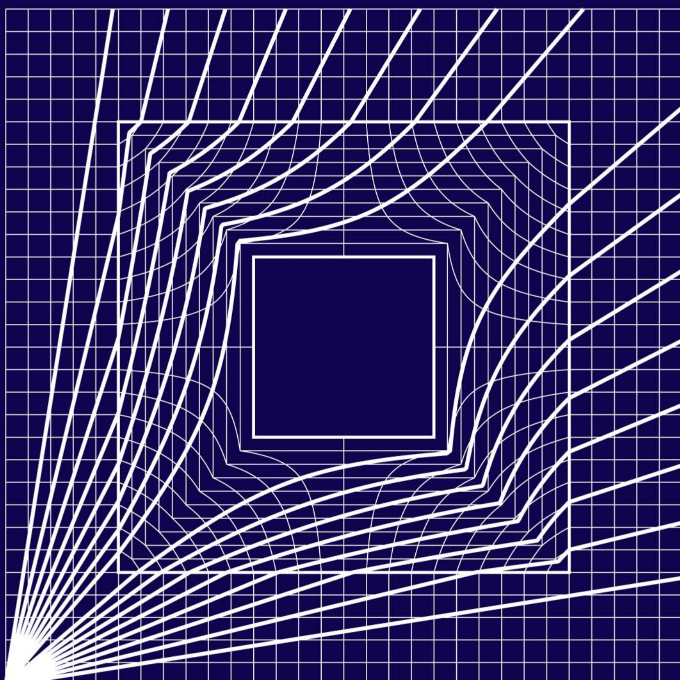
MONOGRAPHS AND RESEARCH NOTES IN MATHEMATICS

Mathematical Modelling of Waves in Multi-Scale Structured Media



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Mathematical Modelling of Waves in Multi-Scale Structured Media

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Preface

The study of wave propagation in structured media can be traced as far back as the seventeenth century with the publication of Newton's *Philosophiæ Naturalis Principia Mathematica*. For instance in *Principia*, Newton examined the, now classical, one-dimensional mass-spring lattice system and derived an expression for the speed of propagation of sound waves. Despite being studied for several centuries, wave propagation in multi-scale structured media remains an active, exciting, and challenging area of research.

Indeed, recent developments toward the control of wave propagation in multi-scale solids have led to considerable progress in the development and application of analytical techniques for the modelling of wave propagation in multi-scale structured media. The principal aim of the present text is to provide a unified account of several of these analytical techniques as applied to a collection of fascinating physical problems.

Following the introduction, Chapter 2 summarises several fundamental methods, concepts, and approaches required for the analysis of wave propagation in multi-scale solids. This chapter is designed to furnish the uninitiated reader with a succinct introduction to the primary methods necessary for the material presented later in the text. It is hoped that graduate students and those researchers unfamiliar with the topics covered in the present monograph will find Chapter 2 particularly useful.

The main body of the present text represents a coherent account of a selection of interesting problems tackled over the last 15 years. We were privileged to have worked with wonderful colleagues with whom we have collaborated over the past years, and we are very grateful to them. The book is based on a series of research papers cited throughout the text.

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Chapter 1

Introduction

The study of wave propagation in multi-scale structured media has proven a rich area of research, attracting the attention of researchers from a broad range of disciplines including mathematics, physics, and engineering. From a mathematical standpoint, the modelling of dynamic phenomena in structured media is both interesting and challenging and has captivated the applied mathematics community for many decades. In the last century, the field of waves in structured media has undergone a rapid series of developments with both theoretical and experimental advances actively driving the field forward; see, for example, [12, 43, 58, 98, 106, 119–121, 137, 143, 158, 174, 175]. More recently, advances in fabrication methods and materials technology, coupled with the desire and ability to create "designer materials" have led to a significant expansion in experimental research for waves in multi-scale structured media.

Elastic structured media presents a unique challenge in the analysis of wave propagation in multi-scale solids. In the case of electromagnetism, all waves travel at the same speed (the speed of light) in a given medium; the same is true of acoustic waves and water waves. In contrast, elastic bodies generally support the propagation of two different types of wave: pressure waves and shear waves. These two waves travel at different speeds, even in this same medium, and are coupled at boundaries and interfaces; this makes the analysis of wave propagation in structured media, where there are many interfaces and boundaries, singularly difficult. In turn, this unique coupling leads to particularly fascinating dispersion properties and, not only creates several new directions in the mathematical modelling of the dynamic response of multi-scale structured solids, but also allows for the development of many interesting applications in mechanics and engineering.

One particularly interesting area of research relates to *"high-contrast"* structures. In the case of mechanical systems, *"high-contrast"* refers to materials of structures with large spatial variations in density and/or stiffness. Such structures are of particular interest because they can support very low frequency standing waves and localised waveforms. Among the plethora of engineering applications, earthquake protection systems are, understandably, of particular interest and multi-scale systems of resonators can be used to efficiently filter, reflect, polarise, mode convert, and otherwise control elastic waves over a desired frequency range (see, for example, [44, 46]).

In the present book, we consider a broad range of theoretical approaches to the challenging problems of modelling wave propagation and localisation in structured elastic media. In particular, we will focus on multi-scale materials, with a dynamic response that may appear unusual and often counterintuitive; using the rigorous methods developed here, we will show that these, apparently strange, phenomena can be easily understood and are, in fact, entirely natural. Examples of such phenomena include dynamic anisotropy, focussing by flat interfaces and cloaking.

1.1 Bloch–Floquet waves

There are many classical monographs, such as [26], [85], [91], and [187], which address wave propagation in structured media; a topic that has proven to be an exceptionally rich and attractive area of research. Motivated, not only by contemporary theoretical and technological advances, but also by work going back to the seventeenth century [135], there has been a particularly high level of scientific interest in the analysis of systems exhibiting photonic and phononic band gaps for Bloch–Floquet waves. The scholarly interest in the analysis of such phenomena is underlined by the extensive bibliography summarised in [60]. In particular focussing and diffraction, which are well known for optical systems (see, for example, [25] and [145]), can have new and interesting aspects in problems of vector elasticity. A systematic analysis of the scattering of waves in solids with periodic arrays of defects was presented in [35]. For problems of the mathematical theory of elasticity, the paper [166] extended the Rayleigh method [169] to examine the scattering of elastic waves in doubly periodic structures.

For the case of lattice dynamics, *localised primitive waveforms* have been identified [8, 151] and studied in detail. In particular, it was demonstrated that these localised vibrations are associated with stationary points on the dispersion surfaces. For both, time-harmonic and transient excitations, the papers [92] and [93] have analysed the dynamic response of a square lattice, with a particular emphasis on the nature of the caustics, which require careful consideration when applying the method of steepest descent.

For systems of elastic rods, which include periodic arrays of trusses and frames, a generic algorithm for treating the dispersive properties of vector lattices and the design of structures possessing band gaps in desired ranges of frequencies was developed in [108].

1.2 Structured interfaces and localisation

The concept of structured interfaces of finite thickness, which join two continuous regions of an elastic medium, was introduced in [21]; wherein the authors note that the inertial properties of the interface significantly affect the dynamic response of the structure and lead to unusual filtering properties for elastic waves. Extending the work of [21], a structured interface between two continuous domains was employed in [29] to demonstrate the focussing of elastic waves via negative refraction. Such structured interfaces in solid media are sometimes referred to as "flat lenses" for elastic waves. Similar effects have also been demonstrated in acoustics, as shown, for example, in [76]. The effects of focussing and filtering of elastic waves have been extended to entirely discrete structures: a diatomic interface lattice embedded in a monatomic ambient lattice of the same geometry was considered in [49]; and it was shown that, for certain frequencies, the interface lattice acts as a flat elastic lens. Dynamic homogenisation, which also incorporates directional preferences and effects of negative refraction, was addressed in the recent papers [54–56, 139].

Anisotropic inhomogeneous interfaces are useful, for example, for the reduction of stress concentration in solids. Figure 1.1 gives a simple example for a uni-axially loaded static solid containing a set of coated inclusions. The material of the ring-coating is orthotropic and the Young's modulus in the tangential direction within the coating increases toward the centre, similar to [58, 98]. As shown in Figures 1.1c and 1.1d, the stress concentration has been reduced dramatically compared to the cases in Figures 1.1a and 1.1b.

The presence of localised waveforms was previously illustrated for scalar lattices [8, 92, 93, 151]. The resulting anisotropy, diffraction patterns and aberrations are often explained using the dispersion surfaces and slowness contours. Two geometrically identical lattices, with the same elastic stiffness may provide different dynamic responses due to different distributions of inertia and embedded resonators of different kind. In statics such lattices, which we denote as type I and type II, may be geometrically identical. However, embedding an interface composed of a lattice of type I into an ambient lattice of type II will result in highly fascinating dynamic features. Although in statics we would have a uniform lattice which is homogenous, its dynamic response may exhibit the total reflection of waves by the interface, as well as negative refraction and focussing.

Imperfect interfaces may influence the overall properties of the homogenised composites. In particular, there are interesting examples concerning the notion of neutral inclusions and, in a static setting, these were studied in [14, 23, 100, 122].

Dynamic composite structures containing coated inclusions have been analysed in [160, 161], where the notion of neutrality was linked to the effective refractive index in the long-wave limit; the coated inclusions were considered

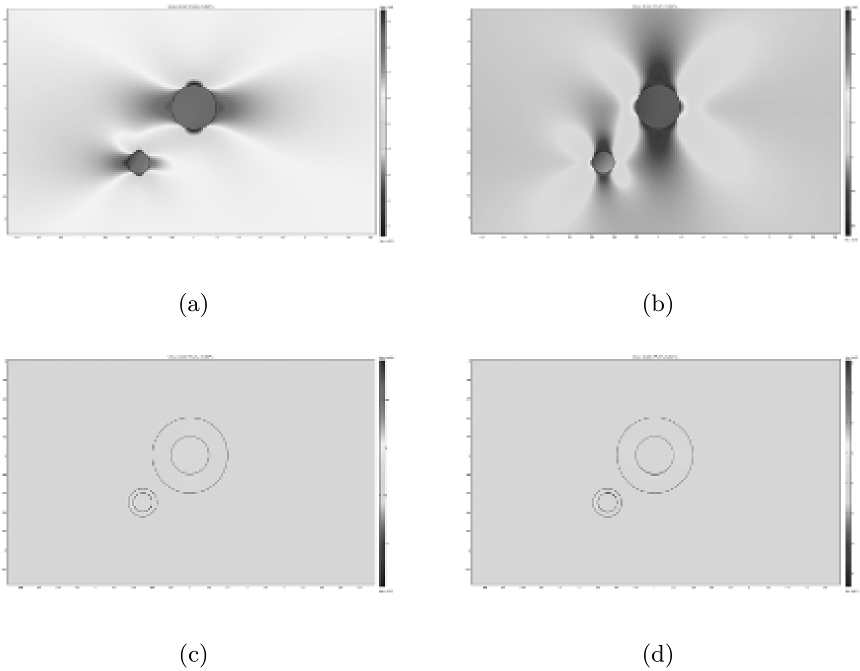


FIGURE 1.1: Panels (a) and (b) show the stress σ_{11} and σ_{22} field plots around an inclusion in a uni-axially loaded solid; (c) and (d) panels show the stress σ_{11} and σ_{22} field plots around an inclusion surrounded by an orthotropic inhomogeneous coating in the same uni-axially loaded solid. The stress concentration has been dramatically reduced.

as neutral if the effective refractive index of the composite medium was the same as that of the homogenous solid without inclusions. In these papers, the elastic coating around an inclusion was considered as anisotropic, and it was demonstrated that the geometrical and physical parameters of the coatings could be chosen to match the effective refractive index of the doubly periodic array of coated inclusions with the refractive index of the unperturbed homogeneous medium. Acoustic band gaps for waves in media with neutral inclusions were considered in [77].

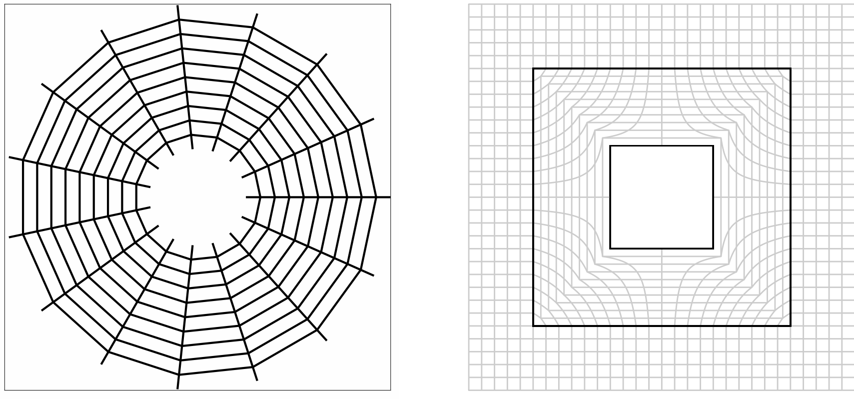
In statics one can introduce so-called *imperfect interfaces* in order to describe high-contrast coatings with elastic properties that are significantly different from those of the ambient elastic medium. In the context of homogenisation and bounds on effective moduli and neutrality, static models of composites containing coated inclusions have been analysed in [13, 15, 78, 79, 101]; including both stiff and soft, as well as highly anisotropic imperfect interfaces in two-dimensional elasticity and problems of torsion.

A model of *non-local structured interfaces* was developed in [21] for static and dynamic problems in elastic composite media. Special features, highlighted in that paper, include strong anisotropy and a finite-width interface. In particular, for dynamic problems the structured interface may possess trapped waveforms, which enhance transmission. The papers [16–18] studied new static models of non-local structured interfaces, and addressed evaluation of their effective mechanical properties and their influence on the elastic stress concentration around defects. Structured interfaces possess inertia, and hence, tractions may become discontinuous across such interfaces in dynamic problems of wave propagation.

The interaction between periodic discrete and continuous systems was studied in [21, 29] wherein the existance of "slow waves" was noted, and the transmission properties of a finite-width dynamic structured interface was analysed leading to unusual dynamic effects such as negative refraction.

1.3 Multi-physics problems and phononic crystal structures

The interaction of electromagnetic waves with photonic crystal structures has been the subject of rapid development in the last decade. In particular, cloaking of electromagnetic waves has been analysed both experimentally and theoretically; some of the earliest investigations in this area of wave studies were reported in [58, 174]. Being natural and well-studied for problems of electromagnetism and acoustics, the concept of wave cloaking was also addressed for other classes of physical problems, which include linear water waves [63] and flexural vibration of elastic plates [64]. The equations of vector elasticity



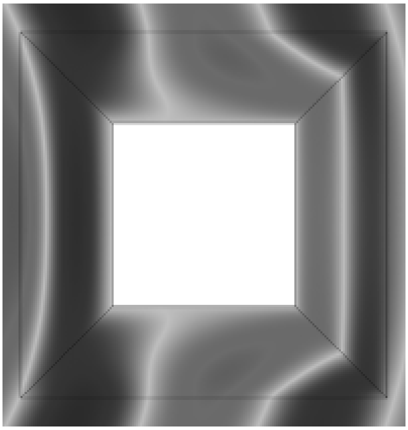
(a) (b)

FIGURE 1.2: Structured interface coating: (a) the radial anisotropic structure, (b) the square anisotropic coating.

bring additional challenges, compared to the Maxwell system or the equations describing flexural waves in elastic structures; the tensor structure of the governing equations of mathematical elasticity incorporates the notion of shear. As a result, pressure and shear waves propagate with different speeds and couple through the boundary conditions. This makes the modelling of elastic metamaterial systems, which possess cloaking properties, difficult as outlined, for example, in [119, 143].

The dynamic response of vector elastic lattice systems, interacting with waves, has been the subject of classical investigations, as discussed, for example in [91], as well as [49, 177–179]. We note substantial differences between scalar problems involving the vibration of systems of harmonic springs and vector problems of elasticity, referring to elastic rods and beams, which connect a system of finite solids or point masses. The notion of shear stress and shear strain is absent in scalar lattice systems and there is no such analogue in models of electromagnetism. The misconception of predictability of dynamic properties of vector elastic lattices is sometimes based on an intuitive extrapolation of results available for the scalar systems or for problems of electromagnetism. As illustrated in [49], results observed for micro-polar elastic lattice structures do not always follow from the simpler scalar physical models or intuitive assumptions.

The papers [99, 129, 160–162, 166, 190] have addressed the problems of dispersion of Bloch–Floquet waves and phononic band gaps for elastic waves in doubly periodic solids containing arrays of voids or inclusions. Transmission problems for waves interacting with arrays of elastic gratings of inclusions were studied in [163, 164], including a comparative analysis of the filtering proper-



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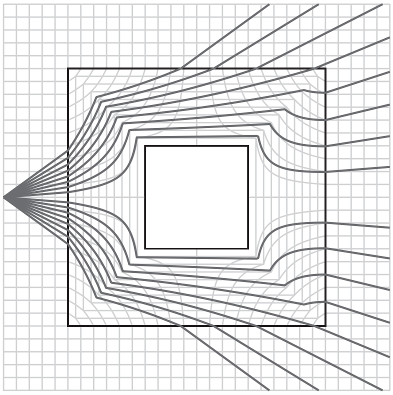


(b)

FIGURE 1.3: Wave-cloaking of a square scatterer: (a) a distorted wave front within the cloaking layer; (b) the ray diagram showing the distorted metrics in the cloaking region.

ties of elastic waves in doubly periodic media and the transmission properties for the corresponding singly periodic gratings consisting of circular inclusions.

An elegantly designed elastic coating, with anisotropic inertial properties, has been proposed in [123] to model an elastic "invisibility cloak", which takes an incident wave around a scatterer. The paper [28] shows an alternative design of an elastic cloaking coating which engages a micro-polar inhomogenous composite structure. Structured interfaces, and their discrete approximations, can be effectively used to construct dynamic invisibility cloaks, with two examples shown in Figure 1.2, which illustrate a radially symmetric structure (part(a)) and a square structured interface (part (b)). The radially symmetric wave cloaks are discussed, for example, in [43, 58, 98], and the square cloak has been modelled in [48].

The papers [30, 39, 40] have addressed propagation of elastic waves, their dispersion properties, and their interaction with defects in a two-dimensional structured medium endowed with micro-rotations associated with spinning gyroscopes embedded into the lattice system. The analysis of gyroscopic motion of an individual spinner was incorporated into the system of conservation of linear and angular momenta within the lattice systems. Consequently, it has led to a special design of chiral media, which are characterised by "handedness" in their microstructures. Such media possess special properties related to shielding, polarisation and filtering of elastic waves.

1.4 Designer multi-scale materials

The term *"metamaterials"* is often used for multi-scale materials, which are engineered to possess specially designed properties that are not found in nature. Examples of interesting designs include structures which exhibit negative refraction [176], filtering, polarisation and focussing of waves [62, 74, 118]. In particular, metamaterial wave cloaking structures have been designed and implemented in many physical applications; the example in Figure 1.3 shows a distorted wave within a square inhomogeneous and anisotropic cloaking interface and the ray diagram illustrating "diversion" of an incident wave [48].

Another term, "hyperbolic metamaterials", is used for a special class of structured multi-scale media, which exhibit a strong dynamic anisotropy accompanied by star-shaped wave forms. Such materials were designed and studied in problems of optics [165], as well as in acoustics and linear elasticity [8,50,92,94,131,151,170]. In particular, in the framework of time-harmonic fields, the waves in hyperbolic metamaterials are modulated by functions, which satisfy hyperbolic partial differential equations.

The paper [52] introduced the concept of "parabolic metamaterials" as multi-scale structures which support uni-directionally localised waveforms. This work also demonstrated interesting connections with the Dirac points on the dispersion surfaces for Bloch–Floquet waves in a special frequency range. We also note that Dirac cones on the dispersion surfaces occur as the result of degeneracies in the dispersion equations characterising Bloch–Floquet waves in photonic [117], phononic [42], and platonic crystals [112].

The dynamic homogenisation scheme developed and demonstrated in [5, 6, 47, 55, 56, 103] is also useful in the description of hyperbolic and parabolic metamaterials. Classical two-scale asymptotic homogenisation approaches for microstructured media (see, for example, [12, 154]) usually involve the study of a class of model problems on an elementary cell and can be used effectively in the long-wave quasi-static regime. In contrast, the approach of dynamic homogenisation addresses perturbations away from known resonances and may be used to study dynamic anisotropy and localisation of waves in dispersive media.

1.5 Dynamic anisotropy and defects in lattice systems

Classical applications in the theory of defects in crystals and dislocations follow from the fundamental work [105], where explicit closed-form solutions were derived for a heterogeneous lattice system when two distant particles of different masses are interchanged. The envelope function-based perturbation approach was developed in [59, 102] for analysis of waveguides in photonic crystal structures. In the latter case, an array of cylindrical inclusions embedded into an ambient medium represents a waveguide, where the frequencies of the guided modes are close to the band edge of the unperturbed doubly periodic system.

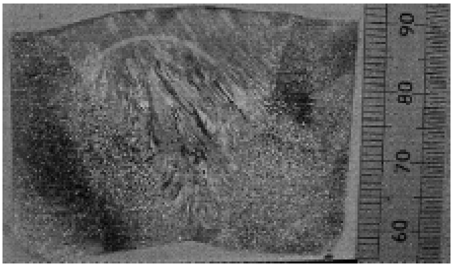
Periodic lattices may exhibit very different behaviour in dynamics compared to their static response. In particular, dynamic anisotropy and different classes of localisation have been identified in periodic lattices, as outlined in [8, 50, 92, 94, 131, 151, 170]. In some frequency regimes, corresponding to neighbourhoods of resonances, hyperbolic metamaterials can be viewed as multi-scale structures, where Bloch–Floquet waves exhibit behaviour associated with locally hyperbolic dispersion surfaces in the neighbourhood of certain resonances. Strong spatial localisation may occur where wave propagation is permitted only along directions associated with the principal curvatures of the hyperbolic dispersion surface. Important connections between the dispersion properties of Bloch–Floquet waves in a periodic lattice and the solutions of forced problems are discussed in [50].

The dynamic lattice Green's function describes the vibration of a lattice with a single-point defect, or applied point force. Green's functions have been studied in [107] for two-dimensional square lattices in pass band regimes. The paper [131] addressed continuous and discrete models for exponentially localised waveforms, with various forcing or defect types. Localised waveforms have been identified for the cases when the forcing frequency and/or the natural frequency of the defect are placed in the band gap. Such defect modes can be linked to the stop-band Green's functions.

Using an asymptotic approach, the papers [20, 68] considered the effect of a pre-stress on the propagation of flexural waves through an elastic beam on a Winkler foundation. Special attention was devoted to dynamically localised waveforms and to control the position of band gaps via pre-stress. It was found that a tensile pre-stress can increase the frequency at which a particular band gap occurs.

The monograph [178] presents a detailed discussion of applications for dynamic lattice problems involving cracks modelled as semi-infinite faults, advancing in two-dimensional periodic elastic lattices. For a structured interface and a crack propagating with an average constant speed through a square lattice, localised modes were analysed in [124]. The article [138] presented the study of a semi-infinite dynamic crack in a non-uniform elastic lattice. The crack stability was analysed and a connection has been established between the unstable crack growth and the energy of waves emanating from the crack-tip in the steady-state regime.

A comprehensive mathematical theory developing an asymptotic analysis of fields in multi-structures is presented in the monograph [90], where boundary layers near junctions are shown to have special importance in both static problems as well as in problems of time-harmonic vibrations.



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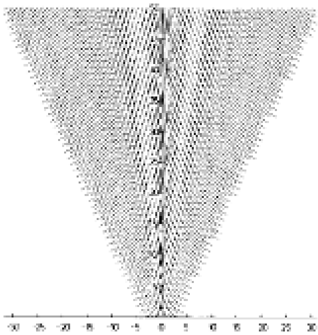

(b)

FIGURE 1.4: Panel (a) shows a photograph of a typical weld which exhibits the characteristic columnar grain alignments. Panel (b) is a plot of the grain orientations generated by a scanning electron microscope from an actual weld. *Images courtesy of Amec Foster Wheeler plc.*

1.6 Models and physical applications in materials science

Studies of the propagation of elastic waves in inhomogeneous and anisotropic materials are often motivated by the requirement for effective ultrasonic non-destructive examination of strongly anisotropic steel welds and other components. In particular, an inhomogeneity may arise when the casting or weld is formed and the metallic crystals align such that their symmetry axes are parallel to the maximal thermal gradient. Figure 1.4 shows a typical weld and grain map, illustrating the inhomogeneity.

In some applications, ultrasonic probes and sensors are used to detect defects in welded structures, and this in turn may require measurements of the scattered fields. A numerical ray-tracing model was developed in [146] to account for the inhomogeneous and anisotropic nature of the weld. The paper [140] presented an investigation into the effect of beam distortion in anisotropic, but homogeneous materials. Further studies [2, 186] suggested that the weld may be modelled as a material with constant density and crystalline moduli, but with the orientations of the crystals smoothly varying with position. In addition, the paper [144] developed a ray theory for elastic waves propagating through the inhomogeneous and anisotropic medium, with the emphasis on the special case of SH-waves.

Applications of the Bloch–Floquet wave theory to the study of the dynamic localisation in welds has been illustrated in [51], which has surpassed the traditional ray-tracing approaches and revealed new features linked to dispersion and localisation of elastic waves in a weld with a granular structure. In par-

FIGURE 1.5: An illustration of the effect of grain structure on the path of ultrasonic waves through granular welds. Here we plot the magnitude of the dilatation.

ticular, the authors of [51] developed an analytical and computational model capable of using data taken from actual welds (such as the grain structure shown in Figure 1.4) to analyse the propagation of ultrasonic waves through granular welds. Figure 1.5 shows an illustrative computation highlighting the localisation and complex wave behaviour exhibited in such welds.

1.7 Structure of the book

This text is based on recent theoretical advances leading to modelling of dynamic anisotropy, localisation and design of multi-scale heterogeneous materials, which possess unusual dynamic properties such as negative refraction, filtering, and focussing of waves by flat interfaces as well as the creating of invisibility cloaks. Some background material and a theoretical introduction are also included in the book. Some of the key topics presented here include Green's functions and localised waveforms, dynamic response of dynamic elastic structures with thin ligaments, dynamic anisotropy and localisation in lattice systems.

Furthermore, the state-of-the-art is presented for models of hyperbolic and parabolic modes and uni-directional or star-shaped wave forms, disintegrating elastic solids and effective junction conditions for multi-structures in dynamics. Multi-resonator systems, which also include tuneable thermo-elastic dynamic systems, are studied using asymptotic approximations, combined with numerical simulations.

Asymptotic analysis is also used here to study continuous and discrete media, dynamic inclusions and cracks in lattices versus Bloch–Floquet waves, platonic crystals and localised defect modes, chiral systems and models of vibrating gyro-lattices, approximate cloaking in lattice structures, as well as scattering reduction in structured plates.

Modelling foundations and methods of analysis of waves in multi-scale media are discussed in Chapter 2. Chapter 3 includes models of waves in so-called "disintegrating solids", which are multi-scale structures with thin ligaments and disintegrating junctions. Dynamic anisotropy, wave localisation and defect waveforms in lattice structures are presented in Chapter 4. Chapter 5 deals with the topic of cloaking via the introduction of specially designed multi-scale heterogeneous coatings. Finally, in Chapter 6, the reader will find models of structured interfaces and special chiral media developed for multi-scale elastic systems.

The book is aimed at a wide audience, including applied mathematicians, physicists and engineers. It may be especially useful to research students who would like to study dispersion and localisation of waves in structured media.

Enjoy reading!

Chapter 2

Foundations, methods of analysis of waves and analytical approaches to modelling of multi-scale solids

This preliminary chapter introduces some fundamental concepts and methods required for the analysis of wave propagation in multi-scale solids. We begin in Section 2.1 by introducing the concept of dispersion, initially for the classical example of linear water waves, then followed by one-dimensional mass-spring chains. The elements of Bloch–Floquet theory for infinite periodic media are summarised in Section 2.2, followed by a discussion of asymptotic approximations of high-contrast continua by discrete lattices in Section 2.3. We then proceed to study a one-dimensional transmission problem where two dissimilar lattices are joined by a *"structured interface"* in Section 2.4; we also introduce the notion of the transmission matrix. An important connection is established between formally distinct Bloch–Floquet problem and the transmission problem.

In Sections 2.5 and 2.6 we move on to problems for defects in otherwise periodic media and analyse the associated localised waves. Finally, in Section 2.7, we briefly summarise the approach of Craster et al. [47, 55] for the asymptotic homogenisation of periodic media at finite frequencies, which has proven extremely useful in the analysis of dynamic problems in multi-scale media.

The present chapter is intended to provide the uninitiated reader with a concise introduction to the principal methods required for the material presented in later chapters. The interested reader is also referred to the references provided for further information. In contrast, the reader familiar with this area of research may consider proceeding directly to Chapter 3.

2.1 Wave dispersion

This introductory section discusses some classical examples of dispersive waves, i.e. waves whose speeds are different for different frequencies. At the beginning we refer to linear water wave theory, which is well known in models of fluid flow as discussed, for example, in [24]. Another example, included in Section 2.2, is a one-dimensional lattice system, known from classical texts, such as [85] and [26]. The dispersion relations, that is the equations relating the permissible values of frequency and wavenumber, are written in explicit form and formation of a stop band, i.e. ranges of frequency with which no waves can propagate, is discussed for heterogeneous systems. An asymptotic model is given for a high-contrast stratified structure in Section 2.2. Following [128] we discuss the lattice approximation of such a system, which is capable of reproducing the dynamic response of the heterogeneous continuum system in the low frequency range.

2.1.1 Elementary considerations for linear water waves

One of the straightforward examples of dispersion is based on the linear theory of waves propagating along the surface of an incompressible inviscid fluid of uniform density. For convenience, we call such a fluid "water". The continuity equation and the equation of motion for the velocity $\mathbf{u}$ and pressure p have the form [3]

$$\nabla \cdot \mathbf{u} = 0, \tag{2.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{F}, \tag{2.2}$$

where $\mathbf{F}$ is the body force density, ρ is the mass density, and t is time. In particular, if the body force represents gravity we have $\mathbf{F} = -g\mathbf{e}^{(3)}$, and

$$\mathbf{F} = \nabla \Xi \quad \text{with} \quad \Xi = -gx_3, \tag{2.3}$$

where g is a positive constant, normalised gravitational acceleration, the upward vertical x_3-axis is orthogonal to the unperturbed water surface, and $\mathbf{e}^{(3)}$ is the unit vector along this axis.

Assuming that the fluid flow is irrotational and using the notation ϕ for the velocity potential we have $\mathbf{u} = \nabla\phi$. Hence according to (2.1) the function ϕ is harmonic

$$\nabla^2 \phi = 0, \tag{2.4}$$

within the fluid. The non-linear term in (2.2) becomes

$$\{(\mathbf{u} \cdot \nabla)\mathbf{u}\}_i = \sum_j u_j \frac{\partial u_i}{\partial x_j} = \sum_j \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{1}{2} \frac{\partial}{\partial x_i}\big(|\nabla\phi|^2\big). \tag{2.5}$$

Thus, (2.2) may be written as

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gx_3\right) = 0,$$

which leads to Bernoulli's equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gx_3 = f(t), \tag{2.6}$$

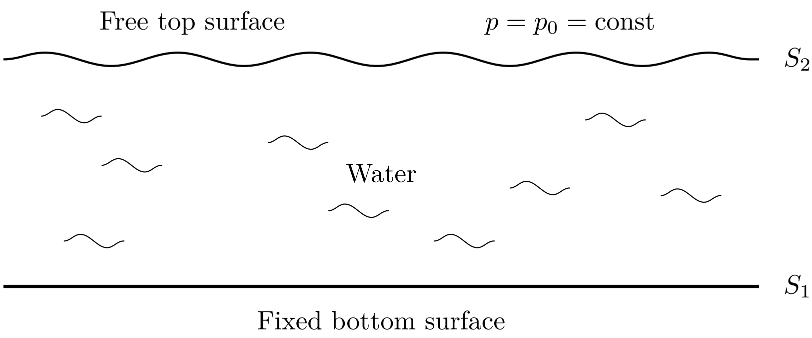

FIGURE 2.1: Surface water waves in a flow of finite depth.

where $f(t)$ is a function of t only.

The boundary conditions are set at the bottom and upper surfaces S_1 and S_2, respectively, as shown in Figure 2.1. It is assumed that the bottom surface S_1 of the channel is fixed and hence the normal component of the velocity is zero, that is,

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on} \quad S_1. \tag{2.7}$$

It is also assumed that the upper surface S_2 is characterised by the equation

$$x_3 = \zeta(x_1, x_2, t),$$

and there is no variation in pressure on S_2:

$$p = p_0 = \text{const} \quad \text{on} \quad S_2.$$

Hence, according to (2.6) on the free surface we have

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + \frac{p_0}{\rho} + g\zeta(x_1, x_2, t) = f(t), \quad \text{as } x_3 = \zeta(x_1, x_2, t). \tag{2.8}$$

This is accompanied by the identity

$$u_3 = \frac{dx_3}{dt} = \frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + u_1 \frac{\partial \zeta}{\partial x_1} + u_2 \frac{\partial \zeta}{\partial x_2} \quad \text{on} \quad x_3 = \zeta(x_1, x_2, t). \tag{2.9}$$

In the linear approximation, we simplify the problem by addressing the case when $|\mathbf{u}|$, the surface fluctuations, and the derivatives $\partial\zeta/\partial x_1$ and $\partial\zeta/\partial x_2$ are all small.

In particular, if the unperturbed surface is $x_3 = h = \text{const}$, then for the free surface we can write

$$\begin{aligned}\phi(\mathbf{x}, t) &= \phi(x_1, x_2, \zeta(x_1, x_2, t), t) = \\ &= \phi(x_1, x_2, h, t) + (\zeta - h)\frac{\partial \phi}{\partial x_3}(x_1, x_2, h, t) + \ldots\end{aligned}$$