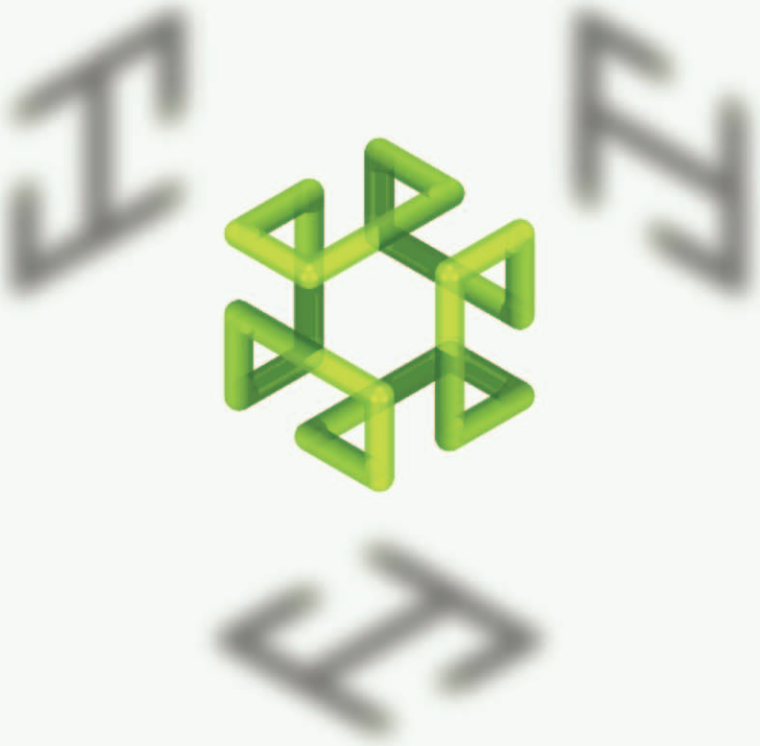
Mathematical Mind-Benders



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The real authors of this book are the people around the world who sent me these wonderful puzzles, including many readers of my previous book and new friends and colleagues in New England. Special thanks are due to the founders of the Albert Bradley Third Century Professorship in the Sciences at Dartmouth.

But whatever part I am entitled to dedicate goes to my parents,

Drs. Bernard and Miriam Winkler,

who must have hoped their first-born would become a useful, contributing member of society, only to watch him grow up to be a mathematician.

Contents

Preface

> Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost.
>
> —W. S. Anglin

This book is for lovers of mathematics, lovers of puzzles, lovers of challenge. Most of all, it is for those who think that the world of mathematics is orderly, logical, and intuitive—and are ready to learn otherwise!

To appreciate the puzzles, and to solve them, it is necessary—but not sufficient—to be comfortable with mathematics. You will need to know what a point and a line are, what a prime number is, and what the probability is of rolling a double-six when you need it. Most importantly, you will need to know what it means to *prove* something.

You will *not* need a professional acquaintance with mathematics. Your computer, calculator, and calculus text can stay in their boxes; but your thinking-cap will have to be on. In some cases, the more courses you have taken in mathematics, the less likely it is that you will find the answer. In some cases, you will read and understand the answer and still not believe it.

The puzzles themselves come from all over the world, and from people of all walks of life. Since publication of my previous puzzle book,[1] many more people have been sending me puzzles, both new and old. I was shocked but delighted to find, a short time before writing these words, that my collection of unpublished puzzles had reached, in both size and quality, what had previously taken me twenty years or so to accumulate.

Readers of my previous book will find some differences. The puzzles themselves lean a bit more toward the surprising; a few, in fact, come from my article "Seven Mathematical Puzzles You Think You Must Not Have Heard Correctly" for the Seventh Gathering for Gardner. I've paid a bit more attention to source than before, with the result that *some* of the information about the origin of the puzzle may actually be correct. Except in the case of puzzles I devised myself, however, all I can promise is "best effort." At the suggestion

[1] *Mathematical Puzzles: A Connoisseur's Collection*, A K Peters, Ltd., 2004.

of some readers, I have tried harder when presenting solutions to indicate how they could have been arrived at; but, alas, in many cases I have either failed to do so convincingly or have no idea myself.

The wording of the puzzles, and of their solutions, is my own, and I must take full responsibility for errors and ambiguities. And there will be some, I can assure you.

Puzzles selected for this book are supposed to be elegant and entertaining; to have easy but illuminating solutions that are challenging to find; to embody some mathematical idea, but not to require advanced mathematics to appreciate or solve. Most of all, for this collection especially, they are supposed to confound the intuition and stimulate the brain. Do they all meet all these criteria? Not on your life. But there are some real gems here, any one of which could bring you joy and enlightenment far beyond the meager price of this volume. Check out Curves on Potatoes, p. 2; or Roulette for the Unwary, p. 3; or Love in Kleptopia, p. 9; or Worms and Water, p. 9; or The Faulty Combination Lock, p. 11; or Names in Boxes, p. 12; or Chameleons, p. 22; or Uniformity at the Bakery, p. 23; or Steadfast Blinkers, p. 23; or Red and Blue Dice, p. 23; or Falling Alice, p. 36; or Alice on the Circle, p. 36; or Coins on the Table, p. 51; or Box in a Box, p. 53; or Impressionable Thinkers, p. 67; or Lemming on a Chessboard, p. 67; or Hats and Infinity, p. 91; or Tower of Bricks, p. 93; or Ice Cream Cake, p. 111; or Curve and Three Shadows, p. 112; or Collapsing a Polygon, p. 114; or. . .

A word on format. The puzzles are organized into chapters for convenience, classified loosely by mathematical area. The solutions are presented at the end of each chapter (except the last); in the hope that readers will think at least a *little* about each puzzle, I did not wish to make it easy to read a puzzle whose solution follows on the page. Information about the background and source of a puzzle is presented with its solution.

These puzzles are hard. You can be justly proud of solving any of them, and in some cases even of just appreciating the solution.

Good luck, and as they say also in the world of mechanical puzzles, happy puzzling!

Peter Winkler

Warm-Ups

Brain (n.) An apparatus with which we think we think.
—Ambrose Bierce (1842–1914), *The Devil's Dictionary*

We begin with some (relatively) easy problems, just to give you a chance to stretch your brain. These require no fancy mathematics, just a bit of logical thinking.

Half Grown

At what age is the average child half the height that he or she will be as an adult?

Bags of Marbles

You have 15 bags. How many marbles do you need so that you can have a different number of marbles in each bag?

Powers of Two

How many people is "two pairs of twins twice"?

Rolling Pencil

A pencil with pentagonal cross-section has the maker's logo imprinted on one of its five faces. If the pencil is rolled on the table, what is the probability that it stops with the logo facing up?

The Portrait

A visitor points to a portrait on the wall and asks who it is. "Brothers and sisters have I none," says the host, "but that man's father is my father's son." Who is pictured?

Strange Sequence

What symbol should come next in the sequence pictured below?

Language Parameter

For Spanish, Russian, or Hebrew, it's 1. For German, 7. For French, 14. What is it for English?

Attention Paraskevidekatriaphobes

Is the 13th of the month more likely to be a Friday than any other day of the week, or does it just *seem* that way?

OK, time to get a *little* more serious.

Fair Play

How can you get a 50-50 decision by flipping a bent coin?

Curves on Potatoes

Prove that, given two potatoes, you can draw a closed curve on the surface of each so that the two curves are identical as curves in three-dimensional space.

You can finish your warm-up with three probability problems; these do require a *modicum* of calculation.

Winning at Wimbledon

As a result of temporary magical powers, you have made it to the singles' finals at Wimbledon and are playing Serena Williams or Roger Federer for all the marbles. However, your powers cannot last the whole match. What score do you want it to be when they disappear to maximize your chances of hanging on to notch an upset win?

Spaghetti Loops

The 100 ends of 50 strands of cooked spaghetti are paired at random and tied together. How many pasta loops should you expect to result from this process, on average?

Roulette for the Unwary

Elwyn is in Las Vegas for a mathematics meeting and finds himself in a casino with some time before the next talk and $105 in his pocket. He saunters over to the roulette table, noting that there are 38 numbers (0, 00, and 1 through 36) on the wheel. If he bets $1 on a single number, he will win with probability 1/38 and collect $36 (in return for his $1 stake, which still goes to the bank). Otherwise, of course, he just loses the dollar.

Elwyn has just enough time to make 105 such one-dollar bets, so he goes ahead with the plan. What, approximately, do you think is the probability that Elwyn will come out ahead? Is it better than, say, 10%?

Sources and Solutions

Half Grown

Parents of young kids will know this one: two years old! (That is, between the second and third birthdays.) Yes, human growth is highly nonlinear. Puzzle suggested by Jeff Steif, of Chalmers University in Sweden.

Bags of Marbles

Fourteen marbles will do the trick. Put an empty bag into a bag containing one marble, then the second bag into a third containing an additional marble, then the third into a fourth containing an additional marble, etc. so that the ith bag contains altogether $i - 1$ marbles (and $i - 1$ bags).

If you didn't think of putting bags in bags—or thought of it but considered it cheating—you would've needed $0 + 1 + \cdots + 14 = 15 \times 7 = 105$ marbles.

Puzzle contributed by Dick Plotz, of Providence RI.

Powers of Two

Eight. It looks like there are four multiples of two in the phrase: "two," "pairs," "twins," and "twice," leading some folks to guess $2^4 = 16$ people. But a twin is only one person. A classic riddle.

Rolling Pencil

My colleague Laurie Snell caught me on this one; did you fall for it too? Sounds like it should be $\frac{1}{5}$, but of course since 5 is odd, the pencil will stop with a face down and an *edge* up. Thus the answer is zero or perhaps $\frac{2}{5}$, depending on your interpretation of "up," but definitely not $\frac{1}{5}$.

The puzzle appears in Chamont Wang's provocative book, *Sense and Nonsense of Statistical Inference* [58].

The Portrait

This one is an *ancient* classic, which appears in Raymond Smullyan's classic *What is the Name of This Book* [55]. Owing to his lack of siblings, "my father's son" can only mean the host himself, hence the portrait is of the host's son.

Strange Sequence

This one was forwarded to me by Keith Cohon, a lawyer with the Environmental Protection Agency. The sequence is intended to represent the beginning of a reversed alphabet, that is, ZYXW, but with the Z turned $90°$ (either clockwise or counterclockwise) and each successive letter turned an additional $90°$. The next symbol would therefore be $<$ or $>$, representing a sideways V.

Language Parameter

Seven. This curious conundrum was devised by Teena Carroll, a Georgia Tech graduate student, and actually is (slightly) mathematical. The question answered by the intended parameter is what, in the given language, is the first multi-syllabic positive integer?

Attention Paraskevidekatriaphobes

Amazingly, this is true, and as far as I can tell was first observed by Bancroft Brown (a Dartmouth math professor, like your author), who published his calculation in the *American Mathematical Monthly* [11]. My present-day colleague Dana Williams is the one who brought this fact to my attention.

It is not hard to verify that in 688 out of 4800 months in the 400-year cycle of our Gregorian calendar, the 13th falls on a Friday. Sunday and Wednesday claim 687 each, Monday and Tuesday 685 each, and Thursday and Saturday only 684 each. To check this you need to remember that years which are multiples of 100 are not leap years unless (like 2000) they are divisible by 400.

The origin of superstition concerning Friday the 13th is usually traced to the date of an order given by King Philip IV of France (Philip the fair), dismantling the Knights Templar.

Incidentally, with some training a person (at least, a person like Princeton's redoubtable John H. Conway) can quickly determine the day of the week of any date in history—even accounting for past calendar glitches. For lazier or more present-time-oriented mortals, a useful fact to remember is that in any year 4/4, 6/6, 8/8, 10/10, 12/12, 9/5, 5/9, 7/11, 11/7, and the last day of February all fall on the same day of the week. (This is even easier to remember if you happen to play craps daily from 9 to 5.) That day of the week is Wednesday for 2007, and advances one each year, two before a leap year.

Fair Play

Flip the bent coin *twice* hoping to get a head and a tail; if the head comes first, call the result HEADS; if the tail comes first, call it TAILS. If the result is two heads or two tails, repeat the experiment.

I was reminded of this puzzle by Tamas Lengyel, of Macalester College; its solution is attributed to the late, great mathematician and pioneer computer scientist John von Neumann, and is in fact sometimes called "von Neumann's trick." It relies on the fact that even if the coin is bent, successive flips are (or at least should be) independent events. Of course, it also relies on it being at least *possible* that the bent coin can land on either side!

If you want to minimize the number of flips to get your decision, the above scheme can be improved upon. For example, if you get HH for the first pair of flips and TT for the second, you can quit and call the result HEADS (obviously, TT followed by HH would then be called TAILS.)

More improvements are possible and in fact an article by Şerban Nacu and Yuval Peres [44] shows how to get the last drop of blood out of the process,