MATHEMATICS
IN SCIENCE
AND
ENGINEERING

Volume 48

Comparison and Oscillation Theory of Linear Differential Equations

This is Volume 48 in
MATHEMATICS IN SCIENCE AND ENGINEERING
A series of monographs and textbooks
Edited by RICHARD BELLMAN, *University of Southern California*

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Comparison and Oscillation Theory of Linear Differential Equations

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ACADEMIC PRESS New York and London 1968

ACADEMIC PRESS INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS INC. (LONDON) LTD.
Berkeley Square House, London W.1

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 68-23477

PRINTED IN THE UNITED STATES OF AMERICA

Preface

This book is concerned primarily with the zeros of solutions of linear differential equations: second order ordinary equations in Chapters 1 and 2, fourth order ordinary equations in Chapter 3, other ordinary equations and systems of differential equations in Chapter 4, and partial differential equations in Chapter 5. The term "comparison theorem" originated with Sturm's classical theorem (Theorem 1.1) but is now used in the following more general sense: If a solution of a differential equation 1 has a property P, here generally connected with its oscillatory behavior, then the solutions of a second differential equation 2 have property P or some related property under some stated connection between 1 and 2. Sturm's classical theorem and many modern analogs of it state roughly this: If 1 has a nontrivial solution with zeros, then every solution of 2 has zeros provided the coefficients of 2 majorize those of 1.

Chapter 1 deals with comparison theorems for second order equations and related topics. Chapter 2 treats oscillation and nonoscillation theorems for second order equations; i.e., conditions (either necessary ones or sufficient ones or both) for the solutions to have (or not to have) an infinite number of zeros in the interval $(0, \infty)$. Although Kneser [95] found some oscillation criteria as early as 1893, completely satisfactory theorems were not provided until the work of Hartman [69–78], Hille [81], Leighton [112–115], Nehari [143], Wintner [69–78, 211–213] and others in the period 1947–1957.

No significant analog of Sturm's theory for higher order equations was found until Birkhoff's paper [21] on third order equations in 1911. Reynolds [173] extended some of Birkhoff's work to nth order equations in 1921, and Fite [45] also gave some early results for higher order equations in 1917. The theory of Morse [136,137] dealing with variational methods gave considerable impetus to the modern theory, and the subject started to grow quickly in the late 1940's. The third order theory has been developed largely since 1955 by Greguš [52–57], Hanan [61], Lazer [111], Ráb [159–162], Švec [187–189], Villari [203,204], and Zlamal [222]. These results are summarized in Chapter 4.

The fourth order theory received a considerable unification with the definitive paper of Leighton and Nehari [117] in 1958. Later contributions were made by Howard [84], Barrett [8–12], Kreith [104], and others. Separation, comparison, and oscillation theorems for fourth order equations are included in Chapter 3.

The nth order case was hardly studied at all from the time of Reynolds until recently, when a number of mathematicians [1,2,5,6,32,42,50,60,94,96, 99–102,119,120] began studying the general case and generalizing some of the results obtained by Hille, Wintner, Leighton, and others already mentioned. The general case is considered in Chapter 4, Sections 5 and 6.

Reid [169–172] and Sternberg [183] extended the theory to systems of second order equations, again using some of the ideas developed by Morse [136], Bliss and Schoenberg [23]. Earlier, Whyburn [210] had considered a special system which is equivalent to a single fourth order equation. These results are described briefly in Chapter 4, Sections 7 and 8.

The first analog of a Sturm–type comparison theorem for an elliptic partial differential equation was obtained by Hartman and Wintner [79] in 1955. Various recent extensions of this result and related theorems are covered in Chapter 5.

It has not been possible to give complete proofs of all results: in general, the proofs are included in Chapters 1, 2, and 5, partially included in Chapter 3, but largely excluded in Chapter 4. Most of the detailed results have appeared in the journals since 1947, although a few of the older results also are included. Some of the results in Chapters 1 and 5 are new. The writer has attempted to trace all results to their original author(s); however, a certain amount of overlapping of authorship occurs on account of the constant generalization in recent years. Any errors or omissions in this regard will be gratefully received by the undersigned.

This book is about *linear* differential equations although, curiously, nonlinear methods are used and seem to be indispensable in some cases; e.g., Chapter 2, Section 5 and Chapter 4, Section 7. In particular, the strongest results in some sections of Chapter 2 are based on the Riccati equation. Eigenvalue problems are thoroughly connected with the theory of oscillation, as recognized by Barrett, Nehari, and others, and accordingly Courant's minimax principle has an important position in the theory of all even order equations. Some properties of eigenvalues and eigenfunctions are presented in Chapter 1, Section 5, and similar results are scattered throughout the sequel.

A large part of the book can be read profitably by a college senior or beginning graduate student well-acquainted with advanced calculus, complex analysis, linear algebra, and linear differential equations. It is hoped also that the book will be helpful to mathematicians working in this subject. Most of the material has never appeared in book form.

Exercises are given after many sections both to test the material and to extend the results. Some of the exercises are large projects which involve consultation of the references given. The bibliography is meant to be fairly complete since 1950, although some of the earlier references are omitted.

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Vancouver
April, 1968

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Chapter 3. **Fourth Order Ordinary Equations**

Chapter 4. **Third Order Ordinary Equations, *n*th Order Ordinary Equations and Systems**

Chapter 5. **Partial Differential Equations**

Chapter 1

Sturm-Type Theorems for Second Order Ordinary Equations

1. Comparison Theorems for Self-Adjoint Equations

The existence and location of the zeros of the solutions of ordinary differential equations are of central importance in the theory of boundary value problems for such equations, and accordingly an immense literature on this subject has arisen during the past century. The first important result was the celebrated comparison theorem of Sturm [184], dealing with the second order self-adjoint equations

$$lu \equiv \frac{d}{dx}\left[a(x)\frac{du}{dx}\right] + c(x)u = 0, \tag{1.1}$$

$$Lv \equiv \frac{d}{dx}\left[A(x)\frac{dv}{dx}\right] + C(x)v = 0 \tag{1.2}$$

on a bounded open interval $\alpha < x < \beta$, where a, c, A, and C are real-valued continuous functions and $a(x) > 0$, $A(x) > 0$ on $[\alpha, \beta]$. These equations define the *differential operators*, l, L, i.e., mappings whose domains consist roughly of twice differentiable functions. Specifically, the domain $\mathfrak{D}_l$ of l is defined as the set of all real-valued functions $u \in \mathscr{C}^1[\alpha, \beta]$ such that† $au' \in \mathscr{C}^1(\alpha, \beta)$, and $\mathfrak{D}_L$ is the analog of $\mathfrak{D}_l$ with A replacing a. A "solution" of (1.1) is a function $u \in \mathfrak{D}_l$ satisfying $lu = 0$ at every point in (α, β). It is not necessary that a' and A' be continuous or even exist at every point, but if $a \in \mathscr{C}^1(\alpha, \beta)$ (or $A \in \mathscr{C}^1(\alpha, \beta)$) the above definition requires only that a solution of $lu = 0$ (or $Lu = 0$) be of

† A prime on a function denotes differentiation with respect to the argument of the function.

class $\mathscr{C}^1[\alpha, \beta] \cap \mathscr{C}^2(\alpha, \beta)$. In what follows, we shall exclude "trivial" solutions, i.e., solutions which are identically zero on the interval under consideration. Sturm's original theorem can be stated as follows:

Sturm's Comparison Theorem 1.1 *Suppose $a(x) = A(x)$ and $c(x) < C(x)$ in the bounded interval $\alpha < x < \beta$. If there exists a nontrivial real solution u of $lu = 0$ such that $u(\alpha) = u(\beta) = 0$, then every real solution of $Lv = 0$ has at least one zero in (α, β).*

Proof Suppose to the contrary that v does not vanish in (α, β). It may be supposed without loss of generality that $v(x) > 0$ and also $u(x) > 0$ in (α, β). Multiplication of (1.1) by v, (1.2) by u, subtraction of the resulting equations, and integration over (α, β) yields

$$\int_\alpha^\beta [(au')'v - (av')'u]\,dx = \int_\alpha^\beta (C - c)uv\,dx. \tag{1.3}$$

Since the integrand on the left side is the derivative of $a(u'v - uv')$ and $C(x) - c(x) > 0$ by hypothesis, it follows that

$$\Big[a(x)(u'(x)v(x) - u(x)v'(x))\Big]_\alpha^\beta > 0. \tag{1.4}$$

However, $u(\alpha) = u(\beta) = 0$ by hypothesis, and since $u(x) > 0$ in (α, β), $u'(\alpha) > 0$ and $u'(\beta) < 0$. Thus the left member of (1.4) is negative, which is a contradiction.

Sturm obtained Theorem 1.1 in 1836 [184] but it was not until 1909 that Picone [149] disposed of the case $a(x) \neq A(x)$. The modification due to Picone is as follows:

Sturm–Picone Comparison Theorem 1.2 *Suppose $a(x) > A(x)$ and $c(x) < C(x)$ in the interval $\alpha < x < \beta$. Then the conclusion of Theorem 1.1 is valid.*

The proof will be deferred since this is a special case of a theorem of Leighton (Theorem 1.4) to be proved later. The original proof of Picone made use of a modification of the formula (1.3) [90].

A significant improvement of the Sturm–Picone theorem was obtained by Leighton in 1962 [116] from a variational lemma depending only on an elementary identity. This lemma will be stated in terms of the quadratic functional defined by the equation

$$J[u] = \int_\alpha^\beta (Au'^2 - Cu^2)\,dx. \tag{1.5}$$

The domain $\mathfrak{D}$ of J is defined to be the set of all real-valued functions $u \in \mathscr{C}^1[\alpha, \beta]$ such that $u(\alpha) = u(\beta) = 0$. A standard integration by parts yields Green's first formula

$$\int_\alpha^\beta uLu\, dx + J[u] = \Big[A(x)u(x)u'(x)\Big]_\alpha^\beta \tag{1.6}$$

for $u \in \mathfrak{D}_L$, or even under less stringent conditions which ensure the meaningfulness of the formula. The following lemma is similar to a theorem from the calculus of variations [24].

Lemma 1.3 *If there exists a function $u \in \mathfrak{D}$, not identically zero, such that $J[u] \le 0$, then every real solution of $Lv = 0$ except a constant multiple of u vanishes at some point of (α, β).*

Proof Suppose to the contrary that there exists a solution $v \neq 0$ in (α, β). For all such v and all $u \in \mathfrak{D}$, the following identity is valid in (α, β):

$$Av^2\left[\left(\frac{u}{v}\right)'\right]^2 + \left(\frac{Au^2v'}{v}\right)' = Au'^2 - Cu^2 + \frac{u^2}{v}\, Lv. \tag{1.7}$$

Indeed, the left member is equal to

$$A\,\frac{(vu' - uv')^2}{v^2} + \frac{2Auu'v'}{v} + \frac{u^2}{v^2}\,[v(Av')' - Av'^2]$$

$$= Au'^2 - Cu^2 + \frac{u^2}{v}\,[(Av')' + Cv].$$

Since $Lv = 0$ in (α, β), integration of (1.7) gives

$$\int_y^z (Au'^2 - Cu^2)\, dx = \int_y^z A\left[v\left(\frac{u}{v}\right)'\right]^2 dx + \left[\frac{Au^2v'}{v}\right]_y^z, \tag{1.8}$$

for arbitrary y and z satisfying $\alpha < y < z < \beta$.

If $v(\alpha) \neq 0$ and $v(\beta) \neq 0$, it follows from (1.5) and the hypotheses $u(\alpha) = u(\beta) = 0$ that

$$J[u] = \int_\alpha^\beta A[v(u/v)']^2\, dx. \tag{1.9}$$

Since $A > 0$, $J[u] \ge 0$, equality if and only if $(u/v)'$ is identically zero, i.e., u is a constant multiple of v. The latter cannot occur since $u(\alpha) = 0$ and $v(\alpha) \neq 0$, and hence $J[u] > 0$. The contradiction shows that v must have a zero in (α, β) in the case $v(\alpha) \neq 0$, $v(\beta) \neq 0$.

Now consider the case $v(\alpha) = v(\beta) = 0$. Since the solutions of second order ordinary linear differential equations have only simple zeros [35], $v'(\alpha) \neq 0$ and