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W. K. CHEN

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GRAPHS AND ELECTRICAL NETWORKS

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APPLIED GRAPH THEORY

GRAPHS AND ELECTRICAL NETWORKS

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To Shiao-Ling

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PREFACE

In the past four decades, we have witnessed a steady development of graph theory and its applications which in the last five to ten years have blossomed out into a new period of intense activity. Some measure of this rapid expansion is indicated by the observation that, over a period of only one and a half years, more than 500 new papers on graph theory and its applications were published. The main reason for this accelerated interest in graph theory is its demonstrated applications. Because of their intuitive diagrammatic representation, graphs have been found extremely useful in modeling systems arising in physical science, engineering, social science, and economic problems. The fact is that any system involving a binary relation can be represented by a graph.

An effort has been made to introduce the subject matter in the book as simply as possible. Thus, all unnecessary definitions are avoided in favor of a little longer statement. For example, an edge-disjoint union of circuits may be defined as a *circ*, but I prefer not to do so, since the list of definitions has already been too long. Since the terminology and symbolism currently in use in graph theory are far from standardized, the choice of terms is dictated by their applications in the five key areas covered in the book. Thus, the node is preferred to vertex or point, circuit to cycle, parallel edges to multiple edges, etc. As a result, one saving feature of the book is that many of the terms used have nearly the same meaning as in everyday English and very little conscious effort is required to remember them.

The guide light throughout the book has been mathematical precision. Thus, all the assertions are rigorously proved; many of these proofs are believed to be new and novel. An attempt has been made to present the key topics in a complete and logical fashion, to indicate the historical background, and to credit to the original contributors as far as I can determine. I have tried to present the material in a concise manner, using discussions and examples to illustrate the concepts and principles involved. The book also contains some of the personal contributions of the author that are not available elsewhere in the literature.

Depending only on chapter 1, each of the following five chapters, although they are not completely independent, is virtually self-contained, so that the material may be useful to the persons who are interested in only a single topic.

Chapter 1 establishes the basic vocabulary for describing graphs and provides a number of results that are needed in the subsequent analysis. In order to shorten the monotone of these necessary preliminaries, only the essential terms are introduced; the others are defined when they are needed in the later chapters. Thus, the reader is urged to study the convention of this chapter carefully before proceeding to the other chapters.

Chapters 2, 3, and 4, constituting about two-thirds of the book, discuss the various applications to electrical network theory, which happens to be the major field of interest of the author. As a matter of fact, the most important application of graph theory in the physical science is its use in the formulation and solution of the electrical network problem. Although the techniques discussed may easily be extended to other disciplines, the dominant theme is nevertheless the electrical network theory. In each of these chapters, the reader is assumed to be familiar with the elementary aspects of the subject and the discussions are devoted to those aspects of the theory that are strongly dependent on the theory of graphs.

A special feature of the book is that almost all the results are documented in relationship to the known literature, and all the references which have been cited in the text are listed in the bibliography. Thus, the book is especially suitable for those who wish to continue with the study of special topics and to apply graph theory to other fields.

Although basically intended as a reference text for serious researchers, the book may be used equally well as a text for graduate level courses on network topology and linear systems and circuits. Some of the later chapters are suitable as topics for advanced seminars. The only prerequisite for this book is really mathematical maturity.

A rich variety of problems has been presented at the end of each chapter.

Much of the material in the book was developed in the past six years from the research grants extended to the author by the National Science Foundation, the National Aeronautics and Space Administration, and the Ohio University Research Committee. During this time, I have enjoyed the hospitality of Purdue University which I have had the opportunity to visit. To this I am particularly indebted to Professors L. O. Chua and B. J. Leon for making this visit possible. The writing of this book could not have been possible without the constant encouragement and assistance of Provost R. L. Savage, Dean B. Davison, and Dr. J. C. Gilfert of Ohio University. I wish to express my gratitude to Professor W. Mayeda of University of Illinois and Professor M. E.

Van Valkenburg of Princeton University for their invaluable inspiration. Thanks are also due to many friends and colleagues who gave useful suggestions; among them are Professors K. E. Eldridge, G. V. S. Raju, H. C. Chen and F. Y. Chen and my students Dr. S. K. Mark and Mr. H. C. Li. Mr. Li assisted me in plotting the preliminary drawings of all the illustrations. In particular, I would like to single out Professor K. E. Eldridge and Dr. S. K. Mark who kindly read both the manuscript and page proofs critically and made valuable suggestions. Considerable assistance was also contributed by Professor P. M. Lin of Purdue University who gave the complete book a careful reading. I also wish to thank Dr. C. Korswagen and the North-Holland Publishing Company for their patience and cooperation in all aspects of the production of this book. Finally, I would like to thank my wife, Shiao-Ling, for her careful proof-reading of the book and for her infinite patience and understanding, to whom this book is dedicated.

April, 1971 W.K.C.
West Lafayette, Indiana

PREFACE TO THE SECOND EDITION

During the past four years since the publication of the first edition, we have witnessed the appearance of more than thirty new books in graph theory and its applications (see Bibliography). The predictions are that it will continue to grow at a rapid rate for some time to come. This accelerated pace of development has confirmed the value of the general approach adopted in the first edition by giving a reasonably deep account of a relatively small part of material that is closely related to engineering applications.

In revising the first edition, I can think of many items that should be added. Judging from the interest of readers, I have decided to concentrate on an area in linear systems and networks that has received wide attention in recent years and that depends heavily on the theory of graphs: State Equations of Networks. This topic will be presented as a key area in chapter 7. In order to keep the book down to a reasonable length, the solutions of the state equations are not discussed since they can be found in many excellent books on ordinary differential equations.

This edition contains a significant number of corrections that have been incorporated throughout the text. In addition, the Bibliography has been updated, including a detailed listing of books.

One inevitable result in adding a key area is that the book has grown longer. It contains more material than can be adequately presented in a one-semester or two-quarter three-hours-per-week course in linear systems and networks. This added flexibility will allow instructors to select subjects and sections to meet his particular needs and environment.

Since the publication of the first edition, many people have been kind enough to give me the benefit of their comments and suggestions, often at the expense of a very considerable amount of their time and energy. As a result, the title of the new edition has been expanded to reflect more accurately its contents. In particular, I am indebted to my students and those users of the book who have contributed to the improvement of this edition. Special thanks are due my doctoral student S. Chandra who gave the complete book a careful reading. Also, I wish to thank Dr. E. Fredriksson of the North-

Holland Publishing Company for making available a student edition at a considerably lower price. Finally, I express my appreciation to my family for their patience and understanding during the preparation of the book.

Athens, Ohio, 1975 W. K. Chen

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CONTENTS

CHAPTER 1

BASIC THEORY

The chapter establishes the basic vocabulary for describing graphs and provides a number of basic results that are needed in the subsequent analysis, omitting those aspects of graph theory that are unrelated to the applications discussed in this book. Since the terminology and symbolism currently in use in graph theory are far from standardized, the reader is urged to study the conventions of this chapter carefully before proceeding to the other chapters.

§ 1. Introduction

The term "graph" used in this book denotes something quite different from the graphs that one may be familiar with from analytic geometry or function theory. The graphs that we are about to discuss are simple geometrical figures consisting of points (nodes) and lines (edges) which connect some of these points; they are sometimes called "linear graphs". Because of this diagrammatic representation, graphs have been found extremely useful in modeling systems arising in physical science (BUSACKER and SAATY [1965], and HARARY [1967]), engineering (SESHU and REED [1961], and ROBICHAUD et al. [1962]), social science (HARARY and NORMAN [1953], and FLAMENT [1963]), and economic problems (AVONDO-BODINO [1962], and FORD and FULKERSON [1962]). The fact is that any system involving a binary relation can be represented by a graph.

The first paper on graphs was written by the famous Swiss mathematician Leonhard Euler (1707–1783). He started with a famous unsolved problem of his day called the *Königsberg Bridge Problem.* The city of Königsberg (now Kaliningrad) in East Prussia is located on the banks and on two islands of the river Pregel. The various parts of the city were connected by seven bridges as shown in fig. 1.1. The problem was to cross all seven bridges, passing over each one only once. One can see immediately that there are many ways of trying the problem without solving it. EULER [1736] solved the problem by showing that it was impossible, and laid the foundations of graph theory. We mention here only the formulation, rather than the details.

Replace each part of the city by a point and each bridge by a line joining the points corresponding to these parts. The result is a graph as shown in fig. 1.2. Euler then showed that, no matter at which point one begins, one cannot cover the graph completely and come back to the starting point without retracing one's steps.

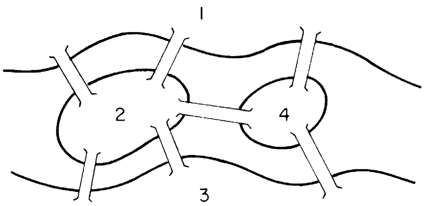

Fig. 1.1. The Königsberg bridge problem.

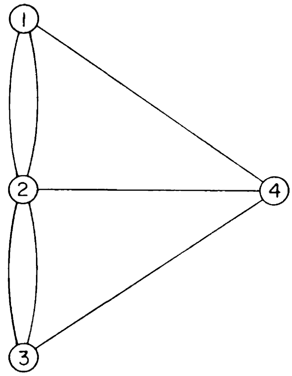

Fig. 1.2. The graph of the Königsberg bridge problem.

The most famous unsolved problem in graph theory is perhaps the celebrated *Four Color Conjecture*. Many centuries ago, makers of maps discovered empirically that in coloring a map of a country, divided into counties, only four distinct colors are required, so that no two adjacent counties should be painted in the same color. At first the problem does not seem to have been taken seriously by the mathematicians until it has withstood every assault by some of the world's most capable mathematicians. HEAWOOD [1890] showed, however, that the conjecture becomes true when "four" is replaced by "five". A counterexample, if ever found, will necessarily be extremely large and complicated, for the conjecture was proved most recently by ORE and STEMPLE [1970] for all maps with fewer than 40 counties.

The problem can easily be transformed into a problem in graph theory because every map yields a graph in which the counties including the exterior region are represented by the points, and two points are joined by a line if and only if their counties have a common boundary.

The most important application of graph theory in the physical science, from our point of view, is its use in the formulation and solution of the electrical network problem by KIRCHHOFF [1847]. His contributions will be treated in great detail in this book; chapters 2 and 4 contain most of his contributions to electrical network theory.

While many of the examples of the graphs arising in applications are geometric, the essential structure in the context of graph theory is combinatorial in nature. In the following sections, we shall introduce the concept of abstract graphs. Aside from stripping the incidental geometric features away from the essential combinatorial characteristics of a graph, the concept enlarges the prospects of applications.

§ 2. Basic concepts of abstract graphs

Like every mathematical theory, we have to begin with a long list of definitions, since we must have a few words to talk about, and in the interest of precision these have to be formally defined. Fortunately, many of these terms that we will define have nearly the same intuitive meaning as in everyday English and so very little conscious effort is required to remember them. In order to relieve the monotony of these necessary preliminaries, we will use diagrams to illustrate our points.

2.1. *General definitions*

DEFINITION 1.1: *Abstract graph.* An *abstract graph* $G(V, E)$, or simply a *graph* G, consists of a set V of elements called *nodes* together with a set E of *unordered* pairs of the form (i, j) or (j, i), i, j in V, called the *edges* of G; the nodes i and j are called the *endpoints* of (i, j).

Other names commonly used for a node are *vertex*, *point*, *junction*, 0-*simplex*, 0-*cell*, and *element*; and for edges *line*, *branch*, *arc*, 1-*simplex*, and *element*. We say that the edge (i, j) is *connected* between the nodes i and j, and that (i, j) is *incident* with the nodes i and j or conversely that i and j are *incident* with (i, j). In the applications, a graph is usually represented equivalently by a *geometric diagram* in which the nodes are indicated by small circles or dots, while any two of them, i and j, are joined by a continuous curve, or even a straight line, between i and j if and only if (i, j) is in E. This definition of a graph is sufficient for many problems in which graphs make their appearance. However, for our purpose, it is desirable to enlarge the graph concept somewhat.

We extend the graph concept by permitting a pair of nodes to be connected by several distinct edges as indicated by the symbols $(i, j)_1$, $(i, j)_2, \ldots, (i, j)_k$; they are called the *parallel edges* of G if $k \geqq 2$. If no particular edge is specified, (i, j) denotes any one, but otherwise fixed, of the parallel edges connected between i and j. We also admit edges for which the two endpoints are identical. Such an edge (i, i) shall be called a *self-loop*. If there are two or more self-loops at a node of G, they are also referred to as the parallel edges of G. In the geometric diagram the parallel edges may be represented by continuous lines connected between the same pair of nodes, and a self-loop (i, i) may be introduced as a circular arc returning to the node i and passing through no other nodes.

As an illustration, consider the graph $G(V, E)$ in which

$$V = \{1, 2, 3, 4, 5, 6, 7\},$$

$$E = \{(1, 1), (1, 2), (1, 4), (4, 4)_1, (4, 4)_2, (4, 3), (2, 3)_1, (2, 3)_2, (6, 7)_1, (6, 7)_2, (6, 7)_3\}.$$

The corresponding geometric graph is as shown in fig. 1.3 in which we have a self-loop at node 1, two self-loops at node 4, two parallel edges connected between the nodes 2 and 3, and three parallel edges between the nodes 6 and 7. We emphasize that in a graph the order of the nodes i and j in (i, j) is immaterial. In fact we consider $(i, j) = (j, i)$, e.g., $(1, 2) = (2, 1)$ and $(6, 7)_2 = (7, 6)_2$.

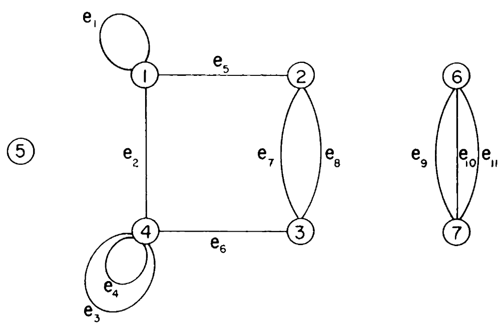

Fig. 1.3. A geometric graph.

A graph $G(V, E)$ is said to be *finite* if both V and E are finite. In this book, we only consider finite graphs. Infinite graphs have some very interesting properties. For interested readers, we refer to KÖNIG [1950] and ORE [1962].

DEFINITION 1.2: *Subgraph*. A *subgraph* of a graph $G(V, E)$ is a graph $G_s(V_s, E_s)$ in which V_s and E_s are subsets of V and E, respectively. If V_s or E_s is a proper subset, the subgraph is called a *proper subgraph* of G. If $V_s = V$, the subgraph is referred to as a *spanning subgraph* of G. If V_s or E_s is empty, the subgraph is called the *null graph*. The null graph is considered as a subgraph of every graph, and is denoted by the symbol $\emptyset$.

DEFINITION 1.3: *Isolated node*. A node not incident with any edge is called an *isolated node*.

In fig. 1.3, for example, the node 5 is an isolated node. Some examples of subgraphs are presented in fig. 1.4. Fig. 1.4(a) is a spanning subgraph since it contains all the nodes of the given graph. Figs. 1.4(b) and (c) are examples of proper subgraphs. A graph itself is also its subgraph.

We say that two subgraphs are *edge-disjoint* if they have no edges in common, and *node-disjoint* if they have no nodes in common. Clearly, two subgraphs are node-disjoint only if they are edge-disjoint, but the converse is not valid in general. For example, in fig. 1.3 the subgraphs (1, 2) and (3, 4) are node-disjoint, and thus they are also edge-disjoint. On the other hand, the subgraphs as shown in figs. 1.4(b) and (c) are edge-disjoint but they are not node-disjoint.

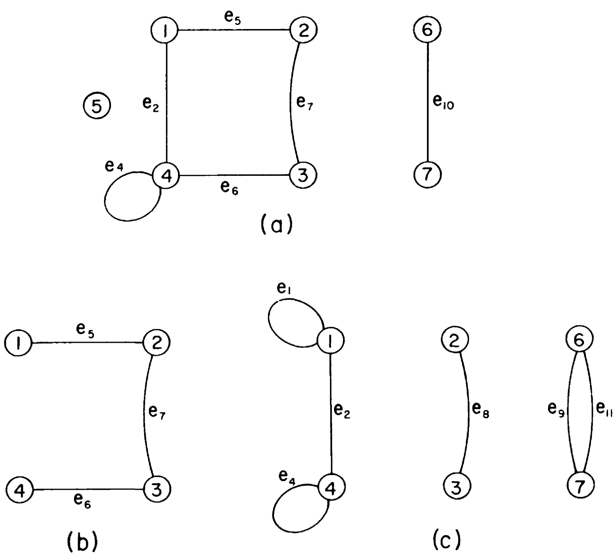

Fig. 1.4. Some examples of the subgraphs of the graph of fig. 1.3.

In a graph G we say that the nodes i and j are *adjacent* if (i, j) is an edge of G. If G_s is a subgraph of G, by the *complement* $\bar{G}_s$ of G_s in G we mean the subgraph of G consisting of all the edges $\bar{E}_s$ that do not belong to G_s and all the nodes of G except those that are in G_s but not in $\bar{E}_s$. Clearly, G_s and $\bar{G}_s$ are edge-disjoint but not necessarily node-disjoint, and their node sets may not be complementary. Thus, the complement of the null graph in G is the graph G itself, and the complement of G in G is the null graph. We also say that G_s and $\bar{G}_s$ are *complementary subgraphs* of G. For example, figs. 1.5(a) and (b) are complementary subgraphs of the graph as shown in fig. 1.3.

In practical applications, it is sometimes convenient to represent the edges of a graph by letters e_i. In this way, a subgraph having no isolated nodes may be expressed by the "product" or by juxtaposition of its edge-designation symbols. For example, in fig. 1.3 the edges of the graph are also represented by the letters e_i: $e_1=(1, 1)$, $e_2=(1, 4)=(4, 1),\ldots$, and $e_{11}=(6, 7)_3=(7, 6)_3$. The subgraphs of figs. 1.4(b) and (c) may be denoted by the products of their edge-designation symbols as $e_5e_6e_7$ and $e_1e_2e_4e_8e_9e_{11}$, respectively. Of course, we can also use this technique to represent subgraphs having isolated nodes, but then an ambiguity involving the null graph will arise.

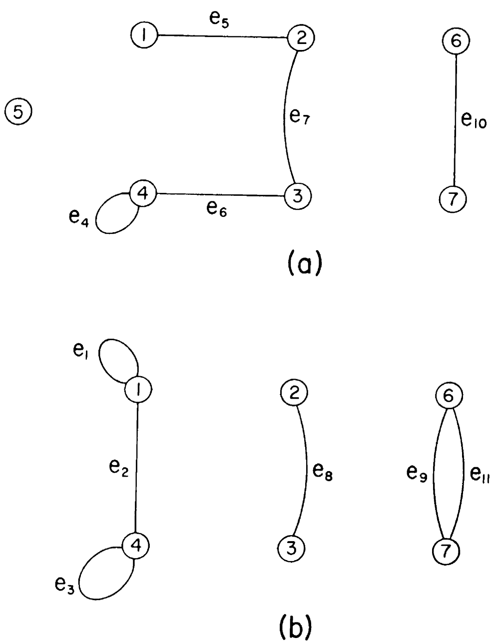

Fig. 1.5. A pair of complementary subgraphs of the graph of fig. 1.3.

2.2. *Isomorphism*

In the preceding section, we have already pointed out that in drawing the geometric diagram of a graph we have great freedom in the choice of the location of the nodes and in the form of the lines joining them. This may make the diagrams of the same graph look entirely different. In such cases we would like to have a precise way of saying that two graphs are really the same even though they are drawn or labeled differently. The next definition provides the terminology necessary for this purpose.

DEFINITION 1.4: *Isomorphism*. Two graphs G_1 and G_2 are said to be *isomorphic*, denoted by $G_1 = G_2$, if there exist a one-to-one correspondence between the elements of their node sets and a one-to-one correspondence between the elements of their edge sets and such that the corresponding edges are incident with the corresponding nodes.

In other words, in two isomorphic graphs the corresponding nodes are connected by the edges in one if, and only if, they are also connected by the same number of edges in the other. Definition 1.4 as stated places two requirements on isomorphism of two graphs. First, they must have the same number of nodes and edges. Second, the incidence relationships must be preserved. The latter is usually difficult to establish.

As an illustration, consider the graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ as shown in fig. 1.6. These two graphs look quite different, but they are isomorphic. The

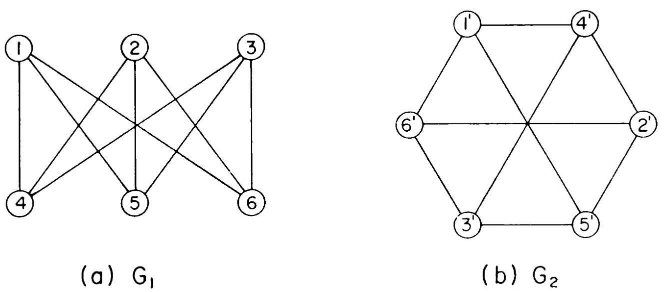

Fig. 1.6. Two isomorphic graphs.

isomorphism of these two graphs can be established by considering the nodes i of V_1 and i' of V_2, $i = 1, 2, 3, 4, 5, 6$, as the corresponding elements of their node sets. It is easy to verify that the corresponding edges are incident with the corresponding nodes. In other words, the incidence relationships are preserved.

As another example, the two graphs given in fig. 1.7 are not isomorphic even though there exists a one-to-one correspondence between their node sets which preserves adjacency. The reason for this is that they do not contain the same

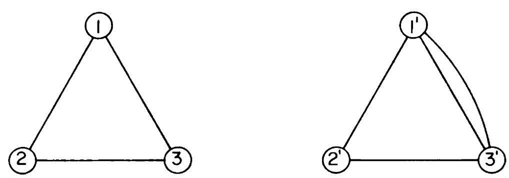

Fig. 1.7. Two non-isomorphic graphs.

number of edges. If we consider the nodes i and i', $i=1, 2, 3$, as the corresponding nodes of the node sets, we have two edges connecting the nodes $1'$ and $3'$ while we have only one connecting the corresponding nodes 1 and 3.

A graph is said to be a *labeled graph* if the nodes or edges of the graph are properly labeled. In this book, the terms *graph* and *labeled graph* are used as synonyms. The graphs that we have witnessed so far are all labeled graphs, the nodes being labeled by the integers 1, 2,... or $1'$, $2'$,..., and the edges by e_1, e_2,....
A *weighted graph* is a graph in which every edge has been assigned a weight.

2.3. *Connectedness*

Sequences of edges which form continuous routes play an important role in graph theory. In a geometric graph, a sequence of edges can be visualized as a series of edges connected in a continuous manner. More formally, we define the following.

DEFINITION 1.5: *Edge sequence.* An *edge sequence of length* $k-1$ in a graph G is a finite sequence of edges of the form

$$(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k), \tag{1.1}$$

$k \geqq 2$, in G. The edge sequence is said to be *closed* if $i_1 = i_k$, and *open* otherwise. In an open edge sequence, the node i_1 is called the *initial node*, and node i_k the *terminal node* of the edge sequence. Together they are called the *endpoints* of the edge sequence.

We mention specifically that not all the nodes in (1.1) are necessarily distinct, and the same edge may appear several times in the edge sequence. For example, in fig. 1.6(a) the sequence of edges

$$(1, 6), (6, 3), (3, 5), (5, 3), (3, 4), (4, 1), (1, 6), (6, 2), (2, 5)$$

is an open edge sequence of length 9. Node 1 is the initial node and node 5 the terminal node of the edge sequence. Similarly, the sequence of edges

$$(4, 2), (2, 6), (6, 3), (3, 5), (5, 2), (2, 6), (6, 3), (3, 5), (5, 2), (2, 4)$$

forms a closed edge sequence of length 10.

We also say that the edge sequence (1.1) is *connected* between its initial and terminal nodes or between the nodes i_1 and i_k, and that for $k>2$, (i_{x-1}, i_x) and (i_x, i_{x+1}), $1<x<k$, are *successive edges* in the edge sequence. Sometimes it is convenient to define an isolated node as an edge sequence.

DEFINITION 1.6: *Edge train.* If all the edges appearing in an edge sequence are distinct, the edge sequence is called an *edge train.*

Thus, an edge train can go through a node more than once but cannot retrace parts of itself, as an edge sequence can. An example of an edge train in fig. 1.6(a) is

$$(1, 6), (6, 2), (2, 5), (5, 1), (1, 4), (4, 2).$$

The edge train is open and is of length 6. Clearly, an edge sequence or an edge train is also contained in a subgraph. If in addition we require that all the nodes in an edge train except the initial and the terminal nodes be distinct, we have the usual concepts of a path and a circuit.

DEFINITION 1.7: *Path.* An open edge train, as shown in (1.1), in which all the nodes $i_1, i_2, \ldots, i_k$ are distinct is called a *path of length* $k-1$. An isolated node is considered as a path of zero length.

DEFINITION 1.8: *Circuit.* A closed edge train, as shown in (1.1), in which all the nodes $i_1, i_2, \ldots, i_{k-1}$ are distinct, in this case $i_1 = i_k$, is called a *circuit of length* $k-1$.

Thus, a self-loop is also a circuit of length 1. In the literature, the term *circuit*

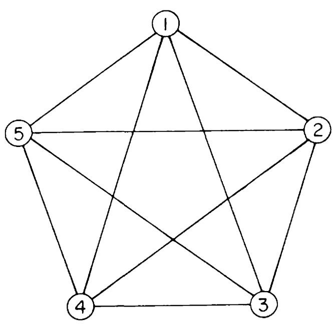

Fig. 1.8. The complete pentagon.

is frequently referred to as a *cycle* or *loop*. The term we adopt here is more commonly used in applications. In fig. 1.8, the open edge train

$$(1, 2), (2, 3), (3, 4), (4, 5)$$

is a path of length 4, and the closed edge train

$$(1, 3), (3, 2), (2, 5), (5, 4), (4, 1)$$

is a circuit of length 5.

DEFINITION 1.9: *Connected graph.* A graph is said to be *connected* if every pair of its nodes are connected by a path.

In other words, a connected graph, intuitively speaking, has only one piece. Fig. 1.5(a) or (b) is an example of a graph which is *not* connected, and fig. 1.6 shows two connected graphs.

DEFINITION 1.10: *Component.* A *component* of a graph is a connected subgraph containing the maximal number of edges. An isolated node is a component.

Thus, if a graph is not connected it must contain a number of components. One or many of these components may each consist of an isolated node. For example, in fig. 1.5(a) the graph has three components; one of them is an isolated node.

Suppose that we define a binary relation R between certain pairs of nodes of a graph G as follows: The relation iRj holds if and only if there is a path connected between the nodes i and j in G where an isolated node is considered as a path of zero length. Then R is an equivalence relation; it uniquely decomposes the node set of G into mutually exclusive equivalence classes of nodes. Each of these classes of nodes, together with the edges of G incident with these nodes, constitutes a component of G. Thus, a graph is connected if and only if it consists of only one component.

We shall speak of circuit and noncircuit edges. A *circuit edge* of a graph is an edge which can be made part of a circuit; otherwise, an edge is called a *noncircuit edge.* Clearly, the deletion of a circuit edge from a connected graph leaves a connected subgraph (Problem 1.3).

THEOREM 1.1: The deletion of a circuit edge from a connected graph leaves a spanning connected subgraph.

It is also clear that the deletion of a noncircuit edge from a graph G results in a graph which has one more component than G. For example, in fig. 1.5(b) if we delete the noncircuit edge e_2 or e_8, the resulting graph has four components, that is one more than the original graph. On the other hand, in fig. 1.7 or 1.8 all the edges are circuit edges; the deletion of any one of these edges results in a spanning connected subgraph.

THEOREM 1.2: Let G be a graph having no parallel edges and self-loops. If G has n nodes and c components, then the maximal number of edges in G is given by

$$\tfrac{1}{2}(n-c)(n-c+1). \qquad (1.2)$$

Proof. Let G_i, $i=1, 2, \ldots, c$, be the components of G, each having n_i nodes. Since the maximal number of edges in each component G_i is $\frac{1}{2}n_i(n_i-1)$, the maximal number of edges in G is given by

$$\frac{1}{2}\sum_{i=1}^{c} n_i(n_i-1). \tag{1.3}$$

If G has $c-1$ isolated nodes, (1.3) reduces to (1.2) and our proof is complete. Thus, we assume that there are two distinct components G_i and G_j with more than an isolated node. Let $n_i \geqq n_j > 1$. If in G we increase the number of nodes in G_i by one and at the same time if we reduce the number of nodes in G_j by one, we obtain a new graph G^* which has the same numbers of nodes and components as G. It is easy to verify that the maximal number of edges in G^* is greater than that in G. Continuing this process, we can show that (1.2) is indeed an upper bound for all such graphs.

COROLLARY 1.1: Let G be an n-node graph having no parallel edges and self-loops. If G has more than $\frac{1}{2}(n-1)(n-2)$ edges, then G must be connected.

2.4. *Rank and nullity*

Rank and nullity are two numbers that are frequently encountered in graph theory. As we shall see in the next chapter, they represent the number of independent "cutsets" and circuits of a graph.

DEFINITION 1.11: *Rank.* The *rank r* of a graph with n nodes and c components is defined as the number $r=n-c$.

DEFINITION 1.12: *Nullity.* The *nullity m* of a graph with b edges, n nodes, and c components is defined as the number $m=b-n+c(=b-r)$.

The term *nullity* is also known by the names of *circuit rank*, *cyclomatic number*, *cycle rank*, *connectivity*, and *first Betti number*. The reason that we choose the above names *rank* and *nullity* is that, as we shall see in the next chapter, they are the rank and nullity of the "incidence matrix" associated with the graph. In fig. 1.8 the rank and nullity of the graph are $4(=5-1)$ and $6(=10-5+1)$, respectively. In fig. 1.5(b) we have $r=6-3=3$ and $m=6-6+3=3$. We notice that all these numbers are nonnegative. This is indeed the case as we can see from the following theorem (Problem 1.5).

THEOREM 1.3: For a given graph, its rank and nullity are both nonnegative. The graph is of nullity 0 if and only if it contains no circuit, and is of nullity 1 if and only if it contains a single circuit.