INTRODUCTION TO

MATHEMATICAL MODELING

MAYER HUMI

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CHAPTER 1

The Process of Mathematical Modeling

CONTENTS

1.1 WHAT IS MODEL BUILDING?

Definition: modeling is the art of describing in symbolic language a real life system so that approximately correct predictions can be made regarding the behavior or evolution of the system under varied circumstances of interest.

We now elaborate on this definition.

First, note that in this definition "modeling" is referred to as an art. As such one cannot develop rigid preset rules for this task. What can be done, however, is to point out a pattern that is found to be useful in many cases and can help the practitioner to avoid many pitfalls.

Furthermore, a model is described as being able to make "correct predictions" about the system. This usually does not mean 100% accuracy. Predictions of many models have a rather wide error margin. The pertinent question, therefore, is whether these margins are acceptable to the user or not. Moreover, it might turn out that several models are capable of describing the same phenomena with different degrees of accuracy (and complexity).

Another important aspect of the definition is that a model should be "solvable." A sophisticated but "insolvable" model might be less useful from a practical point of view than a simple and straightforward one which is capable of making predictions with acceptable error margins.

One should also bear in mind that every model is constructed with certain limitations on its validity, and these should be borne in mind by the prospective user. Thus, in many practical applications it is not that "the model is incorrect" but it is the application which violates the basic assumptions of the model used.

A classical example to illustrate these points is given by gravitation theory. Here we do have at present two concurrent theories which pertain to modeling the same phenomena viz. Newton's Law of Gravitation and Einstein's theory. Though it is accepted and proven that Einstein's theory is better and more accurate, still it is highly complex and "hard to solve." As a result, in most terrestrial applications we use Newton's Law of Gravitation with "acceptable error margin."

As to the problems which require model construction, their source and scope vary between applied problems in life to attempts to duplicate natural phenomena and (what might seem to be) intellectual curiosity.

Examples and Illustrations:

1. In many cases of daily life, we construct "mini-models" without even paying any attention to these facts; e.g., "How do I get to downtown?" (by car, by bus, by subway, on foot, or otherwise) requires a model which depends on:

   (a) The distance to downtown,

   (b) The time element (How fast do I want to get there?),

   (c) Money considerations,

   (d) Security considerations (Is it safe to ride the subway?),

   (e) Availability of means (How frequently do the buses run?),

   (f) The mood of the person.

2. Consider a truck company operating in the U.S. with "truck depots and service centers" in some major cities.

   A major problem for such a company is how to dispatch trucks to their destinations in the most economical way (saving gas and drivers' time).

3. How can the wheat crop be increased to feed the growing human population?

4. What is the cause of global warming and climate change and how can these effects be mitigated?

5. How can sound and light be recorded in a better way?

6. How can rockets be sent to the Moon or the planets?

7. Why is the sky blue?

1.2 MODELING FRAMEWORK

As we said above, model building is a creative act for which no preset rules apply. We do trace here, however, a series of steps which hopefully will be useful in avoiding costly mistakes and around which one can develop one's skills in this field. We would like to stress, moreover, that model building is a non-sequential process. In some cases, several of these steps will overlap, some might be "missing" (i.e., not needed), and between others "loops" have to be made until one may come up with a reasonable (and acceptable) model for the problem at hand.

We now describe these steps.

STEP 0: Set up as **precisely** as possible the **reasons** for constructing the model and its **objectives**.

We note here that in many projects a clear statement of these reasons and objectives might radically influence the model to be built.

Example: The statement "Build a model to predict the weather" is a rather loose and incomplete statement from a modeling point of view. Thus, if no statement is made for the reasons and the precise objectives of the desired model, the problem should be considered as ill-defined. In fact, as stated above, each of the following might be the actual objective of the model.

1. Predict the weather for the next hour.
   Reason: One wants to go shopping.
   Appropriate model: Just look through the window.

2. Predict the weather tomorrow.
   Reason: Going on a trip one would like to know how to dress.
   Appropriate model: Listen to the weather forecast on the radio or the Internet.
3. Predict the weather next winter.
   Reason: Will it be a good idea to build a ski-motel during the summer?

Discussion of an appropriate model: We note, however, that even after these clarifications the model to be built is not precisely defined since the word “weather” might have different connotations, for example:

1. The model should predict the temperature to within 10° (should I take my sweater with me?).
2. The model should predict the temperature, the condition of the sky, and the possibility of showers (would it be a nice day for a trip? Should I take a raincoat?).
3. The model should predict the possibility of snow (should I take my ski gear with me?).

Another point to remember at this stage is that in many instances a reformulation of the model objectives will be required during the process of model building. This is especially so if the original scope of the model turns out to be too large.

Example: The objective of building a model to “cure cancer” requires many sub-models since there are several types of cancer.

STEP 1: Study the problem as it is in real life.

Example: If one attempts to build a model to improve the performance of a production line, then it is imperative to go to the factory and study *first hand* how this line operates (nothing else will do). In many instances, one might discover that the problem of improved production depends on factors which are independent of the line operation.

STEP 2: Data collection and analysis in real life.

At this stage, one studies the phenomenon and its behavior in real life with the objective of identifying the major factors (i.e., causes) that influence the phenomenon.

Example 1: Analysis of car accidents.

As result of data collection and its statistical analysis, one might conclude that the main factors which have a bearing on the frequency of car accidents are: driver, road, car, and weather. Each of these can be further subdivided into several sub-headings; e.g., to define "driver," we must specify age, sex, height, sight, mental state (e.g., intoxication), etc.

We note that at this step some very crucial decisions have to be made viz. to identify those factors that are most important to the problem at hand. For example, in analyzing car accidents, one might make the (questionable) assumption that the height of the driver is of little importance and hence can be ignored. Once again, it is important to keep in mind the need to strike the balance between model simplicity (i.e., few variables and easy to solve) and effectiveness (i.e., accurate predictions).

Moreover, at this point one must also decide whether to limit the scope of the model to be built or to make its objectives more precise.

Example 1: If we started with the objective of curing cancer, we might decide at this stage to study only the relationship between drug X and lung cancer.

Example 2: If we wanted to study, originally, the performance of a given car, we might decide to limit ourselves to the study of a certain component, e.g., the motor.

STEP 3. Controlled lab studies or simulations.

Studies carried out in the labs enable us to vary the factors that influence the phenomena under study in a controlled manner and thus study the influence of each factor separately.

Example : If the strength of a certain material depends on the temperature and pressure, then lab experiments will enable us to study the strength as a function of one variable only (for a constant value of the other variables), something that is not easy to achieve in real life situations.

STEP 4: Construction of a conceptual qualitative model.

As a result of the studies conducted in the previous steps, one should be able to construct a qualitative model for the phenomena at hand.

Example 1: "The accident rate depends mainly on the driver's age where 25 seems to be the most important in changing driving habits."

Example 2: "Pleasure is the main motivating force in human behavior."

Example 3: "Reliability of a given system decreases with heat and speed but increases with weight."

If such a qualitative model is acceptable to the user, then the author or researcher might feel no further need for a mathematical-quantitative model (or the model might be hard to quantify, e.g., amount of pleasure, motivation, etc.) and therefore may proceed directly to step 8 (bypassing steps 5, 6, and 7, which are needed when a mathematical model is constructed).

In many cases, one is required at this stage to make a creative breakthrough, that is provide a new conceptual framework for the problem under consideration and its solution.

Example 4: "The H_2-molecule can be represented by a rod whose mass is concentrated at the end points." This is Raman Model for the H_2-molecule which earned him the Nobel prize.

STEP 5: Conclusions, predictions, and recommendations that follow for the qualitative model.

Example 1: "Tomorrow will be partly cloudy."
Example 2: "Saccharine is carcinogenic."
Example 3: "Car X is not reliable."
Example 4: "Mr. X has a strong personality."
Example 5: "Travel is a pleasurable experience."

The shortcomings of qualitative models are:

1. Models tend to be limited in scope.
2. Model reliability in making predictions is sometimes questionable.
3. Predictions are qualitative rather than quantitative.

The points of strength of these models, on the other hand, are:

1. Their meaning is clear to almost everybody.
2. They are hard to question and challenge (sometimes these models are accepted solely due to the experience and authority of the person that suggests the model).
3. They are simple in structure.
4. They are based sometimes on long personal experience and intuition.

Example : "Buy stocks in January and sell them in April."

While qualitative models are accepted and used extensively in the social sciences, they are rarely acceptable in the natural sciences or engineering. In these fields, one has to perform the following additional steps which lead to a mathematical-quantitative model and its solutions.

Remark: Not all mathematical models make quantitative predictions, e.g., in the stability theory for solutions (or equilibrium points) of differential equations.

STEP 6: Abstraction and symbolic representation.

This step sometimes requires a lot of insight and creativity as the true variables which control the phenomena might be masked by the data. In the majority of the cases, however, it consists of representing the variables by symbols, naming the functions and identifying the axioms or constraints of the model.

Example 1: "We can treat a car as a point particle whose position is given by the vector x (abstraction and simplification)."

Example 2: "The temperature T is a polynomial function of the speed v and the time t, i.e.,

$$T = p(v, t)$$

where p is a polynomial."

Example 3: "Let the number of trucks at station A be denoted by N_A. Then N_A must satisfy $N_A \leq N$ where N is the total number of trucks operated by the company (symbolic representation and constraint)."

Example 4: "For transportation purposes, it is enough to represent the U.S. map by a set of discrete points whose location coincide with the major cities (abstraction and simplification)."

STEP 7: Derive the equations that govern the phenomena.

Example 1: If $\mathbf{F}$ is acting on point particle of mass m, and the particle acceleration is denoted by $\mathbf{a}$, then

$$\mathbf{F} = m\mathbf{a}.$$

This is Newton's Second Law.

Example 2: Denoting by P, V, and T the pressure, volume, and temperature of a gas, then

$$PV = RT$$

where R is a constant. This is the Ideal Gas Law.

Remarks: Broadly, mathematical models are classified as deterministic versus stochastic (viz. probabilistic). Another possible classification of these models is as continuous (i.e., the variables used are continuous) and discrete. Each of these classifications has its merit within a given context.

STEP 8: Model testing.

To this end, one must solve the model equations and compare the solution with the actual data collected in steps 2 and 3. If there is a bad fit, i.e. non-acceptable deviations, then it will be necessary to redo steps 4, 5, and 6.

In this context, we remark that sometimes new mathematical techniques have been devised to solve a mathematical model. If the model equations remain intractable, then some approximations to the model equations must be made, thereby sacrificing accuracy in favor of easier computability.

Example: If the original model equations are highly nonlinear and hard to solve, then one may find an acceptable linear approximation which might be solved easily.

STEP 9: Model limitations and constraints.

At this point, one must become clearly aware of the limitations that must be imposed on the use of the model and the permissible range of the variables.

Example 1: One cannot use the equations of classical mechanics to predict the motion of a particle whose speed is close to the speed of light.

Example 2: The ideal gas law is a good model for some gases but not for others.

STEP 10: Predictions and sensitivity analysis.

Once a model has been tested and found acceptable, then it can be used to make predictions. Whenever such a prediction is found to be correct, the model is considered to be more reliable (in a way every such prediction is a further test of the model).

One should bear in mind, however, that sensitivity analysis of many models is required before their actual use; i.e., one has to find the extent to which the model predictions are sensitive to small variations of the model parameters. We note that some models are "required" to be highly sensitive while in others insensitivity to such variations is necessary (e.g., if the data contains inherent errors).

Example 1: Ballistic tables.

To construct an "exact" ballistic table, one has to know the exact atmospheric conditions, amount of charge, geographic altitude, and state of the cannon to be fired. In field conditions, however, these variables are known approximately at best. Hence, a good ballistic model must be somewhat insensitive to small variations in these parameters while giving a reasonable prediction about the range of the shot.

Example 2: Models for physical resonances.

Here, sensitivity is highly desirable especially when several such "close by" resonances are involved.

Example 3: Chaotic systems.

When the evolution of a system under consideration displays high sensitivity to the initial conditions, we say that the system is "chaotic." Under these circumstances, it is possible to make only "short time" predictions about the state of the system. This is why weather forecasts are accurate only for a "few days" (at best).

STEP 11: Extensions and refinements.

If a model is found to be correct in some instances but less accurate in others, then a refinement of it is needed to take care of these exceptions.

Example: The ideal gas law needs such a refinement when the gas molecules are "large" (e.g., diatomic gases). The refined model is given by

$$\left(P + \frac{\alpha}{V^2}\right) V = RT$$

where α is a parameter which depends on the gas.

STEP 12: Compounding.

Once a correct and reliable model has been established for some phenomena, then related problems can be modeled by a process of compounding.

Example: Once the equations of motion for the spring-mass system are found, one can compound the model to systems of several masses and springs.

Finally, we present here a schematic overview of the modeling process.

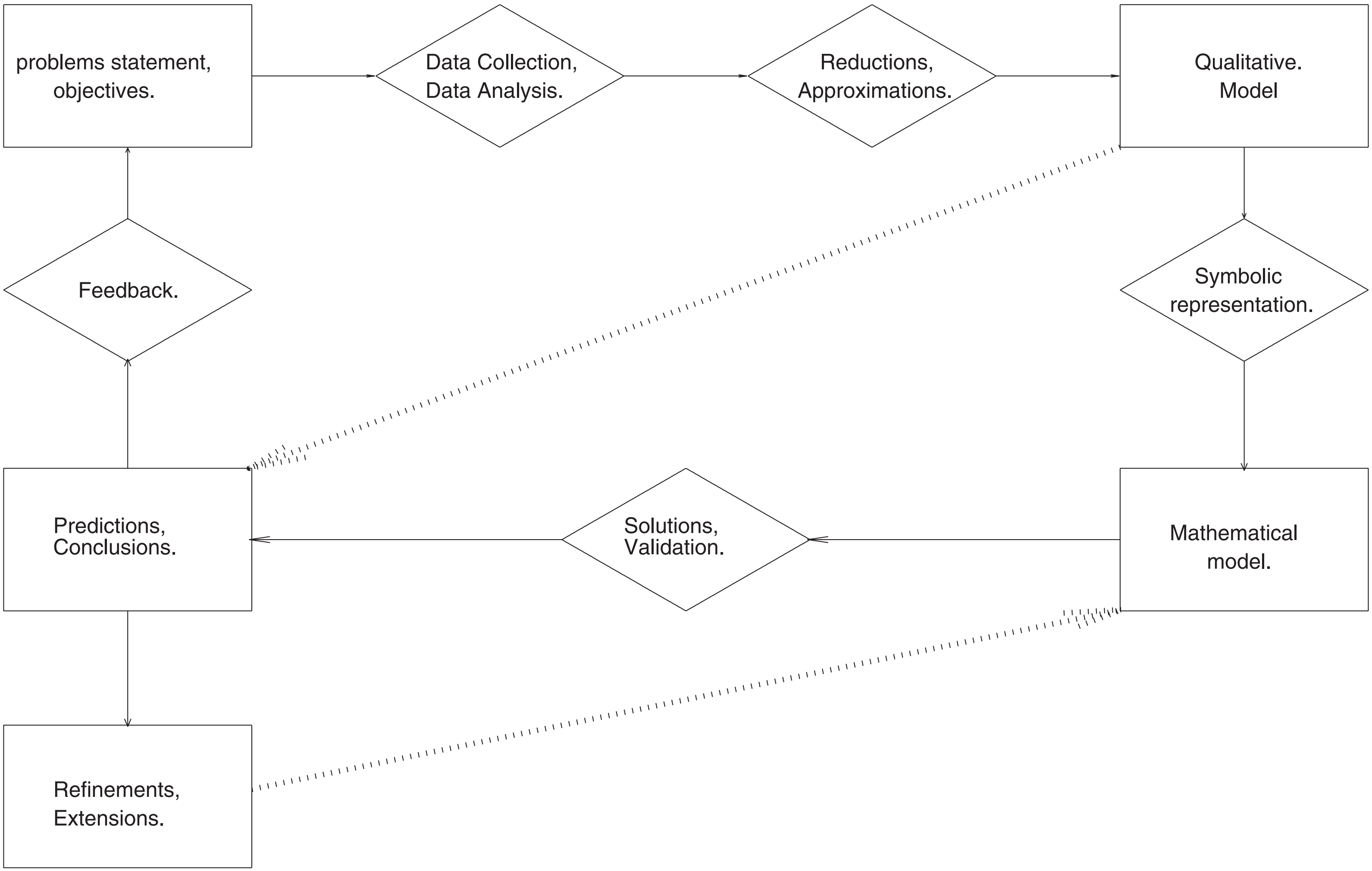

Figure 1.1 A flow chart of the modeling process.

EXERCISES

E. Fermi was one of the great theoretical physicists in the twentieth century. Some of the following "mini-model" questions are attributed to him. We offer these here to sharpen the reader's skill in the modeling of "real world problems."

1. Ignoring oceans and such, how long would it take to walk entirely around the world?
2. How much water per year flows in the Mississippi river?
3. How many dump-truck loads would it take to move Mt. Washington in New Hampshire, USA?
4. Find the dimension of a box that can contain all of the human race (five billion approximately).
5. What is the linear velocity of Earth around the Sun?
6. How many drops of water are in the Pacific ocean?
7. How many books are in a bookstore?
8. How many atoms are in a cell?
9. How many cells are in the human body?
10. How many light-bulbs burn out in one minute throughout the world?
11. What is the actual volume of material in a solid cubic meter of metal (remember atoms are made of nuclei and electrons)?
12. How do you buy the best car for your money?
13. How do you buy the best computer for your money?

1.3 GENES AND BIOLOGICAL REPRODUCTION

Most of the models that we consider in this book will use "continuous variables." Moreover, it will be advantageous in some cases to convert a discrete variable problem into a problem with continuous variables. However, to illustrate the modeling process we consider in this section a model which reduces a "continuous variable problem" into a discrete one.

Motivation and Objective:

A company grows and sells various types of beans (Lima-beans,kidney-beans etc.). It is a known fact that green and smooth texture (Lima-)beans are preferred by the consumers and hence command a higher price. However, on the farms the company grows beans which vary in color (white, red, yellow, green, and anything in between) and texture (from smooth to wrinkled).

It is the objective of this project to understand this phenomenon so that the company will be able to produce larger quantities of the desirable beans and enhance its profits.

Data from the farm:

1. Beans self fertilize.
2. Beans can be divided to a good approximation into the following groups: Green-Smooth (G-S), Green-Wrinkled (G-W), White-Smooth (W-S), White-Wrinkled (W-W), Yellow-Smooth (Y-S), Red-Smooth (R-S), and Red-Wrinkled (R-W).

Remarks: Note that by using the simplification in (2) above, we converted a continuous set of variables for the color and texture into a discrete one. This, sometimes, simplifies the problem considerably. However, sometimes the reverse is true (e.g., in population models).

Experimental Data:

Observations of bean plants grown in seclusion are being made in regard to their crop and their descendants.

Results:

1. It is possible to obtain pure lines of beans, i.e., beans which by self fertilization will always produce descendants of the same type. However, these pure lines are prone to disease and therefore not very desirable from a commercial point of view.

2. Sometimes, a plant from a line producing green smooth beans will produce by self fertilization some descendants which are wrinkled, etc. (so that appearances might be deceptive).

3. If we cross pure lines of G-S beans with G-W, we obtain first generation G-S beans only. However, in the second generation, these G-S beans will give both G-S and G-W beans by approximate ratio of 3:1.

Subproblem:

Build a model to explain texture only; i.e., assume all beans are Green.

Qualitative Model:

1. A bean carries entities which we shall call "genes" which determine whether it is smooth or wrinkled. These will be denoted by S and W.

2. Each bean contains two such genes.

3. A bean is smooth if the combination of genes is SS or SW and wrinkled if WW.

4. In cross fertilization, one gene is accepted (independently) from each parent.

Remark: In such a situation where the combination SW is smooth, we shall say that S is "dominant" with respect to W.

Mathematical Model

Let $R(*, *)$ denote the reproduction function, i.e., the probability distribution of the descendants for a given pair of parents

$$R(p_1, p_2) = (R_1, R_2, R_3)$$

where p_1 and p_2 represent the parents and R_1, R_2, and R_3 are the probabilities of SS, SW, and WW descendants respectively. If $P(*)$ is the probability that a given bean carries a certain gene, we then have

$$R_1(p_1, p_2) = P(S \mid p_1)P(S \mid p_2)$$

$$R_2(p_1, p_2) = P(S \mid p_1)P(W \mid p_2) + P(W \mid p_1)P(S \mid p_2)$$

$$R_3(p_1, p_2) = P(W \mid p_1)P(W \mid p_2).$$

(In these equations $P(S \mid p_1)$ represents the conditional probability that the parent p_1 contributes the S gene to the descendant and so on.)

Model Predictions: In the cross fertilization experiment, we started with two pure lines, i.e. $p_1 = SS, \quad p_2 = WW$. As a result, our model predicts for first generation descendants:

$$R_1 = 0 \quad R_2 = 1 \quad R_3 = 0,$$

i.e. all first generation beans are G-S which corresponds to the experimental results. For the second generation, we therefore have

$$p_1 = p_2 = SW$$

and hence,

$$R_1 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}, \quad R_2 = \frac{1}{2}, \quad R_3 = \frac{1}{4},$$

i.e. $\frac{3}{4}$ of the beans are smooth and $\frac{1}{4}$ are wrinkled, i.e., a ratio of 3:1.

EXERCISES

1. Predict the results of cross fertilization between
   (a) SW and WW beans.
   (b) SW and SS beans.
2. (Compounding) Devise a model for beans which takes color into account.
   Hint: Each bean will now have four genes; S, W for texture and C, c for color.

3. Suppose that beans with (SS, CC) are crossed with (ww, cc) (and C is also dominant with respect to c) and the first generation descendants reproduce by self fertilization. Predict the results for color and texture of the crop.

4. The following are well known facts regarding blood types in humans:

   (a) There are four (major) blood types denoted by A, B, AB and O.

   (b) Each blood cell contains two genes which determine the blood type.

   (c) The O-gene is regressive with respect to the A and B genes, i.e., AO and BO bloods are A and B bloods respectively.

   (d) A and B genes are of "equal strength."

   (e) In the process of reproduction, each parent donates one gene to determine the blood type of the descendant.

Use this data to:

1. Give an explicit representation for the reproduction function of this system.

2. Predict the blood type distribution for the descendants to parents with blood types AO and BO.

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CHAPTER 2

Modeling with Ordinary Differential Equations

CONTENTS

The behavior and evolution of many scientific and engineering systems are described by equations which involve unknown functions and their derivatives. These are called differential equations, and methods for their solution play a central role in many disciplines.

Differential equations are classified as ordinary differential equations (ODEs) and partial differential equations (PDEs). ODEs are equations which involve only one independent variable while PDEs involve several independent variables.

To motivate the study of these equations we consider in this chapter problems in various areas which are modeled naturally by ODEs. For some of these models a solution is possible by elementary integration methods. For others more elaborate methods are needed.

For all the models presented in this chapter we illustrate the modeling process by adhering as closely as possible to the modeling framework that was introduced in the previous chapter.

2.1 THE MOTION OF A PROJECTILE

Model Objective and Motivation: Build a prototype model which describes the motion of a small particle in the gravity field of the Earth. Neglect all other forces and the rotation of the Earth.

This study is motivated by the fact that the motion of a projectile in the atmosphere is important in many applications (e.g rockets, cannon shells, etc). As per usual in the modeling process we first consider this problem in its "bare bones" setting and derive a prototype model.

Background: To derive the equations of motion for this problem we need Newton's second law, which states that the external force acting on a point mass is proportional to its acceleration. Thus

$$\mathbf{F} = m\mathbf{a}$$

where $\mathbf{F}, m, \mathbf{a}$ denote respectively the force, mass, and acceleration of the particle.

2.1.1 Approximations and Simplifications

1. We assume that the speed of the particle is small compared to the speed of light. Hence relativistic corrections to Newton's second law can be neglected.

2. The projectile is considered to be a point particle and the motion around its center of gravity is neglected.

3. We assume that the distance covered by the projectile is small compared to the Earth's radius. Consequently the spherical shape of the Earth can be neglected and we consider the flight to be over a flat plane.

4. We neglect the variation of the gravitational force with height and location (which is due to the fact that the Earth is not a perfect sphere). Hence we approximate g - the acceleration due to gravity by a constant.

5. We neglect the influence of the atmosphere on the motion of the projectile. These include air drag and variations in temperature, density, and pressure.

6. We neglect the effect of the Earth's rotation on the projectile motion.

2.1.2 Model:

With the approximations delineated above it follows from Newton's second law that the equation of motion of the projectile is

$$m\frac{d^2\mathbf{x}}{dt^2} = -mg\mathbf{j}. \tag{2.1}$$

where $\mathbf{j}$ is a unit vector in the upward vertical direction. Since the only force acting on the projectile is in the $\mathbf{j}$-direction, we infer also that its motion is constrained to a plane. Without loss of generality we can choose this plane to be the x-y plane with $\mathbf{x} = (x, y)$, (see Fig. 2.1). Equation (2.1) is equivalent then to two scalar equations

$$\ddot{x} = \frac{d^2x}{dt^2} = 0, \quad \ddot{y} = \frac{d^2y}{dt^2} = -g. \tag{2.2}$$

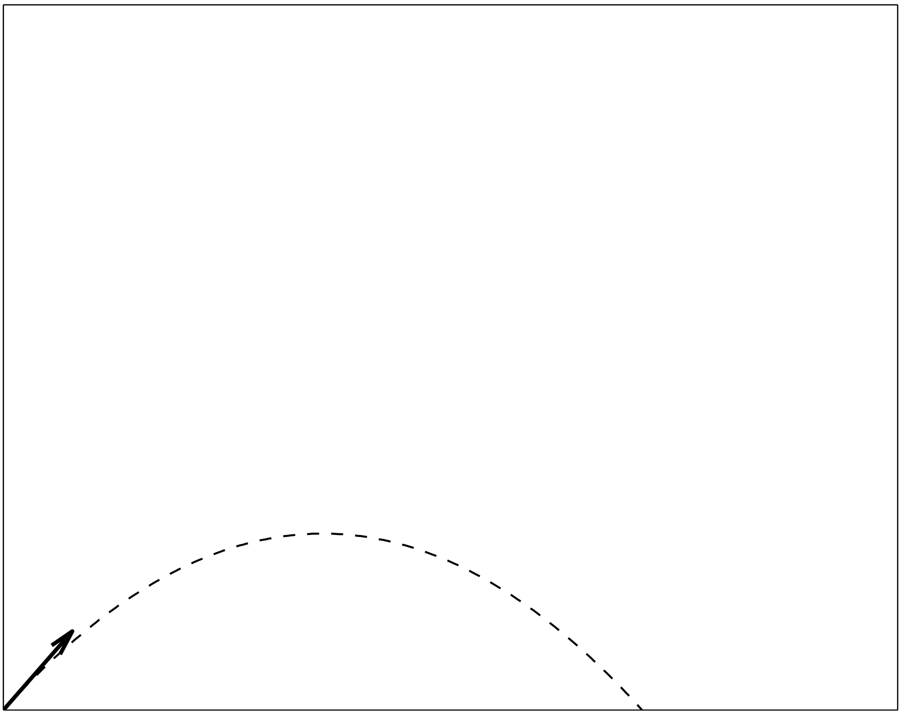

Figure 2.1 Trajectory of a projectile

Since g is constant, we can readily integrate these equations twice to obtain

$$\dot{x} = c_1, \quad \dot{y} = -gt + c_2, \tag{2.3}$$

$$x = c_1 t + c_3, \quad y = -\frac{gt^2}{2} + c_2 t + c_4, \tag{2.4}$$

where $c_i, \;\; i = 1, 2, 3, 4$, are constants. To determine these integration constants we need some "initial conditions" which (in this case) must specify the position and velocity of the projectile at some (initial) time. Thus if we assume that at time $t = 0$ the projectile is at the origin and its velocity $\mathbf{v}_0 = (v_0 \cos\theta, v_0 \sin\theta)$, then

$$x(0) = 0, \;\; y(0) = 0, \;\; \dot{x}(0) = v_0 \cos\theta, \;\; \dot{y}(0) = v_0 \sin\theta. \tag{2.5}$$

To use these conditions we substitute $t = 0$ in Equations (2.3), (2.4) to obtain

$$\begin{aligned} c_3 &= c_4 = 0 \\ c_1 &= v_0 \cos\theta, \quad c_2 = v_0 \sin\theta. \end{aligned} \tag{2.6}$$

It follows then that the parametric representation of the trajectory is

$$x = (v_0 \cos\theta)t, \quad y = -\frac{gt^2}{2} + (v_0 \sin\theta)t. \tag{2.7}$$

The nonparametric representation of the trajectory is obtained by eliminating t from Equation (2.7). This leads to

$$y = x\tan\theta - \frac{g}{2}\left(\frac{x}{v_0\cos\theta}\right)^2. \tag{2.8}$$

Example 2.1.1 *Find the relation between the range of a projectile on Earth and the Moon if they satisfy the same initial conditions.*

Solution 2.1.1 *The range of a projectile is the distance to where it returns to ground zero, i.e., $y = 0$. To find the range R_e on Earth we set $y = 0$ in Equation (2.8) and solve for x.*

We obtain

$$R_e = \frac{v_0^2 \sin 2\theta}{g_e} \tag{2.9}$$

where g_e is the gravitational acceleration on Earth. On the Moon the projectile satisfies the same equation of motion, but the gravitational acceleration is g_m. Hence the range of the projectile on the Moon is

$$R_m = \frac{v_0^2 \sin 2\theta}{g_m}. \tag{2.10}$$

Therefore

$$\frac{R_m}{R_e} = \frac{g_e}{g_m}. \tag{2.11}$$

2.1.3 Model Compounding:

We now compound the prototype model derived above by removing some of its constraints.

Example 2.1.2 *Derive the equation of motion of the projectile when air resistance (drag) has to be taken into consideration.*

Solution 2.1.2 *When the velocity of the projectile is not large, the drag force* $\mathbf{F}_d$ *is (to a good approximation) proportional to the velocity of the projectile*

$$\mathbf{F}_d = -\alpha \mathbf{v}. \tag{2.12}$$

The necessary modifications to Equation (2.1) are given by

$$m\ddot{\mathbf{x}} = -mg\mathbf{j} - \alpha\dot{\mathbf{x}} \tag{2.13}$$

or in scalar form

$$m\ddot{x} = -\alpha\dot{x} \tag{2.14}$$

$$m\ddot{y} = -mg - \alpha\dot{y}. \tag{2.15}$$

Eq. (2.14), (2.15) can be solved by direct integration

$$m\dot{x} = -\alpha x + c_1 \tag{2.16}$$

$$x = \frac{\left[c_2 e^{-bt} + \frac{c_1}{m}\right]}{b}, \quad b = \frac{\alpha}{m}, \quad \alpha \neq 0. \tag{2.17}$$

To solve for y we introduce $\dot{y} = u$. Equation (2.15) becomes

$$m\dot{u} = -mg - \alpha u$$

which then leads to

$$\dot{y} = u = c_3 e^{-bt} - \frac{g}{b} \tag{2.18}$$

$$y = -\frac{1}{b}\left(c_3 e^{-bt} + gt\right) + c_4. \tag{2.19}$$

Once again we need initial conditions in order to solve for the integration constants $c_i, i = 1, \ldots, 4$. We observe that at least formally the solution, Equations (2.16)-(2.19), “looks” totally different from the one obtained when $\alpha = 0$ (however, see ex. 4).

Exercises

1. Derive the equations of motion for a projectile if the variation of the gravitational force with height is to be taken into consideration.

2. Find the maximum height that a projectile will achieve as a function of the initial speed and firing angle. If $v_0 = 1$ km/sec, what will be the maximal change in g along such a trajectory?

3. Solve for the constants $c_i, i = 1, \ldots, 4$, in Equations (2.16)-(2.19) using the initial conditions in Equation (2.5).

4. Use the results of exercise 3 and a first order Taylor expansion for e^{-bt} to show that as $b \to 0$ the solution, Equations (2.16)-(2.19), converges to the one given by Equation (2.7).

5. Write down the equation of motion and initial conditions for the motion of a projectile if there is a wind blowing with velocity $\mathbf{w} = (w_1, w_2)$ where w_1, w_2 are constants (w_1, w_2 are the wind components in the x, y directions). Solve your model.

6. How many firing angles can be used to achieve a given range for a projectile with initial velocity $\mathbf{v}_0$?

7. For a fixed initial speed at what firing angle will a projectile achieve its maximum range?

8. A plane with speed $\mathbf{u}$ is flying from city S to city N, which is at a distance d exactly north of S. A wind of speed $\mathbf{v}$ is blowing in the eastern direction. Find differential equations for the position of the plane if its pilot makes sure that the plane is always aimed towards N. (See Fig. 2.2).

   Hint: Find differential equations for $\frac{dx}{dt}, \frac{dy}{dt}$ in terms of the position (x, y) and u, v.

2.2 SPRING-MASS SYSTEMS

In this section we model spring-mass systems as well as systems with torsion. Per usual we start with a prototype problem and then compound it to model related systems.

Objective: Build a prototype model which describes the motion (in one-dimension) of a mass attached to a spring whose other end is rigidly fixed (see Fig. 3.3). Neglect gravity and all other external forces.

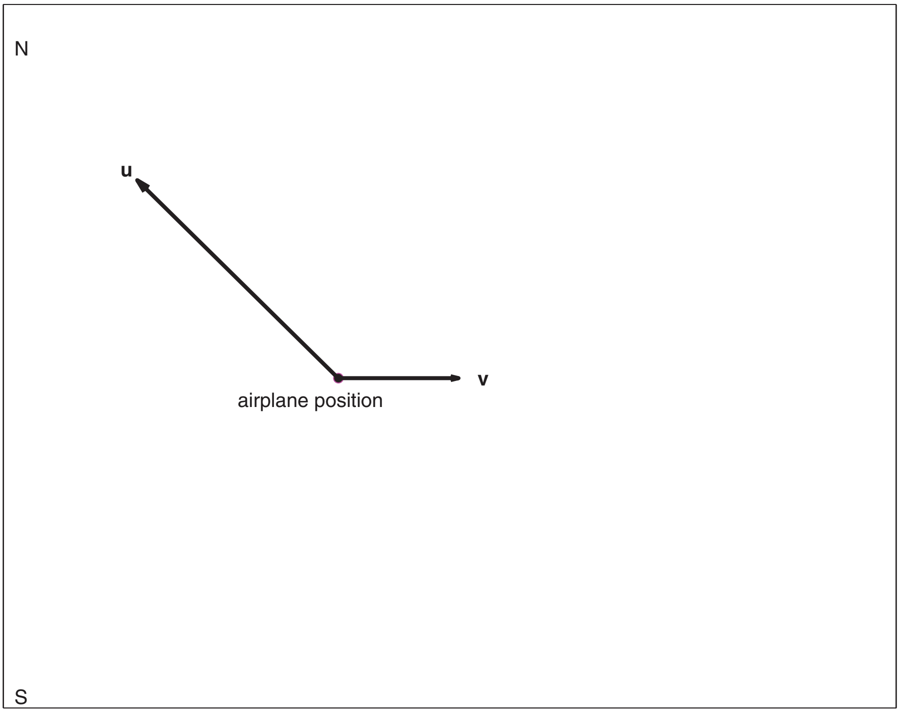

Figure 2.2 A diagram for the plane position and velocity

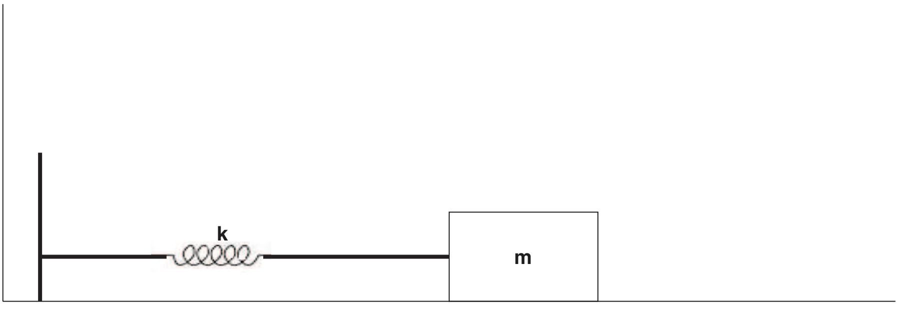

Figure 2.3 Spring Mass system

Background: To model the motion of such a system one usually applies Hooke's law. It states that for small displacements x from the natural length L of the spring the force exerted by it is given by

$$F = -kx, \quad \left| \frac{x}{L} \right| \ll 1$$

where $k > 0$ is called the spring-stiffness. However, to understand the limitations and approximations made in its derivation, we state here some of the experiments needed to establish the law.

2.2.1 Data Collection

1. Experiments to measure the force that various springs exert for positive and negative displacements (stretchings and contractions respectively).

2. Experiments to determine to what extent the force exerted by a spring varies with its use.

3. Experiments to find the effects of environmental factors such as temperature, pressure, and location on various springs.

4. Experiments to determine how the force that is exerted by the spring varies as a function of the material, size of the coil, and number and diameter of the loops.

5. Experiments to find how various imperfections in the structure of a spring, e.g. variations in the diameters of the loops and coil or deviations from circular symmetry, affect its performance.

2.2.2 Approximations and Simplifications

As a result of the data collected in the experiments listed above, one is justified in making the following approximations and simplifications for *metal* springs:

1. Small deviations in the structure of a spring minimally affect its performance. Henceforth, we only consider "ideal springs" which are made of homogeneous material, circular coil, and loops whose diameters are constant.

2. The force exerted by a spring depends very weakly on environmental factors and the number of times that the spring is used. Hence we shall neglect the influence of these factors on the performance of the spring.

3. For equal but opposite displacements the magnitude of the force exerted by the spring is equal but in opposite directions.

4. For *small* displacements the force is proportional to the displacement with a negative proportionality constant. The determination of this proportionality constant for a given spring from first principles (i.e. as function of the material, number of loops, etc.) requires a major modeling effort and is in most cases impractical.

2.2.3 Mathematical Model

Let

F	= force exerted by the spring
m	= mass of body attached to the spring
x	= displacement of the mass (or the center of mass) from equilibrium
a	= acceleration of the mass.

Using the approximations to the data introduced above we can now write for $| x | << 1$ that

$$F(x) = -kx \text{ (Hooke's Law)} \tag{2.20}$$

where $k > 0$ is called the stiffness of the spring.

Using Newton's second law we find that the equation of motion for the mass attached to the spring is given by

$$m\ddot{x} = -kx \tag{2.21}$$

or

$$m\frac{d^2x}{dt^2} = -kx. \tag{2.22}$$

2.2.4 Remarks and Refinements

1. If we displace a spring from x to $x + \Delta x$, we infer from Equation (2.20) that

$$\Delta F = F(x + \Delta x) - F(x) = -k\Delta x. \tag{2.23}$$

This observation, that the additional force exerted by the spring due to a displacement Δx from x is independent of x, is important in many applications.

2. (In preparation for 3) Let there be given an infinite series

$$p(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2.24}$$

which converges for $\mid x \mid< R, \quad R > 0$. If $p(x)$ is an odd function, i.e.

$$p(x) = -p(-x), \quad \mid x \mid< R, \tag{2.25}$$

then $a_{2m} = 0, \quad m = 0, 1, \ldots$. In fact we infer from Equations (2.24), (2.25) that

$$2\sum_{m=0}^{\infty} a_{2m} x^{2m} = 0$$

and since this must be true for all $\mid x \mid< R$, it follows that $a_{2m} = 0$. Similarly if $p(x) = p(-x)$,i.e. $p(x)$ is even, one infers that $a_{2m+1} = 0, \quad m = 0, 1, \ldots$.

3. In many instances engineers and scientists are called upon to solve or model systems in a short period of time. Under these constraints it is impossible to conduct a thorough set of experiments to establish the laws governing the system's behavior. Instead "mathematical approximations" must be used. We now illustrate this procedure.

Assume that the only information given about the force exerted by the spring is:

   (a) $F = f(x)$, i.e. the force is a function of the displacement only.
   (b) $F(0) = 0$

(c) $F(x) = -F(-x)$, i.e. F is an odd function of x, where F is some unknown but smooth (i.e. analytic) function.

Since F is analytic, we can expand it in a Taylor expansion around $x = 0$

$$F(x) = F(0) + \frac{F'(0)}{1!}x + \frac{F''(0)}{2!}x^2 + \ldots = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!}x^n.$$

However, from the fact that F is odd it follows (using the previous observation) that $F^{(2m)}(0) = 0, \quad m = 0, 1, 2, \ldots$. Hence

$$F(x) = k_1 x + k_3 x^3 + \ldots \qquad .$$

If $| \; x \; |$ is small and k_1 is assumed to be nonzero, then we can approximate $F(x)$ by

$$F(x) = -kx, \quad k > 0 \tag{2.26}$$

(the sign can be determined by a simple experiment). Equation (2.26) is called the linear (or first order) approximation to F. It is valid when $F'(0) \neq 0$ and $| \; x \; |$ is small. Hooke's law can be interpreted then as representing this approximation. Our analysis, however, goes one step beyond this law. In fact it shows that the next order approximation to the force exerted by the spring (under present assumptions) is not proportional to x^2 but to x^3, i.e.

$$F(x) = -kx \pm k_3 x^3. \tag{2.27}$$

Compounding: Inclusion of external forces.

If an external force besides that of the spring acts on the mass we have from Newton's second law

$$ma = -kx + F_{ext}. \tag{2.28}$$

However if the external force F_{ext} contains frictional forces F_f then it is customary to separate this force from the other external forces so that

$$ma = -kx + F_{ext} + F_f. \tag{2.29}$$

In three dimensions one can obtain the following data about the frictional force: $\mathbf{F}_f$ always acts in the direction opposite of the velocity $\mathbf{v}$ and is a function of $\mathbf{v}$, the material and shape of the body, and the medium in which the body moves.

For a given body moving on a uniform surface, $\mathbf{F}_f = \mathbf{F}_f(v)$, and we infer from the "data" above that

$$\mathbf{F}_f(v) = -\mathbf{F}_f(-v), \quad v = |\mathbf{v}|. \tag{2.30}$$

Hence in one dimension the "first two term approximation" for F_f is given by

$$\mathbf{F}_f(v) = -bv - rv^3 \tag{2.31}$$

where b, r are positive constants.

In three dimensions the equivalent approximation for $\mathbf{F}_f$ is

$$\mathbf{F}_f(\mathbf{v}) = -b\mathbf{v} - r(\mathbf{v} \cdot \mathbf{v})\mathbf{v}. \tag{2.32}$$

For small $\mid \mathbf{v} \mid$ we therefore obtain the following equation of motion for the spring mass system in one-dimension.

$$ma + bv + kx = F_{ext} \tag{2.33}$$

or

$$m\ddot{x} + b\dot{x} + kx = F_{ext} \tag{2.34}$$

where dots denote differentiation with respect to time ("standard" notation). The equivalent equation of motion of this system in three dimensions is given by

$$m\ddot{\mathbf{x}} + b\dot{\mathbf{x}} + k\mathbf{x} = \mathbf{F}_{ext}. \tag{2.35}$$

As a particular application of the nonlinear frictional forces given by eq. (2.31) we mention the vibrations in the clarinet tube. Lord Rayleigh, who investigated this problem in the 19th century, modeled these vibrations by the equation

$$m\ddot{x} + kx = b\dot{x} - c(\dot{x})^3, \quad b, c > 0 \tag{2.36}$$

or equivalently

$$m\ddot{x} + [c(\dot{x})^2 - b]\dot{x} + kx = 0\ . \tag{2.37}$$

This nonlinear equation is called *Rayleigh equation*.

Solution of Equation (3.84) without friction.

When friction can be neglected and there are no external forces , Equation (3.84) reduces to

$$m\ddot{x} + kx = 0. \tag{2.38}$$

In Equation (2.38) the highest order derivative is $\ddot{x}$, and therefore this is a second order differential equation. This equation is also linear; i.e., if we consider the equation as a polynomial in $x, \dot{x}, \ddot{x}$, etc., then each term is of the first order. Also we observe that the coefficients of the equation are constant.

We now show how Equation (2.38) can be solved by elementary techniques of integration.

Multiplying Equation (2.38) by $\dot{x}$ and observing that $\dot{x}\ddot{x} = \frac{1}{2}\frac{d}{dt}(\dot{x}^2)$. This yields

$$\frac{m}{2}\frac{d}{dt}(\dot{x}^2) + k\dot{x}x = 0. \tag{2.39}$$

Integrating this with respect to t leads to

$$\dot{x}^2 + \omega^2 x^2 = c^2, \quad \omega = \sqrt{k/m} \tag{2.40}$$

where c^2 is a constant of integration (observe that this constant must be non-negative since the left hand side of (2.40) is a sum of squares). Hence

$$\dot{x} = \sqrt{c^2 - \omega^2 x^2}. \tag{2.41}$$

Equation (2.41) can be easily integrated, and we obtain the solution in the form

$$x = A\cos(\omega t + \phi) \tag{2.42}$$

where A, ϕ are constant. Thus the general solution of Equation (2.38) contains two arbitrary constants. These can be determined if the initial conditions $x(0),\ \dot{x}(0)$ are known.

As expected, the solution, Equation (2.42), represents vibrations with fixed amplitude as there is no friction to damp the motion.

Related Systems:

Example 2.2.1 *Derive the equations of motion for two masses m_1, m_2 which are attached to a spring with stiffness k as in Fig. 2.4.*

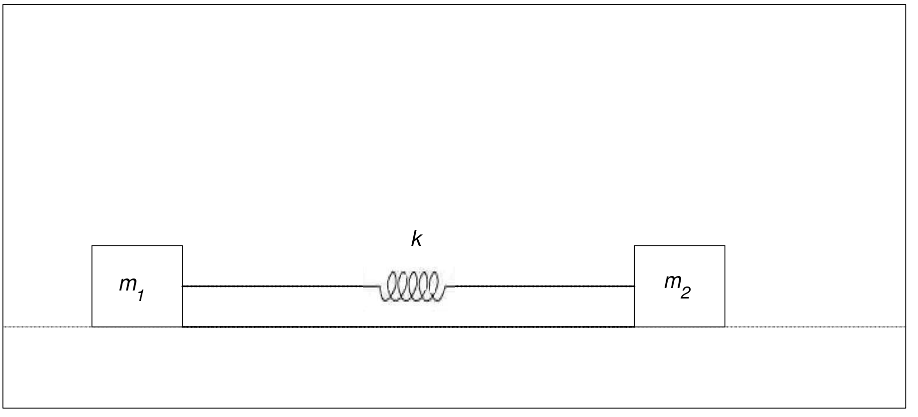

Figure 2.4 Two masses attached to a spring

Solution 2.2.1 *Let the distance between the center of mass of m_1, m_2 at equilibrium be L (if we idealize the system and treat m_1, m_2 as point particles, then L is the natural length of the spring). If these centers of mass at time t are at x_1, x_2 respectively, then either a. $x_2 - x_1 - L > 0$ or b. $x_2 - x_1 - L < 0$.*

In the first case (a) the spring is stretched beyond its natural length, and hence m_1 is pulled to the right and m_2 to the left (by Newton's third law these two forces are equal but in opposite directions.) Hence, using Equation (2.23), we have

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= k(x_2 - x_1 - L) \\ m_2\frac{d^2x_2}{dt^2} &= -k(x_2 - x_1 - L). \end{aligned} \tag{2.43}$$

Similarly in case (b) m_1 is pushed to the left and m_2 to the right. Since $x_2 - x_1 - L < 0$, we infer once again that the equations of motion are given by (2.43). Thus the differential equations which govern the system are the same in both cases.

We observe that this system is modeled by a system of *coupled* ordinary differential equations.

Example 2.2.2 *Derive the equation of motion for a mass in between two springs which are attached to rigid walls whose distance from each other is L, as shown in Fig. 2.5.*

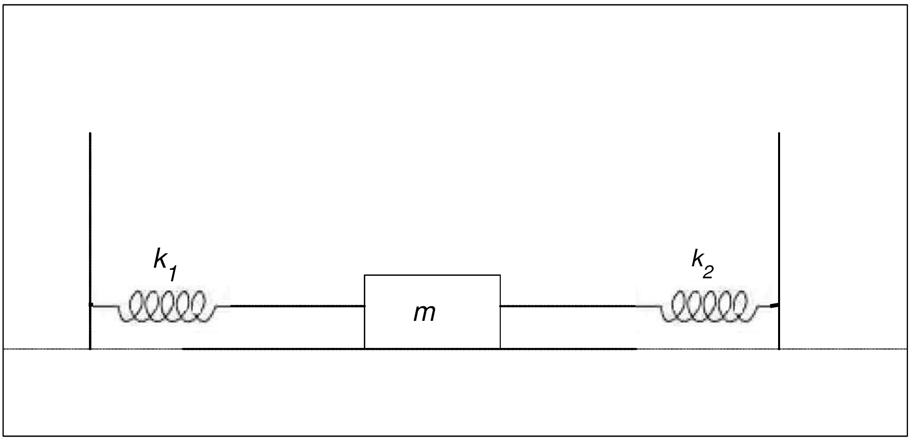

Figure 2.5 A mass and two springs enclosed by rigid walls

Solution 2.2.2 *In problems of this type it is natural to use a coordinate system whose origin coincides with the equilibrium position of the mass (which does not have to be calculated) and obtain a differential equation for the displacement from this position as a function of time. In fact for such a displacement x the* **change** *in the forces acting on m is given by (using Equation (2.23))*

$$F = -k_1 x - k_2 x$$

(regardless of the sign of x). Hence the desired equation of motion is

$$m\ddot{x} = -(k_1 + k_2)x. \tag{2.44}$$

Remark 2.2.1 *To evaluate the position x_{eq} of m at equilibrium we use the fact that in this state $F_{ext} = 0$. Hence if ℓ_1, ℓ_2 are the natural lengths of the springs and m is treated as a point particle we have*

$$k_1(x_{eq} - \ell_1) = k_2(L - x_{eq} - \ell_2)$$

(where we used a coordinate system whose origin is at the left wall of the system). However, note again that Equation x_{eq} is not needed for the derivation of Equation (2.44).

Example 2.2.3 *Derive a model equation for the motion of a mass which is attached to a thin elastic bar and subject to torsional forces ("twists").*

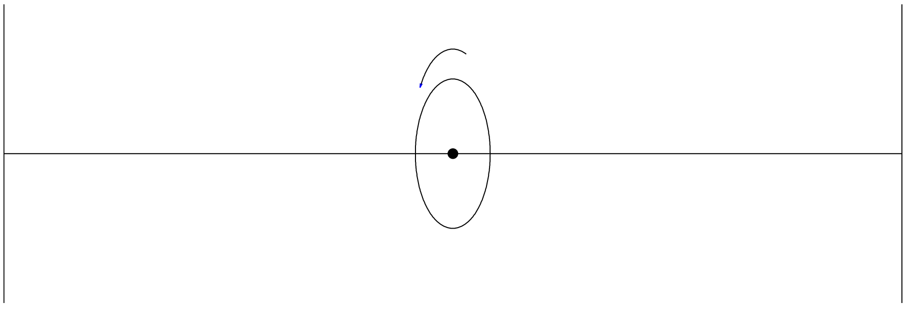

Figure 2.6 A mass attached to a thin elastic bar

Solution 2.2.3 *To model this problem one must conduct the same type of experiments and make the same approximations as in the spring-mass system. For small "twists," i.e., when the twist angle θ is small, the elastic restoring torque due to the bar can be approximated by*

$$T = -k\theta. \tag{2.45}$$

Using Newton's second law for rotating bodies we then have

$$I\ddot{\theta} + k\theta = T_{ext} \tag{2.46}$$

where T_{ext} is the external torque and I is the moment of inertia of m around the axis of rotation which is defined as

$$I = \int_V r^2 \rho(\mathbf{x}) d\mathbf{x}. \tag{2.47}$$

Here r is the distance of $\mathbf{x}$ from the axis of rotation and $\rho(x)$ is the density of the mass attached to the bar.

When frictional forces are also present then for $|\ \dot{\theta}\ | \ll 1$, we have

$$F_f = -b\dot{\theta}, \tag{2.48}$$

and the equation of motion for m becomes

$$I\ddot{\theta} + b\dot{\theta} + k\theta = T_{ext}. \tag{2.49}$$

Exercises

1. Find the differential equation which governs the motion of the system shown in Fig. 2.7:

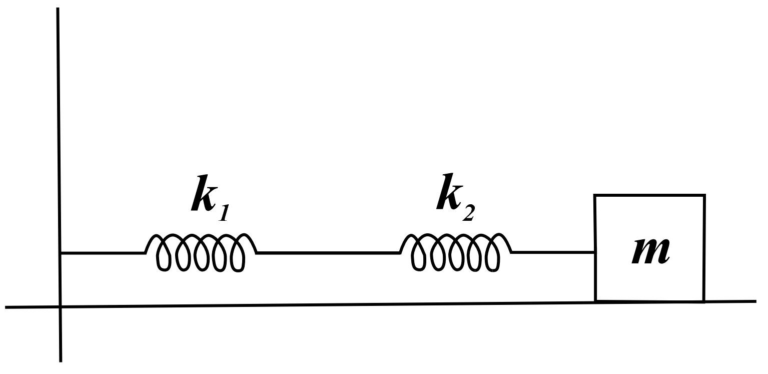

Figure 2.7 A mass attached to two springs "in series"

   Hint: Apply Newton's second law to the massless point P at which the two springs are connected.

2. Repeat Ex. 1 for the system shown in Fig. 2.8.

   Assume that m is always "gliding" on the x-axis.

3. What is the equivalent stiffness for the two springs in the systems of ex. 1,2;i.e., if one wants to replace the two springs by one, what should its stiffness be to yield exactly the same equation of motion for a mass m attached to it? Compare these results to the addition of resistors in series and parallel in an electric circuit.

4. Find the moment of inertia for a thin homogeneous rod of length L and linear density ρ (i.e. mass/unit length) which is rotating around its

   (a) mid-point