THERMO-ELASTICITY

2ND EDITION

W. NOWACKI

WARSAW, POLAND

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THERMOELASTICITY

by

WITOLD NOWACKI

Translated by

HENRYK ZORSKI

Second edition
Revised and enlarged

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CONTENTS

PREFACE TO THE FIRST EDITION

In the postwar years we have seen a rapid development of thermoelasticity, stimulated by various engineering sciences. A considerable progress in the field of aircraft and machine structures, mainly with gas and steam turbines, and the emergence of new topics in chemical and nuclear engineering have given rise to numerous problems in which thermal stresses play an important and frequently even a primary role.

Thermoelasticity embraces a wide field of phenomena. It contains the theory of heat conduction and the theory of strains and stresses due to the flow of heat, when coupling of temperature and strain fields occurs. Thermoelasticity makes it possible to determine the stresses produced by the temperature field, and, moreover, to calculate the distribution of temperature due to the action of internal forces which vary with time.

The particular case of thermoelasticity in which the influence of the coupling of temperature and strain fields is neglected is the basis of most quasi-static and dynamic problems dealt with in this monograph; the influence of the coupling is considered in detail in the Chapters IV and VIII.

We shall investigate mainly the thermoelastic problems which occur in isotropic and homogeneous bodies, under the assumption of small displacements and linear stress-strain laws. We assume, moreover, that the material constants are independent of temperature. These assumptions are peculiar to the classical theory of elasticity and obviously limit the applicability of the solutions obtained to certain ranges of temperature. They make it possible, however, to investigate a large class of engineering problems.

An attempt has been made in this monograph to unify the exposition. The solutions have been constructed consisting of two parts, the first representing the thermoelastic displacement potential while the second one is an additional solution expressed by the components of the Galerkin vector or the stress functions. Wherever it was convenient, Maysel's method was employed.

In many problems investigated in this monograph the principal point is the determination of the Green functions for stresses. Therefore the integral transforms of Fourier, Hankel and Laplace have consistently been used. The latter is particularly useful in solving non-stationary thermoelastic problems.

The problems have been divided into two large groups — the spatial and plane problems, each group containing both stationary and non-stationary solutions. The detailed examination of the problems is preceded by an extensive introduction to thermoelasticity (Chapter I) which contains the basic relations and equations, and the methods of solution.

In Chapter XI we deal with the non-classical problem of thermal stresses in viscoelastic bodies with linear material relations. Finally, the last chapter contains a review of new trends in thermoelasticity. We also consider briefly thermal stresses in non-homogeneous, isotropic and anisotropic bodies.

I wish here to express my gratitude to Doc. habil. Dr. H. Zorski, Dr. J. Ignaczak and Dr. W Piechocki, for reading the manuscript and for many helpful suggetions.

W. Nowacki

Warsaw

PREFACE TO THE SECOND EDITION

The first edition of *Thermoelasticity* appeared 20 years ago, many changes, deletions and additions therefore have been made, first of all in Chapter I dealing with the fundamental relations and equations of thermoelasticity and in Chapter II devoted to the new trends and new developments in thermoelasticity.

It is assumed in the static elasticity that during a slow growth of the loadings and hence also the deformations, a complete exchange of heat with the surroundings takes place. Furthermore, we assume that thc temperature in the whole body is equal to the temperature in the natural state. In classical dynamic elasticity the corresponding assumption is that the heat exchange occurring by means of the heat conduction, is very slow, the process therefore is adiabatic. The theory of thermal stresses is based on a different assumption, namely we take into account the action of heat sources and surface heating but we neglect the changes of the temperature field due to the deformations of the body.

The above three basic assumptions lead to three different sets of differential equations describing the fields of strain and temperature. The creation of the coupled thermoelasticity is due to the tendency to obtain one system of differential equations describing all thermodynamic processes. Chapter I treats thermoelasticity as a synthesis of the theory of elasticity and the theory of heat conduction; some particular cases of thermoelasticity are also investigated, e.g. stationary problems, the theory of thermal stresses, classical dynamic elasticity, etc.

Chapter XII has been completed by a brief examination of micropolar thermoelasticity, magnetothermoelasticity and thermopiezoelectricity. In order to reduce the volume of the book we omitted Section VI.8 concerning the application of the complex variable method to two-dimensional problems of thermoelasticity. Finally, we reduced the list of references to the papers which played a fundamental role in the development of thermoelasticity.

W. Nowacki

Warsaw, 1983

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CHAPTER I

BASIC RELATIONS AND EQUATIONS OF THERMOELASTICITY

I.1. Principle of energy conservation. Entropy balance

A deformation of a body is inseparably connected with a change of its heat content and therefore with a change of the temperature distribution in the body. A deformation of a body which varies in time leads to temperature changes, and conversely. The internal energy of the body depends on both the temperature and the deformation. The science which deals with the investigation of the above coupled processes, is called thermoelasticity.

It should be emphasized that investigations in the field of thermoelasticity were preceded by extensive investigations in the so-called theory of thermal stresses. The latter is a theory of the state of strain and stress in an elastic body, due to a heating, under the simplifying assumption that the influence of the deformation on the temperature field may be neglected.

In the theory of thermal stresses which goes back to the beginnings of the theory of elasticity, the classical heat conduction equation was used, which does not contain the term due to the deformation of the body. Knowing the temperature distribution from the solution of the heat conduction equation, the displacement equations of the theory of elasticity were solved, the latter containing the known terms proportional to the temperature gradient (1).

At the same time the classical dynamic elasticity was being developed, also under the simplifying assumption that the heat exchange between parts of the body due to the heat conduction occurs very slowly and therefore the motion may be regarded as adiabatic.

Thermoelasticity deals with a wide class of phenomena. It contains the generalized theory of heat conduction, the generalized theory of thermal stresses, describes the temperature distribution produced by deformation and finally it contains a description of the phenomenon of thermoelastic dissipation. The cognitive merits of this theory are very large indeed. In spite of its mathematical complexity, the thermoelasticity enables us to examine deeper than before, the mechanism of the deformation and thermal processes occurring in elastic bodies.

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(1) J. M. C. Duhamel, *Second mémoire sur les phénomènes thermo-mécaniques*, J. de l'Ecole Polytechnique, 1837, Vol. 15, p. 1.

Thermoelasticity is a new field. Although the coupling between the deformation and temperature fields was postulated already by J. M. C. Duhamel and the extended heat conduction equation was given by W. Voigt (1) and H. Jeffreys (2), the real development of the theory occurred in the last twenty years. The starting point was the paper by M. A. Biot (3) in which on the basis of the thermodynamics of irreversible processes the basic relations and equations were derived and the variational theorems of thermoelasticity were formulated.

In this chapter we present in detail the derivation of the basic relations and equations of thermoelasticity and the methods of their solution, as well as the energy theorems and the integration methods they imply.

We begin our considerations by deducing the constitutive relations, i.e. the relations between the stress tensor and the entropy, and the strain tensor and the temperature.

We confine ourselves in principle to elastic isotropic homogeneous bodies. An elastic deformation is a state such that when the forces producing the deformation are removed, the body returns to its initial undeformed state. The isotropy means that the elastic properties of the body are independent of the direction and the homogeneity means the independence of the elastic properties of the position.

Assume that the body in its undeformed stress-free state (no external forces acting) has a certain constant temperature T_0. This initial state is called the natural state of the body. Due to the action of external forces, i.e. body forces and surface tractions, and to the action of heat sources inside the body or a heating (or cooling) of its surface, the body undergoes a deformation. There arise in the body the displacements $\mathbf{u}$ and the temperature undergoes a change $T = \mathscr{T} - T_0$, where $\mathscr{T}$ is the absolute temperature. The deformation of the body is accompanied by a production of the strain ε_{ij} and the stress σ_{ij}. These quantities depend on position and time. We assume that the change of temperature $T = \mathscr{T} - T_0$ accompanying the deformation is small and that the temperature increase T does not produce any appreciable changes of the material coefficients, both elastic and thermal. These coefficients will be regarded as quantities independent of T.

The assumption $|T/T_0| \ll 1$ is now completed by another one concerning the magnitude of the deformation; we assume that the squares and products of the components of the strain ε_{ij} can be neglected as compared with ε_{ij}. Thus, we confine ourselves to the geometrically linear thermoelasticity. The relation between the strain and displacement has then the linear form

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})\,. \tag{1}$$

It is known that the strain components cannot be arbitrary and must satisfy six so-called compatibility relations (4)

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{jl,ik} - \varepsilon_{ik,jl} = 0, \qquad i, j, k, l = 1, 2, 3\,. \tag{2}$$

(1) W. Voigt, *Lehrbuch der Kristallphysik*, Teubner, Leipzig, 1910.

(2) H. Jeffreys, *The thermodynamics of an elastic solid* in Proc. Cambr. Phill. Soc., 1930.

(3) M. A. Biot, *Thermoelasticity and irreversible thermodynamics*, J. Appl. Phys., 1956, Vol. 27, p. 240.

(4) J. S. Sokolnikoff, *Mathematical theory of elasticity*, New York, 1956.

The basic problem consists in deducing the constitutive relations, i.e. the relations connecting the components of the stress tensor σ_{ij}, the entropy S, the components of the strain tensor ε_{ij} and the temperature $\mathscr{T}$.

Observe that the mechanical and thermal state of the medium at a certain instant of time, is completely described by the distribution of the deformations ε_{ij} and the temperature $\mathscr{T}$. Thus, in an isothermal change of state we are faced with a process both elastically and thermally reversible. However, in processes in which the temperature varies in time we deal with two coupled processes, namely the reversible elastic process and the irreversible thermodynamic process. The latter is due to a spontaneous and hence irreversible process of heat transfer by means of heat conduction.

The thermoelastic changes cannot be described by means of the classical thermodynamics valid for equilibrium states; we must use the relations of the thermodynamics of irreversible processes.([1]) ([2])

The point of departure of our considerations is the first and second law of thermodynamics. The first law, the law of energy conservation, has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{U}+\mathscr{K}) = \mathscr{L}+\frac{\mathrm{d}Q}{\mathrm{d}t}. \tag{3}$$

Here $\mathscr{U}$ is the internal energy, $\mathscr{K}$ the kinetic energy, $\mathscr{L}$ the power of external forces and $\dot{Q}$ is the increment in time of the quantity of heat absorbed by the body. Equation (3) is the energy balance. It states that the increment in time of the sum of the internal and kinetic energies is equal to the sum of the power of the external forces and the heat absorbed by the body.

The power of the external forces is given by the formula

$$\mathscr{L} = \int_V X_i v_i \,\mathrm{d}V + \int_A p_i v_i \,\mathrm{d}A, \tag{4}$$

where $\mathbf{X}$ is the vector of the body forces, $\mathbf{p}$ is the vector of surface tractions occurring on the surface A bounding the body, $\mathbf{v} = \mathrm{d}\mathbf{u}/\mathrm{d}t$ is the vector of the displacement velocity. The components p_i of the vector $\mathbf{p}$ are connected with the stress vector by the relations

$$p_i = \sigma_{ji} n_j. \tag{5}$$

Here n_i is the component of normal — of the surface A. We assume that the normal $\mathbf{n}$ is directed towards the exterior of the body.

The kinetic energy appearing in (3) is expressed by the integral

$$\mathscr{K} = \frac{\varrho}{2}\int_V v_i v_i \,\mathrm{d}V, \tag{6}$$

([1]) S. R. de Groot, *Thermodynamics of irreversible processes*, Amsterdam, 1952.
([2]) L. Prigogine, *Etude thermodynamique des phénomenès irreversibles*, Liège, 1947.

and the non-mechanical power $\dot{Q}$ has the form

(7) $$\dot{Q} = -\int_A q_i n_i \, dA + \int_V W \, dV.$$

Here **q** is the vector of the heat flux and W the quantity of heat generated in unit time and unit volume.

Making use of the Gauss-Ostrogradski theorem we obtain from (7)

(8) $$\dot{Q} = -\int_V (q_{i,i} - W) \, dV.$$

We introduce now the internal energy U per unit volume

(9) $$\mathscr{U} = \int_V U \, dV.$$

Substituting (4)–(9) into the energy balance (3) we obtain

(10) $$\int_V (\varrho v_i \dot{v}_i + \dot{U}) \, dV = \int_V X_i v_i \, dV + \int_A p_i v_i \, dA - \int_V (q_{i,i} - W) \, dV.$$

Since

$$p_i = \sigma_{ji} n_j, \qquad \int_A \sigma_{ji} n_j v_i \, dA = \int_V (\sigma_{ji} v_i)_{,j} \, dV,$$

Equation (10) takes the form

(11) $$\int_V [\dot{U} - (\sigma_{ji,j} + X_i - \varrho \dot{v}_i) v_i - \sigma_{ji} v_{i,j} + q_{i,i} - W] \, dV.$$

Equation (11) should be valid for every part of the body; thus we arrive at the local principle of energy conservation

(11′) $$\dot{U} = (\sigma_{ji,j} + X_i - \varrho \dot{v}_i) v_i + \sigma_{ji} v_{i,j} - q_{i,i} + W.$$

We now demand that the expression (11′) be invariant with respect to a rigid motion of the body.(1)

Consider first the rigid displacement

(12) $$v_i \to v_i + b_i,$$

where b_i is an arbitrary constant vector. We assume that during a rigid displacement the quantities U, ϱ, σ_{ij}, X_i, q_i, W remain constant. Introducing (12) into (11′) we obtain

(13) $$\dot{U} = (v_i + b_i)(\sigma_{ji,j} + X_i - \varrho v_i) + \sigma_{ji} v_{i,j} - q_{i,i} + W.$$

Subtracting (11′) from (13) we have

$$b_i(\sigma_{ji,j} + X_i - \varrho v_i) = 0.$$

(1) A. E. Green, R. S. Rivlin, Arch. Rat. Mech. Analysis, **17** (1964), 113.

This equations should be satisfied for an arbitrary value of b_i. Thus, we arrive at the equations of motion

$$\sigma_{ji,j}+X_i-\varrho\dot{v}_i = 0\,. \tag{14}$$

Taking into account equation (14) we obtain a considerably simplified law of energy balance (11′), namely

$$\dot{U} = \sigma_{ji}\,v_{i,j}-q_{i,i}+W. \tag{15}$$

This expression should be invariant with respect to a rigid rotation of the body. We assume that

$$v_i \rightarrow v_i-\varepsilon_{ikl}\,x_k\,\Omega_l, \qquad v_{i,j} \rightarrow v_{i,j}-\varepsilon_{ijl}\,\Omega_l\,, \qquad \Omega_l = \text{const.} \tag{16}$$

Substituting from (16) to (15) we have

$$\dot{U} = \sigma_{ji}(v_{i,j}-\varepsilon_{ijl}\,\Omega_l)-q_{i,i}+W. \tag{17}$$

Subtracting (15) from (17) and taking into account the invariance of the quantities U, σ_{ji}, q_i, W we obtain

$$\Omega_l\,\varepsilon_{ijl}\,\sigma_{ji} = 0\,, \qquad \Omega_l \neq 0\,.$$

This result proves that the stress tensor is symmetric, i.e.

$$\sigma_{ji} = \sigma_{ij}\,. \tag{18}$$

Observe that

$$v_{i,j} = \tfrac{1}{2}\,(\dot{u}_{i,j}+\dot{u}_{j,i})+\tfrac{1}{2}\,(\dot{u}_{i,j}-\dot{u}_{j,i}) = \dot{\varepsilon}_{ij}+\dot{\omega}_{ij}\,.$$

Here $\dot{\varepsilon}_{ij} = \partial\varepsilon_{ij}/\partial t$, ε_{ij} is defined by the relation (1), and $\dot{\omega}_{ij} = \partial\omega_{ij}/\partial t$ is the time derivative of the rotation tensor ω_{ij}. Since the stress tensor σ_{ji} is symmetric and the tensor ω_{ji} is antisymmetric, we have $\sigma_{ji}\,\dot{\omega}_{ji} = 0$.

Thus, we have the local relation

$$U = \sigma_{ij}\,\varepsilon_{ij}-q_{i,i}+W. \tag{19}$$

Consider now the local entropy balance

$$\mathscr{T}\dot{S} = -q_{i,i}+W, \tag{20}$$

where S is the entropy per unit volume and unit time. Let us integrate (20) over the volume of the body. Then

$$\int\limits_V \dot{S}\,\mathrm{d}V = -\int\limits_V \left(\frac{q_i}{\mathscr{T}}\right)_{,i}\mathrm{d}V-\int\limits_V \frac{q_i\,\mathscr{T}_{,i}}{\mathscr{T}^2}\,\mathrm{d}V+\int\limits_V \frac{W}{\mathscr{T}}\,\mathrm{d}V,$$

or

$$\int\limits_V \dot{S}\,\mathrm{d}V = -\int\limits_A \frac{q_i\,n_i}{\mathscr{T}}\,\mathrm{d}A-\int\limits_V \frac{q_i\,\mathscr{T}_{,i}}{\mathscr{T}^2}\,\mathrm{d}V+\int\limits_V \frac{W}{\mathscr{T}}\,\mathrm{d}V. \tag{21}$$

The increment of the entropy in time consists of two basic parts. The first is described by the surface integral constituting the increase (or decrease) of the entropy due to the heat flux through the surface A. Thus, this integral describes the heat exchange with the surroundings. The volume integrals are due to the entropy production in the volume V. The first integral is the entropy produced in V by the heat exchange while the second is the entropy produced by the action of the heat sources.

Let us now return to the relation (20) written in the form

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\left(\frac{q_i}{\mathcal{T}}\right)_{,i} - \frac{q_i \mathcal{T}_{,i}}{\mathcal{T}^2} + \frac{W}{\mathcal{T}}. \tag{22}$$

Here, too, the first term refers to the heat exchange with the surroundings while the two remaining ones describe the entropy production in an elementary volume of the body.

The local statement of the second law of thermodynamics of irreversible processes leads to the Claussius–Duhem inequality

$$-\frac{q_i \mathcal{T}_{,i}}{\mathcal{T}^2} \geqslant 0, \tag{23}$$

or

$$\frac{\mathrm{d}S}{\mathrm{d}t} + \left(\frac{q_i}{\mathcal{T}}\right)_{,i} - \frac{W}{\mathcal{T}} \geqslant 0. \tag{24}$$

In what follows it will be more convenient to introduce the Helmholtz free energy $F = U - S\mathcal{T}$ which depends on the variables ε_{ij} and $\mathcal{T}$. Thus, we are led to the equation

$$\dot{F} = \dot{U} - S\dot{\mathcal{T}} - \dot{S}\mathcal{T} = \sigma_{ij}\dot{\varepsilon}_{ij} - S\dot{\mathcal{T}} - \dot{S}\mathcal{T} - q_{i,i} + W. \tag{25}$$

Eliminating the heat sources from the inequality (24) and equation (25) we obtain

$$-(\dot{F} + S\dot{\mathcal{T}}) + \sigma_{ij}\dot{\varepsilon}_{ij} - \frac{q_i \mathcal{T}_{,i}}{\mathcal{T}} \geqslant 0. \tag{26}$$

We have

$$\dot{F} = \frac{\partial F}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} + \frac{\partial F}{\partial \mathcal{T}}\dot{\mathcal{T}} + \frac{\partial F}{\partial \mathcal{T}_{,i}}\dot{\mathcal{T}}_{,i}, \tag{27}$$

where we assumed that $F \equiv F(\varepsilon_{ij}, \mathcal{T}, \mathcal{T}_{,i})$. Introducing (27) into (26) we arrive at the inequality

$$\left(\sigma_{ij} - \frac{\partial F}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} - \left(S + \frac{\partial F}{\partial \mathcal{T}}\right)\dot{\mathcal{T}} + \frac{\partial F}{\partial \mathcal{T}_{,i}}\dot{\mathcal{T}}_{,i} - \frac{q_i \mathcal{T}_{,i}}{\mathcal{T}} \geqslant 0. \tag{28}$$

This inequality should be satisfied for all rates $\dot{\varepsilon}_{ij}$, $\dot{\mathcal{T}}$, $\dot{\mathcal{T}}_{,i}$. Hence the coefficients of the above variables must vanish:

$$\sigma_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}}, \quad S = -\frac{\partial F}{\partial \mathcal{T}}, \quad \frac{\partial F}{\partial \mathcal{T}_{,i}} = 0. \tag{29}$$

The relation $(29)_3$ implies that the free energy is independent of the temperature gradient. We are therefore left with the inequality

$$-\frac{q_i \mathscr{T}_{,i}}{\mathscr{T}} \geqslant 0, \tag{30}$$

which is satisfied if we assume that

$$q_i = -\lambda_{ij} \mathscr{T}_{,j}. \tag{31}$$

The above is the Fourier law of heat conduction for an anisotropic body. The quantity $\Omega = -q_i \mathscr{T}_{,i}$ should be a positive definite quadratic form

$$\Omega = -q_i \mathscr{T}_{,i} = \lambda_{ij} \mathscr{T}_{,i} \mathscr{T}_{,j}. \tag{32}$$

In view of Silvester's theorem, the considered inequality imposes additional restrictions on the symmetric tensor of heat conduction λ_{ij}. For an isotropic body we have

$$q_i = -\lambda_0 \mathscr{T}_{,i}, \quad \lambda_0 > 0. \tag{33}$$

Let us now expand the free energy $F(\varepsilon_{ij}, \mathscr{T})$ into the Taylor series in the vicinity of the natural state ($\varepsilon_{ij} = 0$, $\mathscr{T} = T_0$)

$$F(\varepsilon_{ij}, \mathscr{T}) = F(0, T_0) + \frac{\partial F(0, T_0)}{\partial \varepsilon_{ij}} \varepsilon_{ij} + \frac{\partial F(0, T_0)}{\partial \mathscr{T}} (\mathscr{T} - T_0) + \tag{34}$$

$$+ \frac{1}{2}\left(\frac{\partial^2 F(0, T_0)}{\partial \varepsilon_{ij}\, \partial \varepsilon_{kl}} \varepsilon_{ij} \varepsilon_{kl} + 2 \frac{\partial^2 F(0, T_0)}{\partial \varepsilon_{ij}\, \partial \mathscr{T}} \varepsilon_{ij}(\mathscr{T} - T_0) + \frac{\partial^2 F(0, T_0)}{\partial \mathscr{T}^2} (\mathscr{T} - T_0)^2\right) + \ldots$$

Here $F(0, T_0)$ is the energy of the natural state, which we assume to be equal to zero. Moreover, we assume that the quantities $\partial F(0, T_0)/\partial \varepsilon_{ij}$ and $\partial F(0, T_0)/\partial \mathscr{T}$ vanish; they constitute the stress and the entropy in the natural state, respectively. Thus

$$F(\varepsilon_{ij}, \mathscr{T}) = \tfrac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \beta_{ij} \varepsilon_{ij} T - \tfrac{1}{2} m T^2 + \ldots, \tag{35}$$

where

$$T = \mathscr{T} - T_0, \quad c_{ijkl} = \frac{\partial^2 F(0, T_0)}{\partial \varepsilon_{ij}\, \partial \varepsilon_{kl}}, \quad \beta_{ij} = -\frac{\partial^2 F(0, T_0)}{\partial \varepsilon_{ij}\, \partial \mathscr{T}}, \quad m = -\frac{\partial^2 F(0, T_0)}{\partial \mathscr{T}^2}.$$

Retaining in the expansion (35) the quadratic terms only we obtain from (29)

$$\sigma_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}} = c_{ijkl} \varepsilon_{kl} - \beta_{ij} T. \tag{36}$$

Observe that

$$\left(\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}\right)_T = c_{ijkl}, \quad \left(\frac{\partial \sigma_{ij}}{\partial \mathscr{T}}\right)_\varepsilon = -\beta_{ij}. \tag{37}$$

Equation (36) is the law of elasticity, the so-called Duhamel–Neumann relation for an anisotropic body. The constants c_{ijkl} and β_{ij} refer here to the natural state of the body, i.e. the istothermal state. The constants c_{ijkl} are the elasticity constants for an anisotropic body and β_{ij}, as we shall see later, relate to its elastic and thermal properties.

For an isotropic body the free energy has a far simpler form, namely

$$F(\varepsilon_{ij}, \mathcal{T}) = \mu\varepsilon_{ij}\,\varepsilon_{ij}+\tfrac{1}{2}\,\lambda\varepsilon_{kk}\,\varepsilon_{nn}-\gamma\varepsilon_{kk}\,T-\tfrac{1}{2}\,mT^2\,. \tag{38}$$

This relation follows from the following considerations. Since the free energy is a scalar, every term in the right-hand side of (38) is a scalar, but from the components of the strain tensor only two independent second order scalars can be constructed, namely $\varepsilon_{ij}\,\varepsilon_{ij}$ and the square of the dilatation $\varepsilon_{kk}\,\varepsilon_{mm}$. The third term in the right-hand side of (38) contains the invariant ε_{kk}. This is due to the fact that from the tensor ε_{ij} only one scalar can be constructed, namely the dilatation ε_{kk}. The material constants μ, λ appearing in equation (28) (for a homogeneous body) are the Lamé constants. The quantities γ and m will be defined later.

The form (38) of the free energy can also be derived introducing into (35) the following isotropic tensor quantities

$$c_{ijkl} = \mu\,(\delta_{ik}\,\delta_{jl}+\delta_{il}\,\delta_{jk})+\lambda\delta_{ij}\,\delta_{kl}\,, \qquad \beta_{ij} = \gamma\delta_{ij}\,. \tag{39}$$

Observe that the relation (38) leads to the Duhamel–Neumann relation

$$\sigma_{ij} = \frac{\partial F}{\partial\varepsilon_{ij}} = 2\mu\varepsilon_{ij}+(\lambda\varepsilon_{kk}-\gamma T)\,\delta_{ij}\,, \tag{40}$$

valid for a homogeneous isotropic body. Performing the contraction of the stress tensor we obtain

$$\sigma_{kk} = 3\,(K-\gamma T)\,, \qquad K = \lambda+\tfrac{2}{3}\,\mu\,. \tag{41}$$

Solving equation (40) with respect to ε_{ij} and (41) with respect to ε_{kk} we have

$$\varepsilon_{ij} = \frac{\gamma}{3K}\,T\delta_{ij}+\frac{1}{2\mu}\left(\sigma_{ij}-\frac{\lambda\sigma_{kk}}{2\mu+3\lambda}\,\delta_{ij}\right), \tag{42}$$

$$\varepsilon_{kk} = \frac{\gamma}{K}\,T+\frac{\sigma_{kk}}{3K}\,. \tag{43}$$

Consider now a free expansion of a volume element with no tractions on its boundary. Equation (42) and (43) lead to the relations

$$\varepsilon^0_{ij} = \frac{\gamma}{3K}\,T\delta_{ij}\,, \qquad \varepsilon^0_{kk} = \frac{\gamma}{K}\,T. \tag{44}$$

It is evident that a change of volume is proportional to the temperature T, the proportionality coefficient being the coefficient of the volume expansion $\alpha = \gamma/K$. Denoting by α_t the coefficient of linear volume expansion, we obtain the following form of the relation (44):

$$\varepsilon^0_{ij} = \alpha_t\,T\delta_{ij}\,, \qquad \varepsilon^0_{kk} = 3\alpha_t\,T. \tag{44'}$$

These relations express the property of an isotropic body, namely an elementary volume of a body undergoes during increase of the temperature a volume change only, the shape of the volume remaining the same.

We have thus defined the quantity $\gamma = (3\lambda+2\mu)\,\alpha_t = 3K\alpha_t$ appearing in the expression (38) for the free energy and in the Duhamel–Neumann relations (40). Observe that the formulae (40) and (41) imply the relations

$$(45)\qquad \left(\frac{\partial\sigma_{ij}}{\partial\mathscr{T}}\right)_\varepsilon = -\gamma\delta_{ij}\,, \quad \left(\frac{\partial\varepsilon_{kk}}{\partial\sigma_{kk}}\right)_T = \frac{1}{3K}\,, \quad \left(\frac{\partial\varepsilon_{kk}}{\partial T}\right)_\varepsilon = 3\alpha_t\,.$$

It should be emphasized that the quantities μ, λ, K appearing in the formulae (40)–(45) refer to the isothermal state.

Frequently the following form of equations (42) and (43) is used:

$$(42')\qquad \varepsilon_{ij} = \alpha_t T\delta_{ij} + \frac{1}{2G}\left(\sigma_{ij} - \frac{\nu\sigma_{kk}}{1+\nu}\,\delta_{ij}\right),$$

$$(43')\qquad \varepsilon_{kk} = 3\alpha_t\, T + \frac{1-2\nu}{E}\,\sigma_{kk}\,.$$

Solving (42′) with respect to the stresses we have

$$(40')\qquad \sigma_{ij} = 2G\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\left(\varepsilon_{kk} - \frac{1+\nu}{\nu}\,\alpha_t T\right)\delta_{ij}\right],$$

where $G = \mu$ is the shear modulus, ν is the Poisson ratio and E is the Young modulus; we have

$$G = \mu\,, \quad E = \frac{\mu\,(3\lambda+2\mu)}{\lambda+\mu}\,, \quad \nu = \frac{\lambda}{2\,(\lambda+\mu)}$$

and

$$\lambda = \frac{\nu E}{(1+\nu)\,(1-2\nu)}\,, \quad \mu = \frac{E}{2\,(1+\nu)}\,.$$

Let us now return to the expression for the free eneregy (38). We neglect here all powers higher than the second in ε_{ij} and collect all terms depending on the temperature in one term $C(T)$; thus we confine ourselves to the expression

$$(46)\qquad F(\varepsilon_{ij}, \mathscr{T}) = \mu\varepsilon_{ij}\,\varepsilon_{ij} + \frac{\lambda}{2}\,\varepsilon_{kk}\,\varepsilon_{nn} - \gamma\varepsilon_{kk}\,T + C(T)\,, \qquad T = \mathscr{T} - T_0\,.$$

Making use of the relations (29), in view of the expression (46) for the free energy we obtain

$$(47)\qquad \sigma_{ij} = \frac{\partial F}{\partial\varepsilon_{ij}} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma T)\,\delta_{ij}\,.$$

$$(48)\qquad S = -\frac{\partial F}{\partial\mathscr{T}} = \gamma\varepsilon_{kk} - \frac{\partial C(T)}{\partial\mathscr{T}}\,.$$

It remains to determine the second term of the relation (48). Since $S = S(\varepsilon_{ij}, \mathcal{T})$ we have

$$\mathrm{d}S = \left(\frac{\partial S}{\partial \varepsilon_{ij}}\right)_{\mathcal{T}} \mathrm{d}\varepsilon_{ij} + \left(\frac{\partial S}{\partial \mathcal{T}}\right)_{\varepsilon} \mathrm{d}\mathcal{T}. \tag{49}$$

The quantity $\mathcal{T}\left(\frac{\partial S}{\partial \mathcal{T}}\right)_{\varepsilon}$ is the measure of heat generated in a unit volume of the body during a change of the temperature at a constant strain. We denote this quantity by c_ε[1] and call it the specific heat at constant strain.

On the other hand, in view of (48)

$$\mathrm{d}S = \gamma\, \mathrm{d}\varepsilon_{kk} - \frac{\mathrm{d}^2 C(T)}{\partial \mathcal{T}^2} \mathrm{d}\mathcal{T}, \tag{50}$$

whence

$$c_\varepsilon = \mathcal{T}\left(\frac{\partial S}{\partial \mathcal{T}}\right)_{\varepsilon} = -\mathcal{T} \frac{\partial^2 C(T)}{\partial \mathcal{T}^2}. \tag{51}$$

We now integrate the above expression twice with respect to $\mathcal{T}$. Since in the natural state $F = 0$ and $S = 0$ the integration constants vanish and we obtain the integral

$$C(T) = -\int_{T_0}^{T} \mathrm{d}\mathcal{T} \int_{T_0}^{T} \frac{c_\varepsilon\, \partial \mathcal{T}}{\mathcal{T}}. \tag{52}$$

The term $\frac{\partial C(T)}{\partial \mathcal{T}}$ appearing in (48) takes the form

$$-\frac{\mathrm{d}C(T)}{\partial \mathcal{T}} = -c_\varepsilon \log \frac{\mathcal{T}}{T_0}.$$

Hence

$$S = \gamma \varepsilon_{kk} + c_\varepsilon \log\left(1 + \frac{T}{T_0}\right), \qquad \mathcal{T} = T + T_0. \tag{48'}$$

We have introduced before a restriction for the temperature increase, $|T/T_0| \ll 1$. Expanding $\log(1 + T/T_0)$ into an infinite power series and retaining the first term only, we arrive at the following formulae for the free energy and the entropy:

$$F = \mu \varepsilon_{ij} \varepsilon_{ij} + \frac{\lambda}{2} \varepsilon_{kk} \varepsilon_{nn} - \gamma \varepsilon_{kk} T - \frac{c_\varepsilon}{2T_0} T^2, \tag{53}$$

$$S = \gamma \varepsilon_{kk} + \frac{c_\varepsilon}{T_0} T. \tag{54}$$

Comparing (38) and (53) we find that $m = c_\varepsilon / T_0$.

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[1] Observe that $c_\varepsilon = \varrho c$ where c is the specific heat referred to the unit mass of the body.

I.2. The fundamental differential equations of thermoelasticity

In a solid the heat transfer occurs by means of the heat conduction understood as the heat transfer from places with higher temperature to places with lower temperature. This process is spontaneous and irreversible, connected with an entropy production. The equation of heat conduction can be derived from the entropy balance

$$\mathscr{T}\dot{S} = -q_{i,i}+W. \tag{1}$$

Introducing into (1) the Fourier law of heat conduction [see (33) in I. 1]

$$q_i = -\lambda_0 \mathscr{T}_{,i} = -\lambda_0 T_{,i}, \tag{2}$$

and the constitutive relation for the entropy [the formula (54) in I. 1]

$$S = \gamma\varepsilon_{kk}+\frac{c_\varepsilon}{T_0}T, \tag{3}$$

we obtain the equation

$$\mathscr{T}\left(\gamma\dot{\varepsilon}_{kk}+\frac{c_\varepsilon}{T_0}\dot{T}\right) = \lambda_0 T_{,ii}+W. \tag{4}$$

The above equation is non-linear. We linearize it by replacing in the left-hand side $\mathscr{T}$ by T_0 which is permissible in view of the assumption $|T/T_0| \ll 1$ and thus we arrive at the heat conduction equation in the form

$$T_{,ii}-\frac{1}{\varkappa}\dot{T}-\eta\dot{\varepsilon}_{kk} = -\frac{W}{\lambda_0}, \quad \varkappa = \frac{\lambda_0}{c_\varepsilon}, \quad \eta = \frac{\gamma T_0}{\lambda_0}. \tag{5}$$

In what follows, for convenience we shall use the following form of the heat conduction equation:

$$\left(\nabla^2-\frac{1}{\varkappa}\frac{\partial}{\partial t}\right)T-\eta\dot{\varepsilon}_{kk} = -\frac{Q}{\varkappa}, \quad Q = \frac{\varkappa W}{\lambda_0}. \tag{6}$$

Observe that the extended heat conduction equation contains a term coupling the temperature increase with the rate of dilatation of the body.

The heat conduction equation is completed by the equation of motion in displacements. Let us introduce into the equations of motion

$$\sigma_{ji,j}+X_i = \varrho\ddot{u}_i \tag{7}$$

the Duhamel–Neumann relations

$$\sigma_{ij} = 2\mu\varepsilon_{ij}+(\lambda\varepsilon_{kk}-\gamma T)\,\delta_{ij}. \tag{8}$$

Expressing the strain in terms of the displacement

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}) \tag{9}$$

we arrive at the following form of equation (7):

$$\mu u_{i,jj}+(\lambda+\mu)\,u_{j,ji}+X_i = \gamma T_{,i}+\varrho\ddot{u}_i. \tag{10}$$

Equations (5) and (10) constitute a complete set of the differential equations of thermoelasticity. The equations in this set are coupled. The equations of motion contain besides the displacement u_i the temperature increase T whereas the heat conduction equation contains besides the temperature T the rate of dilatation $\dot{\varepsilon}_{kk}$.

Equations (5) and (10) can also be written in the vector form

$$\mu\,\nabla^2\mathbf{u}+(\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u}+\mathbf{X} = \gamma\operatorname{grad}T+\varrho\,\ddot{\mathbf{u}}\,, \tag{11}$$

$$\left(\nabla^2-\frac{1}{\varkappa}\frac{\partial}{\partial t}\right)T-\eta\operatorname{div}\dot{\mathbf{u}} = -\frac{Q}{\varkappa}\,. \tag{12}$$

The sources of the deformation and of the coupled temperature field are body forces and heat sources, or the external forces and the thermal interactions of the solid with its surrounding medium. The strain field and the temperature field, arise as a result of the action of any of these sources. We have yet to state the boundary and initial conditions for the system of equations (11) and (12).

The boundary conditions may be of various types. Thus, the thermal boundary condition expresses the action of the surrounding heat environment on the body and may take on the bounding surface A the following forms:

1) The temperature T is prescribed on A in terms of position and time

$$T = k(\mathbf{x}, t)\,,\quad \mathbf{x}\in A,\ t>0\,. \tag{13}$$

2) The temperature gradient $\partial T/\partial n$ is given on A in terms of position and time

$$\frac{\partial T}{\partial n} = k(\mathbf{x}, t)\,,\quad \mathbf{x}\in A\,,\ t>0\,. \tag{14}$$

3) The function

$$\left(\frac{\partial}{\partial n}+\alpha\right)T(\mathbf{x}, t) = f(\mathbf{x}, t)\,,\quad \mathbf{x}\in A\,,\ t>0 \tag{15}$$

is given; here α is a constant.

Condition 2) corresponds to a flow of heat through the surface A. The case $\partial T/\partial n = 0$ describes a thermal insulation over the surface bounding the body. Finally, condition 3) refers to a free heat exchange over the surface A. There occur also cases of mixed boundary conditions in which, over different parts of the surface, different boundary conditions take place.

Furthermore, on the surface A external loadings p_i ($i = 1, 2, 3$) may be given, or else the displacements u_i may be prescribed in terms of position and time. In the former case we have

$$p_i(\mathbf{x}, t) = \sigma_{ji}\,n_j = \mu\,(u_{i,j}+u_{j,i})\,n_j+(\lambda\varepsilon_{kk}-\gamma T)\,n_i\,,\quad \mathbf{x}\in A\,,\ t>0\,, \tag{16}$$

while in the latter case the boundary conditions take the form

$$u_i = U_i(\mathbf{x}, t)\,,\quad \mathbf{x}\in A\,,\ t>0\,. \tag{17}$$

As before, mixed boundary conditions may occur, on a part of the surface A tractions being given, while on the remaining part the displacement being prescribed. In solving the differential equations of thermoelasticity we must state the initial conditions. For the temperature the initial condition consists in prescribing its distribution at the initial instant $t = 0$ in terms of position, i.e.

$$T(\mathbf{x}, 0) = l(\mathbf{x}), \quad \mathbf{x} \in B, \ t = 0. \tag{18}$$

The deformation of the body is determined at the initial instant by the functions

$$u_i(\mathbf{x}, 0) = g_i(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = f_i(\mathbf{x}), \quad \mathbf{x} \in B, \ t = 0, \ i = 1, 2, 3, \tag{19}$$

where l, g_i, f_i are known functions.

I.3. The wave equations of thermoelasticity

Consider now the system of differential equations of thermoelasticity in the vector form

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mathbf{X} = \gamma \operatorname{grad} T + \varrho \ddot{\mathbf{u}}, \tag{1}$$

$$\nabla^2 T - \frac{1}{\varkappa} \dot{T} - \eta \operatorname{div} \dot{\mathbf{u}} = -\frac{Q}{\varkappa}. \tag{2}$$

Assume first that the heat sources and the body forces are absent. Then equations (1) and (2) are homogeneous. Performing the operation of rotation on equation (1) and denoting by $\mathbf{\Omega}$ the rotation of the vector $\mathbf{u}$, we arrive at the wave equation

$$\left(\nabla^2 - \frac{1}{c_2^2} \partial_t^2\right) \mathbf{\Omega} = 0, \quad \mathbf{\Omega} = \operatorname{curl} \mathbf{u}, \quad c_2 = \left(\frac{\mu}{\varrho}\right)^{1/2}. \tag{3}$$

Evidently, the vector $\mathbf{\Omega}$ is propagated with the velocity c_2 of the transverse wave without producing a temperature change. Volume elements of the body do not undergo dilatation; only shears occur and they do not lead to any temperature change. The rotational wave is not damped and does not undergo dispersion.

Let us now apply to the homogeneous equation (1) the operation of divergence. Introducing the notation $e = \operatorname{div} \mathbf{u}$ we obtain the wave equation

$$\left(\nabla^2 - \frac{1}{c_1^2} \partial_t^2\right) e = m \nabla^2 T, \quad c_1 = \left(\frac{\lambda + 2\mu}{\varrho}\right)^{1/2}, \quad m = \frac{\gamma}{\lambda + 2\mu}, \tag{4}$$

which describes the dilatational wave. It follows from (4) that the latter is coupled with the temperature T and hence with the homogeneous equation (2)

$$\left(\nabla^2 - \frac{1}{\varkappa} \partial_t\right) T - \eta \dot{e} = 0. \tag{5}$$

The coupling of the above equations indicates that the propagation of the dilatational wave is accompanied by a generation of heat. In a dilatational wave the energy is partly converted into heat producing temperature changes.

Eliminating from equations (4) and (5) the function T we arrive at the following wave equation describing the propagation of dilatational waves in a thermoelastic medium

$$\left[\left(\nabla^2-\frac{1}{\varkappa}\partial_t\right)\left(\nabla^2-\frac{1}{c_1^2}\partial_t^2\right)-\eta m\,\nabla^2\partial_t\right]e=0. \tag{6}$$

We shall see later that the dilatational wave undergoes damping and dispersion.

Let us now return to the non-homogeneous equations (1) and (2) and let us decompose the displacement and the body force vectors into their potential and solenoidal parts

$$\mathbf{u}=\operatorname{grad}\Phi+\operatorname{curl}\mathbf{\Psi}, \quad \operatorname{div}\mathbf{\Psi}=0, \tag{7}$$

$$\mathbf{X}=\varrho\,(\operatorname{grad}\vartheta+\operatorname{curl}\chi), \quad \operatorname{div}\chi=0. \tag{8}$$

The assumptions (7) and (8) served in the classical dynamic elasticity for the decomposition of the vibrations into their longitudinal and transverse modes. The potential Φ refers to longitudinal vibrations connected with volume changes in the body; the motion of the particles taking place in the direction parallel to the direction of the wave propagation. The vector $\mathbf{\Psi}$ on the other hand refers to the propagation of transverse waves producing changes of shape only. The assumptions (7) and (8) in the thermoelastic medium lead also to the decomposition of waves into longitudinal and transverse, for substituting the relations (7) and (8) into equations (1) and (2) leads to the relations

$$\left(\nabla^2-\frac{1}{\varkappa}\partial_t\right)T-\eta\,\nabla^2\dot{\Phi}=-\frac{Q}{\varkappa}, \tag{9}$$

$$\left(\nabla^2-\frac{1}{c_1^2}\partial_t^2\right)\Phi=mT-\frac{1}{c_1^2}\vartheta, \tag{10}$$

$$\left(\nabla^2-\frac{1}{c_2^2}\partial_t^2\right)\mathbf{\Psi}=-\frac{1}{c_2^2}\chi. \tag{11}$$

Introducing the following notations for the differential operators

$$\Box_\alpha^2=\nabla^2-\frac{1}{c_\alpha^2}\partial_t^2, \quad \alpha=1,2, \qquad D=\nabla^2-\frac{1}{\varkappa}\partial_t$$

we represent the system of equations (7)–(9) in the form

$$DT-\eta\,\nabla^2\dot{\Phi}=-\frac{Q}{\varkappa}, \tag{12}$$

$$\Box_1^2\,\Phi=mT-\frac{1}{c_1^2}\vartheta, \tag{13}$$

$$\Box_2^2\,\mathbf{\Psi}=-\frac{1}{c_2^2}\chi. \tag{14}$$

Eliminating from the above system the temperature T we obtain two wave equations

$$(\Box_1^2 D - \eta m \partial_t \nabla^2)\Phi = -\frac{mQ}{\varkappa} - \frac{1}{c_1^2} D\vartheta, \tag{15}$$

$$\Box_2^2 \boldsymbol{\Psi} = -\frac{1}{c_2^2} \boldsymbol{\chi}. \tag{16}$$

The first of the above equations describes the longitudinal dilatational wave, while the second describes the transverse wave.

Consider first wave propagation in an infinite thermoelastic medium. Assume that there act heat sources and body forces in a bounded region, described by the potential ($\mathbf{X} = \varrho \operatorname{grad} \vartheta$, $\boldsymbol{\chi} = 0$). only. In this particular case $\boldsymbol{\Psi} = 0$ in the entire medium; in an infinite thermoelastic body there are dilatational waves only, described by equation (15) and the potential Φ.

After having determined a particular solution of equation (15), we find the displacement and the strain from the relations

$$u_i = \partial_i \Phi, \quad \varepsilon_{ij} = \partial_i \partial_j \Phi, \quad e = \nabla^2 \Phi. \tag{17}$$

The knowledge of the function Φ makes it possible to calculate the temperature T from the formula (10). The stresses are found from the Duhamel–Neumann relations if we express the strain entering these relations by means of (17) and make use of equation (10). Thus, we obtain

$$\sigma_{ij} = 2\mu(\partial_i \partial_j \Phi - \delta_{ij} \nabla^2 \Phi) + \varrho(\ddot{\Phi} - \vartheta)\delta_{ij}. \tag{18}$$

Observe that the right-hand side of equation (15) contains the causes of generation of the longitudinal waves, i.e. the heat source and the potential part of the body forces. The action of the forces $\mathbf{X} = \varrho \operatorname{grad} \vartheta$ produces an effect analogous to that of heat sources. We shall find later when investigating in detail longitudinal waves, that these waves are damped and undergo a dispersion.

If in an infinite thermoelastic space heat sources are absent ($Q = 0$) and, moreover, $\vartheta = 0$, $\mathbf{X} = \varrho \operatorname{curl} \boldsymbol{\chi}$ then there exist transverse waves only, described by equation (16), propagated with velocity c_2; these waves are not damped and do not produce any temperature changes. Since in this case $\Phi = 0$, $T = 0$ and $e = \nabla^2 \Phi = 0$ the entropy S also vanishes. The displacements connected with the transverse wave are given by the formulae

$$u_1 = \partial_2 \psi_3 - \partial_3 \psi_2, \quad u_2 = \partial_3 \psi_1 - \partial_1 \psi_3, \quad u_3 = \partial_1 \psi_2 - \partial_2 \psi_1. \tag{19}$$

The stresses are calculated by means of the formulae

$$\sigma_{ij} = 2\mu \varepsilon_{ij}. \tag{20}$$

In principle, in a bounded body there exist simultaneously longitudinal and transverse waves. In this case the solution of equations (12)–(14) consists of two parts, namely the particular solutions T', Φ', ψ' of the non-homogeneous equations

$$DT' - \eta \nabla^2 \dot{\Phi}' = -\frac{Q}{\varkappa}, \tag{21}$$

$$\Box_1^2 \Phi' = mT' - \frac{1}{c_1^2}\vartheta, \tag{22}$$

$$\Box_2^2 \mathbf{\Psi}' = -\frac{1}{c_2^2}\chi, \tag{23}$$

and the general solutions T'', Φ'', ψ'' of the homogeneous equations

$$DT'' - \eta \nabla^2 \dot{\Phi}'' = 0, \tag{24}$$

$$\Box_1^2 \Phi'' - mT'' = 0, \tag{25}$$

$$\Box_2^2 \mathbf{\Psi}'' = 0. \tag{26}$$

The functions T', Φ', ψ' can be assumed to be the particular solutions in an infinite space or else to be functions which satisfy a part of the boundary conditions on the surface A bounding the body. The solution T'', Φ'', ψ'' is chosen in such a way that the functions $T = T'+T''$, $\Phi = \Phi'+\Phi''$, $\psi = \psi'+\psi''$ satisfy all boundary conditions.

Consider again the homogeneous system of the differential equations of thermoelasticity (1) and (2)

$$\mu \nabla^2 \mathbf{u} + (\lambda+\mu) \operatorname{grad} \operatorname{div} \mathbf{u} = \gamma \operatorname{grad} T + \varrho \ddot{\mathbf{u}}, \tag{27}$$

$$\nabla^2 T - \frac{1}{\varkappa}\dot{T} = \eta \operatorname{div} \dot{\mathbf{u}}. \tag{28}$$

The heat conduction equation has been written in such a form that the term containing the time derivative of the dilatation is in the right-hand side of the equation. We shall confine our considerations to an infinite thermoelastic medium. We regard the function $\eta \operatorname{div} \dot{\mathbf{u}}$ as a heat source in the classical heat conduction equation. Making use of the Green function of the latter we can represent the solution of equation (28) with the homogeneous initial condition in the form

$$T(P, t) = \frac{\eta}{8\pi^{3/2}\varkappa^{5/2}} \int_0^t \mathrm{d}\tau \int_B (t-\tau)^{-3/2} \exp\left(-\frac{\varrho^2}{4\varkappa(t-\tau)}\right) \frac{\partial}{\partial\tau} \operatorname{div} \mathbf{u}\, \mathrm{d}V(\xi), \tag{29}$$

$$\varrho^2 = (x_i - \xi_i)(x_i - \xi_i), \quad \mathbf{x} \equiv (x_1, x_2, x_3), \quad B = E^3.$$

Integrating by parts the expression (29) and substituting (27) we arrive at the following integro-differential equation for $\mathbf{u}$

$$\mu \nabla^2 \mathbf{u} + (\lambda+\mu) \operatorname{grad} \operatorname{div} \mathbf{u} - \varrho \ddot{\mathbf{u}} \tag{30}$$

$$= -\frac{\eta\gamma}{8\pi^{3/2}\varkappa^{5/2}} \operatorname{grad} \int_0^t \mathrm{d}\tau \int_B \operatorname{div} \mathbf{u} \frac{\partial}{\partial\tau}\left[(t-\tau)^{-3/2} \exp\left(-\frac{\varrho^2}{4\varkappa(t-\tau)}\right)\right] \mathrm{d}V(\xi).$$