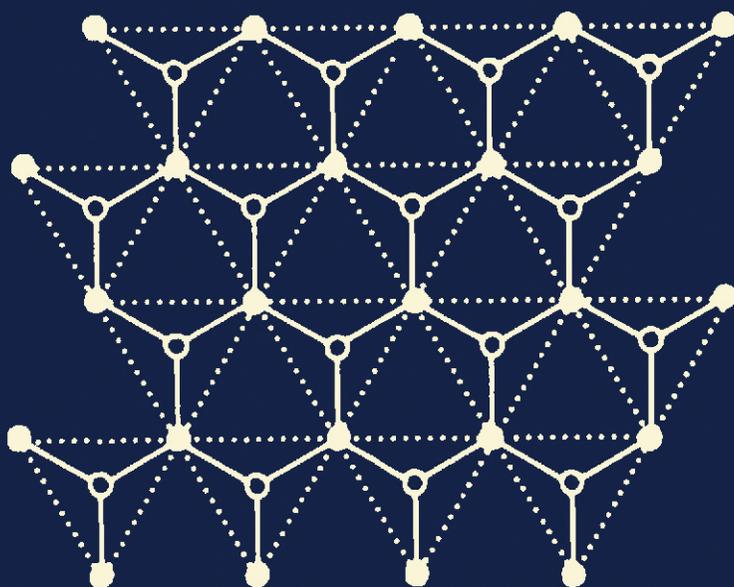


EXACTLY
SOLVED MODELS
IN STATISTICAL
MECHANICS



R. J. BAXTER

Exactly Solved Models in Statistical Mechanics

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PREFACE

This book was conceived as a slim monograph, but grew to its present size as I attempted to set down an account of two-dimensional lattice models in statistical mechanics, and how they have been solved. While doing so I have been pulled in opposite directions. On the one hand I remembered the voice of the graduate student at the conference who said ‘But you’ve left out all the working—how *do* you get from equation (81) to (82)?’ On the other hand I knew from experience how many sheets of paper go into the waste-paper basket after even a modest calculation: there was no way they could all appear in print.

I hope I have reached a reasonable compromise by signposting the route to be followed, without necessarily giving each step. I have tried to be selective in doing so: for instance in Section 8.13 I discuss the functions $k(\alpha)$ and $g(\alpha)$ in some detail, since they provide a particularly clear example of how elliptic functions come into the working. Conversely, in (8.10.9) I merely quote the result for the spontaneous staggered polarization P_0 of the F -model, and refer the interested reader to the original paper: its calculation is long and technical, and will probably one day be superseded when the eight-vertex model conjecture (10.10.24) is verified by methods similar to those used for the magnetization result (13.7.21).

There are ‘down-to-earth’ physicists and chemists who reject lattice models as being unrealistic. In its most extreme form, their argument is that if a model can be solved exactly, then it must be pathological. I think this is defeatist nonsense: the three-dimensional Ising model is a very realistic model, at least of a two component alloy such as brass. If the predictions of universality are corrected, then they should have exactly the same critical exponents. Admittedly the Ising model has been solved only in one and two dimensions, but two-dimensional systems do exist (see Section 1.6), and can be quite like three-dimensional ones. It is true that the two-dimensional Ising model has been solved only for zero magnetic

field, and that this case is quite unlike that of non-zero field; but physically this means Onsager solved the most interesting and tricky case. His solution vastly helps us understand the full picture of the Ising model in a field.

In a similar way, the eight-vertex model helps us understand more complicated systems and the variety of behaviour that can occur. The hard hexagon model is rather special, but needs no justification: It is a perfectly good lattice gas and can be compared with a helium monolayer adsorbed onto a graphite surface (Riedel, 1981).

There is probably also a feeling that the models are 'too hard' mathematically. This does not bear close examination: Ruelle (1969) rightly says in the preface to his book that if a problem is worth looking at at all, then no mathematical technique is to be judged too sophisticated.

Basically, I suppose the justification for studying these lattice models is very simple: they are relevant and they can be solved, so why not do so and see what they tell us?

In the title the phrase 'exactly solved' has been chosen with care. It is not necessarily the same as 'rigorously solved'. For instance, the derivation of (13.7.21) depends on multiplying and diagonalizing the infinite-dimensional corner transfer matrices. It ought to be shown, for instance, that the matrix products are convergent. I have not done this, but believe that they are (at least in a sense that enables the calculation to proceed), and that as a result (13.7.21) is exactly correct.

There is of course still much to be done. Barry McCoy and Jacques Perk rightly pointed out to me that whereas much is now known about the correlations of the Ising model, almost nothing is known about those of the eight-vertex and hard hexagon models.

There are many people to whom I am indebted for the opportunity to write this book. In particular, my interest in mathematics and theoretical physics was nurtured by my father, Thomas James Baxter, and by Sydney Adams, J. C. Polkinghorne and K. J. Le Couteur. Elliott Lieb initiated me into the complexities of the ice-type models. Louise Nicholson and Susan Turpie worked wonders in transforming the manuscript into immaculate typescript. Paul Pearce has carefully read the proofs of the entire volume. Most of all, my wife Elizabeth has encouraged me throughout, particularly through the last turbulent year of writing.

R. J. Baxter

Canberra, Australia
February 1982

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1

BASIC STATISTICAL MECHANICS

1.1 Phase Transitions and Critical Points

As its name implies, statistical mechanics is concerned with the average properties of a mechanical system. Obvious examples are the atmosphere inside a room, the water in a kettle and the atoms in a bar magnet. Such systems are made up of a huge number of individual components (usually molecules). The observer has little, if any, control over the components: all he can do is specify, or measure, a few average properties of the system, such as its temperature, density or magnetization. The aim of statistical mechanics is to predict the relations between the observable macroscopic properties of the system, given only a knowledge of the microscopic forces between the components.

For instance, suppose we knew the forces between water molecules. Then we should be able to predict the density of a kettleful of water at room temperature and pressure. More interestingly, we should be able to predict that this density will suddenly and dramatically change as the temperature is increased from 99°C to 101°C: it decreases by a factor of 1600 as the water changes from liquid to steam. This is known as a *phase transition*.

Yet more strange effects can occur. Consider an iron bar in a strong magnetic field, H , parallel to its axis. The bar will be almost completely magnetized: in appropriate units we can say that its magnetization, M , is $+1$. Now decrease H to zero: M will decrease, but not to zero. Rather, at zero field it will have a *spontaneous magnetization* M_0 .

On the other hand, we expect molecular forces to be invariant with respect to time reversal. This implies that reversing the field will reverse the magnetization, so M must be an odd function of H . It follows that

$M(H)$ must have a graph of the type shown in Fig. 1.1(a), with a discontinuity at $H = 0$.

This discontinuity in the magnetization is very like the discontinuity in density at a liquid – gas phase transition. In fact, in the last section of this chapter it will be shown that there is a precise equivalence between them.

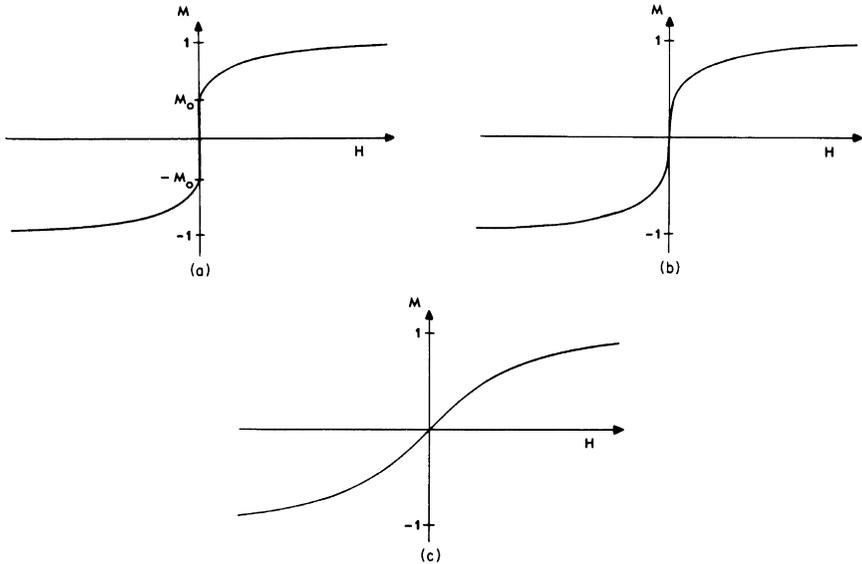


Fig. 1.1. Graphs of $M(H)$ for (a) $T < T_c$, (b) $T = T_c$, (c) $T > T_c$.

The iron bar can be regarded as undergoing a phase transition at $H = 0$, changing suddenly from negative to positive magnetization. In an actual experiment this discontinuity is smeared out and the phenomenon of hysteresis occurs: this is due to the bar not being in true thermodynamic equilibrium. However, if the iron is soft and subject to mechanical disturbances, a graph very close to that of Fig. 1.1(a) is obtained (Starling and Woodall, 1953, pp. 280–281; Bozorth, 1951, p. 512).

The above remarks apply to an iron bar at room temperature. Now suppose the temperature T is increased slightly. It is found that $M(H)$ has a similar graph, but M_0 is decreased. Finally, if T is increased to a critical value T_c (the Curie point), M_0 vanishes and $M(H)$ becomes a continuous function with infinite slope (susceptibility) at $H = 0$, as in Fig. 1.1(b).

If T is further increased, $M(H)$ remains a continuous function, and becomes analytic at $H = 0$, as in Fig. 1.1(c).

These observations can be conveniently summarized by considering a (T, H) plane, as in Fig. 1.2. There is a cut along the T axis from 0 to T_c . The magnetization M is an analytic function of both T and H at all points in the right-half plane, *except those on the cut*. It is discontinuous across the cut.

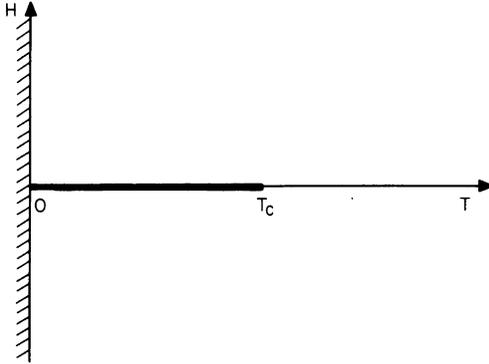


Fig. 1.2. The (T, H) half-plane, showing the cut across which M is discontinuous. Elsewhere M is an analytic function of T and H .

The cut is a line of phase transitions. Its endpoint $(T_c, 0)$ is known as a *critical point*. Clearly the function $M(H, T)$ must be singular at this point, and one of the most fascinating aspects of statistical mechanics is the study of this singular behaviour near the critical point.

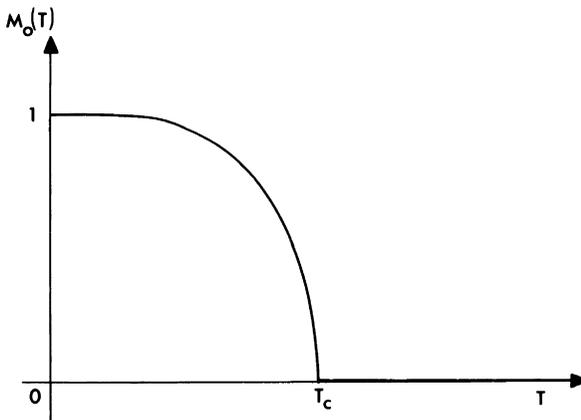


Fig. 1.3. The spontaneous magnetization M_0 as a function of temperature.

The spontaneous magnetization is a function of T and can be defined as

$$M_0(T) = \lim_{H \rightarrow 0^+} M(H, T), \quad (1.1.1)$$

the limit being taken through positive values of H . It has a graph of the type shown in Fig. 1.3, being positive for $T < T_c$ and identically zero for $T > T_c$.

Critical Exponents

The *susceptibility* of a magnet is defined as

$$\chi(H, T) = \frac{\partial M(H, T)}{\partial H}. \quad (1.1.2)$$

When considering critical behaviour it is convenient to replace T by

$$t = (T - T_c)/T_c. \quad (1.1.3)$$

Then the thermodynamic functions must have singularities at $H = t = 0$. It is expected that these singularities will normally be simple non-integer powers; in particular, it is expected that

$$M_0(T) \sim (-t)^\beta \quad \text{as } t \rightarrow 0^-, \quad (1.1.4)$$

$$M(H, T_c) \sim H^{1/\delta} \quad \text{as } H \rightarrow 0, \quad (1.1.5)$$

$$\chi(0, T) \sim t^{-\gamma} \quad \text{as } t \rightarrow 0^+, \quad (1.1.6)$$

$$\chi(0, T) \sim (-t)^{-\gamma'} \quad \text{as } t \rightarrow 0^-. \quad (1.1.7)$$

Here the notation $X \sim Y$ means that X/Y tends to a non-zero limit. The power-law exponents β , δ , γ , γ' are numbers, independent of H and T : they are known as *critical exponents*.

For brevity, the phrase 'near T_c ' will be frequently used in this book to mean 'near the critical point', it being implied that H is small, if not zero.

1.2 The Scaling Hypothesis

It is natural to look for some simplified form of the thermodynamic functions that will describe the observed behaviour near T_c . Widom (1965) and Domb and Hunter (1965) suggested that certain thermodynamic functions might be homogeneous. In particular, Griffiths (1967) suggested that H

might be a homogeneous function of $M^{1/\beta}$ and t . Since H is an odd function of M , this means that near T_c

$$H/kT_c = M|M|^{\delta-1} h_s(t|M|^{-1/\beta}), \quad (1.2.1)$$

where β and δ are numbers (as yet undefined), k is Boltzmann's constant, and $h_s(x)$ is a dimensionless *scaling function*. A typical graph of $h_s(x)$ is shown in Fig. 1.4: it is positive and monotonic increasing in the interval $-x_0 < x < \infty$, and vanishes at $-x_0$.

Note that (1.2.1) implies that H is an odd function of M , as it should be.

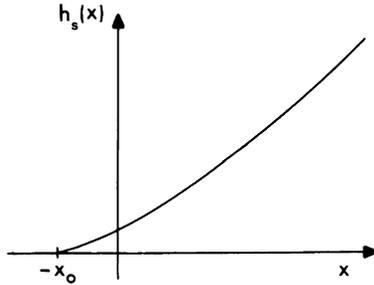


Fig. 1.4. The scaling function $h_s(x)$ for the square-lattice Ising model (Gaunt and Domb, 1970).

The scaling hypothesis predicts certain relations between the critical exponents. To see this, first consider the behaviour on the cut in Fig. 1.2. Here $H = 0$, $t < 0$ and $M = \pm M_0$. From (1.2.1) the function $h_s(x)$ must be zero, so $x = -x_0$, i.e.

$$t = -x_0 |M|^{1/\beta}. \quad (1.2.2)$$

The relation (1.1.4) follows, so β in (1.2.1) is the critical exponent defined in (1.1.4).

Now set $t = 0$ (1.2.1). Since $h_s(0)$ is non-zero, this implies that near T_c

$$H \sim M^\delta, \quad (1.2.3)$$

in agreement with (1.1.5). Hence the δ in (1.2.1) is the same as that in (1.1.5).

Differentiate (1.2.1) with respect to M , keeping t fixed. From (1.1.2) this gives

$$(kT_c \chi)^{-1} = |M|^{\delta-1} [\delta h_s(x) - \beta^{-1} x h'_s(x)] \quad (1.2.4)$$

where

$$x = t|M|^{-1/\beta}. \quad (1.2.5)$$

Again consider the behaviour on the cut in Fig. 1.2. Here x has the fixed value $-x_0$, so

$$\chi^{-1} \sim |M|^{\delta-1} \sim (-t)^{\beta(\delta-1)}. \quad (1.2.6)$$

This agrees with (1.1.7), and predicts that the critical exponent γ' is given by

$$\gamma' = \beta(\delta - 1). \quad (1.2.7)$$

To obtain (1.1.6) from the scaling hypothesis, we need the large x behaviour of the scaling function $h_s(x)$. This can be obtained by noting that for fixed positive t , we must have

$$H \sim M \quad \text{as } M \rightarrow 0. \quad (1.2.8)$$

Comparing this with (1.2.1), we see that

$$h_s(x) \sim x^{\beta(\delta-1)} \quad \text{as } x \rightarrow \infty. \quad (1.2.9)$$

From (1.2.1) and (1.2.9), it follows that for arbitrary small positive t ,

$$H \sim t^{\beta(\delta-1)}M \quad \text{as } M \rightarrow 0, \quad (1.2.10)$$

so from (1.1.1),

$$\chi(0, T) \sim t^{-\beta(\delta-1)} \quad \text{as } t \rightarrow 0^+. \quad (1.2.11)$$

Comparing this with (1.1.6), and using (1.2.7), we see that the scaling hypothesis predicts the exponent relations

$$\gamma = \gamma' = \beta(\delta - 1). \quad (1.2.12)$$

Other exponents α , ν , ν' , η , μ will be defined in Section 1.7, but for completeness the various scaling predictions are listed here:

$$\alpha + 2\beta + \gamma' = 2, \quad (1.2.13)$$

$$\nu = \nu', (2 - \eta) \nu = \gamma, \quad (1.2.14)$$

$$\mu + \nu = 2 - \alpha, \quad (1.2.15)$$

$$d\nu = 2 - \alpha, \quad (1.2.16)$$

where d is the dimensionality of the system.

A partial derivation of (1.2.14) will be given in Section 1.7, but it is beyond the scope of this book to attempt to justify all these relations: the interested reader is referred to the articles by Widom (1965), Fisher (1967), Kadanoff *et al.* (1967), Hankey and Stanley (1972), Stanley (1971) and

Vicentini-Missoni (1972). Their relevance here is that exactly solved models can be used to test the relations, and indeed we shall find that scaling passes every possible test for the models to be discussed.

The scaling relations (1.2.12)–(1.2.15) are in good agreement with available experimental and theoretical results, and the scaling function $h_s(x)$ has been obtained approximately for a number of systems (see for example Gaunt and Domb, 1970).

The last relation (1.2.16) involves the dimensionality d . It is derived by making further assumptions, known as ‘strong scaling’ or ‘hyperscaling’. It is expected to be valid for $d \leq 4$, but there is some question whether it is consistent with available numerical results for three- and four-dimensional models (Baker, 1977). The total set of equations (1.2.12)–(1.2.16) is sometimes known as ‘two exponent’ scaling, since if two independent exponents (such as δ and β) are given, then all other exponents can be obtained from the equations.

1.3 Universality

Consider a system with conservative forces. Let s denote a state (or configuration) of the system. Then this state will have an energy $E(s)$, where the function $E(s)$ is the *Hamiltonian* of the system.

The thermodynamic properties, such as $M(H, T)$ and T_c , are of course expected to depend on the forces in the system, i.e. on $E(s)$. However, it is believed (Fisher, 1966; Griffiths, 1970) that the critical exponents are ‘universal’, i.e. independent of the details of the Hamiltonian $E(s)$.

They will, of course, depend on the dimensionality of the system, and on any symmetries in the Hamiltonian. To see the effect of these, suppose $E(s)$ can be written as

$$E(s) = E_0(s) + \lambda E_1(s), \quad (1.3.1)$$

where $E_0(s)$ has some symmetry (such as invariance under spatial reflection) and $E_1(s)$ has not. The critical exponents are then supposed to depend on λ only in so far as they have one value for $\lambda = 0$ (symmetric Hamiltonian), and another fixed value for $\lambda \neq 0$ (non-symmetric). For example, there would be two numbers β_0, β_1 such that

$$\begin{aligned} \beta &= \beta_0 && \text{if } \lambda = 0 \\ &= \beta_1 && \text{if } \lambda \neq 0, \end{aligned} \quad (1.3.2)$$

β being discontinuous at $\lambda = 0$.

On the other hand, if $E_0(s)$ is some simple Hamiltonian and $E_1(s)$ is very complicated, but they have the same dimensionality and symmetry, then β should be completely constant, even at $\lambda = 0$. The implications of this are far reaching. One could take a realistic and complicated Hamiltonian $E(s)$, 'strip' it to a highly idealized Hamiltonian $E_0(s)$, and still obtain exactly the same critical exponents. For instance, on these grounds it is believed that carbon dioxide, xenon and the three-dimensional Ising model should all have the same critical exponents. To within experimental error, this appears to be the case (Hocken and Moldover, 1976).

There are some difficulties: there is usually more than one way of describing a system, in particular of labelling its states. In one of these there may be an obvious symmetry which occurs for some special values of the parameters. In another formulation this symmetry may not be obvious at all. Thus if the second formulation were used, and these special values of the parameters were accidentally chosen, then the critical exponents could be unexpectedly different from those appropriate to other values.

Also, in this book the solution of the two-dimensional 'eight-vertex' model will be presented. This has exponents that vary continuously with the parameters in the Hamiltonian. This violates the universality hypothesis, but it is now generally believed that such violations only occur for very special classes of Hamiltonians.

It should be noted that scaling and universality, while commonly grouped together, are independent assumptions. One may be satisfied and the other not, as in the case of the eight-vertex model, where universality fails but scaling appears to hold.

1.4 The Partition Function

How do we calculate thermodynamic functions such as $M(H, T)$ from the microscopic forces between the components of the system? The answer was given by John Willard Gibbs in 1902. Consider a system with states s and Hamiltonian $E(s)$. Form the partition function

$$Z = \sum_s \exp[-E(s)/kT], \quad (1.4.1)$$

where k is Boltzmann's constant and the summation is over all allowed states s of the system. Then the free energy F is given by

$$F = -kT \ln Z. \quad (1.4.2)$$

Also, the probability of the system being in a state s is

$$Z^{-1} \exp[-E(s)/kT], \quad (1.4.3)$$

so if X is some observable property of the system, such as its total energy or magnetization, with value $X(s)$ for state s , then its observed average thermodynamic value is

$$\langle X \rangle = Z^{-1} \sum_s X(s) \exp[-E(s)/kT]. \quad (1.4.4)$$

In particular, the internal energy is

$$\begin{aligned} U &= \langle E \rangle \\ &= Z^{-1} \sum_s E(s) \exp[-E(s)/kT], \end{aligned} \quad (1.4.5)$$

and by using the above definitions (1.4.1) and (1.4.2) we can verify that

$$\begin{aligned} U &= kT^2 \frac{\partial}{\partial T} \ln Z \\ &= -T^2 \frac{\partial}{\partial T} (F/T), \end{aligned} \quad (1.4.6)$$

in agreement with standard thermodynamics.

The basic problem of equilibrium statistical mechanics is therefore to calculate the sum-over-states in (1.4.1) (for continuum systems this sum becomes an integral, for quantum mechanical ones a trace). This will give Z and F as functions of T and of any variables that occur in $E(s)$, such as a magnetic field. The thermodynamic properties can then be obtained by differentiation.

Unfortunately, for any realistic interacting system of macroscopic size, including the examples mentioned above, the evaluation of Z is hopelessly difficult. One is therefore forced to do one or both of the following:

- A. Replace the real system by some simple idealization of it: this idealization is known as a *model*. Mathematically, it consists of specifying the states s and the energy Hamiltonian function $E(s)$.
- B. Make some approximation to evaluate the sum-over-states (1.4.1).

1.5 Approximation Methods

Let us consider the step (B) above. Some of the better-known approximation schemes are:

- (i) Cell or cluster approximations. In these the behaviour of the whole system is extrapolated from that of a very few components inside

some 'cell', approximations being made for the interaction of the cell with the rest of the system. Examples are the mean-field (Bragg and Williams, 1934; Bethe, 1935), quasi-chemical (Guggenheim, 1935) and Kikuchi (1951) approximations. They have the advantage of being fairly simple to solve; they predict the correct qualitative behaviour shown in Figs. 1.1 to 1.3, and are reasonably accurate except near the critical point (Domb, 1960, pp. 282–293; Burley, 1972).

- (ii) Approximate integral equations for the correlation functions, notably the Kirkwood (1935), hyper-netted chain (van Leeuwen *et al.*, 1959) and Percus-Yevick (Percus and Yevick, 1958; Percus, 1962) equations. These give fairly good numerical values for the thermodynamic properties of simple fluids.
- (iii) Computer calculations on systems large on a microscopic scale (e.g. containing a few hundred atoms), but still not of macroscopic size. These calculations evaluate Z by statistically sampling the terms on the RHS of (1.4.1), so are subject to statistical errors, usually of a few per cent. For this reason they are really 'approximations' rather than 'exact calculations'.
- (iv) Series expansions in powers of some appropriate variable, such as the inverse temperature or the density. For very realistic models these can only be obtained to a few terms, but for the three-dimensional Ising model expansions have been obtained to as many as 40 terms (Sykes *et al.*, 1965, 1973a).

The approximation schemes (i) to (iii) can give quite accurate values for the thermodynamic properties, *except near the critical point*. There is a reason for this: they all involve neglecting in some way the correlations between several components, or two components far apart. However, near T_c the correlations become infinitely-long ranged, all components are correlated with one another, and almost any approximation breaks down. This means that approximations like (i), (ii) and (iii) are of little, if any, use for determining the interesting cooperative behaviour of the system near T_c .

Method (iv) is much better: if sufficient terms can be obtained then it is possible, with considerable ingenuity, to obtain plausible guesses as to the nature of the singularities of the thermodynamic functions near the critical point. In particular, the best estimates to date of critical exponents in three dimensions have been obtained by the series expansion method. However, an enormous amount of work is required to obtain the series, and the resulting accuracy of the exponents is still not as good as one would like.

- (v) There is another approach, due to Kadanoff (1966) and Wilson (1971) (see also Wilson and Kogut, 1974; Fisher, 1974): this is the so-called *renormalization group*. In this method the sum over states (1.4.1) is evaluated in successive stages, a 'renormalized' Hamiltonian function $E(s)$ being defined at each stage. This defines a mapping in Hamiltonian space. If one makes some fairly mild assumptions about this mapping, notably that it is analytic, then it follows that the thermodynamic functions do have branch-point singularities such as (1.1.4) at T_c , that the scaling hypothesis (1.2.1) and the relations (1.2.12)–(1.2.16) are satisfied, and that the exponents of the singularities should normally be universal (Fisher, 1974, p. 602).

In principle, the renormalization group approach could be carried through exactly. However, this is more difficult than calculating the partition function directly, so to obtain actual numerical results some approximation method is needed for all but the very simplest models. The fascinating result is that quite crude cell-type approximations give fairly accurate values of the critical exponents (Kadanoff *et al.*, 1976). The reason for this is not yet fully understood.

To summarize: approximate methods (step B) either fail completely near T_c , or require considerable acts of faith in the assumptions made.

1.6 Exactly Solved Models

Another approach is to use step A to the fullest, and try to find models for which $E(s)$ is sufficiently simple that the partition function (1.4.1) can be calculated exactly. This may not give useful information about the values of the thermodynamic functions of real systems, but it will tell us qualitatively how systems can behave, in particular near T_c . In fact if we could solve a model with the same dimensionality and symmetry as a real system, universality asserts that we should obtain the *exact* critical exponents of the real system.

There is a further condition for universality, which was not mentioned in Section 1.3. In most physical systems the intermolecular forces are effectively short ranged: in inert gases they decay as r^{-7} , r being the distance between molecules; in crystals it may be sufficient to regard each atom as interacting only with its nearest neighbour. The infinite-range correlations that occur at a critical point are caused by the cooperative behaviour of the system, not by infinite-range interactions.

If, on the other hand, sufficiently long-range interactions are included in $E(s)$, they clearly can affect the way the correlations become infinite near T_c , and it comes as no surprise that critical exponents can be altered in this way. Thus universality only applies to systems with the same range of interactions. To obtain the correct critical behaviour, a model of a real system should not introduce non-physical long-range interactions.

Unfortunately no short-range genuinely three-dimensional model has been solved. The simplest such model is the three-dimensional Ising model (which will be defined shortly): this has been extensively investigated using the series expansion method (Gaunt and Sykes, 1973), but no exact solution obtained.

The models of interacting systems for which the partition function (1.4.1) has been calculated exactly (at least in the limit of a large system) can generally be grouped into the following four classes.

One-Dimensional Models

One-dimensional models can be solved if they have finite-range, decaying exponential, or Coulomb interactions. As guides to critical phenomena, such models with short-range two-particle forces (including exponentially decaying forces) have a serious disadvantage: they do not have a phase transition at a non-zero temperature (van Hove, 1950; Lieb and Mattis, 1966). The Coulomb systems also do not have a phase transition, (Lenard, 1961; Baxter, 1963, 1964 and 1965), though the one-dimensional electron gas has long-range order at all temperatures (Kunz, 1974).

Of the one-dimensional models, only the nearest-neighbour Ising model (Ising, 1925; Kramers and Wannier, 1941) will be considered in this book. It provides a simple introduction to the transfer matrix technique that will be used for the more difficult two-dimensional models. Although it does not have a phase transition for non-zero temperature, the correlation length does become infinite at $H = T = 0$, so in a sense this is a 'critical point' and the scaling hypothesis can be tested near it.

A one-dimensional system can have a phase transition if the interactions involve infinitely many particles, as in the cluster interaction model (Fisher and Felderhof, 1970; Fisher, 1972). It can also have a phase transition if the interactions become infinitely long-ranged, but then the system really belongs to the following class of 'infinite-dimensional' models.

'Infinite Dimensional' Models

To see what is meant by an 'infinite dimensional' system, one needs a working definition of the effective dimensionality of a Hamiltonian. For