MONOGRAPHS AND RESEARCH NOTES IN MATHEMATICS

Computational Aspects of Polynomial Identities

Volume 1
Kemer's Theorems
2nd Edition

Alexei Kanel-Belov
Yakov Karasik
Louis Halle Rowen

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Volume l

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Foreword

The motivation of this second edition is quite simple: As proofs of PI-theorems have become more technical and esoteric, several researchers have become dubious of the theory, impinging on its value in mathematics. This is unfortunate, since a closer investigation of the proofs attests to their wealth of ideas and vitality. So our main goal is to enable the community of researchers and students to absorb the underlying ideas in recent PI-theory and confirm their veracity.

Our main purpose in writing the first edition was to make accessible the intricacies involved in Kemer's proof of Specht's conjecture for affine PI-algebras in characteristic 0. The proof being sketchy in places in the original edition, we have undertaken to fill in all the details in the first volume of this revised edition.

In the first edition we expressed our gratitude to Amitai Regev, one of the founders of the combinatoric PI-theory. In this revision, again we would like to thank Regev, for discussions resulting in a tighter formulation of Zubrilin's theory. Earlier, we thanked Leonid Bokut for suggesting this project, and Klaus Peters for his friendly persistence in encouraging us to complete the manuscript, and Charlotte Henderson at AK Peters for her patient help at the editorial stage.

Now we would also like to Rob Stern and Sarfraz Khan of Taylor and Francis for their support in the continuation of this project. Mathematically, we are grateful to Lance Small for the more direct proof (and attribution) of the Wehrfritz–Beidar theorem and other suggestions, and also for his encouragement for us to proceed with this revision. Eli Aljadeff provided much help concerning locating and filling gaps in the proof of Kemer's difficult PI-representability theorem, including supplying an early version of his write-up with Belov and Karasik. Uzi Vishne went over the entire draft and provided many improvements. Finally, thanks again to Miriam Beller for her invaluable assistance in technical assistance for this revised edition.

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Preface

An **identity** of an associative algebra A is a noncommuting polynomial that vanishes identically on all substitutions in A. For example, A is commutative iff $ab - ba = 0$, $\forall a, b \in A$, iff $xy - yx$ is an identity of A. An identity is called a **polynomial identity** (PI) if at least one of its coefficients is ± 1. Thus in some sense PIs generalize commutativity.

Historically, PI-theory arose first in a paper of Dehn [De22], whose goal was to translate intersection theorems for a Desarguian plane to polynomial conditions on its underlying division algebra D, and thereby classify geometries that lie between the Desarguian and Pappian axioms (the latter of which requires D to be commutative). Although Dehn's project was only concluded much later by Amitsur [Am66], who modified Dehn's original idea, the idea of PIs had been planted.

Wagner [Wag37] showed that any matrix algebra over a field satisfies a PI. Since PIs pass to subalgebras, this showed that every algebra with a faithful representation into matrices is a PI-algebra, and opened the door to representation theory via PIs. In particular, one of our main objects of study are **representable** algebras, i.e., algebras that can be embedded into an algebra of matrices over a suitable field.

But whereas a homomorphic image of a representable algebra need not be representable, PIs do pass to homomorphic images. In fact, PIs also can be viewed as the atomic universal elementary sentences satisfied by algebras. Consider the class of all algebras satisfying a given set of identities. This class is closed under taking subalgebras, homomorphic images, and direct products; any such class of algebras is called a **variety** of algebras. Varieties of algebras were studied in the 1930s by Birkhoff [Bir35] and Mal'tsev [Mal36], thereby linking PI-theory to logic, especially through the use of constructions such as ultraproducts.

In this spirit, one can study an algebra through the set of all its identities, which turns out to be an ideal of the free algebra, called a **T-ideal**. Specht [Sp50] conjectured that any such T-ideal is a consequence of a finite number of identities. Specht's conjecture turned out to be very difficult, and became the hallmark problem in the theory. Kemer's positive solution [Kem87] (in characteristic 0) is a tour de force that involved most of the theorems then known in PI-theory, in conjunction with several new techniques such as the use of superidentities. But various basic questions remain, such as finding an explicit set of generators for the T-ideal of 3×3 matrices!

Another very important tool, discovered by Regev, is a way of describing identities of a given degree n in terms of the group algebra of the symmetric group S_n. This led to the asymptotic theory of codimensions, one of the most active areas of research today in PI-theory.

Motivated by an observation of Wagner [Wag37] and M. Hall [Ha43] that the polynomial $(xy-yx)^2$ evaluated on 2×2 matrices takes on only scalar values, Kaplansky asked whether arbitrary matrix algebras have such "central" polynomials; in 1972, Formanek [For72] and Razmyslov [Raz72] discovered such polynomials on arbitrary $n \times n$ matrices. This led to the introduction of techniques from commutative algebra to PI-theory, culminating in a beautiful structure theory with applications to central simple algebras, and (more generally) Azumaya algebras.

While the interplay with the commutative structure theory was one of the main focuses of interest in the West, the Russian school was developing quite differently, in a formal combinatorial direction, often using the polynomial identity as a tool in word reduction. The Iron Curtain and language barrier impeded communication in the formative years of the subject, as illustrated most effectively in the parallel histories of Kurosh's problem, whether or not finitely generated (i.e., affine) algebraic algebras need be finite dimensional. This problem was of great interest in the 1940's to the pioneers of the structure theory of associative rings — Jacobson, Kaplansky, and Levitzki — who saw it as a challenge to find a suitable class of algebras which would be amenable to their techniques. Levitzki proved the result for algebraic algebras of bounded index, Jacobson observed that these are examples of PI-algebras, and Kaplansky completed the circle of ideas by solving Kurosh's problem for PI-algebras. Meanwhile Shirshov, in Russia, saw Kurosh's problem from a completely different combinatorial perspective, and his solution was so independent of the associative structure theory that it also applied to alternative and Jordan algebras. (This is evidenced by the title of his article, "On some nonassociative nil-rings and algebraic algebras," which remained unread in the West for years.)

A similar instance is the question of the nilpotence of the Jacobson radical J of an affine PI-algebra A, demonstrated in Chapter 2. Amitsur had proved the local nilpotence of J, and had shown that J is nilpotent in some cases. There is an easy argument to show that J is nilpotent when A is representable, but the general case is much harder to resolve. By a brilliant but rather condensed analysis of the properties of the Capelli polynomial, Razmyslov proved that J is nilpotent whenever A satisfies a Capelli identity, and Kemer [Kem80] verified that any affine algebra in characteristic 0 indeed satisfies a Capelli identity. Soon thereafter, Braun found a characteristic-free proof that was mostly structure theoretical, employing a series of reductions to Azumaya algebras, for which the assertion is obvious.

There is an analog in algebraic geometry. Whereas affine varieties are the subsets of a given space that are solutions of a system of algebraic equations, i.e., the zeroes of a given ideal of the algebra $F[\lambda_1, \dots, \lambda_n]$ of commutative polynomials, PI-algebras yield 0 when substituted into a given T-ideal of noncommutative polynomials. Thus, the role of radical ideals of $F[\lambda_1, \dots, \lambda_n]$ in commutative algebraic geometry is analogous to the role of T-ideals of the free algebra, and the coordinate algebra of algebraic geometry is analogous to the relatively free PI-algebra. Hilbert's Basis theorem says that every ideal of the polynomial algebra $F[\lambda_1, \dots, \lambda_n]$ is finitely generated as an ideal, so Specht's conjecture is the PI-analog viewed in this light.

The introduction of noncommutative polynomials vanishing on A intrinsically involves a sort of noncommutative algebraic geometry, which has been studied from several vantage points, most notably the coordinate algebra, which is an affine PI-algebra. This approach is described in the seminal paper of Artin and Schelter [ArSc81].

Starting with Herstein [Her68] and [Her71], many expositions already have been published about PI-theory, including a book [Ro80] and a chapter in [Ro88b, Chapter 6] by one of the coauthors (relying heavily on the structure theory), as well as books and monographs by leading researchers, including Procesi [Pro73], Jacobson [Jac75], Kemer [Kem91], Razmyslov [Raz89], Formanek [For91], Bakhturin [Ba91], Belov, Borisenko, and Latyshev [BelBL97], Drensky [Dr00], Drensky and Formanek [DrFor04], and Giambruno and Zaicev [GiZa05].

Our motivation in writing the first edition was that some of the important advances in the end of the 20th century, largely combinatoric, still remained accessible only to experts (at best), and this limited the exposure of the more advanced aspects of PI-theory to the general mathematical community. Our primary goal in the original edition was to present a full proof of Kemer's solution to Specht's conjecture (in characteristic 0) as quickly and comprehensibly as we could.

Our objective in this revision is to provide further details for these breakthroughs. The motivating result is Kemer's solution of Specht's conjecture in characteristic 0; the first seven chapters of this book are devoted to the theory needed for its proof, including the featured role of the Grassmann algebra and the translation to superalgebras (which also has considerable impact on the structure theory of PI-algebras). From this point of view, the reader will find some overlap with [Kem91]. Although the framework of the proof is the same as for Kemer's proof, based on what we call the **Kemer index** of a PI-algebra, there are significant divergences; in the proof given here, we also stay more within the PI context. This approach enables us to develop Kemer polynomials for arbitrary varieties, as a tool for proving diverse theorems in later chapters, and also lays the groundwork for analogous theorems that have been proved recently for Lie algebras and alternative algebras, to be handled in Volume II. ([Ilt03] treats the Lie case.) In this revised edition, we add more explanation and detail, especially concerning Zubrilin's theory in Chapter 2 and Kemer's PI-representability theorem in Chapter 6. In Chapter 9, we present counterexamples to Specht's conjecture in characteristic p, as well as their underlying theory.

More recently, positive answers to Specht's conjecture along the lines of Kemer's theory have been found for graded algebras (Aljadeff-Belov [AB10]), algebras with involution, graded algebras with involution, and, more generally, algebras with a Hopf action, which we include in Volume II.

Other topics are delayed until after Chapter 9. These topics include Noetherian PI-algebras, Poincaré–Hilbert series, Gelfand-Kirillov dimension, the combinatoric theory of affine PI-algebras, and description of homogeneous identities in terms of the representation theory of the general linear group GL. In the process, we also develop some newer techniques, such as the "pumping procedure." Asymptotic results are considered more briefly, since the reader should be able to find them in the book of Giambruno and Zaicev [GiZa05].

Since most of the combinatorics needed in these proofs do not require structure theory, there is no need for us to develop many of the famous results of a structural nature. But we felt these should be included somewhere in order to provide balance, so we have listed them in Section 1.6, without proof, and with a different indexing scheme (Theorem A, Theorem B, and so forth). The proofs are to be found in most standard expositions of PI-theory.

Although we aim mostly for direct proofs, we also introduce technical machinery to pave the way for further advances. One general word of caution is that the combinatoric PI-theory often follows a certain Heisenberg principle — complexity of the proof times the manageability of the quantity computed is bounded below by a constant. One can prove rather quickly that affine PI-algebras have finite Shirshov height and satisfy a Capelli identity (thereby leading to the nilpotence of the radical), but the bounds are so high as to make them impractical for making computations. On the other hand, more reasonable bounds now available are for these quantities, but the proofs become highly technical.

Our treatment largely follows the development of PI-theory via the following chain of generalizations:

1. Commutative algebra (taken as given)
2. Matrix algebras (references quoted)
3. Prime PI-algebras (references usually quoted)
4. Subrings of finite dimensional algebras
5. Algebras satisfying a Capelli identity
6. Algebras satisfying a sparse system
7. Algebras satisfying R-Z identities
8. PI-algebras in terms of Kemer polynomials (the most general case)

The theory of Kemer polynomials, which is embedded in Kemer's proof of Specht's conjecture, shows that the techniques of finite dimensional algebras are available for all affine PI-algebras, and perhaps the overriding motivation of this revision is to make these techniques more widely accepted.

Another recurring theme is the Grassmann algebra, which appears first in Rosset's proof of the Amitsur-Levitzki theorem, then as the easiest example of a finitely based T-ideal (generated by the single identity $[[x_1, x_2], x_3]$), later in the link between algebras and superalgebras, and finally as a test algebra for counterexamples in characteristic p.

Enumeration of Results

The text is subdivided into chapters, sections, and at times subsections. Thus, Section 9.4 denotes Section 4 of Chapter 9; Section 9.4.1 denotes subsection 1 of Section 9.4. The results are enumerated independently of these subdivisions. Except in Section 1.6, which has its own numbering system, all results are enumerated according to chapter only; for example, Theorem 6.13 is the thirteenth item in Chapter 6, preceded by Definition 6.12. The exercises are listed at the end of each chapter. When referring in the text to an exercise belonging to the same chapter we suppress the chapter number; for example, in Chapter 9, Exercise 9.12 is called "Exercise 12," although in any other chapter it would have the full designation "Exercise 9.12."

Symbol Description

Due to the finiteness of the English and Greek alphabets, some symbols have multiple uses. For example, in Chapters 2 and 11, μ denotes the Shirshov height, whereas in Chapter 6 and 7, μ is used for the number of certain folds in a Kemer polynomial. We have tried our best to recycle symbols only in unambiguous situations. The symbols are listed in order of first occurrence.

Chapter 1

p. 4: $\mathbb{N}$	The natural numbers (including 0)
$\mathbb{Z}/n$	The ring $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n
$\mathrm{Cent}(A)$	The center of an algebra A
$[a,b]$	The ring commutator $ab-ba$
S_n	The symmetric group
$\mathrm{sgn}(\pi)$	The sign of the permutation π
p. 5: $C[\lambda]$	The commutative algebra of polynomials over C
$C[a]$	The C-subalgebra of A generated by a
$M_n(A)$	The algebra of $n\times n$ matrices over A
p. 6: δ_{ij}	The Kronecker delta
tr	The trace
A^{op}	The opposite algebra
p. 7: $\mathrm{Jac}(A)$	The Jacobson radical of A
p. 9: $S^{-1}A$	The localization of A at a central submonoid A
p. 12: $\sqrt{S}$	The radical of a subset S of A
p. 13: $\mathcal{M}\{X\}$	The word monoid on the set of letters X
$f(x_1,\dots,x_m)$, $f(\vec{x})$	The polynomial f in indeterminates $x_1,\dots,x_m$
p. 14: $f(A)$	The set of evaluations of a polynomial f in an algebra A
$\mathrm{id}(A)$	The set of identities of A
p. 15: $\deg f$	The degree of a polynomial f
$\mathrm{UT}(n)$	The set of upper triangular $n\times n$ matrices
p. 16: $\Delta_i f$	The multilinearization step of f in x_i
p. 18: $\tilde{s}_n$	The symmetric polynomial in n letters
$A_1\sim_{\mathrm{PI}} A_2$	A_1 and A_2 satisfy the same identities
p. 20: s_t	The standard polynomial (on t letters)
c_t	The Capelli polynomial (on t letters)
p. 22: πf	The left action of a permutation π on a polynomial f
p. 23: $f_{\mathcal{A}(i_1,\dots,i_t;X)}$	The alternator of f with respect to the indeterminates $x_{i_1},\dots,x_{i_t}$
$\tilde{f}$	The symmetrizer of a multilinear polynomial f
p. 24: A_g	The g-component of the graded algebra A

p. 25: $F[\Lambda], F[\lambda_1, \dots, \lambda_n]$	The commutative polynomial algebra in several indeterminates
$T(V)$	The tensor algebra of a vector space V
$T^n(V)$	The n-homogeneous component of $T(V)$
p. 29 G	The Grassmann algebra, usually in an infinite set of letters
$e_1, e_2, \dots$	The standard base of the Grassmann algebra G
G_0	The odd elements of G
G_1	The even elements of G
p. 38: $\mathrm{Nil}(A)$	The sum of the nil left ideals of A
p. 39: $\mathcal{M}_{n,F}$	The identities of $M_n(F)$
$\mathcal{M}_n$	The identities of $M_n(\mathbb{Q})$
p. 45: $\mathrm{UT}(n_1, \dots, n_q)$	The $(n_1, \dots, n_q)$-block upper triangular matrices
p. 54: $\mathrm{id}(\mathcal{S})$	The identities common to a class $\mathcal{S}$ of algebras
p. 56: $U_{\mathcal{I}}$	The relatively free algebra of a T-ideal $\mathcal{I}$
p. 57: U_A	The relatively free algebra of an algebra A
p. 59: $F\{Y\}_n$	The algebra of generic $n \times n$ matrices
$F(\Lambda)$	The field of fractions of $F[\Lambda]$
$\mathrm{UD}(n, F)$	The generic division algebra of degree n
$A *_C B$	The free product of A and B over C
$A\langle X \rangle$	The free product $A *_C C\{X\}$
$A\langle X \rangle_{\mathcal{I}}$	The relatively free product modulo a T-ideal

Chapter 2

p. 78: $\|w\|$	The length of a word w
p. 79: $\widehat{W}_\mu$	The Shirshov words of height $\leq \mu$ over W
$\succ$	The lexicographic order on words
$\bar{w}$	The image of a word w in $C\{a_1, \dots, a_\ell\}$, under the canonical specialization $x_i \mapsto a_i$
p. 80: $\mu = \mu(A)$	The Shirshov height of an affine PI-algebra
p. 81: $\beta(\ell, k, d)$	The Shirshov bound for an affine algebra $C\{a_1, \dots, a_\ell\}$ of PI-degree d
p. 83: u^∞	The infinite periodic hyperword with period u
p. 84: $\beta(\ell, k, d, h)$	The Shirshov bound for a given hyperword h evaluated on the algebra A
p. 88, 92: $\hat{A}$	The trace ring of a representable algebra A
p. 96: $\delta(xv)$	The cyclic shift
p. 99: $\bar{h} = 0$	The image of a hyperword being 0
p. 108–110: Ω, $B^p(i)$, $L(j)$, $\psi(p)$	Used in the proof of Theorem 2.8.3
p. 111–112: Ω', $C^q(i)$, $\phi(q)$	Used in the proof of Theorem 2.8.4

p. 113: $\Phi(d,\ell)$	Used in the proof of Theorem 2.8.5

Chapter 3

p. 124: V_n	The space of multilinear polynomials of degree n
$M_\sigma(x_1,\dots,x_n)$	The monomial corresponding to a given permutation
σM_π	The left action of a permutation σ on a monomial M_π
$M_\sigma \pi$	The right action of a permutation π on a monomial M_σ
p. 125: Γ_n	The space of multilinear identities of A having degree n
p. 126: $f^*(x_1,\dots,x_n;x_{n+1},\dots)$	Capelli-type polynomial
p. 128: $\lambda=(\lambda_1,\dots,\lambda_k)$	A partition
p. 129: $\mu>\lambda$	Partial order on partitions
s^λ	Number of standard tableaux of shape λ
p. 132: $\chi^\lambda\uparrow$	The induced character
$\chi^\mu\downarrow$	The restricted character
p. 133: $g_d(n)$	The number of d-good permutations in S_n
p. 134: $\mathrm{Disc}_k(\xi)$	The discriminant
$E_{n,k}$	$\mathrm{End}_F(T^n(V))$
p. 135: $\hat\sigma$	The operator of $E_{n,k}$ corresponding to σ
$\varphi_{n,k}$	The map $\sigma\mapsto\hat\sigma$
$A(n,k)$	The image of $F[S_n]$ under $\hat\sigma$
p. 137: $c_n(A)$	The n codimension of A
p. 142: $H(k,\ell;n)$	The collection of shapes whose $k+1$ row have length $\le\ell$
p. 144: L	The multilinearization operator

Chapter 4

p. 154: $\overline{C\{X,Y,Z\}}$	The relatively free algebra of c_{n+1}
p. 156: $\delta^{(\vec{x},n)}$	Zubrilin's operator
p. 159: DCap_n	The double Capelli polynomial
s	
p. 165: $\mathcal{M}$	The module of doubly alternating polynomials
p. 168: $\mathrm{Obst}_n(A)$	The obstruction to integrality
p. 172: $\mathcal{DCAP}_n$	The module generated by double Capelli polynomials
φ_w	A map containing w in the image

Chapter 5

p. 178: V_n	The space spanned by all monomials in $y_1,\dots,y_n,t$ which are linear in $y_1,\dots,y_n$

$V_{n,\pi}$	The subspace in which the variables $y_1, \dots, y_n$ occur in the order $y_{\pi(1)}, \dots, y_{\pi(n)}$
$\mathrm{Ad}^t_{\ell k}$	The transformation $V_n \to V_n$ used to define the identity of algebraicity
p. 179: D^t	The identity of algebraicity
p. 188: $\mathcal{C}_T, \mathcal{R}_T$	The set of column (resp. row) permutations of the tableau T

Chapter 6

p. 206: $A = R_1 \oplus \cdots \oplus R_q \oplus J$	The Wedderburn decomposition of a f.d. algebra A over an algebraically closed field
t_A	The dimension of the semisimple part of a finite dimensional algebra A
s_A	The nilpotence index of the Jacobson radical of a finite dimensional algebra A
p. 214: $\beta(A)$	The general combinatorial analog of t_A
p. 216: $\tilde{f}_{X_1,\dots,X_\mu}$	The μ-fold alternator of a polynomial f
p. 218: $\gamma(A)$	The general combinatorial analog of s_A
$\mathrm{index}(W), (\beta(W), \gamma(W))$	The Kemer index of a PI-algebra W
$\mathrm{index}(\Gamma)$	The Kemer index of a T-ideal Γ
p. 221: $f_{\mathcal{A}(I_1)\dots\mathcal{A}(I_s)\mathcal{A}(I_{s+1})\dots\mathcal{A}(I_{s+\mu})}$	The μ-fold multiple alternator
p. 222: $\hat{A}_u, \hat{A}_{u,\nu}, \hat{A}_{u,\nu;\Gamma}$	The u-generic algebra

Chapter 7

p. 249: p^*_I	The Grassmann involution
p. 250: $G(A)$	The Grassmann envelope
p. 252: $\mathrm{Odd}(x)$	The number of odd components of a vector
$\sigma \bullet (x_1 \cdots x_n)$	The odd action on the Grassmann algebra
$\varepsilon(\sigma, I)$	Used to compute the odd action
p. 260: $\mathrm{index}_2\, A$	The Kemer superindex
p. 261: $\hat{A}_{u,\nu;\Gamma}$	The u-generic superalgebra of A

Chapter 8

p. 277: tr	The formal trace symbol
p. 279: V^*	The dual space

Chapter 9

p. 295: G^+	The extended Grassmann algebra
p. 303: P_n	The polynomials generating a non-finitely based T-space in characteristic 2
p. 309: $\tilde{A}$	The test space
p. 317: $\hat{A}$	The test algebra
p. 318: Q_n	The polynomials generating a non-finitely based T-ideal in odd characteristic

Chapter 10

p. 332: $F[S_n]$	The group algebra
Δ	The subgroup of elements of G having finitely many conjugates.
p. 334: $U(L)$	The enveloping algebra of a Lie algebra L

Chapter 11

p. 338: H_A, H_M	The Hilbert series of an algebra or module
p. 340: GKdim	The Gelfand-Kirillov dimension
p. 350: $H_{A;V}$,$H_{M;V}$	The Hilbert series with respect to V

Chapter 12

p. 364: $\chi_n(A)$	The cocharacter
p. 366: $GL(V)$	The general linear group

Part I

Basic Associative PI-Theory

Chapter 1

Basic Results

In this chapter, we introduce PI-algebras and review some well-known results and techniques, most of which are associated with the structure theory of algebras. In this way, the tenor of this chapter is different from that of the subsequent chapters. The emphasis is on matrix algebras and their subalgebras (called **representable** PI-algebras) .

1.1 Preliminary Definitions

$\mathbb{N}$ denotes the natural numbers (including 0). $\mathbb{Z}/n$ denotes the ring of integers modulo n. Throughout, C denotes a commutative ring (often a field). Finite dimensional algebras over a field are so important that we often use the abbreviation **f.d.** for them. For any algebra A, $\mathrm{Cent}(A)$ denotes the center of A. Given elements a, b of an algebra A, we define $[a, b] = ab-ba$. S_n denotes the symmetric group, i.e., the permutations on $\{1, \dots, n\}$, and we denote typical permutations as σ or π. We write $\mathrm{sgn}(\pi)$ for the sign of a permutation π.

We often quote standard results about commutative algebras from [Ro05]. We also assume that the reader is familiar with prime and semiprime algebras, and prime ideals. Although the first edition dealt mostly with algebras over a field, the same proofs often work for algebras over a commutative ring C, so we have shifted to that generality.

Remark 1.1.1. *There is a standard way of adjoining* 1 *to a C-algebra A without* 1*, by replacing A by the C-module $A_1 := A \oplus C$, made into an algebra by defining multiplication as*

$$(a_1, c_1)(a_2, c_2) = (a_1 a_2 + c_1 a_2 + c_2 a_1, c_1 c_2).$$

We can embed A as an ideal of A_1 via the identification $a \mapsto (a, 0)$, and likewise every ideal of A can be viewed as an ideal of A_1.

This enables us to reduce most of our major questions about associative algebras to algebras with 1. Occasionally, we will discuss this procedure in more detail, since one could have difficulties with rings without 1; *clearly, if $A^2 = 0$ we do not have $A_1^2 = 0$.*

In this volume, unless otherwise indicated, an algebra A over C is assumed to be associative with a unit element 1. We will be more discriminating in Volume II, which deals with nonassociative algebras such as Lie algebras.

An element $a \in A$ is **algebraic** (over C) if a is a root of some nonzero polynomial $f \in C[\lambda]$; we say that $a \in A$ is **integral** if f can be taken to be monic. In this case $C[a]$ is a finite module over C. The algebra A is **integral** over C if each element of A is integral.

An element $a \in A$ is **nilpotent** if $a^k = 0$ for some $k \in \mathbb{N}$. An ideal $\mathcal{I}$ of A is **nil** if each element is nilpotent; $\mathcal{I}$ is **nilpotent** of **index** k if $\mathcal{I}^k = 0$ with $\mathcal{I}^{k-1} \neq 0$. One of the basic questions addressed in ring theory is which nil ideals are nilpotent.

Definition 1.1.2. *An element $e \in A$ is **idempotent** if $e^2 = e$; the **trivial idempotents** are $0, 1$.*

*Idempotents e_1 and e_2 are **orthogonal** if $e_1 e_2 = e_2 e_1 = 0$. An idempotent $e = e^2$ is **primitive** if e cannot be written $e = e_1 + e_2$ for orthogonal idempotents $e_1, e_2 \neq 0$.*

Remark 1.1.3. *Given a nontrivial idempotent e of A, and letting $e' = 1 - e$, we recall the **Peirce decomposition***

$$A = eAe \oplus eAe' \oplus e'Ae \oplus e'Ae'. \qquad (1.1)$$

Note that eAe, $e'Ae'$ are algebras with respective multiplicative units e, e'. If $eAe' = e'Ae = 0$, then $A \cong eAe \times e'Ae'$.

The Peirce decomposition can be extended in the natural way, when we write $1 = \sum_{i=1^t} e_i$ as a sum of orthogonal idempotents, usually taken to be primitive. Now $A = \oplus_{i=1}^t e_i A e_j$. The Peirce decomposition is formulated for algebras without 1 in Exercises 1 and 6.8.

1.1.1 Matrices

$M_n(A)$ denotes the algebra of $n \times n$ matrices with entries in A, and e_{ij} denotes the **matrix unit** having 1 in the i, j position and 0 elsewhere. The set of $n \times n$ matrix units $\{e_{ij} : 1 \leq i, j \leq n\}$ satisfy the properties:

$$\sum_{i=1}^{n} e_{ii} = 1,$$

$$e_{ij} e_{k\ell} = \delta_{jk} e_{i\ell},$$

where δ_{jk} denotes the **Kronecker delta** (which is 1 if $j = k$, 0 otherwise). Thus, the e_{ii} are idempotents.

One of our main tools in matrices is the trace function.

Definition 1.1.4. *For any C-algebra A, and fixed n, a* ***trace function*** *is a C-linear map* $\operatorname{tr} : A \to \operatorname{Cent}(A)$ *satisfying*

$$\operatorname{tr}(ab) = \operatorname{tr}(ba), \qquad \operatorname{tr}(a \operatorname{tr}(b)) = \operatorname{tr}(a) \operatorname{tr}(b), \qquad \forall a, b \in A.$$

It follows readily that

$$\operatorname{tr}(a_1 \dots a_k) = \operatorname{tr}((a_1 \dots a_{k-1}) a_k) = \operatorname{tr}(a_k a_1 \dots a_{k-1})$$

for any k.

Of course the main example is $\operatorname{tr} : M_n(C) \to C$ given by $\operatorname{tr}((c_{ij})) = \sum c_{ii}$; here $\operatorname{tr}(1) = n$.

Remark 1.1.5. *The trace satisfies the "nondegeneracy" property that if* $\operatorname{tr}(ab) = 0$ *for all $b \in A$, then $b = 0$.*

Definition 1.1.6. *Over a commutative ring C, the* ***Vandermonde matrix*** *of elements $c_1, \dots, c_n \in C$ is the matrix (c_i^{j-1}).*

Remark 1.1.7. *When $c_1, \dots, c_n$ are distinct, the Vandermonde matrix is nonsingular, with determinant $\prod_{1 \leq i < k \leq n}(c_k - c_i)$, cf. [Ro05, Example 0.9]. This gives rise to the famous* ***Vandermonde argument****, which says that if $\sum_{j=0}^{n-1} c_i^j a_j = 0$ for each $1 \leq i \leq n$, then each $a_j = 0$. The Vandermonde argument occurs repeatedly in proofs in PI theory.*

A^{op} denotes the **opposite algebra**, which has the same algebra structure except with the new multiplication $\cdot$ in A reversed, i.e., $a \cdot b = ba$. In particular, $C^{\mathrm{op}} = C$, and $M_n(C) \cong M_n(C)^{\mathrm{op}}$ via the transpose map.

1.1.2 Modules

We assume the basic properties of modules. We often consider the submodule of an A-module M **spanned** or **generated** by a given subset of M. We say that M is **finitely generated**, denoted by **f.g.**, if $M = \sum_{i=1}^{t} A w_i$ for suitable $w_i \in M$, $t \in \mathbb{N}$. In this case, to avoid confusion with other notions of "generated," we usually say that M is **finite over** A. A module is **finitely presented** over A if it has the form M/N, where M and N are both finite over A.

For C-algebras A_1 and A_2, an A_1, A_2 **bimodule** is a (left) A_1-module M which is also a right A_2-module and a module over C, satisfying the extra associativity condition

$$(a_1y)a_2 = a_1(ya_2), \quad \forall a_i \in A_i,\ y \in M,$$

as well as the scalar condition

$$cy = (c1)y = y(c1), \quad \forall c \in C,\ y \in M.$$

Thus, the A_1, A_2 bimodules correspond to the $A_1 \otimes_C A_2^{\mathrm{op}}$-modules. In particular, the sub-bimodules of an algebra A are precisely its ideals.

1.1.3 Affine algebras

Our main interest arises in the following important class of algebras:

Definition 1.1.8. *An algebra A is* ***affine*** *over the commutative ring C if A is generated as an algebra over C by a finite number of elements $a_1, \dots, a_\ell$; in this case we write $A = C\{a_1, \dots, a_\ell\}$. A commutative affine algebra is notated $C[a_1, \dots, a_\ell]$.*

In most cases, we shall be considering affine algebras over a field F, so unless specified otherwise, "affine" will mean "affine over a field."

Commutative affine algebras are precisely the coordinate algebras of affine algebraic varieties, and thus play a crucial role in classical algebraic geometry. One of the main thrusts of PI-theory is to generalize commutative affine theory to affine PI-algebras.

1.1.4 The Jacobson radical and Jacobson rings

Definition 1.1.9. *The* ***Jacobson radical*** $\mathrm{Jac}(A)$ *of an algebra A is the intersection of the "primitive" ideals of A. (These are the maximal ideals when A is commutative; also see Corollary 1.5.1.)*

Remark 1.1.10. $\mathrm{Jac}(A/J) = \mathrm{Jac}(A)/J$*, whenever* $J \subseteq \mathrm{Jac}(A)$*, cf. [Ro08, Exercise 15.28].*

We quote a celebrated result of Amitsur [Ro05, Theorem 2.5.23]:

Theorem 1.1.11. *If A has no nonzero nil ideals, then* $\mathrm{Jac}(A[\lambda]) = 0$.

Lemma 1.1.12. *If* $\mathrm{Jac}(C) = 0$ *and A is a commutative integral domain affine and faithful over C, then* $\mathrm{Jac}(A) = 0$.

Proof. Write $A = C[a_1, \dots, a_\ell]$, and let $C_1 = C[a_\ell]$. It is enough to show that $\mathrm{Jac}(C_1) = 0$, since then we apply induction on ℓ.

So write $a = a_\ell$ and assume that $A = C[a]$. If a is transcendental over C, then the assertion is clear by Theorem 1.1.11 (since $C[a]$ is isomorphic to a polynomial ring); an easy direct argument is given in Exercise 2.

Thus we may assume that a is algebraic over C, so A is algebraic over C, and by [Ro05, Lemma 6.29] it is enough to show that $C \cap \mathrm{Jac}(A) = 0$. Write $\sum_{i=0}^{t} c_i a^i = 0$ for $c_t \neq 0$, and let $S = \{c_t^i : i \in \mathbb{N}\}$. Let $\mathcal{P}$ be the set of maximal ideals of C not containing c_t, and $J = \cap\{P \in \mathcal{P}\}$. Then $c_t J$ is contained in every maximal ideal of C and thus is 0, implying $J = 0$. On the other hand $S^{-1}A$ is integral over $S^{-1}C$. If $P \in \mathcal{P}$, then $S^{-1}P$ is a prime ideal of $S^{-1}C$, which then is contained in a prime ideal $S^{-1}Q$ of $S^{-1}A$, for some prime ideal Q of A containing P (in view of [Ro05, Proposition 8.11]), implying the integral domain A/Q is a finite extension of the field C/P, and is thus a field. Hence Q is a maximal ideal of A whose intersection with C is P, implying that $C \cap \mathrm{Jac}(A) \subseteq J = 0$, as desired. □

Definition 1.1.13. *An integral domain C is **local** if it has a unique maximal ideal, which thus is* $\mathrm{Jac}(C)$.

An equivalent formulation [Ro05, Corollary 8.20]: If $a + b = 1$, then either a or b is invertible. One key notion in commutative algebra is localization, treated in [Ro05, Chapter 8].

Definition 1.1.14. *A ring is **Jacobson** (called **Hilbert** in [Kap70b]) if the Jacobson radical of every prime homomorphic image is* 0.

In other words, in a Jacobson ring, any prime ideal is the intersection of primitive ideals of A. Obviously any field is Jacobson, since its only prime ideal 0 is maximal.

Lemma 1.1.15. *Suppose a field $K = C[a_1, \dots, a_t]$ is affine over a commutative Jacobson subring C. Then C also is a field, and $[K : C] < \infty$.*

Proof. C is an integral domain, and thus $\mathrm{Jac}(C) = 0$. The field K is affine over the field of fractions L of C, implying K is algebraic over C, by [Ro05, Theorem 5.11]. Letting c_i be the leading coefficient of the minimal polynomial of a_i over C, and $c = c_1 \cdots c_t$, we see that each a_i is integral over $C[c^{-1}]$, and thus K is integral over $C[c^{-1}]$, implying $C[c^{-1}]$ is a field, by the easy [Ro05, Proposition 5.31]. Hence any nonzero prime ideal of C contains a power of c, and thus c, implying $c \in \mathrm{Jac}(C) = 0$, a contradiction unless C is already a field, i.e., $L = C$ and thus K is finite over C. □

We also have a result in the opposite direction.

Lemma 1.1.16. *Any commutative affine algebra $A = C[a_1, \dots, a_t]$ over a commutative Jacobson ring C is Jacobson.*

Proof. For any prime ideal P of A, $\mathrm{Jac}(A/P) = 0$ by Lemma 1.1.12. □

This often is called the "weak Nullstellensatz."

1.1.5 Central localization

The localization procedure can be generalized directly from the commutative situation to $S^{-1}A$ whenever S is a (multiplicative) submonoid of $\mathrm{Cent}(A)$. In particular the ideals of $S^{-1}A$ are precisely those subsets $S^{-1}\mathcal{I}$ where $\mathcal{I} \triangleleft A$. We say that an element $s \in A$ is **regular** when $sa, as \neq 0$ for all $a \neq 0$ in A. When A is prime, then every submonoid of $\mathrm{Cent}(A)$ is regular. Here is an easy but useful result.

Proposition 1.1.17. *Suppose S is a submonoid of* $\mathrm{Cent}(A)$ *which is regular in A. Then $S^{-1}A$ is prime iff A is prime.*

Proof. ($\Rightarrow$) If $\mathcal{I}_1, \mathcal{I}_2 \triangleleft A$ with $\mathcal{I}_1\mathcal{I}_2 = 0$, then $(S^{-1}\mathcal{I}_1)(S^{-1}\mathcal{I}_2) = 0$, implying $S^{-1}\mathcal{I}_1 = 0$ or $S^{-1}\mathcal{I}_2 = 0$, so $\mathcal{I}_1 = 0$ or $\mathcal{I}_2 = 0$.

($\Leftarrow$) If $S^{-1}\mathcal{I}_1, S^{-1}\mathcal{I}_2 \triangleleft S^{-1}A$ with $S^{-1}\mathcal{I}_1\mathcal{I}_2 = 0$, then $\mathcal{I}_1 = 0$ or $\mathcal{I}_2 = 0$, implying $S^{-1}\mathcal{I}_1 = 0$ or $S^{-1}\mathcal{I}_2 = 0$, so $\mathcal{I}_1 = 0$ or $\mathcal{I}_2 = 0$. □

Corollary 1.1.18. *Suppose A is a prime algebra, and S is a submonoid of* $\mathrm{Cent}(A)$, *and $A \subseteq B \subseteq S^{-1}A$. Then B is prime.*

Proof. $S^{-1}A$ is prime, but $S^{-1}A = S^{-1}B$, implying B is prime. □

1.1.6 Chain conditions

A partially ordered set $\mathcal{S}$ is said to satisfy the ACC (**ascending chain condition**) if every infinite ascending chain

$$S_1 \subseteq S_2 \subseteq \dots$$

stabilizes in the sense that there is some k such that $S_i = S_{i+1}$ for all $i \geq k$. In particular, a commutative ring is **Noetherian** if it satisfies the ACC on ideals. The Hilbert Basis Theorem implies that every commutative affine algebra over a Noetherian ring (in particular, over a field) is Noetherian, thereby elevating the Noetherian theory to a central role in algebra and geometry.

Recall three noncommutative generalizations of "Noetherian," in increasing strength:

Definition 1.1.19. *(i) A ring R is* ***weakly Noetherian*** *if it satisfies the ACC on two-sided ideals. (Equivalently, R is a Noetherian $R \otimes R^{\mathrm{op}}$-module.)*

(ii) A ring R is ***left Noetherian*** *if it satisfies the ACC (ascending chain condition) on left ideals.*

(iii) R is ***Noetherian*** *if it is left and right Noetherian, i.e., satisfies the ACC on left ideals and also satisfies the ACC on right ideals.*

Any finite module over a left Noetherian ring is left Noetherian. Any weakly Noetherian ring obviously has a unique maximal nilpotent ideal, which is the intersection of its prime ideals.

Remark 1.1.20. *We recall the important technique of "Noetherian induction": To prove a theorem about weakly Noetherian rings, we suppose on the contrary that we have a counterexample R, and take an ideal $\mathcal{I}$ maximal with respect to the theorem failing for $R/\mathcal{I}$. Replacing R by $R/\mathcal{I}$, we may assume that R is a counterexample, but $R/\mathcal{J}$ is not a counterexample for every $0 \neq \mathcal{J} \triangleleft R$.*

Noetherian induction can also be used for proving theorems about Noetherian modules, in an analogous fashion.

We can pass the Noetherian property to the center by means of the following result.

Proposition 1.1.21 (Artin-Tate Lemma)**.** *Suppose that A is an affine C-algebra, finite over its center Z. If C is Noetherian, then Z is affine, and thus is Noetherian.*

Proof. For the reader's convenience, we reproduce the easy proof given in [Ro88b, Proposition 6.2.5]. Namely, write $A = C\{a_1, \dots, a_t\}$ and $A = \sum_{\ell=1}^{q} Zb_\ell$. Writing $b_i b_j = \sum_{m=1}^{q} z_{ijm} b_m$ for $z_{ijm} \in Z$, and $a_k = \sum_{\ell=1}^{q} z'_{k\ell} b_\ell$, we let

$$Z_1 = C[z_{ijm}, z'_{k\ell} : 1 \leq i, j, \ell, m \leq t,\ 1 \leq k \leq q],$$

which is affine over C, and thus Noetherian. But $\sum_{\ell=1}^{q} Z_1 b_\ell$ is an algebra over Z_1 containing $C\{a_1, \dots, a_t\} = A$, and thus is a Noetherian Z_1-module, proving that its submodule Z is finite over Z_1, and thus is affine as an algebra. □

A related result due to Eakin-Formanek (Exercise 3) is that if a ring is Noetherian and finite over its center Z, then Z is Noetherian.

Definition 1.1.22. *Suppose that some set S acts on an algebra A from the right. For any subset $T \subset S$ one defines the* ***left annihilator***

$$\operatorname{Ann} T = \{a \in A : aT = 0\},$$

a left ideal of A. ***ACC(Left annihilators)*** *denotes the ACC on $\{$left annihilators$\}$. When $\operatorname{Ann} T$ is a 2-sided ideal of A, we call $\operatorname{Ann} T$ an* ***annihilator ideal****. In this case, $\operatorname{Ann} T$ is the left annihilator of a 2-sided ideal, namely of its right annihilator.*

Lemma 1.1.23 (Fitting-type Lemma)**.** *Given a module M over a commutative ring Z, with $z \in Z$ and $k \in \mathbb{N}$, let $N = \{a \in M : z^k a = 0\}$. If N satisfies the property that $z^{2k} a = 0$ implies $a \in N$, then $z^k M \cap N = 0$.*

Proof. If $z^k a \in N$, then $z^{2k} a = 0$, implying $a \in N$, so $z^k a = 0$. □

1.1.7 Subdirect products and irreducible algebras

Definition 1.1.24. *A is a* ***subdirect product*** *of the algebras $\{A_i : i \in I\}$ if there is an injection $\psi : A \to \prod A_i$ for which $\pi_j \psi : A \to A_j$ is onto for each $j \in I$, where π_j denotes the natural projection $\prod A_i \to A_j$.*

In this case, $\cap \ker \pi_j = 0$. Conversely, if $A_i = A/\mathcal{I}_i$ for each $i \in I$ and $\cap_i \mathcal{I}_i = 0$, then A is a subdirect product of the A_i in the obvious way.

The following concept often fits in with Noetherian.

Definition 1.1.25. *An algebra A is* ***irreducible*** *if the intersection of two nonzero ideals is always nonzero.*

By induction, the intersection of finitely many nonzero ideals of an irreducible algebra is always nonzero.

Lemma 1.1.26. *Any weakly Noetherian algebra A is a finite subdirect product of irreducible algebras.*

Proof. The usual Noetherian induction argument. Otherwise, take a counterexample A, and take $\mathcal{I} \triangleleft A$ maximal with respect to $A/\mathcal{I}$ not being a counterexample. Passing to $A/\mathcal{I}$, we may assume that A is a counterexample to the lemma, but $A/\mathcal{I}$ is not a counterexample, for all $0 \neq \mathcal{I} \triangleleft A$.

In particular, A itself is reducible, so has nonzero ideals $\mathcal{I}_1, \mathcal{I}_2$ such that $\mathcal{I}_1 \cap \mathcal{I}_2 = 0$. But by hypothesis $A/\mathcal{I}_1$ is a finite subdirect product of irreducible algebras $A/\mathcal{I}_{1,1}, \dots, A/\mathcal{I}_{1,t}$ and $A/\mathcal{I}_2$ is a finite subdirect product of irreducible algebras $A/\mathcal{I}_{2,1}, \dots, A/\mathcal{I}_{2,u}$, implying A is a subdirect product of $A/\mathcal{I}_{1,1}, \dots, A/\mathcal{I}_{1,t}, A/\mathcal{I}_{2,1}, \dots, A/\mathcal{I}_{2,u}$. □

1.1.7.1 ACC for classes of ideals

This subsection contains basic material about chain conditions on classes of ideals of a given ring R, with an eye on applications to ideals of noncommutative algebras. The reason we include it is that Kemer's solution of Specht's problem, given in Chapters 6 and 7, has thrust open the door to a new application of this material, and we might as well present it here to have it available for other purposes (such as for the structure of affine PI-algebras). We skip some proofs, when they are formal and in direct analogy to the well-known proofs in commutative algebra. Throughout, we fix a monoid $\mathcal{S}$ of ideals of R, satisfying the following properties:

(i) The intersection of members of $\mathcal{S}$ is in $\mathcal{S}$;

(ii) If $\mathcal{I}, \mathcal{J} \in \mathcal{S}$, then $\mathcal{I} + \mathcal{J} \in \mathcal{S}$.

Definition 1.1.27. *Given $S \subseteq \mathcal{S}$, the member of $\mathcal{S}$* ***generated*** *by S is defined as $\cap\{\mathcal{I} \in \mathcal{S} : S \subseteq \mathcal{I}\}$. $\mathcal{I} \in \mathcal{S}$ is* ***finitely generated*** *in $\mathcal{S}$ if $\mathcal{I}$ is generated by some finite set S.*

(This generalizes the notion of a finite module.)

Remark 1.1.28. *The following are equivalent:*

(i) $\mathcal{S}$ *satisfies the ACC.*

(ii) Every member of $\mathcal{S}$ *is finitely generated in* $\mathcal{S}$.

(iii) Every subset of $\mathcal{S}$ *has a maximal member.*

Definition 1.1.29. *A member* P *of* $\mathcal{S}$ *is* ***prime*** *if, for all* $\mathcal{I}, \mathcal{J} \in \mathcal{S}$ *not contained in* P*, we have* $\mathcal{I}\mathcal{J} \not\subseteq P$*. For any* $S \subseteq A$*, a prime* P *of* $\mathcal{S}$ *containing* S *is* ***minimal****prime over* S *if* P *does not properly contain a prime of* $\mathcal{S}$ *containing* S.

Lemma 1.1.30. *Every prime of* $\mathcal{S}$ *containing* S *contains a minimal prime containing* S.

Proof. In view of Zorn's lemma, we need to show that for any chain $\mathcal{P}$ of primes, that $P = \cap\{P \in \mathcal{P}\}$ is also prime. But this is standard: If $\mathcal{I}\mathcal{J} \subseteq P$ with $\mathcal{I} \not\subseteq P$, then $\mathcal{I} \not\subseteq P_{j_0}$ for some P_{j_0} in $\mathcal{P}$, implying $\mathcal{J} \subseteq P_j$ for each $P_j \subset P_{j_0}$ in $\mathcal{P}$, implying $\mathcal{J} \subseteq P$. □

Theorem 1.1.31. *Suppose that* $\mathcal{S}$ *satisfies the ACC. Then for any* $\mathcal{I} \in \mathcal{S}$*, there are only finitely many primes* $P_1, \dots, P_n$ *in* $\mathcal{S}$ *minimal over* $\mathcal{I}$*, and some finite product of the* P_i *is contained in* $\mathcal{I}$.

Proof. By Noetherian induction. Otherwise, there is $\mathcal{I} \in \mathcal{S}$ maximal with respect to being a counterexample. Certainly $\mathcal{I}$ is not itself prime, so take $\mathcal{J}_1, \mathcal{J}_2 \supset I$ in $\mathcal{S}$ such that $\mathcal{J}_1\mathcal{J}_2 \subseteq \mathcal{I}$. (We can replace $\mathcal{J}_i$ by $\mathcal{J}_i + \mathcal{I}$ if necessary.) By hypothesis, the conclusion of the theorem holds for $\mathcal{J}_1$ and $\mathcal{J}_2$, i.e., there are primes P_{ik} minimal over $\mathcal{J}_k$ with some finite product contained in $\mathcal{J}_k$. But then the product together is contained in $\mathcal{J}_1\mathcal{J}_2$ and thus, in $\mathcal{I}$. Any prime P containing $\mathcal{J}_1\mathcal{J}_2$ contains some minimal prime over $\mathcal{J}_1\mathcal{J}_2$, which in turn must contain some P_{ik} and thus must equal P_{ik}. □

Definition 1.1.32. *The* ***radical*** $\sqrt{S}$ *of* $S \subseteq A$ *is the intersection of all primes of* $\mathcal{S}$ *containing* S.

The foregoing results did not involve associativity of the multiplication of $\mathcal{S}$, although the subsequent ones do, in order that $P_1 \cdots P_n$ is well-defined. (The subtleties of the nonassociative case are treated in Volume II.)

Corollary 1.1.33. *Suppose that* $\mathcal{S}$ *satisfies the ACC. If* $\mathcal{I} \in \mathcal{S}$*, then* $\sqrt{\mathcal{I}}$ *is a finite intersection of primes of* $\mathcal{S}$*, each minimal over* $\sqrt{\mathcal{I}}$.

Corollary 1.1.34. *If* $\mathcal{S}$ *satisfies the ACC, then* $\sqrt{\mathcal{I}}^t \subseteq \mathcal{I}$ *for some* t.

Proof. Write $\sqrt{\mathcal{I}} = P_1 \cap \cdots \cap P_n$, and then note that some product of t of the P_i are in $\mathcal{I}$, implying

$$(\sqrt{\mathcal{I}})^t \subseteq P_1 \cdots P_t \subseteq \sqrt{\mathcal{I}}.$$

□

Corollary 1.1.35. *Suppose that $\mathcal{S}$ satisfies the ACC, and $0 \in \mathcal{S}$. If $\mathcal{I} \in \mathcal{S}$ is contained in every prime, then $\mathcal{I}$ is nilpotent.*

Proof. $\mathcal{I} \subseteq \sqrt{0}$, so apply the previous corollary. □

Corollary 1.1.36. *Any nil subset N of a commutative (associative) Noetherian ring C is nilpotent.*

Proof. N is contained in every prime ideal P, since C/P is an integral domain. □

(This fails for noncommutative rings, even for $\{e_{12}, e_{21}\} \subset M_2(F)$.)

1.2 Noncommutative Polynomials and Identities

In order to get to our subject, we need the noncommutative analog of polynomials.

1.2.1 The free associative algebra

Recall that the free (associative) monoid $\mathcal{M}\{X\}$ in $X = \{x_i : i \in I\}$ is the monoid of **words** $\{x_{i_1} x_{i_2} \cdots x_{i_t} : t \in \mathbb{N}\}$ permitting duplication of subscripts, and whose unit element is the blank word $\emptyset$; the monoid operation is given in terms of juxtaposition of words.

$C\{X\}$, often denoted $C\langle X\rangle$ in the literature, denotes the free associative algebra (with 1) in the set $X = \{x_i : i \in I\}$ of noncommuting indeterminates. (Usually $I = \mathbb{N}$, but often I is taken to be finite.) In other words, $C\{X\}$ is the monoid algebra of $\mathcal{M}\{X\}$. The elements of $C\{X\}$ are called **polynomials**. $C\{X\}$ is free as a C-module, with base consisting of $\mathcal{M}\{X\}$, the set of words; thus, any $f \in C\{X\}$ is written uniquely as $\sum c_j h_j$ where $h_j \in \mathcal{M}(X)$. We call these $c_j h_j$ the **monomials** of f.

Given $f \in C\{X\}$ we write $f(x_1, \ldots, x_m)$ to denote that $x_1, \ldots, x_m$ are the only indeterminates occurring in f. Sometimes we write $f(\vec{x})$ for short. Later, when the notation becomes more cumbersome, we shall have occasion to use Y (and at times Z) to denote extra sets of indeterminates that do not enter the computations as actively as the x_i. In this case we write $C\{X, Y\}$ or $C\{X, Y, Z\}$ in place of $C\{X\}$, and we write $f(\vec{x}, \vec{y})$ or $f(\vec{x}, \vec{y}, \vec{z})$ accordingly.

The main feature of $C\{X\}$ is the following.

Remark 1.2.1. *Given a C-algebra A and elements $\{a_i : i \in I\} \subseteq A$, there is a unique algebra homomorphism $\phi : C\{X\} \to A$, called the* ***substitution homomorphism****, such that $\phi(x_i) = a_i$, $\forall i \in I$. Indeed, one defines*

$$\phi(x_{i_1} \cdots x_{i_m}) = a_{i_1} \cdots a_{i_m}$$

and extends this linearly to all of $C\{X\}$.

The **evaluation** $f(a_1, \dots, a_m)$ denotes the image of f under the homomorphism of Remark 1.2.1. We also say that f **specializes** to $f(a_1, \dots, a_m)$, and $a_1, \dots, a_m$ are **substitutions** in f.

1.2.2 Polynomial identities

We write $f(A)$ for the set of evaluations $\{f(a_1, \dots, a_m) : a_i \in A\}$.

Definition 1.2.2. *An element $f \in C\{X\}$ is an* ***identity*** *of a C-algebra A if $f(A) = 0$, i.e., $f \in \ker \phi$ for every homomorphism $\phi : C\{X\} \to A$.*

Identities pass to related algebras as follows.

Remark 1.2.3. *If f is an identity of an algebra A, then f is an identity of any homomorphic image of A and also of any subalgebra of A. Furthermore if f is an identity of each C-algebra A_i, $i \in I$, then f is an identity of $\prod_{i \in I} A_i$.*

Remark 1.2.3 provides an alternate approach to identities, cf. §1.7 below.

Definition 1.2.4. *For a monomial h we define $\deg_i h$ to be the number of occurrences of x_i in h, and the* ***degree*** *$\deg h = \sum_i \deg_i h$; for a polynomial f, we define $\deg f$ to be the maximum degree of the monomials of f. For example $\deg(x_1x_2 + x_3x_4) = 2$.*

One needs some way of excluding the identity px_1, which only says that A has characteristic p. Toward this end, we formulate the main definition of this book.

Definition 1.2.5. *An identity f is a* ***PI*** *(polynomial identity) for A if at least one of its coefficients is 1. An algebra A is a* ***PI-algebra*** *of PI-****degree*** *d if A satisfies a PI of degree d.*

This definition might seem restrictive, but in fact is enough to encompass the entire PI-theory, cf. [Am71]. Since PI-algebras are the subject of our study, let us address a subtle distinction in terminology. A ring R is a PI-**ring** when it is a PI-algebra for $C = \mathbb{Z}$. Although most of the general structure theory holds for PI-rings in general, our focus in this book is usually on a particular base ring C, sometimes a field which we denote as F rather than C; often we require $\operatorname{char}(F) = 0$, for reasons to be discussed shortly.

Definition 1.2.6. *We write $\operatorname{id}(A)$ for the set of identities of A.*

Here is a notion closely related to PI.

Definition 1.2.7 (Central polynomials). *A polynomial $f(x_1, \dots, x_n)$ is A-* ***central*** *if $0 \neq f(A) \subseteq \mathrm{Cent}(A)$.*

In other words, $f(x_1, \dots, x_n)$ is A-central iff $[y, f]$ (but not f) is in $\mathrm{id}(A)$.

The most basic examples of PI-algebras are the matrix algebra $M_n(C)$ for arbitrary n, f.d. algebras over a field, and the Grassmann algebra G, cf. Definition 1.3.26. Since these examples require a bit more theory, we first whet the reader's appetite with some easier examples.

Example 1.2.8.

(i) The polynomial x is central for any commutative algebra.

(ii) Let $\mathrm{UT}(n)$ denote the algebra of upper triangular matrices over a given base ring C. Any product of n strictly upper triangular $n \times n$ matrices is 0. Since $[a, b]$ is strictly upper triangular, for any upper triangular matrices a, b, we conclude that the algebra $\mathrm{UT}(n)$ satisfies the identity

$$[x_1, x_2][x_3, x_4] \cdots [x_{2n-1}, x_{2n}].$$

(iii) (Wagner's identity) If F is a field, then $M_2(F)$ satisfies the identity $[[x, y]^2, z]$ or, equivalently, the central polynomial $[x, y]^2$, cf. Exercise 19.

(iv) Fermat's Little Theorem translates to the fact that any field F of n elements satisfies the identity $x^n - x$. (See Exercise 27 for a generalization.)

(v) Any Boolean algebra satisfies the identity $x^2 - x$.

When dealing with arbitrary PIs it is convenient to work with certain kinds of polynomials. We say that a polynomial $f(x_1, \dots, x_m)$ is **homogeneous** in x_i if x_i has the same degree in each monomial of f. We say that f is **homogeneous** if f is homogeneous in every indeterminate. (Sometimes this is called "completely homogeneous" or "multi-homogeneous" in the literature.) In this case, if x_i has degree d_i in f_i for $1 \le i \le m$, we say that f has **multi-degree** $(d_1, \dots, d_m)$, where $\deg f = d_1 + \cdots + d_m$. Here is a very important special case.

Definition 1.2.9. *A monomial h is* ***linear*** *in x_i if $\deg_i h = 1$. A polynomial f is* ***linear*** *in x_i if each monomial of f is linear in x_i; f is t-**linear** if f is linear in each of $x_1, \dots, x_t$.*

A polynomial $f(x_1, \dots, x_m)$ is ***multilinear*** *if f is m-linear. In other words, each indeterminate of f appears with degree exactly 1 in each monomial of f.*

Thus, $x_1x_2 - x_2x_1$ is multilinear. However, $x_1x_2x_3 - x_2x_1$ is not multilinear, since x_3 does not appear in the second monomial.

Given a multilinear polynomial $f(x_1, \dots, x_m)$, we pick any nonzero monomial h, and renaming the indeterminates appropriately, we may assume that $h = cx_1x_2 \dots x_m$ for some $c \in C$. Thus, the general form for a multilinear polynomial is

$$f(x_1, \dots, x_m) = c_1 x_1 x_2 \cdots x_m + \sum_{1 \neq \sigma \in S_m} c_\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)}. \tag{1.2}$$

Furthermore, if C is a field, then we can divide by c_1 and assume that $c_1 = 1$. The main reason we focus on multilinear identities is because of Proposition 1.2.18 below. However, the linearity property already is quite useful:

Remark 1.2.10. *If f is linear in x_i, then*

$$f(a_1, \dots, \sum_j c_j a_{ij}, \dots, a_m) = \sum_j c_j f(a_1, \dots, a_{ij}, \dots, a_m)$$

for all $c_j \in C$, $a_{ij} \in A$.

Lemma 1.2.11. *Suppose A is spanned over C by a set B. Then a multilinear polynomial f is an identity of A iff f vanishes on all substitutions to elements of B; f is A-central iff every substitution of f on B is in* Cent(A) *but some substitution on B is nonzero.*

Proof.

$$f\left(\sum_{i_1} c_{i_1} b_{i_1}, \dots, \sum_{i_m} c_{i_m} b_{i_m}\right) = \sum_{i_1,\dots,i_m} c_{i_1} \cdots c_{i_m} f(b_{i_1}, \dots, b_{i_m}),$$

in view of Remark 1.2.10. □

1.2.3 Multilinearization

These observations raise the question of how to go back and forth from arbitrary identities (or central polynomials) to multilinear ones. The answer is in the process of **multilinearization**, also called **polarization**. This will be tied to group actions in §3.5 (also cf. Exercise 6), but can be described briefly as follows:

Definition 1.2.12 (Multilinearization). *Suppose the polynomial $f(x_1, \dots, x_m)$ has degree $n_i > 1$ in x_i. We focus on one of the indeterminates, x_i, and define the **partial linearization***

$$\begin{aligned} &\Delta_i f(x_1, \dots, x_{i-1}, x_i, x_i', x_{i+1}, \dots, x_m) \\ &\quad = f(\dots, x_i + x_i', \dots) - f(\dots, x_i, \dots) - f(\dots, x_i', \dots) \end{aligned} \tag{1.3}$$

where x_i' is a new indeterminate. Clearly $\Delta_i f$ remains an identity for A when $f \in \mathrm{id}(A)$, but all monomials of degree n_i in x_i cancel out in $\Delta_i f$. The remaining monomials have x_i' replacing x_i in some (but not all) instances, and thus have degree $< n_i$ in x_i, the maximum degree among them being $n_i - 1$.

Remark 1.2.13. *Since this procedure is so important, let us rename the indeterminates more conveniently, writing x_1 for x_i and y_j for the other indeterminates.*

(i) Now our polynomial is $f(x_1; \vec{y})$ and our partial linearization may be written as

$$\begin{aligned}\Delta_1 f(x_1, x_2; \vec{y}) = \\ f(x_1 + x_2; \vec{y}) - f(x_1; \vec{y}) - f(x_2; \vec{y}),\end{aligned} \tag{1.4}$$

where x_2 is the new indeterminate.

(ii) Before we get started, we must cope with the situation in which x_1 does not appear in each monomial. For example, if we want to multilinearize $f = x_1 y + y$ in x_1, then the only way would be to apply Δ_1, but

$$\Delta_1 f = (x_1 + x_2)y + y - (x_1 y + y) - (x_2 y + y) = -y,$$

and we have lost x_1 altogether. This glitch could complicate subsequent proofs.

Fortunately, we can handle this situation by defining $g = f(0; \vec{y})$, the sum of those monomials in which x_1 does not appear. If $f \in \mathrm{id}(A)$, then also $f - g \in \mathrm{id}(A)$, so we can replace f by $f - g$ and thereby assume that any indeterminate appearing in f appears in each monomial of f, as desired. We call such a polynomial ***blended****.*

(iii) Let $n = \deg_1 f$. Iterating the linearization procedure $n - 1$ times (each time introducing a new indeterminate x_i) yields an n-linear polynomial $\bar{f}(x_1, \dots, x_n; \vec{y})$ which preserves only those monomials h originally of degree n in x_1. For each such monomial h in f we now have $n!$ monomials in $\bar{f}$ (according to the order in which $x_1, \dots, x_n$ appears), each of which specializes back to h when we substitute x_1 for each x_i. Thus, when f is homogeneous in x_1, we have

$$\bar{f}(x_1, \dots, x_1; \vec{y}) = n! f\,. \tag{1.5}$$

We call $\bar{f}$ the ***linearization*** *of f in x_1. In characteristic 0 this is about all we need, since $n!$ is invertible and we have recovered f from $\bar{f}$. This often makes the characteristic 0 PI-theory easier than the general theory.*

(iv) Repeating the linearization process for each indeterminate appearing in f yields a multilinear polynomial, called the ***multilinearization****, or* ***total multilinearization****, of f.*