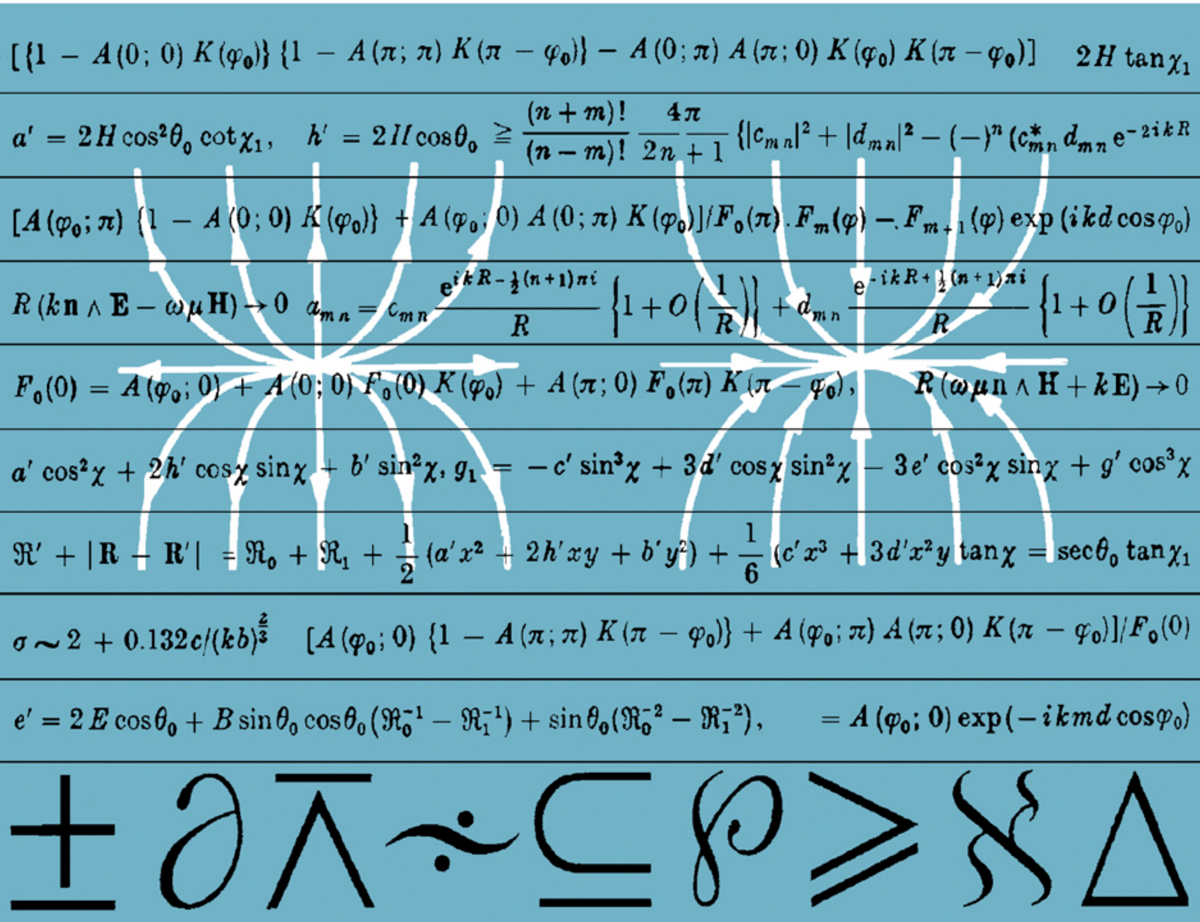
Abelian Groups

L. Fuchs



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by

L. FUCHS

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PREFACE

In recent researches on group theory the theory of abelian (i. e. commutative) groups[1] has come to play an increasingly important rôle. It is sufficient to cite, for instance, the theorems which determine completely or approach better and better continually growing classes of abelian groups or to refer to newly discovered ideas and methods the exploitation of which is going on with considerable success.

The development of abelian group theory can be divided into three periods. Already at the first stage of dealing with (finite) groups it was recognized that the assumption of commutativity considerably simplifies the group structure and implies (in our present terminology) that the group is a direct sum of finite cyclic groups — a fact which was implicitly contained in the researches of C. F. GAUSS on decompositions of quadratic forms (1801) and was proved explicitly by L. KRONECKER (1870). That the orders of the cyclic components may be taken to be prime powers seems not to have been discovered as late as 1878 when it was proved by G. FROBENIUS and L. STICKELBERGER. A long time has passed again without any essential result in the field of abelian groups. It was already in the early years of our century when attention was directed to infinite (discrete) groups and in the abelian case the structure of finitely generated groups has been determined. Actually, the "Habilitationsschrift" of F. LEVI, published in 1917, was the first work which contained deeper results on infinite abelian groups and gave wider aspects to a general theory. The second period begins with H. PRÜFER whose significant papers (1921—24) have laid the foundations of modern abelian group theory and led to its slow but direct development. The most outstanding result is the structure theory of countable torsion groups developed by H. PRÜFER (1923), H. ULM (1933) and L. ZIPPIN (1935). In the theory of torsion free abelian groups a considerable approach towards the structure problem (especially in the case of finite rank) has been made by R. BAER (1937), A. KUROŠ (1937) and D. DERRY (1937), but up to now no satisfactory theory is known except for rather special cases. Mixed groups and various important questions were discussed in a series of penetrating papers by R. BAER. The third period may be regarded to open with the appearance of two papers of L. KULIKOV (1941, 1945). The new ideas contained in them have induced no doubt a new epoch in the theory of primary abelian groups of arbitrary power, although no complete structural characterization is yet

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[1] Today commutative groups are generally called abelian, named after N. H. ABEL, the famous Norwegian mathematician, who investigated a class of solvable algebraic equations related to commutative groups.

known beyond the countable power. The investigations published within the last decade comprise a large variety of different problems on abelian groups, leading to interesting new results, to several simplifications in the subject matter, and deepening our knowledge of abelian groups in a notable manner (T. SZELE, L. KULIKOV etc.). Recently, many of these results have been extended to various classes of operator groups.

In writing this monograph the author's aim was to give a fairly complete and detailed account of the present status of the theory with special emphasis on results concerning structure problems. Of course, if a book intends to encompass practically everything of importance and of interest that is known about its central theme, then either the size of the book becomes so immense that a young reader gets no real idea of what is essential in the theory and what is not, what is worth while or necessary to keep in mind and what is not; or else the author has to adhere to an „Ergebnisse-style" not suitable for beginners. In order to follow a form of exposition that will make this monograph useful as an up-to-date source book for young research workers and as a text-book for students who intend to familiarize themselves with this branch of algebra, the author had no choice but to elaborate in the text the most essential parts of the material in detail and to leave the remaining parts to the reader as exercises or omit them completely whenever they were far from the theme of the text. Thus the text is accompanied by more than five hundred exercises of varying degrees of difficulty, with and without hints.[2] Some of the exercises illuminate the theorems of the text by providing alternative developments, proofs or counterexamples of generalizations, others contain simple illustrative examples, corollaries and variant formulations not included in the text. I hope that an adequately prepared reader who takes the trouble to work through many of these exercises will hardly fail to acquire a feeling for the subject. The 86 unsolved problems mentioned at the end of the chapters will perhaps influence one or another of the more advanced students to begin scientific researches in this field.

The author's intention was to present the ideas and concepts and to organize the material in such a manner that it be possible for a student with only a moderate knowledge in modern algebra, especially in group theory, to have no serious difficulty in comprehending the subject matter. Actually, a certain mathematical maturity is required of the reader. The spread of ideas of set theory in the theory of abelian groups has been so fruitful that today many results carry its mark, in particular, some theorems need in their formulation transfinite numbers, so that in our discussions the transfinite methods are indispensable and some previous acquaintance with them must be taken for granted.

To assist the reader, the first chapter is devoted to collecting the main facts on which the sequel is based; concepts such as factor group or homomorphism are defined but not explained in detail. The proper material begins with Chapters II—V which are concerned with direct sums of cyclic groups, divisible groups, direct summands and pure subgroups, KULIKOV's basic sub-

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[2] Difficult exercises are marked by an asterisk.

groups; they constitute what may be called the rudiments of the theory. Chapters VI—VIII form the focal point of the whole development: they discuss the structure theory of the three main classes of abelian groups, namely, the primary, the torsion free and the mixed groups. The subsequent chapters are in the main independent of each other and contain applications of the theory or are devoted to specific topics of importance, such as homomorphism groups and endomorphism rings (without entering into proper ring-theoretical problems), the Schreier extension theory with a discussion of the group of extensions, the structure of the tensor product, the theory of the additive group of rings and the multiplicative group of fields, BAER's theory of the lattice of subgroups, HAJÓS' result on the factorization of finite abelian groups with generalizations to infinite groups etc.

The theory of abelian groups is universally known as a branch of algebra that has succeeded in obtaining several satisfactory structure theorems. As this is a rather rare phenomenon in algebra, a thorough study of abelian groups may be recommanded to all those who should like to get to know what is called structure theory. To such a reader I may advice to study the following sections: §§ 1—12, 15—25, 29—37, 42—44, 50 (and possibly one or another of Chapters IX—XVI according to his taste).

Experts perhaps may not agree with me in having chosen the most essential material and the clearest or most direct method of proof of some of the theorems discussed here. What is essential: the standard results and methods are not missing, and the others are, in the author's opinion, largely a matter of taste. However, some words must be inserted here about what is not but might have been incorporated in this book: the theory of modules. Modules will be dealt with if necessary and they occur even in the course of the exercises (these are marked by •), nevertheless a basically module-theoretic treatment would imply either almost trivial generalizations and at the same time unnecessary complications in the discussions, or deeper extensions together with hard problems which are of ring-theoretic rather than group-theoretic character. The theory of modules could be the theme of a separate monograph. Although topological methods have not been excluded from the discussions, the algebraic theory of topologized groups is omitted, and so are the partially ordered abelian groups and the representation theory of abelian groups which are parts of other general theories.

A new book must prove its existence in competition with books published formerly on the same subject, that is, in our case, with two excellent works: Part II of KUROŠ' *Group theory* (1953) and KAPLANSKY's *Infinite abelian groups* (1954). The present work covers entirely the material of KUROŠ' and about three quarters of KAPLANSKY's book, discusses the subject matter in greater detail, includes a number of themes not touched in these former books and collects much material which has hitherto been available only in the original papers. I hope that my book will not prove useless even in this heavy competition.

I feel bound to remember here my late colleague and beloved friend Prof. T. SZELE. It was mostly his enthusiastic personality and fruitful ideas that influenced me to turn my attention to problems in abelian group theory.

He had the intention to write a monograph on abelian groups, but his early death in 1955 prevented him in the fulfilment of this plan. I express here my gratitude to him for a lot of suggestions and advices made during my former researches.

I wish to express my deep gratitude of the valuable assistence rendered me by Prof. R. BAER and Dr. A. KERTÉSZ who have been very kind in undertaking to criticize the manuscript. A number of points of detail have been improved and several inaccuracies corrected as a result of their suggestions. Following the advice of Prof. BAER, the manuscript has been completed by some new sections.

Last but not least my sincere thanks are due to the Hungarian Academy of Sciences for the publication of this book, and to the publisher and the press for having given so pleasant a form to this book.

A work can hardly be hoped to be free of errors and misprints. The author will be grateful to any of his readers who will give him notice of necessary corrections or possible improvements.

L. F.

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NOTE. Since 1958, the publication of this book, great progress has been made in the theory of abelian groups. As the methods of homological algebra have been extensively applied to abelian groups, new aspects of the theory have come to light. The numerous results on the algebraic structure of Hom, Ext, $\otimes$ and Tor, the surprising phenomena discovered in torsion free groups, and the recent achievements in endomorphism rings and automorphism groups etc. show that time has come to revise the material of this book. This cannot be done, however, without fundamental changes in the method of discussion, and therefore, the author has decided to write a new, up-to-date edition of this book in which both the structural aspects and the homological methods will be emphasized. Since the new edition will not be out before 1968, this reprint of the first edition is published in order to meet the urgent demands.

April 1966.

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TABLE OF NOTATIONS

$A, B, \ldots, G, H, \ldots$	groups (or their subsets)
$a, b, c, g, h, u, v, w, x, y, z$	elements of groups
$i, j, k, l, m, n, p, q, r, s, t$	rational integers (p a prime)
$\mathfrak{Z}$	ring of integers
$\mathfrak{P}$	ring of p-adic integers
ω	first infinite ordinal
$\aleph_\alpha$ $(\alpha = 0, 1, \ldots)$	αth infinite cardinal ($\aleph =$ continuum)
$[a, \ldots]$	set of elements $a, \ldots$
$\{a, \ldots\}$	subgroup generated by $a, \ldots$
$\{a, \ldots\}_*$	pure subgroup generated by $a, \ldots$
nG	set of all ng with $g \in G$
$G[n]$	set of all $g \in G$ with $ng = 0$
$\mathfrak{R}^+$	additive group of the ring $\mathfrak{R}$
$\mathfrak{F}^\times$	multiplicative group of the field $\mathfrak{F}$
$O(a)$	order of a
$E(a)$	exponent of a
$H(a)$ $(H_p(a))$	height of a (at the prime p)
$T(a)$	type of a
$r(G), r_0(G), r_p(G), r^*(G)$	ranks of G (fin $r(G) =$ final rank of G)
$\mathfrak{C}(m)$	cyclic group of order $m = 1, 2, \ldots, \infty$
$\mathfrak{C}(p^\infty)$	quasicyclic group
$\mathfrak{C}$	group of all roots of unity
$\mathfrak{R}$	full rational group
$\mathfrak{J}$	group of p-adic integers
$\mathfrak{F}(\mathfrak{m})$ $(\mathfrak{F}_p(\mathfrak{m}))$	free group of rank $\mathfrak{m}$ (free p-group)
$G(\mathfrak{a}), G^*(\mathfrak{a}), G^{**}(\mathfrak{a})$	certain subgroups of G, see p. 147
$A(G)$	automorphism group of G
$\mathfrak{E}(G)$	endomorphism ring of G
$\mathbf{L}(G)$	subgroup lattice of G
$A \subseteq B$ $(A \subset B)$	A is a (proper) subset of B
$A \cap B, A \cup B$	intersection, union of the sets A and B
$\|S\|$	power of S
$[G : H]$	index of H in G
$g(G : H)$	minimal number of generators of G mod H
$G \cong H$	G and H are isomorphic
$G \sim H$	H is a homomorphic image of G
$G + H, \sum G_\lambda \left(\sum^* G_\lambda\right)$	direct sum (complete direct sum)
$\sum_{\mathfrak{m}} G$	direct sum of $\mathfrak{m}$ copies of G
$G (+) H$	a subdirect sum of G and H
$G \otimes H$	tensor product of G and H
$\mathrm{Hom}\,(U, V)$	group of homomorphisms of U into V
$\mathrm{End}\, G$	group of endomorphisms of G
$\mathrm{Ext}\,(L, K)$	group of extensions of K by L
$\mathrm{Mult}\, G$	group of multiplications on G

CHAPTER I

BASIC CONCEPTS. THE MOST IMPORTANT GROUPS

This chapter is of introductory character. Its aim is to lay down the terminologies used throughout this book and to collect the basic facts on group theory which we use without reference later on. Their proofs will be omitted, since they may be found in any text book on modern algebra or in monographs on group theory in general. The most important types of groups, their main properties as well as linear independence and ranks are discussed here in detail. A few groups will occur with extreme frequency in our discussions and we shall save numerous repetitions by the adoption of certain conventions about their notations which will be introduced here.

§ 1. Notation and terminology

By a *group* we understand always an additively written abelian group, i. e. a non-void set G of elements such that

1. with every pair of elements a, b in G there is associated an element g of G written as $g = a + b$ and called the *sum* of a and b;

2. the *associative law* holds: $(a+b)+c = a+(b+c)$ for all a, b, c in G;

3. the *commutative law* holds: $a+b = b+a$ for all a, b in G;

4. there is an element 0 in G, called *zero*, subject to $a+0 = a$ for every $a \in G$;[1]

5. to each $a \in G$ there is an $x \in G$ with $a+x = 0$; this x is called the *inverse* of a and is designated as $x = -a$.

Note that postulates 1, 4 and 5 may be substituted by the single one:

0. in the equation $a+b=c$ any two of the elements $a, b, c \in G$ determine uniquely the third one.

By the associative law, a sum of several group elements may be written without parantheses, and by the commutative law, the terms of a sum may be permuted. Note that, for the sake of brevity, one costumarily writes $a-b$ instead of $a+(-b)$; $-a-b$ is the inverse of $a+b$; $a+\cdots+a$ (n times) is abbreviated as na and $-a-\cdots-a$ (n times) as $(-n)a$ or $-na$. A sum without terms is 0.

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[1] $\in$ means "is an element of".

By the *order* of a group G we mean the cardinal number $|G|$ of the set of its (different) elements.[2] If the order of G is a finite cardinal, we call G a finite group.

A non-void subset H of G is a *subgroup* if the elements of H form a group under the same rule of combination. If G is finite, by LAGRANGE's theorem, $|H|$ is always a divisor of $|G|$. For a non-void subset H of G, in order to form a subgroup, it is necessary and sufficient that

(a) $a, b \in H$ implies $a+b \in H$, and

(b) $a \in H$ implies $-a \in H$,

or, more simply, that

(c) $a, b \in H$ implies $a-b \in H$.

If H is the subgroup consisting of the zero alone[3] or of all elements of G, then H is a *trivial* subgroup of G. If H is a subgroup of G, but different from G, H is called a *proper* subgroup of G. (In general, we shall write $H \subseteq G$ to indicate that H is contained in G; the sign $\subset$ will denote proper inclusion.)

Let H be a subgroup of G. The set $a+H$ (consisting of all elements $a+h$ with h running over all elements of H) is said to be a *coset* of G modulo H. The cosets of G modulo H have the following properties:

(i) $a+H$ is of the same power as H;

(ii) a and b ($\in G$) belong to the same coset if and only if $a-b \in H$; this fact will also be written as $a \equiv b \pmod{H}$;

(iii) two cosets are either identical or disjoint;

(iv) if $H = H_0, H_1, \ldots, H_\lambda, \ldots$ are all the different cosets of G modulo H (λ may range over an arbitrary index set), then G is the set-theoretic union of the pairwise disjoint sets H_λ.

The cardinal number of the set of the different cosets modulo H is called the *index* of H in G, and denoted as $[G:H]$. This may be finite or infinite. If G is a finite group, then $[G:H] = |G| : |H|$.

The cosets of G modulo H themselves form a group G/H known as the *factor group* (of G with respect to H). In G/H the addition may be defined with the aid of representatives: the sum of the cosets H_λ, H_μ is the coset H_ν containing (moreover, consisting of) the elements of the form $h_\lambda + h_\mu$ with $h_\lambda \in H_\lambda, h_\mu \in H_\mu$. The 0 element of G/H is H itself and the inverse of the coset H_λ of G mod H is the coset $-H_\lambda$ consisting of all elements $-x$ with $x \in H_\lambda$.

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[2] The same notation $|\ldots|$ will be used for the power of any set.

[3] Since the chance of confusion is small, we fail to distinguish between the subgroup consisting of the single element 0 and the element 0 itself, and therefore use again the symbol 0 to denote the zero subgroup.

There is an important and natural one-to-one correspondence between the subgroups of $G^* = G/H$ and those subgroups of G which contain H. The elements of G belonging to the elements ($=$ cosets of G) of a subgroup J^* of G^* form a subgroup J in G such that $G \supseteq J \supseteq H$. Conversely, if J is such that $G \supseteq J \supseteq H$, then those cosets of G modulo H which contain at least one element from J, form a subgroup J^* of G^*. These correspondences are inverse to each other, and thus we may write $J^* = J/H$. We observe: $|J^*| = [J:H]$ and $[G^*:J^*] = [G:J]$.

If H and K are subgroups of G, then the intersection $H \cap K$ of H and K (consisting of the elements common to both H and K) is again a subgroup. More generally, if K_ν is a non-empty set of subgroups of G, then the intersection $K = \bigcap_\nu K_\nu$ is a well-defined subgroup of G.

Let S be any subset in G. If its elements are a_λ with λ ranging over an index set Λ, then we indicate this fact by writing $S = [a_\lambda]_{\lambda \in \Lambda}$. By $\{S\}$ we denote the subgroup *generated by* S, i. e. the intersection of all subgroups of G containing S. In particular, if S consists of the elements a_λ $(\lambda \in \Lambda)$, we also write $\{S\} = \{\ldots, a_\lambda, \ldots\}$. This $\{S\}$ consists exactly of all finite linear combinations of the elements of S, i. e. all sums $n_1 a_1 + \cdots + n_k a_k$ with $a_i \in S$, n_i integers and k arbitrary non-negative integer. $\{S\}$ is of the same power as S unless S is finite in which case $\{S\}$ is either finite or countable. If S is void, we put $\{S\} = 0$. In case $\{S\} = G$, we say S is a *generating system* of G and the elements of S are *generators* of G. If there is a finite generating system, G is said to be a *finitely generated* group.

Let $a \in G$; $\{a\}$ is called the *cyclic* group generated by a. The order of $\{a\}$ is also called the *order* of a, in notation: $O(a)$. $O(a)$ is either a natural integer or the symbol ∞ (more precisely, we should have to write $\aleph_0$).[4] If $O(a) = \infty$, all multiples na of a $(n = 0, \pm 1, \pm 2, \ldots)$ are different and exhaust $\{a\}$, while if $O(a)$ is a natural integer m, then $0, a, \ldots, (m-1)a$ are different and every multiple ka of a is equal to some ra with $r = 0, 1, \ldots, m-1$. In the latter case m is the least natural integer with $ma = 0$. It has the familiar property that $na = 0$ implies the divisibility relation $m | n$.

Groups in which every element has a finite order are called *torsion* or *periodical* groups; those in which the elements except 0 are of infinite order are *torsion free*, while those containing non-zero elements of finite order as well as elements of infinite order are said to be *mixed* groups. By a *primary* group or a *p-group* is meant a group in which the orders of the elements are powers of one and the same prime p. If G is any group containing elements of order p, then p is called a *relevant prime* for G.

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[4] It is also costumary to call a of order 0 if $\{a\}$ is infinite. This is in accordance with the definition of order in groups with operators; cf. § 7.

An important fact is included in

THEOREM 1.1. *The set T of all elements of finite order in a group G is a torsion subgroup of G and the factor group G/T is torsion free.*

Let $a, b \in G$ belong to T, i. e. $na = 0$ and $mb = 0$ for some natural integers n, m. $a - b$ again belongs to T, for $nm(a-b) = 0$, proving the first part of the assertion. To verify the second part, suppose $n(a+T) \subseteq T$ for some natural n. Then $na \in T$, i. e. there is a natural m with $m(na) = 0$. Thus a is of finite order, $a \in T$ and $a + T = T$ is the zero in G/T. — We call T *the maximal torsion subgroup of* G. (Note that if U is a torsion subgroup of G, then $U \subseteq T$, and if V is a subgroup of G with a torsion free factor group G/V, then $V \supseteq T$.)

The next two concepts are defined for p-groups. If a is an element of order p^n, we call n the *exponent* of a, and write $n = E(a)$. Let k be the greatest non-negative integer r for which $p^r x = a$ is solvable for some $x \in G$, with a fixed a. This k is called the *height* $H(a)$ of a. If there is no such k, i. e. if $p^r x = a$ has a solution for every positive integer r, we say a is of infinite height and set $H(a) = \infty$. The zero element is of infinite height.

For any group G and natural integer n, nG will denote the set of the elements of the form nx $(x \in G)$, while $G[n]$ the set of all $y \in G$ with $ny = 0$. Thus $a \in nG$ if and only if $nx = a$ is solvable with respect to x and $b \in G[n]$ if and only if $O(b)$ divides n. In case G is a p-group, $H(a) = n$ is equivalent to the statement: $a \in p^n G$ but $a \notin p^{n+1} G$, while $H(a) = \infty$ means the same thing as $a \in p^n G$ for all natural integers n.

A one-to-one mapping $a \longleftrightarrow a'$ between the elements of two groups G and G' is an *isomorphism* if it preserves the operations in G and G', i. e. if $a \longleftrightarrow a'$, $b \longleftrightarrow b'$ imply $a + b \longleftrightarrow a' + b'$. G and G' are then called *isomorphic*, in notation: $G \cong G'$. As usual in abstract algebra, we make no distinction between isomorphic groups with the sole exception when both are distinct subsets of the same larger group considered. If G and G' coincide, an isomorphism between G and G' is usually said to be an *automorphism* of G.

A mapping $G \rightarrow G'$ associating with each element a of G a unique element a' of G' is a *homomorphism* η of G onto G' if it preserves the operations and every $a' \in G'$ is the image of some $a \in G$. We then write $G \sim G'$. If $G' \subseteq G$, η is an *endomorphism* of G. The set of all $x \in G$ mapped by a homomorphism η upon the zero of G' is the *kernel* $G[\eta]$ of η. It is a subgroup K of G and the (complete) inverse image of an element $a' \in G'$ in G is a coset $a + K$ of G. It follows that η induces an isomorphism $G' \cong G/K$. We call the mapping $a \rightarrow a + K$ the *natural homomorphism* of G onto G/K to distinguish it from other possible homomorphisms.

Of great importance are the *isomorphism theorems* (of E. NOETHER):

(I) $\{H, K\}/K \cong H/(H \cap K)$ for all subgroups H and K of G;

(II) $G/H \cong (G/K)/(H/K)$ if H and K are subgroups of G such that $G \supseteq H \supseteq K$.

We mention in passing the fact, used several times, that if H and K are subgroups of G satisfying $H \supseteq K$, then G/H is a homomorphic image of G/K.

Let η be a homomorphism of the group A into some group B. We then write $A \xrightarrow{\eta} B$, or simply $A \rightarrow B$ provided there is no ambiguity. A sequence of homomorphisms

$$A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_k \qquad (k \geqq 3)$$

is called *exact* if the image of A_{i-1} in A_i under $A_{i-1} \rightarrow A_i$ coincides with the kernel of $A_i \rightarrow A_{i+1}$ $(i = 2, \ldots, k-1)$. Thus $0 \rightarrow A \rightarrow B$ is exact if and only if the mapping $A \rightarrow B$ is an isomorphism of A into B, and $A \rightarrow B \rightarrow 0$ is exact if and only if $A \rightarrow B$ is a homomorphism of A onto B.

Let J be a subgroup of G, sent onto itself by every automorphism of G; then J is a so-called *characteristic subgroup* of G. J is *fully invariant* or *fully characteristic* if it is sent into itself by every endomorphism of G.

For other notations we refer to the table of notations on p. 12.

§ 2. Direct sums

This concept is of utmost importance in the theory of abelian groups. Its importance lies in two facts: 1. if we are able to decompose a group into a direct sum of subgroups, then in most cases we can thereby reduce the study of the given group to the consideration of groups which are in general of a simpler structure; 2. new groups can be constructed as direct sums of known groups.

Let A, B be two subgroups of G such that

(i) $$\{A, B\} = G$$

and

(ii)[5] $$A \cap B = 0.$$

Then we call G the *direct sum* of its subgroups A, B, and write $G = A + B$. (i) tells us that each $g \in G$ may be written in the form $g = a + b$ with $a \in A$, $b \in B$, while (ii) is equivalent to asserting the unicity of this representation. For if $g = a + b = a' + b'$ $(a' \in A, b' \in B)$, then $a - a' = b' - b$ lies in $A \cap B = 0$.

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[5] Since there is no danger of confusion, we shall frequently indicate this fact by saying that the subgroups A and B are *disjoint*.

Conversely, the uniqueness of the representation $g=a+b$ excludes the possibility that $a+0=0+b$ is a non-zero element in the intersection $A\cap B$. — A subgroup A of G is called a *direct summand* of G if there is a subgroup B of G so that $G=A+B$. Then B is a *complementary direct summand* of A in G.

If A_λ $(\lambda\in\Lambda)$ is a set of subgroups of G such that (i) $\{\dots,A_\lambda,\dots\}=G$, i. e. the A_λ together generate G, and (ii) for every $\lambda\in\Lambda$, $A_\lambda\cap\{\dots,A_\mu,\dots\}=0$ whenever μ ranges over all indices different from λ, then G is the *direct sum* of its subgroups A_λ,

$$G=\sum_{\lambda\in\Lambda}A_\lambda.$$

The elements g of G may be written in the form $g=a_{\lambda_1}+\dots+a_{\lambda_k}$ with a_{λ_i} $(\neq 0)$ belonging to different *components* A_λ $(i=1,\dots,k$ where $k\geqq 0)$. Since any element of $\{\dots,A_\lambda,\dots\}$ belongs to a subgroup generated by a finite number of the A_λ, (ii) may be substituted by the apparently weaker postulate: (ii′) $A_\lambda\cap\{A_{\mu_1},\dots,A_{\mu_k}\}=0$ where $\mu_i\neq\lambda$ and k is a natural integer.

The main properties of direct sums are enumerated below.

(a) If $G=\sum_{\lambda\in\Lambda}A_\lambda$, then $G=A_\lambda+A'_\lambda$ where $A'_\lambda=\sum_{\mu\neq\lambda}A_\mu$ is a complementary direct summand of A_λ in G.

(b) If $G=\sum_{\lambda\in\Lambda}A_\lambda$ and each A_λ is again a direct sum, $A_\lambda=\sum_\nu B_{\lambda\nu}$, then $G=\sum_\lambda\sum_\nu B_{\lambda\nu}$. This is called a *refinement* or a *continuation* of the initial decomposition.

(c) Conversely, if $G=\sum_\lambda\sum_\nu B_{\lambda\nu}$, then $G=\sum_\lambda A_\lambda$ if A_λ denotes $\sum_\nu B_{\lambda\nu}$.

(d) If $G=\sum_\lambda A_\lambda$ and, for each index λ, B_λ is a subgroup of A_λ, then $\{\dots,B_\lambda,\dots\}$ is the direct sum of the B_λ.

(e) If $G=A+B$ and H is a subgroup of G containing A, then $H=A+B'$ for the subgroup $B'=B\cap H$ of B.

(f) If $G=A+B$, then $B\cong G/A$.

(g) If $g=a+b$ is an element of $G=A+B$ $(a\in A, b\in B)$, then $O(g)=[O(a),O(b)]$, the l. c. m. of $O(a)$ and $O(b)$.[6]

(h) If A_λ $(\lambda\in\Lambda)$ are subgroups of G, then in order that they generate their direct sum $\sum_{\lambda\in\Lambda}A_\lambda$ in G it is necessary and sufficient the fulfilment of requirement (ii) or (ii′). Therefore, if $a_{\lambda_1}+\dots+a_{\lambda_k}=0$ implies $a_{\lambda_1}=\dots=a_{\lambda_k}=0$ whenever a_{λ_i} belong to different A_λ, then the A_λ generate their direct sum.

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[6] If any one of $O(a)$ and $O(b)$ is infinite, then $O(g)=\infty$ too.