – and corresponds point by point to $\mathbb{R}$.

Just this is assumed. And just this makes a line into a copy of $\mathbb{R}$, turning it into the number line. But the general philosophical question remains:

Can the set of reals $\mathbb{R}$ *be* the linear continuum?

Mathematically the decision has been made, but the question has not been answered.

Once again this question arises, in another way. It is associated with the obviousness of the axiom of completeness in which the central mathematical assumption is to be found. Is the axiom of completeness really geometrically obvious?

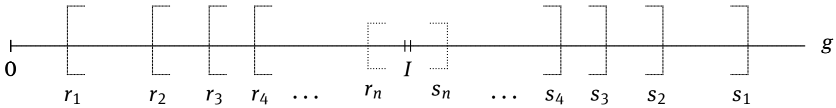
If one considers the nesting of intervals *geometrically* then segments correspond to intervals. Consider the perceptual intersection over all segments of a nesting of intervals. Then, by the axiom of completeness, this intersection is geometrically exactly one point. Why *exactly* one point? Can there be more points? Their distance would then be infinitely small, infinitesimal. This is possible when one does not make an assumption expressed by "exactly one" in the formulation of the axiom of completeness given above. This assumption is the following.

Axiom (Axiom of Archimedes). If ε is a real then there exists a natural number n such that $\frac{1}{n} < \varepsilon$.

This requirement completely excludes infinitesimals and consequently also infinitesimal distances between points on a number line. But beside real numbers there also are non-archimedean "hyperreal" domains of numbers with infinitesimal numbers that are all smaller than all $\frac{1}{n}$. In Section 3.3 and especially in Chapter 6 we shall give an example.

One more observation should be made. Above, from a continuum by an operation of intersection over all intervals arises a point: hence something being not continuous. Can it be so? Is it then not the case that it is possible, or even not the case that it is necessary that the result of the intersection should be an interval, a *continuum*? It should be infinitely small, that is "infinitesimal". Hence we may ask: could it be as follows:

Let $([r_n, s_n])$ be a nesting of intervals with real bounds on a line g. Then there exists a *continuum* I included in all intervals $([r_n, s_n])$. This continuum is infinitesimal.



The idea of something being infinitely small is a challenge. Infinite smallness is today no mathematical problem; it is a philosophical problem – like the problem of the infinity that we face soon.

How is infinite smallness thinkable? What is its relation with numbers?

The idea of infinitesimal magnitudes occurred early in mathematics, and led in the 17th century to the calculus of infinitesimals, blossomed in the 18th century in spite of the problematic nature of infinitesimals and was rejected in the 19th century. In the middle of the 20th century mathematically infinitesimal magnitudes were rehabilitated – with rather little influence on mathematical thinking. We write about this in the third section of Chapter 3 as well as in Chapter 6.

Note that the intersections over intervals that we considered above and that led to the question about the infinitely small contain also an element of the *infinitely large*: in the case of an intersection one has to do with infinitely many intervals.

1.5 On the construction of the reals

Now we come to the question what kind of an object the number x can be claimed in the axiom of completeness. This question is about the construction of those numbers and the construction of the domain of numbers $\mathbb{R}$.

One is doing here something that is very mathematical and that possibly surprises a naive reader or perhaps appears strange. Imagine a nesting of intervals as above that – as it is said – converges to the claimed number $\sqrt{2}$. As one cannot say anything about what this $x = \sqrt{2}$ is, one raises the nesting of intervals given above into the mathematical identity and says – in a first attempt: this nesting of intervals is $\sqrt{2}$. Concisely but illegally said:

$\sqrt{2}$ is the nesting of intervals converging to $\sqrt{2}$.

In such a formulation we have made the very mistake we wanted to avoid: at the end of the sentence we proceed as if $\sqrt{2}$ already there were. In this way the formulation becomes circular. In other words, however it turns laborious and vague: *This* nesting of intervals is $\sqrt{2}$. The idea is to explain the formalterm $\sqrt{2}$ as a given, concrete nesting of intervals. Briefly and less clearly:

$\sqrt{2}$ is a nesting of intervals.

This formulation is also to some extent curious from the psychological and philosophical point of view. The problem is to construct a number postulated by the axiom of completeness. We have at our disposal as an instrument a nesting of intervals, a process. However, the constructing process provides no number. It leads to no result because it is infinite. But since the only thing one has at disposal is the process, it is *declared* to be its own result. So:

The process itself is put in place of its own result.

This seems to be paradoxical. Since the problem of constructing a number is represented by a nesting of intervals, it becomes almost absurd:

The problem is a solution.

When a problem is as concrete as in the example of $\sqrt{2}$ then it is mathematically legitimate. It is a typical theoretical procedure that is however very unusual and difficult for a learner to accept. A nesting of intervals is in fact no number.

What is done here is also noteworthy from the point of view of epistemology. Even Richard Dedekind had problems with this sort of procedure. It was he who in the 19th century participated in a vital way in the construction of the real numbers. In his efforts he used something that was other than the nestings of intervals, but ultimately comparable to them. He would have defended himself against identifying a number *imagined* within all intervals with a nesting itself. In thinking – according to him – something is added to the infinite process. Dedekind spoke about $\sqrt{2}$ as a "mental creation" generated by man on the base of the imagination of the nesting of intervals. It is however a philosophical way of speaking and mathematically not precise. In our example only the nesting of intervals is a mathematical entity.

This mathematical procedure – the declaration that the nesting of intervals is a number and then calculations with such nestings of intervals – illustrates very clearly the mathematical step towards the theoretical. For an unprepared beginner this step is a special challenge. The construction of real numbers is a construction in a new theoretical sense. It is usually – with reason – not mentioned at school. Also in university teaching it is deliberately avoided.

In the provisionally described construction arises an additional problem that makes the first attempt to define $\sqrt{2}$ more complicated though in principle not different. There are various nestings of intervals that can represent $\sqrt{2}$. It suffices in the example to change only the initial conditions of Heron's procedure, given above, and one gets another nesting of intervals that can represent $\sqrt{2}$. Since there is a free choice and there are various procedures, there are unboundedly many nestings of intervals that can be $\sqrt{2}$? What is then to be done? One takes all such "equivalent" nestings of intervals and declares them together to be $\sqrt{2}$. Particular nestings of intervals – such as those described above – *represent* then $\sqrt{2}$ in this new sense. This is a *second and final attempt* to say what $\sqrt{2}$ is.

- $\sqrt{2}$ is a set of nestings of intervals.
- $\mathbb{R}$ is the set of all sets of equivalent nestings of intervals.

There are different constructions of the reals that proceed in a similar way. And all constructions indicate – as the axiom of completeness did – that in the domain of numbers something new and special happens:

> By the *requirement* of reals expressed especially in the axiom of completeness the reference to reality is set aside. The *constructions* provide representatives for the reals that are not single objects but infinite processes.

The axiom of completeness is – as remarked above – *not* obvious from the point of view of numbers.

> The axiom of completeness is for the domain of numbers something abstract, borrowed from the geometry of the line.

The aim was the correspondence between points on a line and numbers in order to represent quantities and points on a line by numbers and then to be able to replace one with the other. This aim forced the unexplained assumption that a line is a completed set of points. Only in this way can one speak about *the* points of a line. This way of speaking became a custom so that we do not notice it any more. A consequence and the explicit intention was the arithmetization of analysis and of mathematics.

Constructions of the reals – we described one of them – are no abstractions from the physical or perceptual reality. In the next subsection we shall say something about their degree and problems of formality.

Axiomatization and construction of reals that accept infinite processes as mathematical entities and capture a line as a set of points makes mathematics ascend to a new level of abstraction and being theoretical.

1.6 On the handling of the infinite

In the construction of reals described above nestings of intervals were considered as clearly determined objects. Nowadays this has become so common that one does not notice what in fact is happening here any more.

A nesting of intervals is an infinite sequence of intervals

$$[b_1, a_1], [b_2, a_2], [b_3, a_3], [b_4, a_4], \dots,$$

that goes with the sequence of natural numbers $1, 2, 3, 4, \dots$. Such a sequence is an open and never closed process. This will be especially clear when we think for example about the construction of the nesting of intervals given above and converging to $\sqrt{2}$. One estimates interval after interval and never comes to an end. The bounds of succeeding intervals cannot be realized in a similar way as in the sequence of natural numbers. Dots "..." in the expression $[b_1, a_1], [b_2, a_2], [b_3, a_3], [b_4, a_4], \ldots$ send a signal that the process always continues.

At first glance one considers such sequences *potentially infinite*. Here sequences are not clearly defined terminated objects like other mathematical objects. They can be scarcely mathematically grasped. Sequences themselves are so unfinished; they are not concrete and cannot be grasped as concrete individual objects. But the latter is in fact necessary when one wants to say what $\sqrt{2}$ is.

What is done in this situation? One acts – and this is again an *assumption* – as if the infinite process of members of a sequence – that proceeds like counting – were closed. In such an approach the sequences of intervals are treated as "actually infinite", as clearly determined wholes. Instead of $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ one writes (a_n) and (b_n), and $([b_n, a_n])$ instead of the corresponding sequence of intervals, and in this way symbolizes the sequence as a completely given object.

What is happening here will become more clear in the case of natural numbers. The term "$1, 2, 3, \ldots$" becomes here "$\{1, 2, 3, \ldots\}$". In this case the assumption is particularly clear: the dots "..." symbolize a never-ending process and by putting the curly bracket "}" the end of this process is symbolized. This is connected with the set-theoretical Axiom of Infinity claiming that one can think about infinite sets as *actually infinite*, i.e., as being given in a way similar to other mathematical objects and usually treated in a similar way as finite sets.

Today such an approach is mathematical daily routine. It is usually practiced without any comment both in lecturing and teaching. A hundred years ago mathematicians discussed whether this is legitimate, whether infinite sets can be thought and applied. We shall see that the decision to accept the actual infinity first promoted by Georg Cantor (1845–1918) involves philosophical as well as mathematical problems. The problem of the infinite will accompany us in all the following chapters.

The question that arises here is the question about the infinity:

Is it legitimate to treat and grasp infinite sequences as ordinary entities?

In Chapter 2 various historical and current positions with respect to this question will be presented. In Section 3.2, "Infinities", we discuss in detail the question of the infinity as well as various answers to it. In Section 4.3 the Axiom of Infinity is the decisive one.

Notice that despite the problems connected with the actual infinity, actual infinite sets – as well as the reals based on them – are reliable instruments in mathematics today. There are indications of a further foundational question. The reals are theoretical constructions or they are claimed in an axiomatic way. We know how well they function in applications. But why is this the case?

– Why do the highly abstract real numbers solve concrete problems?

Why does the concept of an actual infinity – that probably has nothing to do with the real world – appear to be so effective in applications?

– Why is infinite mathematics applicable?

We shall look for answers to this philosophical question about applicability in Section 3.4.

1.7 Infinite non-periodic decimal fractions

The calculation of bounds of intervals to determine $\sqrt{2}$ for example by Heron's algorithm leads to finite sequences of decimal fractions whose difference is still smaller and smaller.

When it is said that the decimal fractions approach root 2 more and more closely, what it means is first of all that the length of the diagonals of the unity square will be approximated even more closely:

$$\sqrt{2} \approx 1,41421356.$$

If one wants to give $\sqrt{2}$ "exactly" then it is written

$$\sqrt{2} = 1,41421356\ldots.$$

The dots "..." say "and so on" and this suggests that one knows how it goes on – as in the case when one writes $1, 2, 3, \ldots$. At each step it is open what the next position will look like. Nevertheless, it is assumed – on the basis of an actual infinite sequence of all possible calculations – that all infinitely many positions of $\sqrt{2}$ are on hand. This indicates again the power of the assumption contained in the *Axiom of Infinity*. Such ways of writing and thinking make great demands on someone learning mathematics for the first time.

It is still more problematic when one speaks about *arbitrary* infinite non-periodic decimal fractions in general that should be a complement of the rational numbers with respect to the reals. In the Handbook [258] indicated in the Introduction one finds the following sentence:

> "It is right to expect that the set of real numbers (i.e., by our definition the totality of all points of a number line) *completely corresponds to the set of all possible decimal fractions* (finite, periodic or non-periodic)."

How should such an arbitrary infinite non-periodic fraction be specified? For example in such a way: 3,33526788 ...? What can "..." mean here? We are not given a procedure that would provide values for position after position that interprets "..." as "and so on". The term "..." cannot be understood. "non-periodic" is only a negative expression that cancels the meaning of "..." and "and so on". Using terms like "infinite non-periodic decimal fractions" is problematic. This comprises an intangible non-denumerable dimension. It is alarming when such problems are omitted in teaching and lecturing.

Already fifty years ago Paul Lorenzen (1915–1994) commented on the “leger-demain” of infinite non-periodic decimal fractions as follows.

> “To speak about infinitely many digits following each other is – supposing that it is not a complete nonsense – at least a big risk. In teaching however is usually no word about this.”[3] ([234, p. 5], quoted from [340, p. 327].)

Till today nothing has changed here.

We would like to give two small examples to what troubles and dilemmas “infinite non-periodic decimal fractions” can lead, even if a calculating procedure is given. The last example comes from the intuitionist L. E. J. Brouwer (1881–1966) who at the beginning of the 20th century vigorously declaimed against the actual infinite. One finds this example, e.g., in [339] depicted in a slightly different form. It indicates how severe the problem was.

We construct on the base of an infinite decimal expansion of π a new number ψ as follows:

ψ_1 begins with 0 or a point. Digits after the point are determined as follows: the n-th position of ψ_1 is 1 if the n-th position of the decimal expansion of π is 0 followed by the series $1, 2, 3, 4, 5, 6, 7, 8, 9$. Otherwise the nth position is equal to 0.

Is

$$\psi_1 = 0 \quad \text{or} \quad \psi_1 \neq 0?$$

Can this be decided? 50 years ago an answer to this question was utopian. Today however this can be decided thanks to computers we have at our disposal – and in which we trust.

The answer:

The 17387594880-th position of ψ_1 is 1.

This means – even if ψ_1 is “in practice” 0 – mathematically it holds that

$$\psi_1 \neq 0.$$

Now we can construct, in a similar way, another number ψ_2 on the base of π:

ψ_2 begins again with 0 and a point.

(i) If the first position after the point in a decimal expansion of π is 7 then put 1 in the 1-st position after the point in ψ_2, otherwise one puts 0.
(ii) If then in the decimal expansion of π there follow two digits 7 then put 1 in the next, 2-nd position of ψ_2, otherwise put 0.

3 “Von einer Aufeinanderfolge unendlich vieler Ziffern zu reden, ist also – wenn es überhaupt nicht Unsinn ist – zumindest ein großes Wagnis. Hierüber wird im mathematischen Unterricht zur Zeit aber meist kein Wort verloren.”

(iii) If then in the decimal expansion of π there follow three digits 7 then put 1 in the next, 3-rd position of ψ_2, otherwise put 0.
(iv) If then in the decimal expansion of π there follow four digits 7 then put 1 in the next, 4-th position of ψ_2, otherwise put 0.
(v) Etc.

Is $\psi_2 = 0$ or $\psi_2 \neq 0$? Can it be decided?

This leads us to a curious situation. One would bet that

$$\psi_2 = 0.$$

However, no computer – neither today nor in the future – will be able to settle this bet. Is the finite calculating ability of all available computers the reason for this? No! If we want to win the bet then we would with the help of our giant but finite computers running in a finite time through infinitely many positions of the decimal expansion of π. But this is *in principle* impossible.

Or does the following hold:

$$\psi_2 \neq 0?$$

Is it really possible to exclude the fact that eventually, beyond all imagined accessibility, somewhere among the infinitely many series of numbers the required series of sevens does appear?

The situation is as follows: We are even not able to decide whether the undecidability of the disjunction "$\psi_2 = 0$ or $\psi_2 \neq 0$" is of principled or practical nature.

It is assumed that the infinite expansion of π is on hand. Then the infinite decimal expansion of ψ_2 will be also on hand. What does the dilemma of ψ_2 about our "infinite non-periodic decimal fractions" say to us? We know as much or as little about the decimal positions of ψ_2 as we know about positions in the decimal expansion of π. Is our belief that the *infinite* decimal expansion of π is on hand – when we consider the above quotation of Lorenzen – maybe after all absurd rather than merely risky?

2 On the history of the philosophy of mathematics

Which kind of philosophy one chooses depends on which kind of human one is.[1]

Johann Gottlieb Fichte

Chapter 1 was an elementary introduction to problems of the philosophy of mathematics. It itself belongs already to the philosophy of mathematics. The latter begins where mathematics and mathematical activity are considered. We asked questions and detected problems we would like to answer and to solve. However, we are still unable to present answers and solutions. Hence we are looking for hints and support in the history of mathematics and philosophy since antiquity till modern times. Therefore we shall follow the historical development, describe it and present various views and positions of famous and important philosophers and mathematicians concerning mathematics and its objects. We will try to describe them in a neutral way and to provide a wide objective basis for possible answers to our questions we will propose in Chapter 3 as well as for possible solutions of problems that will appear in further chapters. The reader will be able to build – according to the motto – his own conception using the variety of presented views. However, there will be no true, right position implying unique true answers.

Our survey is chronological and necessarily sketchy. At first we follow great names of famous philosophers who considered mathematics and its objects. Since the 19th century among them were more and more mathematicians themselves who began to consider their own discipline and to formulate conceptions concerning it. Hence there will appear names of great mathematicians whose conceptions will then orientate our survey. Finally, there arose schools and tendencies represented by various mathematicians. At the turn of the 19th and 20th centuries mathematicians took charge of the foundations of mathematics and consequently also a part of the philosophy of mathematics connected with foundational problems. There are still philosophers dealing with the phenomenon of mathematics but also they are doing this primarily taking into account the foundations.

Positions in the philosophy of mathematics we will consider are influenced by general philosophical systems and conceptions that can be in principle classified as belonging to the neighbourhood of one of three *basic philosophical positions*. Intuitively they can be presented as points in the following triangle whose distance from the vertices indicates how close they are to the basic positions.

1 Was für eine Philosophie man wähle, hängt davon ab, was für ein Mensch man ist.

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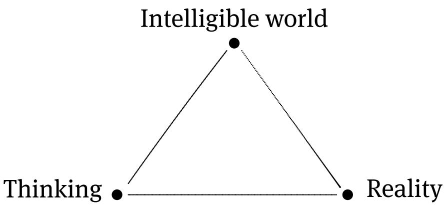

The fundamental question is: Where are the sources of the knowledge and its concepts? Positions represented by the vertices are the classical basic positions:

- The idealistic position claiming that the source of knowledge lies in the spiritual, intelligible world.
- The empirical position according to which the reality through experience gives us knowledge.
- The rationalistic position seeing the foundations of our knowledge in structures of thinking.

A famous representative of the idealistic position is Plato. Positivists, e.g., represent the empirical position. Kant represents what we call – a bit against common philosophical and linguistic usage – "rationalistic". Those positions will be described more exactly when they come in. Another new basic position will be presented when we will speak about new tendencies in the philosophy of mathematics.

Conceptions concerning the sources of knowledge are always connected with ontological questions, i.e., questions on the nature and existence of objects. Investigations of them belong to philosophical doctrines of being, to the so-called ontology where according to positions different answers are given.

As guiding thread we follow the views on the first and simplest mathematical objects, i.e., natural numbers, through the development of the philosophy of mathematics. Questions on their essence and mode of existence are the oldest questions asked in the philosophy of mathematics. We want to characterize briefly and concisely particular conceptions concerning natural numbers and to emphasize

characterizations in italic and as distinguished text.

This will be done in all cases in which representatives of concrete conceptions did formulate their views or we are able to reconstruct them authentically.

2.1 Pythagoras and Pythagoreans

In the prehistory of human beings an elementary mathematics has been developed in order to manage everyday problems. It served to solve economic and practical geometrical problems of measuring the areas of soil and of the early astronomy. Pythagoras and the Pythagoreans seem to be the first who not only applied and developed this elementary mathematics but also reflected on it.[2]

Pythagoras (ca. 570–ca. 500 BC) brought to Greece from his travels and stays in ancient Egypt and Babylon the arithmetical, astronomical and geometrical knowledge of priests. At the end of sixth century B.C, about 530 BC, he founded an ethical-religion secret society of Pythagoreans in Croton (Lower Italy) in west Greek colonies. Croton became soon the center of science at this time and Pythagoreans were the leading philosophers in ancient Greece. They were scientific pioneers in particular in the domain of mathematics and natural sciences. They were interested first of all in mathematics, music and astronomy. Various mathematical results in arithmetic, number theory and geometry are ascribed to them. Characteristic for them was that they connected religious mysticism with scientific principles and exact research methods.

Only few Pythagoreans are known by their names. The reason seems to be the Pythagorean world-view according to which it was forbidden to stress one's personal achievements. Also scientific discoveries have been apparently kept a secret. The blossom of Pythagorean mathematics took place at the turn of the fifth and fourth centuries BC Members of the school were then Archytas of Tarentum, Timaeus, Eudoxus of Cnidus, Philolaus of Tarentum and Eurytas. It is worth noting that – according to Diogenes Laërtius – Pythagoreans were the authors of the first definitions in mathematics. From them probably come the majority of definitions from the first book of Euclid's *Elements*.

At the center of mathematics of the early Pythagorean school around Pythagoras were numbers, arithmetic and elementary number theory. Together with mathematics Pythagoras brought also the mysticism of numbers cultivated by priests in Babylon and Egypt. Numbers were simultaneously measure numbers used to describe astronomical relations and as symbols having mystical meaning used in the astrology. This mysticism of numbers transformed Pythagoras into an early philosophical world-view. "All is number" was the philosophical motto of Pythagoreans. Only what is formed can be known – said Philolaus – and form is based on measure and number. Numbers formed for Pythagoreans a distinct, higher spiritual world according to which the earthly world was built.

This conception is noteworthy at least for two reasons:

– It explains the origin of a *theory*.
  Pythagoreans investigated the higher world of numbers in order to understand the material world. It was excluded to justify laws of this higher world of numbers by experience in the material world. Justification of them *should* be done inside the world of numbers. There possibly appeared the first real theory: number theory.

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2 Texts on mathematical contributions of the Pythagoreans as well as their reconstructions are uncertain; cf. for example H. Boehme [37, 39] and W. Burkert [59]. We follow the common tradition; cf. for example van der Waerden [349].

– This theory was mathematics and philosophy in one.

Numbers in their philosophical role received by the Pythagoreans beside practical and theoretical meaning also metaphorical meaning that partially indicates their mystical and symbolic sources. Even numbers were "feminine", odd ones – "masculine". Five, the sum of the first even and odd number symbolized marriage. Ten being the sum of the first four numbers – *τετράκτυς* (tetractys) – was a "divine" number. There were – as still are in the contemporary number theory – *amicable* and *perfect* numbers. The later are numbers that are equal to the sum of their divisors. A task of number theory was to find principles of obtaining perfect numbers. Number one represented geometrically "point" – the geometrical "unit", two – a line, three – a surface, four – a solid body and three dimensions of the space. Numbers were for Pythagoreans no abstractions. They were *real powers* acting on the nature and in the nature. "Number" was the principle of all being. The structure of the real world was for them an image of the higher world of numbers. Number theory was metaphysics.[3]

Then there were – as Iamblichus reported about 300 BC – attempts to determine philosophically what numbers are. The following phrase comes probably from Pythagoras himself: numbers are an "unfolding" of "the generating principle lying in the unit". This phrase explains why the Pythagoreans did not treat one as an ordinary number but as a principle: as the principle of unit and simultaneously as source and origin of "common numbers".

Characterizing the philosophical position of the Pythagoreans concerning numbers, one can summarize it by saying the following:

Numbers are elements of a higher world – generated by the unit.
They are spiritual powers of form – over/beyond *things*.

The number theory of the Pythagoreans collapsed as the foundation of their world-view and this happened apparently in a dramatic way (cf. [36]). It happened in this way: For the Pythagoreans any segment was a number – relatively to a given unit of measure – or a proportion of numbers, and geometry was a subdiscipline of number theory. The absolute domination of numbers was finished by discovering the incommensurability about 450 BC There were no numbers characterizing the proportion of the side and the diagonal of the regular pentagon (cf. Section 1.2). This discovery led to a genuine crisis of the philosophical world-view. Perhaps it was the birth of mathematics that – after having lost its philosophical function – became now a proper discipline. How dramatic the whole situation was back then is indicated by the legend about the traitor who revealed the secret of the incommensurability – we have told it in Section 1.2.

The discovery of the incommensurability was not only a philosophical challenge. It dared mathematics to look for new mathematical foundations that would replace

3 Metaphysics is the fundamental philosophical discipline investigating origins and conditions of being.

number theory and would solve mathematical problems. Additionally there appeared a new challenge – the paradoxes of Zeno of Elea that indicated further troubles with the infinite.

The best known paradox is probably one of the paradoxes of motion, namely Achilles and the tortoise. Everybody knows that Achilles overtakes the tortoise even if he allows the tortoise a head start. On the other hand it is difficult to disprove the contrary claim that Achilles *cannot* overtake the tortoise. Whenever Achilles reaches somewhere the tortoise has been, he still has farther to go. For that time the tortoise will have advanced farther etc. (cf. [1, Book 6;9]).

New foundations have been then found in the so-called geometrical algebra. It consisted in replacing numbers and operations on them by geometrical magnitudes and operations on them. A definite solution of the crises was provided only by the so-called method of exhaustion coming from Eudoxus of Cnidus – it made possible to omit problems connected with the infinity. A further big step was the theory of magnitudes and the theory of proportions from the first half of the fourth century BC – it can be treated as a predecessor of today's theory of real numbers. All this can be found in Euclid's *Elements* (about 300 BC).

In Euclid's *Elements* geometry stays at the first place. Eudoxus' theory of magnitudes was presented by Euclid in Book V of the *Elements* before the number theory. The latter is since then in principle a part of the theory of magnitudes and their proportions even if it was in the *Elements* strongly separated and developed in an independent way and later on practiced in Greek mathematics.

Speaking about the early – before Plato – philosophy of mathematics, one should necessarily mention still one philosopher: *Socrates* (469–399 BC). He himself did not rather deal with mathematics, however his influence on the further development of mathematics was very great. In his ideas and his methods one can suppose the sources of Plato's theories, of the methodology of Aristotle as well as of the systematic deduction of Euclid. We tell about that in the next sections.

2.2 Plato

Plato (427–347 BC), the founder of the famous Academy (385 BC – 529 AD), a school of philosophers, is one of the most famous and important philosophers in the history. For him the most fundamental, perhaps basic philosophical problem was the problem of distinguishing between the seeming and the real. This distinction was for him not only a theoretical problem that is essential for philosophers and scientists, but it had also great practical and ethical meaning – for example for politics in ancient Athens. The result of this distinction and the background of Plato's philosophy was his theory of ideas.

By Plato ideas built – similarly to numbers in the perception of the Pythagoreans – a distinct higher world that penetrates the actual world of being, the material world. Ideas have their own existence and they determine the existence of real things.

Plato's theory of ideas distinguishes two modes of being forming two separate worlds: the unchanging, constant world of ideas that are clearly determined real entities existing beyond time, space and independently of human cognition. Ideas are unique: there is only one idea of unit, of beauty or of circularity.

On the other hand there are variable things of the material world being experienced by human beings with the help of their senses, possessing a lower level reality that appear as unstable shadows of ideas. Ideas do exist really, physical objects have their source in them. They receive their existence from ideas. Physical objects are for example circular when they participate in the idea of circularity. There are many circular objects. Things are like images of ideas, ideas are prototypes of things.

The relation between ideas and physical things can be seen like the following proportion:

$$\frac{\text{ideas}}{\text{things}} = \frac{\text{things}}{\text{shadows}}.$$

This means that the relation of ideas and things resembles the relation of things and shadows (of them). This has been vividly described by Plato in his famous allegory of the Cave (cf. [275, VII, 1–3]). It can be said that ideas are like forms for earthly things. The order of the material world is a reflection of the order governing the world of ideas. Material things are apprehended by senses and ideas – by concepts.

This ontology, this theory of existence and essence of things, is decisive (normative) when Plato deals with mathematics. Mathematical concepts are like ideas – immaterial and *real* and are characterized like ideas – as being unchangeable, clear and necessary. Ideas exist independently of space and time. The same holds also for mathematical concepts that are close to ideas and – in the interpretation of Aristotle – lie between the world of ideas and the world of physical things. There are the ideas of circularity, of proportion, of number and many mathematical realizations of them: there are many circles, many proportions, many numbers in the mathematical world.

At the end of Book IV of *Republic* Plato discusses the following classes of objects:

Tab. 2.1. Classes of objects by Plato.

immaterial	ideas	unit, beauty, goodness, justice, circularity ...
	mathematical objects	circle, number, relation, proportion ...
material	physical objects	wheel, table, house, tree, ...
	images of physical objects	image of a wheel, image of a tree ...