Edited by F F Mazda

Electronics Engineer's Reference Book

Sixth edition

Butterworth International Editions

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Sixth Edition

Edited by
F F Mazda
DFH, MPhil, CEng, FIEE, DMS, MBIM
With specialist contributors

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Preface

On taking over as editor of the fifth edition of *The Electronics Engineer's Reference Book*, I did what all new editors do; out came my broom and I started sweeping! The changes from the fourth edition to the fifth were dramatic, and it has been grafitying to note the success which the book has subsequently had on both sides of the Atlantic. The whole text was laid out in a larger, more convenient to use format; the information was grouped together into five parts; and much of the old material was removed in order to make way for thirty-two new chapters.

Having completed, what I considered to be, such an Herculean task, I naturally expected to rest on my laurels! So when the time came around to produce the sixth edition, I brought out my pen in readiness to dot the odd i and cross the occasional t. Alas, how mistaken I was! As someone famous (whose name escapes me for the moment) once said, 'Time and Technology wait for no man'. So the son of Hercules was recruited to work on the sixth edition!

For this new edition I have retained the same five groupings, which have proved so successful in the fifth edition. As before the first part contains a synopsis of mathematical and electrical techniques used in the analysis of electronic systems. Part two covers physical phenomena, such as electricity, light and radiation, often met with in electronic systems. The third part of the book contains chapters on basic electronic components and materials, the building blocks of any electronic design. The information presented covers a wide spectrum of devices, from the humble resistor to the glamorous microprocessor.

Part four has chapters on electronic circuit design and on instrumentation. A range of design techniques are covered from linear to digital circuits, and from signal power levels to those in the megawatt region. The fifth part of the book contains topics, such as radar and computers, which form well recognised application areas of electronics.

All the chapters have been revised in going from the fifth to the sixth edition, many of them being extensively rewritten to ensure that they are up to date. Inevitably some of the older material has had to be deleted, to make room for new text. Six new chapters have been added, on topics which have gained considerably in importance since the last edition. These are on: application specific integrated circuits, such as gate arrays and standard cells, which have rapidly become an essential element in any economical electronic design; computer aided design techniques, involving the full range of design tools and methodology from designs on silicon chips to those on printed circuit boards; digital system analysis, covering the instrumentation and measurement techniques used to analyse digital circuits, including those based on microprocessors; software engineering which, with the importance of software in all major projects, plays a vital part in electronic designs; local area networks, describing the latest developments in networks based on the ISO open system interconnect (OSI) standards; and integrated services digital network (ISDN), a topic which no self-respecting modern book on electronics could be without!

A large number of sub-editors from Butterworths worked on the sixth edition of *The Electronics Engineer's Reference Book*. To them go my thanks for the dedication which they have shown in labouring tirelessly in the background, and even though I do not mention them by name they will know the genuine gratitude I have for their hard work and support.

And finally, as always, the last word must be reserved for the many, many authors who made this book possible. Thank you all for the excellent manuscripts you have produced; for accepting the tight schedules you have had to work to; and for the skilful excuses you have employed when late manuscripts were being chased by the sub-editors!

FFM
Bishop's Stortford
March 1989

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Part 1

Techniques

1 Trigonometric Functions and General Formulae

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Contents

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1.1 Mathematical signs and symbols

Sign, symbol	Quantity
$=$	equal to
$\neq$ $\neq$	not equal to
$\equiv$	identically equal to
$\triangleq$	corresponds to
$\approx$	approximately equal to
$\rightarrow$	approaches
$\simeq$	asymptotically equal to
$\sim$	proportional to
∞	infinity
$<$	smaller than
$>$	larger than
$\leq$ $\leqslant$ $\leqq$	smaller than or equal to
$\geq$ $\geqslant$ $\geqq$	larger than or equal to
$\ll$	much smaller than
$\gg$	much larger than
$+$	plus
$-$	minus
$.$ $\times$	multiplied by
$\frac{a}{b}$ a/b	a divided by b
$\lvert a\rvert$	magnitude of a
a^n	a raised to the power n
$a^{1/2}$ $\sqrt{a}$	square root of a
$a^{1/n}$ $\sqrt[n]{a}$	nth root of a
$\bar{a}$ $\langle a\rangle$	mean value of a
$p!$	factorial p, $1\times2\times3\times\ldots\times p$
$\binom{n}{p}$	binomial coefficient, $\dfrac{n(n-1)\ldots(n-p+1)}{1\times2\times3\times\ldots\times p}$
Σ	sum
Π	product
$f(x)$	function f of the variable x
$[f(x)]_a^b$	$f(b)-f(a)$
$\lim\limits_{x\to a} f(x)$; $\lim_{x\to a} f(x)$	the limit to which $f(x)$ tends as x approaches a
Δx	delta x = finite increment of x
δx	delta x = variation of x
$\frac{df}{dx}$; df/dx; $f'(x)$	differential coefficient of $f(x)$ with respect to x
$\frac{d^nf}{dx^n}$; $f^{(n)}(x)$	differential coefficient of order n of $f(x)$
$\frac{\partial f(x,y,\ldots)}{\partial x}$; $\left(\frac{\partial f}{\partial x}\right)_{y\ldots}$	partial differential coefficient of $f(x,y,\ldots)$ with respect to x, when $y,\ldots$ are held constant
df	the total differential of f
$\int f(x)dx$	indefinite integral of $f(x)$ with respect to x
$\int_a^b f(x)dx$	definite integral of $f(x)$ from $x=a$ to $x=b$
e	base of natural logarithms
e^x; $\exp x$	e raised to the power x
$\log_a x$	logarithm to the base a of x
$\lg x$; $\log x$; $\log_{10} x$	common (Briggsian) logarithm of x
$\mathrm{lb}\, x$; $\log_2 x$	binary logarithm of x
$\sin x$	sine of x
$\cos x$	cosine of x
$\tan x$; $\mathrm{tg}\, x$	tangent of x
$\cot x$; $\mathrm{ctg}\, x$	cotangent of x
$\sec x$	secant of x
$\mathrm{cosec}\, x$	cosecant of x
$\arcsin x$	arc sine of x
$\arccos x$	arc cosine of x
$\arctan x$, $\mathrm{arctg}\, x$	arc tangent of x
$\mathrm{arccot}\, x$, $\mathrm{arcctg}\, x$	arc cotangent of x
$\mathrm{arcsec}\, x$	arc secant of x
$\mathrm{arccosec}\, x$	arc cosecant of x
$\sinh x$	hyperbolic sine of x
$\cosh x$	hyperbolic cosine of x
$\tanh x$	hyperbolic tangent of x
$\coth x$	hyperbolic cotangent of x
$\mathrm{sech}\, x$	hyperbolic secant of x
$\mathrm{cosech}\, x$	hyperbolic cosecant of x
$\mathrm{arsinh}\, x$	inverse hyperbolic sine of x
$\mathrm{arcosh}\, x$	inverse hyperbolic cosine of x
$\mathrm{artanh}\, x$	inverse hyperbolic tangent of x
$\mathrm{arcoth}\, x$	inverse hyperbolic cotangent of x
$\mathrm{arsech}\, x$	inverse hyperbolic secant of x
$\mathrm{arcosech}\, x$	inverse hyperbolic cosecant of x
i, j	imaginary unity, $i^2=-1$
$\mathrm{Re}\, z$	real part of z
$\mathrm{Im}\, z$	imaginary part of z
$\lvert z\rvert$	modulus of z
$\arg z$	argument of z
z^*	conjugate of z, complex conjugate of z
$\tilde{A}$, A', A^t	transpose of matrix A
A^*	complex conjugate matrix of matrix A
A^+	Hermitian conjugate matrix of matrix A
$\mathbf{A}$, $\mathbf{a}$	vector
$\lvert\mathbf{A}\rvert$, A	magnitude of vector
$\mathbf{A}\cdot\mathbf{B}$	scalar product
$\mathbf{A}\times\mathbf{B}$, $\mathbf{A}\wedge\mathbf{B}$	vector product
∇	differential vector operator
$\nabla\varphi$, grad φ	gradient of φ
$\nabla\cdot A$, div $\mathbf{A}$	divergence of $\mathbf{A}$
$\nabla\times\mathbf{A}$, $\nabla\wedge\mathbf{A}$, curl $\mathbf{A}$, rot $\mathbf{A}$	curl of $\mathbf{A}$
$\nabla^2\varphi$, $\Delta\varphi$	Laplacian of φ

1.2 Trigonometric formulae

$$\sin^2 A+\cos^2 A=\sin A\,\mathrm{cosec}\,A=1$$

$$\sin A=\frac{\cos A}{\cot A}=\frac{1}{\mathrm{cosec}\,A}=(1-\cos^2 A)^{1/2}$$

$$\cos A=\frac{\sin A}{\tan A}=\frac{1}{\sec A}=(1-\sin^2 A)^{1/2}$$

$$\tan A=\frac{\sin A}{\cos A}=\frac{1}{\cot A}$$

$$1+\tan^2 A=\sec^2 A$$

$$1+\cot^2 A=\mathrm{cosec}^2 A$$

$$1-\sin A=\mathrm{coversin}\,A$$

$$1-\cos A=\mathrm{versin}\,A$$

$$\tan\tfrac{1}{2}\theta=t;\quad \sin\theta=2t/(1+t^2);\quad \cos\theta=(1-t^2)/(1+t^2)$$

$\cot A = 1/\tan A$

$\sec A = 1/\cos A$

$\operatorname{cosec} A = 1/\sin A$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$$

$\sin A \pm \sin B = 2 \sin \tfrac{1}{2}(A \pm B) \cos \tfrac{1}{2}(A \mp B)$

$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)$

$\cos A - \cos B = 2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(B - A)$

$$\tan A \pm \tan B = \frac{\sin(A \pm B)}{\cos A \cos B}$$

$$\cot A \pm \cot B = \frac{\sin(B \pm A)}{\sin A \sin B}$$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$\cos^2 A - \sin^2 B = \cos(A + B)\cos(A - B)$

$\tan 2A = 2 \tan A/(1 - \tan^2 A)$

$$\sin \tfrac{1}{2}A = \left(\frac{1 - \cos A}{2}\right)^{1/2}$$

$$\cos \tfrac{1}{2}A = \pm\left(\frac{1 + \cos A}{2}\right)^{1/2}$$

$$\tan \tfrac{1}{2}A = \frac{\sin A}{1 + \cos A}$$

$\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$

$\cos^2 A = \tfrac{1}{2}(1 + \cos 2A)$

$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$$

$$\tan \tfrac{1}{2}(A \pm B) = \frac{\sin A \pm \sin B}{\cos A + \cos B}$$

$$\cot \tfrac{1}{2}(A \pm B) = \frac{\sin A \pm \sin B}{\cos B - \cos A}$$

1.3 Trigonometric values

Angle	*0°*	*30°*	*45°*	*60°*	*90°*	*180°*	*270°*	*360°*
Radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
Sine	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	0	-1	0
Cosine	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	-1	0	1
Tangent	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞	0	∞	0

1.4 Approximations for small angles

$\sin\theta = \theta - \theta^3/6$; $\cos\theta = 1 - \theta^2/2$; $\tan\theta = \theta + \theta^3/3$;
(θ in radians)

1.5 Solution of triangles

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

where A, B, C and a, b, c are shown in *Figure 1.1*. If $s = \tfrac{1}{2}(a + b +$

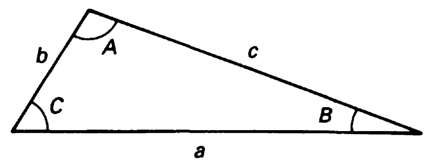

Figure 1.1 Triangle

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \qquad \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$$

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \qquad \tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

1.6 Spherical triangle

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$\cos a = \cos b \cos c + \sin b \sin c \cos A$

$\cos b = \cos c \cos a + \sin c \sin a \cos B$

$\cos c = \cos a \cos b + \sin a \sin b \cos C$

where A, B, C and a, b, c are now as in *Figure 1.2*.

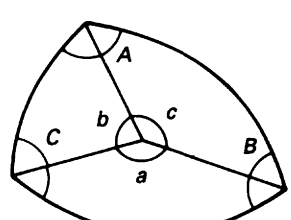

Figure 1.2 Spherical triangle

1.7 Exponential form

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}\qquad \cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$e^{i\theta}=\cos\theta+i\sin\theta\qquad e^{-i\theta}=\cos\theta-i\sin\theta$$

1.8 De Moivre's theorem

$$(\cos A+i\sin A)(\cos B+i\sin B)$$
$$=\cos(A+B)+i\sin(A+B)$$

1.9 Euler's relation

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta=e^{in\theta}$$

1.10 Hyperbolic functions

$$\sinh x=(e^x-e^{-x})/2\qquad \cosh x=(e^x+e^{-x})/2$$

$$\tanh x=\sinh x/\cosh x$$

Relations between hyperbolic functions can be obtained from the corresponding relations between trigonometric functions by reversing the sign of any term containing the product or implied product of two sines, e.g.:

$$\cosh^2 A-\sinh^2 A=1$$

$$\cosh 2A=2\cosh^2 A-1=1+2\sinh^2 A$$
$$=\cosh^2 A+\sinh^2 A$$

$$\cosh(A\pm B)=\cosh A\cosh B\pm\sinh A\sinh B$$

$$\sinh(A\pm B)=\sinh A\cosh B\pm\cosh A\sinh B$$

$$e^x=\cosh x+\sinh x\qquad e^{-x}=\cosh x-\sinh x$$

1.11 Complex variable

If $z=x+iy$, where x and y are real variables, z is a complex variable and is a function of x and y. z may be represented graphically in an Argand diagram (*Figure 1.3*).

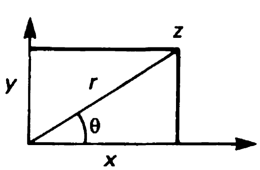

Figure 1.3 Argand diagram

Polar form:

$$z=x+iy=|z|e^{i\theta}=|z|(\cos\theta+i\sin\theta)$$

$$x=r\cos\theta\qquad y=r\sin\theta$$

where $r=|z|$.

Complex arithmetic:

$$z_1=x_1+iy_1\qquad z_2=x_2+iy_2$$

$$z_1\pm z_2=(x_1\pm x_2)+i(y_1\pm y_2)$$

$$z_1\,.\,z_2=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$$

Conjugate:

$$z^*=x-iy\qquad z\,.\,z^*=x^2+y^2=|z|^2$$

Function: another complex variable $w=u+iv$ may be related functionally to z by

$$w=u+iv=f(x+iy)=f(z)$$

which implies

$$u=u(x,y)\qquad v=v(x,y)$$

e.g.,

$$\cosh z=\cosh(x+iy)=\cosh x\cosh iy+\sinh x\sinh iy$$
$$=\cosh x\cos y+i\sinh x\sin y$$

$$u=\cosh x\cos y\qquad v=\sinh x\sin y$$

1.12 Cauchy–Riemann equations

If $u(x,y)$ and $v(x,y)$ are continuously differentiable with respect to x and y,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$w=f(z)$ is continuously differentiable with respect to z and its derivative is

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=\frac{1}{i}\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)$$

It is also easy to show that $\nabla^2 u=\nabla^2 v=0$. Since the transformation from z to w is conformal, the curves $u=$ constant and $v=$ constant intersect each other at right angles, so that one set may be used as equipotentials and the other as field lines in a vector field.

1.13 Cauchy's theorem

If $f(z)$ is analytic everywhere inside a region bounded by C and a is a point within C

$$f(a)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z-a}\,dz$$

This formula gives the value of a function at a point in the interior of a closed curve in terms of the values on that curve.

1.14 Zeros, poles and residues

If $f(z)$ vanishes at the point z_0 the Taylor series for z in the region of z_0 has its first two terms zero, and perhaps others also: $f(z)$ may then be written

$$f(z)=(z-z_0)^n g(z)$$

where $g(z_0)\neq 0$. Then $f(z)$ has a *zero* of order n at z_0. The reciprocal

$$q(z)=1/f(z)=h(z)/(z-z_0)^n$$

where $h(z)=1/g(z)\neq 0$ at z_0. $q(z)$ becomes infinite at $z=z_0$ and is said to have a *pole* of order n at z_0. $q(z)$ may be expanded in the form

$q(z) = c_{-n}(z-z_0)^{n} + \ldots + c_{-1}(z-z_0)^{-1} + c_0 + \ldots$

where c_{-1} is the *residue* of $q(z)$ at $z = z_0$. From Cauchy's theorem, it may be shown that if a function $f(z)$ is analytic throughout a region enclosed by a curve C except at a finite number of poles, the integral of the function around C has a value of $2\pi i$ times the sum of the residues of the function at its poles within C. This fact can be used to evaluate many definite integrals whose indefinite form cannot be found.

1.15 Some standard forms

$$\int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\,d\theta = 2\pi/n!$$

$$\int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx = \pi\operatorname{cosec} a\pi$$

$$\int_0^{\infty} \frac{\sin\theta}{\theta}\,d\theta = \frac{\pi}{2}$$

$$\int_0^{\infty} x\exp(-h^2x^2)\,dx = \frac{1}{2h^2}$$

$$\int_0^{\infty} \frac{x^{a-1}}{1-x}\,dx = \pi\cot a\pi$$

$$\int_0^{\infty} \exp(-h^2x^2)\,dx = \frac{\sqrt{\pi}}{2h}$$

$$\int_0^{\infty} x^2\exp(-h^2x^2)\,dx = \frac{\sqrt{\pi}}{4h^3}$$

1.16 Coordinate systems

The basic system is the rectangular Cartesian system (x, y, z) to which all other systems are referred. Two other commonly used systems are as follows.

1.16.1 Cylindrical coordinates

Coordinates of point P are (x,y,z) or (r,θ,z) (see *Figure 1.4*), where

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

In these coordinates the volume element is $r\,dr\,d\theta\,dz$.

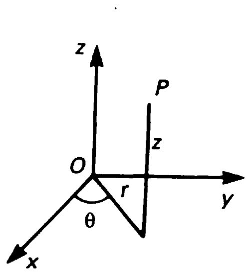

Figure 1.4 Cylindrical coordinates

1.16.2 Spherical polar coordinates

Coordinates of point P are (x,y,z) or (r,θ,φ) (see *Figure 1.5*), where

$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

In these coordinates the volume element is $r^2\sin\theta\,dr\,d\theta\,d\phi$.

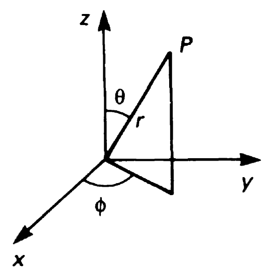

Figure 1.5 Spherical polar coordinates

1.17 Transformation of integrals

$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint \varphi(u,v,w)|J|\,du\,dv\,dw$$

where

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

is the Jacobian of the transformation of coordinates. For Cartesian to cylindrical coordinates, $J = r$, and for Cartesian to spherical polars, it is $r^2\sin\theta$.

1.18 Laplace's equation

The equation satisfied by the scalar potential from which a vector field may be derived by taking the gradient is Laplace's equation, written as:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

In cylindrical coordinates:

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

In spherical polars:

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial\phi}{\partial\theta}\Bigg) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\phi^2}$$

The equation is solved by setting

$\phi = U(u)V(u)W(w)$

in the appropriate form of the equation, separating the variables and solving separately for the three functions, where (u, v, w) is the coordinate system in use.

In Cartesian coordinates, typically the functions are trigonometric, hyperbolic and exponential; in cylindrical coordinates the function of z is exponential, that of θ trigonometric and that of r is a Bessel function. In spherical polars, typically the function of r is a power of r, that of φ is trigonometric, and that of θ is a Legendre function of $\cos\theta$.

1.19 Solution of equations

1.19.1 Quadratic equation

$$ax^2+bx+c=0$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

In practical calculations if $b^2>4ac$, so that the roots are real and unequal, calculate the root of larger modulus first, using the same sign for both terms in the formula, then use the fact that $x_1x_2=c/a$ where x_1 and x_2 are the roots. This avoids the severe cancellation of significant digits which may otherwise occur in calculating the smaller root.

For polynomials other than quadratics, and for other functions, several methods of successive approximation are available.

1.19.2 Bisection method

By trial find x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ have opposite signs (see *Figure 1.6*). Set $x_2=(x_0+x_1)/2$ and calculate $f(x_2)$. If

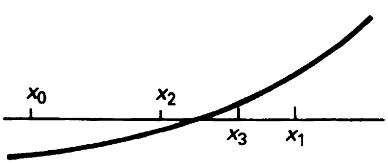

Figure 1.6 Bisection method

$f(x_0)f(x_2)$ is positive, the root lies in the interval (x_1,x_2); if negative in the interval (x_0,x_2); and if zero, x_2 is the root. Continue if necessary using the new interval.

1.19.3 Regula falsi

By trial, find x_0 and x_1 as for the bisection method; these two values define two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$. The straight line joining these two points cuts the x-axis at the point (see *Figure 1.7*)

$$x_2=\frac{x_0f(x_1)-x_1f(x_0)}{f(x_1)-f(x_0)}$$

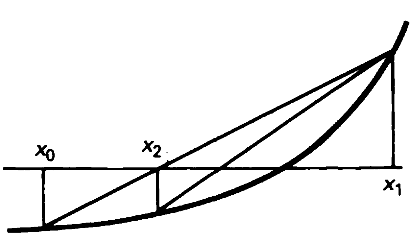

Figure 1.7 Regula falsi

Evaluate $f(x_2)$ and repeat the process for whichever of the intervals (x_0,x_2) or (x_1,x_2) contains the root. This method can be accelerated by halving at each step the function value at the retained end of the interval, as shown in *Figure 1.8*.

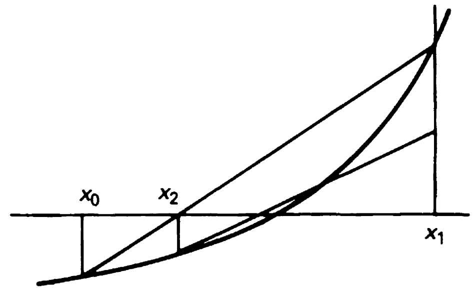

Figure 1.8 Accelerated method

1.19.4 Fixed-point iteration

Arrange the equation in the form

$$x=f(x)$$

Choose an initial value of x by trial, and calculate repetitively

$$x_{k+1}=f(x_k)$$

This process will not always converge.

1.19.5 Newton's method

Calculate repetitively (*Figure 1.9*)

$$x_{k+1}=x_k-f(x_k)/f'(x_k)$$

This method will converge unless: (a) x_k is near a point of inflexion of the function; or (b) x_k is near a local minimum; or (c) the root is

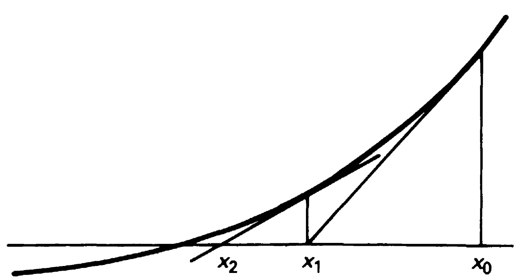

Figure 1.9 Newton's method

multiple. If one of these cases arises, most of the trouble can be overcome by checking at each stage that

$$f(x_{k+1})<f(x_k)$$

and, if not, halving the preceding value of $|x_{k+1}-x_k|$.

1.20 Method of least squares

To obtain the best fit between a straight line $ax+by=1$ and several points (x_1,y_1), (x_2,y_2), …, (x_n,y_n) found by observation, the coefficients a and b are to be chosen so that the sum of the squares of the errors

$e_i = ax_i + by_i - 1$

is a minimum. To do this, first write the set of inconsistent equations

$$ax_1 + by_1 - 1 = 0$$
$$ax_2 + by_2 - 1 = 0$$
$$\vdots$$
$$ax_n + by_n - 1 = 0$$

Multiply each equation by the value of x it contains, and add, obtaining

$$a \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i = 0$$

Similarly multiply by y and add, obtaining

$$a \sum_{i=1}^{n} x_i y_i + b \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} y_i = 0$$

Lastly, solve these two equations for a and b, which will be the required values giving the least squares fit.

1.21 Relation between decibels, current and voltage ratio, and power ratio

$$\mathrm{dB} = 10 \log \frac{P_1}{P_2} = 20 \log \frac{V_1}{V_2} = 20 \log \frac{I_1}{I_2}$$

dB	I_1/I_2 or V_1/V_2	I_2/I_1 or V_2/V_1	P_1/P_2	P_2/P_1
0.1	1.012	0.989	1.023	0.977
0.2	1.023	0.977	1.047	0.955
0.3	1.035	0.966	1.072	0.933
0.4	1.047	0.955	1.096	0.912
0.5	1.059	0.944	1.122	0.891
0.6	1.072	0.933	1.148	0.871
0.7	1.084	0.923	1.175	0.851
0.8	1.096	0.912	1.202	0.832
0.9	1.109	0.902	1.230	0.813
1.0	1.122	0.891	1.259	0.794
1.1	1.135	0.881	1.288	0.776
1.2	1.148	0.871	1.318	0.759
1.3	1.162	0.861	1.349	0.741
1.4	1.175	0.851	1.380	0.724
1.5	1.188	0.841	1.413	0.708
1.6	1.202	0.832	1.445	0.692
1.7	1.216	0.822	1.479	0.676
1.8	1.230	0.813	1.514	0.661
1.9	1.245	0.804	1.549	0.645
2.0	1.259	0.794	1.585	0.631
2.5	1.334	0.750	1.778	0.562
3.0	1.413	0.708	1.995	0.501
3.5	1.496	0.668	2.24	0.447
4.0	1.585	0.631	2.51	0.398
4.5	1.679	0.596	2.82	0.355
5.0	1.778	0.562	3.16	0.316
5.5	1.884	0.531	3.55	0.282
6.0	1.995	0.501	3.98	0.251
6.5	2.11	0.473	4.47	0.224
7.0	2.24	0.447	5.01	0.200
7.5	2.37	0.422	5.62	0.178
8.0	2.51	0.398	6.31	0.158
8.5	2.66	0.376	7.08	0.141
9.0	2.82	0.355	7.94	0.126
9.5	2.98	0.335	8.91	0.112
10.0	3.16	0.316	10.00	0.100
10.5	3.35	0.298	11.2	0.089 1
11.0	3.55	0.282	12.6	0.079 4
15.0	5.62	0.178	31.6	0.031 6
15.5	5.96	0.168	35.5	0.028 2
16.0	6.31	0.158	39.8	0.025 1
16.5	6.68	0.150	44.7	0.022 4
17.0	7.08	0.141	50.1	0.020 0
17.5	7.50	0.133	56.2	0.017 8
18.0	7.94	0.126	63.1	0.015 8
18.5	8.41	0.119	70.8	0.014 1
19.0	8.91	0.112	79.4	0.012 6
19.5	9.44	0.106	89.1	0.011 2
20.0	10.00	0.100 0	100	0.010 0
20.5	10.59	0.094 4	112	0.008 91
21.0	11.22	0.089 1	126	0.007 94
21.5	11.88	0.084 1	141	0.007 08
22.0	12.59	0.079 4	158	0.006 31
22.5	13.34	0.075 0	178	0.005 62
23.0	14.13	0.070 8	200	0.005 01
23.5	14.96	0.066 8	224	0.004 47
24.0	15.85	0.063 1	251	0.003 98
24.5	16.79	0.059 6	282	0.003 55
25.0	17.78	0.056 2	316	0.003 16
25.5	18.84	0.053 1	355	0.002 82
26.0	19.95	0.050 1	398	0.002 51
26.5	21.1	0.047 3	447	0.002 24
27.0	22.4	0.044 7	501	0.002 00
27.5	23.7	0.042 2	562	0.001 78
28.0	25.1	0.039 8	631	0.001 58
28.5	26.6	0.037 6	708	0.001 41
29.0	28.2	0.035 5	794	0.001 26
29.5	29.8	0.033 5	891	0.001 12
30.0	31.6	0.031 6	1 000	0.001 00
31.0	35.5	0.028 2	1 260	7.94×10^{-4}
32.0	39.8	0.025 1	1 580	6.31×10^{-4}
33.0	44.7	0.022 4	2 000	5.01×10^{-4}
34.0	50.1	0.020 0	2 510	3.98×10^{-4}
35.0	56.2	0.017 8		3.16×10^{-4}
36.0	63.1	0.015 8	3 980	2.51×10^{-4}
37.0	70.8	0.014 1	5 010	2.00×10^{-4}

2 Calculus

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Contents

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2.1 Derivative

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

If u and v are functions of x,

$$(uv)' = u'v + uv'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \ldots + {}^nC_p u^{(n-p)}v^{(p)} + \ldots + uv^{(n)}$$

where

$${}^nC_p = \frac{n!}{p!\,(n-p)!}$$

If $z = f(x)$ and $y = g(z)$, then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

2.2 Maxima and minima

$f(x)$ has a stationary point wherever $f'(x) = 0$: the point is a maximum, minimum or point of inflexion according as $f''(x) <, >$ or $= 0$.

$f(x, y)$ has a stationary point wherever

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

Let (a, b) be such a point, and let

$$\frac{\partial^2 f}{\partial x^2} = A, \qquad \frac{\partial^2 f}{\partial x\, \partial y} = H \qquad \frac{\partial^2 f}{\partial y^2} = B$$

all at that point, then:

If $H^2 - AB > 0$, $f(x, y)$ has a saddle point at (a, b).
If $H^2 - AB < 0$ and if $A < 0$, $f(x, y)$ has a maximum at (a, b), but if $A > 0$, $f(x, y)$ has a minimum at (a, b).
If $H^2 = AB$, higher derivatives need to be considered.

2.3 Integral

$$\int_a^b f(x)\,dx = \lim_{N \to \infty} \sum_{n=0}^{N-1} f\left(a + \frac{n(b-a)}{N}\right)\left(\frac{b-a}{N}\right)$$

$$= \lim_{N \to \infty} \sum_{n=1}^{N} f(a + (n-1)\delta x)\,\delta x$$

where $\delta x = (b-a)/N$.

If u and v are functions of x, then

$$\int uv'\,dx = uv - \int u'v\,dx \quad \text{(integration by parts)}$$

2.4 Derivatives and integrals

y	$\frac{dy}{dx}$	$\int y\,dx$
x^n	nx^{n-1}	$x^{n+1}/(n+1)$
$1/x$	$-1/x^2$	$\ln(x)$
e^{ax}	$a\,e^{ax}$	e^{ax}/a
$\ln(x)$	$1/x$	$x[\ln(x) - 1]$
$\log_a x$	$\frac{1}{x}\log_a e$	$x\,log_a\left(\frac{x}{e}\right)$
$\sin ax$	$a\cos ax$	$-\frac{1}{a}\cos ax$
$\cos ax$	$-a\sin ax$	$\frac{1}{a}\sin ax$
$\tan ax$	$a\sec^2 ax$	$-\frac{1}{a}\ln(\cos ax)$
$\cot ax$	$-a\,\mathrm{cosec}^2\, ax$	$\frac{1}{a}\ln(\sin ax)$
$\sec ax$	$a\tan ax\sec ax$	$\frac{1}{a}\ln(\sec ax + \tan ax)$
$\mathrm{cosec}\, ax$	$-a\cot ax\,\mathrm{cosec}\, ax$	$\frac{1}{a}\ln(\mathrm{cosec}\, ax - \cot ax)$

y	$\dfrac{dy}{dx}$	$\int y\,dx$
$\arcsin(x/a)$	$1/(a^2-x^2)^{1/2}$	$x\arcsin(x/a)+(a^2-x^2)^{1/2}$
$\arccos(x/a)$	$-1/(a^2-x^2)^{1/2}$	$x\arccos(x/a)-(a^2-x^2)^{1/2}$
$\arctan(x/a)$	$a/(a^2+x^2)$	$x\arctan(x/a)-\frac{1}{2}a\ln(a^2+x^2)$
$\operatorname{arccot}(x/a)$	$-a/(a^2+x^2)$	$x\operatorname{arccot}(x/a)+\frac{1}{2}a\ln(a^2+x^2)$
$\operatorname{arcsec}(x/a)$	$a(x^2-a^2)^{-1/2}/x$	$x\operatorname{arcsec}(x/a)-a\ln[x+(x^2-a^2)^{1/2}]$
$\operatorname{arccosec}(x/a)$	$-a(x^2-a^2)^{-1/2}/x$	$x\operatorname{arccosec}(x/a)+a\ln[x+(x^2-a^2)^{1/2}]$
$\sinh ax$	$a\cosh ax$	$\frac{1}{a}\cosh ax$
$\coth ax$	$a\sinh ax$	$\frac{1}{a}\sinh ax$
$\tanh ax$	$a\,\mathrm{sech}^2\,ax$	$\frac{1}{a}\ln(\cosh ax)$
$\coth ax$	$-a\,\mathrm{cosech}^2\,ax$	$\frac{1}{a}\ln(\sinh ax)$
$\mathrm{sech}\,ax$	$-a\tanh ax\,\mathrm{sech}\,ax$	$\frac{2}{a}\arctan(e^{ax})$
$\mathrm{cosech}\,ax$	$-a\coth ax\,\mathrm{cosech}\,ax$	$\frac{1}{a}\ln\left(\tanh\frac{ax}{2}\right)$
$\operatorname{arsinh}(x/a)$	$(x^2+a^2)^{-1/2}$	$x\operatorname{arsinh}(x/a)-(x^2+a^2)^{1/2}$
$\operatorname{arcosh}(x/a)$	$(x^2-a^2)^{-1/2}$	$x\operatorname{arcosh}(x/a)-(x^2-a^2)^{1/2}$
$\operatorname{artanh}(x/a)$	$a(a^2-x^2)^{-1}$	$x\operatorname{artanh}(x/a)+\frac{1}{2}a\ln(a^2-x^2)$
$\operatorname{arcoth}(x/a)$	$-a(x^2-a^2)^{-1}$	$x\operatorname{arcoth}(x/a)+\frac{1}{2}a\ln(x^2-a^2)$
$\operatorname{arsech}(x/a)$	$-a(a^2-x^2)^{-1/2}/x$	$x\operatorname{arsech}(x/a)+a\arcsin(x/a)$
$\operatorname{arcosech}(x/a)$	$-a(x^2+a^2)^{-1/2}/x$	$x\operatorname{arcosech}(x/a)+a\operatorname{arsinh}(x/a)$
$(x^2\pm a^2)^{1/2}$		$\frac{1}{2}x(x^2\pm a^2)^{1/2}\pm\frac{1}{2}a^2\operatorname{arsinh}(x/a)$
$(a^2-x^2)^{1/2}$		$\frac{1}{2}x(a^2-x^2)^{1/2}+\frac{1}{2}a^2\arcsin(x/a)$
$(x^2\pm a^2)^p x$		$\begin{cases}\frac{1}{2}(x^2\pm a^2)^{p+1}/(p+1) & (p\neq-1)\\ \frac{1}{2}\ln(x^2\pm a^2) & (p=-1)\end{cases}$
$(a^2-x^2)^p x$		$\begin{cases}-\frac{1}{2}(a^2-x^2)^{p+1}/(p+1) & (p\neq-1)\\ -\frac{1}{2}\ln(a^2-x^2) & (p=-1)\end{cases}$
$x(ax^2+b)^p$		$(ax^2+b)^{p+1}/2a(p+1)\quad(p\neq-1)$ $[\ln(ax^2+b)]/2a\quad(p=-1)$
$(2ax-x^2)^{-1/2}$		$\arccos\left(\frac{a-x}{a}\right)$
$(a^2\sin^2 x+b^2\cos^2 x)^{-1}$		$\frac{1}{ab}\arctan\left(\frac{a}{b}\tan x\right)$
$(a^2\sin^2 x-b^2\cos^2 x)^{-1}$		$-\frac{1}{ab}\operatorname{artanh}\left(\frac{a}{b}\tan x\right)$
$e^{ax}\sin bx$		$e^{ax}\frac{a\sin bx-b\cos bx}{a^2+b^2}$
$e^{ax}\cos bx$		$e^{ax}\frac{(a\cos bx+b\sin bx)}{a^2+b^2}$

y	$\int y\,dx$	
$\sin mx \sin nx$	$\frac{1}{2}\frac{\sin(m-n)x}{m-n}-\frac{1}{2}\frac{\sin(m+n)x}{m+n}$	$(m\neq n)$
	$\frac{1}{2}\left(x-\frac{\sin 2mx}{2m}\right)$	$(m=n)$
$\sin mx \cos nx$	$-\frac{1}{2}\frac{\cos(m+n)x}{m+n}-\frac{1}{2}\frac{\cos(m-n)x}{m-n}$	$(m\neq n)$
	$-\frac{1}{2}\frac{\cos 2mx}{2m}$	$(m=n)$
$\cos mx \cos nx$	$\frac{1}{2}\frac{\sin(m+n)x}{m+n}+\frac{1}{2}\frac{\sin(m-n)x}{m-n}$	$(m\neq n)$
	$\frac{1}{2}\left(x+\frac{\sin 2mx}{2m}\right)$	$(m=n)$

2.5 Standard substitutions

Integral a function of	*Substitute*
a^2-x^2	$x=a\sin\theta$ or $x=a\cos\theta$
a^2+x^2	$x=a\tan\theta$ or $x=a\sinh\theta$
x^2-a^2	$x=a\sec\theta$ or $x=a\cosh\theta$

2.6 Reduction formulae

$$\int \sin^m x\,dx=-\frac{1}{m}\sin^{m-1}x\cos x+\frac{m-1}{m}\int \sin^{m-2}x\,dx$$

$$\int \cos^m x\,dx=\frac{1}{m}\cos^{m-1}x\sin x+\frac{m-1}{m}\int \cos^{m-2}x\,dx$$

$$\int \sin^m x\cos^n x\,dx=\frac{\sin^{m+1}x\cos^{n-1}x}{m+n}+\frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx$$

If the integrand is a rational function of $\sin x$ and/or $\cos x$, substitute $t=\tan\frac{1}{2}x$, then

$$\sin x=\frac{1}{1+t^2},\qquad \cos x=\frac{1-t^2}{1+t^2},\qquad dx=\frac{2dt}{1+t^2}$$

2.7 Numerical integration

2.7.1 Trapezoidal rule (*Figure 2.1*)

$$\int_{x1}^{x2} y\,dx=\tfrac{1}{2}h(y_1+y_2)+O(h^3)$$

2.7.2 Simpson's rule (*Figure 2.1*)

$$\int_{x1}^{x2} y\,dx=2h(y_1+4y_2+y_3)/6+O(h^5)$$

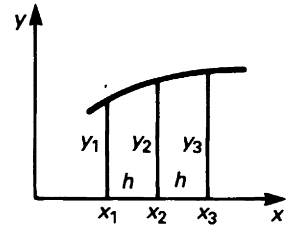

Figure 2.1 Numerical integration

2.7.3 Change of variable in double integral

$$\iint f(x,y)\,dx\,dy=\iint F(u,v)|J|\,du\,dv$$

where

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}$$

is the Jacobian of the transformation.

2.7.4 Differential mean value theorem

$$\frac{f(x+h)-f(x)}{h}=f'(x+\theta h)\qquad 0<\theta<1$$

2.7.5 Integral mean value theorem

$$\int_a^b f(x)g(x)\,dx=g(a+\theta h)\int_a^b f(x)\,dx$$

$h=b-a,\ 0<\theta<1$

2.8 Vector calculus

Let $s(x,y,z)$ be a scalar function of position and let $\mathbf{v}(x,y,z)=\mathbf{i}v_x(x,y,z)+\mathbf{j}v_y(x,y,z)+\mathbf{k}v_z(x,y,z)$ be a vector function of position. Define

$$\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$$

so that

$$\nabla.\nabla=\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

then

$$\text{grad}\,s=\nabla s=\mathbf{i}\frac{\partial s}{\partial x}+\mathbf{j}\frac{\partial s}{\partial y}+\mathbf{k}\frac{\partial s}{\partial z}$$

$$\text{div}\,\mathbf{v}=\nabla.\mathbf{v}=\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}$$

$$\text{curl}\,\mathbf{v}=\nabla\times\mathbf{v}=\mathbf{i}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\mathbf{j}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\mathbf{k}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

The following identities are then true:

$\text{div}\,(s\mathbf{v})=s\,div\,\mathbf{v}+(\text{grad}\,s).\mathbf{v}$

$\operatorname{curl}(s\mathbf{v}) = s \operatorname{curl} \mathbf{v} + (\operatorname{grad} s) \times \mathbf{v}$

$\operatorname{div}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} . \operatorname{curl} \mathbf{u} - \mathbf{u} . \operatorname{curl} \mathbf{v}$

$\operatorname{curl}(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \operatorname{div} \mathbf{v} - \mathbf{v} \operatorname{div} \mathbf{u} + (\mathbf{v} . \nabla)\mathbf{u} - (\mathbf{u} . \nabla)\mathbf{v}$

$\operatorname{div} \operatorname{grad} s = \nabla^2 s$

$\operatorname{div} \operatorname{curl} \mathbf{v} = 0$

$\operatorname{curl} \operatorname{grad} s = 0$

$\operatorname{curl} \operatorname{curl} \mathbf{v} = \operatorname{grad}(\operatorname{div} \mathbf{v}) - \nabla^2 \mathbf{v}$

where ∇^2 operates on each component of $\mathbf{v}$.

$\mathbf{v} x \operatorname{curl} \mathbf{v} + (\mathbf{v} . \nabla)\mathbf{v} = \operatorname{grad} \frac{1}{2}\mathbf{v}^2$

Potentials:

If $\operatorname{curl} \mathbf{v} = 0$, $\mathbf{v} = \operatorname{grad} \varphi$ where φ is a scalar potential.
If $\operatorname{div} \mathbf{v} = 0$, $\mathbf{v} = \operatorname{curl} \mathbf{A}$ where $\mathbf{A}$ is a vector potential.

3 Series and Transforms

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3.1 Arithmetic series

Sum of n terms,

$$S_n = a + (a+d) + (a+2d) + \ldots + [a+(n-1)d]$$
$$= n[2a+(n-1)d]/2$$
$$= n(a+l)/2$$

3.2 Geometric series

Sum of n terms,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} = a(1-r^n)/(1-r)$$

$(|r|<1)$

$$S_\infty = a/(1-r)$$

3.3 Binomial series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$

If p is a positive integer the series terminates with the term in x^p and is valid for all x; otherwise the series does not terminate, and is valid only for $-1<x<1$.

3.4 Taylor's series

Infinite form

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots$$
$$+ \frac{h^n}{n!}f^{(n)}(x) + \ldots$$

Finite form

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots$$
$$+ \frac{h^n}{n!}f^{(n)}(x) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(x+\lambda h)$$

where $0 \leqslant \lambda \leqslant 1$.

3.5 Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots$$

Neither of these series is necessarily convergent, but both usually are for appropriate ranges of values of h and of x respectively.

3.6 Laurent's series

If a function $f(z)$ of a complex variable is analytic on and everywhere between two concentric circles centre a, then at any point in this region

$$f(z) = a_0 + a_1(z-a) + \ldots + b_1/(z-a) + b_2/(z-a)^2 + \ldots$$

This series is often applicable when Taylor's series is not.

3.7 Power series for real variables

	Math	*Comp*
$e^x = 1 + x + \frac{x^2}{2!} + \ldots$	all x	$\lvert x\rvert \leqslant 1$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$	$-1 < x \leqslant 1$	
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$	all x	$\lvert x\rvert \leqslant 1$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$	all x	$\lvert x\rvert \leqslant 1$
$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots$		$\lvert x\rvert < \frac{\pi}{2}$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$		$\lvert x\rvert \leqslant 1$
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$	all x	$\lvert x\rvert \leqslant 1$
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$	all x	$\lvert x\rvert \leqslant 1$

The column headed 'Math' contains the range of values of the variable x for which the series is convergent in the pure mathematical sense. In some cases a different range of values is given in the column headed 'Comp', to reduce the rounding errors which arise when computers are used.

3.8 Integer series

$$\sum_{n=1}^{N} n = 1+2+3+4+\ldots+N = N(N+1)/2$$

$$\sum_{n=1}^{N} n^2 = 1^2+2^2+3^2+4^2+\ldots+N^2 = N(N+1)(2N+1)/6$$

$$\sum_{n=1}^{N} n^3 = 1^3+2^3+3^3+4^3+\ldots+N^3 = N^2(N+1)^2/4$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln(2) \qquad \text{(see } \ln(1+x))$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4} \qquad \text{(see } \arctan x)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{N} n(n+1)(n+2)\ldots(n+r)$$
$$= 1.2.3\ldots + 2.3.4\ldots + 3.4.5\ldots + \ldots$$
$$+ N(N+1)(N+2)\ldots(N+r)$$
$$= \frac{N(N+1)(N+2)\ldots(N+r+1)}{r+2}$$

3.9 Fourier series

$$f(\theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

with

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\Theta)\cos n\Theta \, d\Theta$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\Theta)\sin n\Theta \, d\Theta$$

or

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n \exp(jn\theta)$$

with

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(\Theta)\exp(-jn\Theta)\, d\Theta = \begin{cases} \frac{1}{2}(a_n + jb_n) & n<0 \\ \frac{1}{2}(a_n - jb_n) & n>0 \end{cases}$$

The above expressions for Fourier series are valid for functions having at most a finite number of discontinuities within the period 0 to 2 of the variable of integration.

3.10 Rectified sine wave

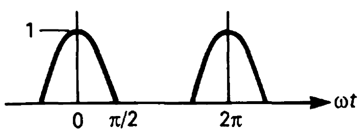

Figure 3.1 Half wave

$$f(\omega t) = \frac{1}{\pi} + \frac{1}{2}\cos \omega t + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos 2n\omega t}{4n^2-1}$$

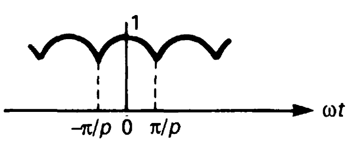

Figure 3.2 *p*-phase

$$f(\omega t) = \frac{\sin(\pi/p)}{\pi/p} + \frac{2p}{\pi}\sin\left(\frac{\pi}{p}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos np\omega t}{p^2n^2-1}$$

3.11 Square wave

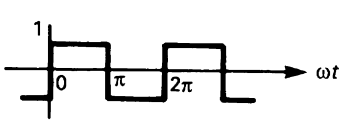

Figure 3.3 Square wave

$$f(\omega t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\omega t}{(2n-1)}$$

3.12 Triangular wave

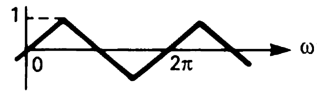

Figure 3.4 Triangular wave

$$f(\omega t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(2n-1)\omega t}{(2n-1)^2}$$

3.13 Sawtooth wave

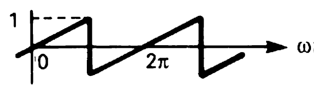

Figure 3.5 Sawtooth wave

$$f(\omega t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\omega t}{n}$$

3.14 Pulse wave

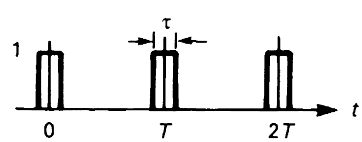

Figure 3.6 Pulse wave

$$f(t) = \frac{\tau}{T} + \frac{2\tau}{T}\sum_{n=1}^{\infty}\frac{\sin(n\omega\tau/T)}{n\pi\tau/T}\cos\left(\frac{2n\pi t}{T}\right)$$

3.15 Fourier transforms

Among other applications, these are used for converting from the time domain to the frequency domain.

Basic formulae:

$$\int_{-\infty}^{\infty} U(f)\exp(j2\pi ft)\,df = u(t) \leftrightarrows U(f) = \int_{-\infty}^{\infty} u(t)\exp(-j2\pi ft)\,dt$$

Change of sign and complex conjugates:

$$u(-t) \leftrightarrows U(-f), \qquad u^*(t) \leftrightarrows U^*(-f)$$

Time and frequency shifts (τ and φ constant):

$$u(t-\tau) \leftrightarrows U(f)\exp(-j2\pi f\tau) \exp(j2\pi\varphi t)u(t) \leftrightarrows U(f-\varphi)$$

Scaling (T constant):

$u(t/T) \leftrightarrows TU(fT)$

Products and convolutions:

$u(t) \dagger v(t) \leftrightarrows U(f)V(f), \qquad u(t)v(t) \leftrightarrows U(f) \dagger V(f)$

Differentiation:

$u'(t) \leftrightarrows \mathrm{j}2\pi f U(f), \qquad -\mathrm{j}2\pi t u(t) \leftrightarrows U'(f)$

$\partial u(t,\alpha)/\partial\alpha \leftrightarrows \partial(U - f, \alpha)/\partial\alpha$

Integration ($U(0)=0$, a and b real constants):

$$\int_{-\infty}^{t} u(\tau)\,\mathrm{d}\tau \leftrightarrows U(f)/\mathrm{j}2\pi f$$

$$\int_{a}^{b} v(t,\alpha)\,\mathrm{d}\alpha \leftrightarrows \int_{a}^{b} V(f,\alpha)\,\mathrm{d}\alpha$$

Interchange of functions:

$U(t) \leftrightarrows u(-f)$

Dirac delta functions:

$\delta(t) \leftrightarrows 1 \qquad \exp(\mathrm{j}2\pi f_0 t) \leftrightarrows \delta(f - f_0)$

Rect(t) (unit length, unit amplitude pulse, centred on $t=0$):

$\mathrm{rect}(t) \leftrightarrows \sin \pi f/\pi f$

Gaussian distribution:

$\exp(-\pi t^2) \leftrightarrows \exp(-\pi f^2)$

Repeated and impulse (delta function) sampled waveforms:

$$\sum_{-\infty}^{\infty} u(t-nT) \leftrightarrows (1/T)U(f) \sum_{-\infty}^{\infty} \delta(f-n/T)$$

$$u(t) \sum_{-\infty}^{\infty} \delta(t-nT) \leftrightarrows (1/T) \sum_{-\infty}^{\infty} U(f-n/T)$$

Parseval's lemma:

$$\int_{-\infty}^{\infty} u(t)v^*(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} U(f)V^*(f)\,\mathrm{d}f$$

$$\int_{-\infty}^{\infty} |u(t)|^2\,\mathrm{d}t = \int_{-\infty}^{\infty} |U(f)|^2\,\mathrm{d}f$$

3.16 Laplace transforms

$$\bar{x}_s = \int_0^{\infty} x(t)\exp(-st)$$

Function	*Transform*	*Remarks*
$e^{-\alpha t}$	$\dfrac{1}{s+\alpha}$	
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$	
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$	
$\sinh \omega t$	$\dfrac{\omega}{s^2-\omega^2}$	
$\cosh \omega t$	$\dfrac{s}{s^2-\omega^2}$	
t^n	$n!/s^{n+1}$	
1	$1/s$	
$H(t-\tau)$	$\dfrac{1}{s}\exp(-s\tau)$	Heaviside step function
$x(t-\tau)H(t-\tau)$	$\exp(-s\tau)\bar{x}(s)$	Shift in t
$\delta(t-\tau)$	$\exp(-s\tau)$	Dirac delta function
$\exp(-\alpha t)\,x(t)$	$\bar{x}(s+\alpha)$	Shift in s
$\exp(-\alpha t)\sin \omega t$	$\dfrac{\omega}{(s+\alpha)^2+\omega^2}$	
$\exp(-\alpha t)\cos \omega t$	$\dfrac{(s+\alpha)}{(s+\alpha)^2+\omega^2}$	
$tx(t)$	$-\dfrac{\mathrm{d}\bar{x}(s)}{\mathrm{d}s}$	
$\dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = x'(t)$	$s\bar{x}(s) - x(0)$	
$\dfrac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = x''(t)$	$s^2\bar{x}(s) - sx(0) - x'(0)$	
$\dfrac{\mathrm{d}^n x(t)}{\mathrm{d}x^n} = x^{(n)}(t)$	$s^n\bar{x}(s) - s^{n-1}x(0) - s^{n-2}x'(0)\ldots - sx^{(n-2)}(0) - x^{(n-1)}(0)$	

Convolution integral

$$\int_0^t x_1(\sigma)x_2(t-\sigma)\,\mathrm{d}\sigma \rightarrow \bar{x}_1(s)\bar{x}_2(s)$$

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4 Matrices and Determinants

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4.1 Linear simultaneous equations

The set of equations

$$a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1$$

$$a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=b_2$$

$$\ldots$$

$$a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n=b_n$$

may be written symbolically

$$\mathbf{Ax}=\mathbf{b}$$

in which $\mathbf{A}$ is the *matrix* of the coefficients a_{ij}, and $\mathbf{x}$ and $\mathbf{b}$ are the *column matrices* (or vectors) $(x_1\ldots x_n)$, and $(b_1\ldots b_n)$. In this case the matrix $\mathbf{A}$ is square $(n\times n)$. The equations can be solved unless two or more of them are not independent, in which case

$$\det\mathbf{A}=|\mathbf{A}|=0$$

and there then exist non-zero solutions x_i only if $\mathbf{b}=0$. If $\det\mathbf{A}\neq 0$, there exist non-zero solutions only if $\mathbf{b}\neq 0$. When $\det\mathbf{A}=0$, $\mathbf{A}$ is *singular*.

4.2 Matrix arithmetic

If $\mathbf{A}$ and $\mathbf{B}$ are both matrices of m rows and n columns they are *conformable*, and

$$\mathbf{A}\pm\mathbf{B}=\mathbf{C}\quad\text{where } C_{ij}=A_{ij}\pm B_{ij}$$

4.2.1 Product

If $\mathbf{A}$ is an $m\times n$ matrix and $\mathbf{B}$ an $n\times l$, the product $\mathbf{AB}$ is defined by

$$(\mathbf{AB})_{ij}=\sum_{k=1}^{n}(\mathbf{A})_{ik}(\mathbf{B})_{kj}$$

In this case, if $l\neq m$, the product $\mathbf{BA}$ will not exist.

4.2.2 Transpose

The transpose of $\mathbf{A}$ is written $\mathbf{A}'$ or $\mathbf{A}^t$ and is the matrix whose rows are the columns of $\mathbf{A}$, i.e.

$$(\mathbf{A}^t)_{ij}=(\mathbf{A})_{ji}$$

A square matrix may be equal to its transpose, and it is then said to be *symmetrical.* If the product $\mathbf{AB}$ exists, then

$$(\mathbf{AB})^t=\mathbf{B}^t\mathbf{A}^t$$

4.2.3 Adjoint

The *adjoint* of a square matrix $\mathbf{A}$ is defined as $\mathbf{B}$, where

$$(\mathbf{B})_{ij}=(A)_{ji}$$

and A_{ji} is the *cofactor* of a_{ji} in $\det\mathbf{A}$.

4.2.4 Inverse

If $\mathbf{A}$ is non-singular, the *inverse* $\mathbf{A}^{-1}$ is given by

$$\mathbf{A}^{-1}=\operatorname{adj}\mathbf{A}/\det\mathbf{A}\quad\text{and}\quad\mathbf{A}^{-1}\mathbf{A}=\mathbf{AA}^{-1}=\mathbf{I}$$

the *unit* matrix.

$$(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$$

if both inverses exist. The original equations $\mathbf{Ax}=\mathbf{b}$ have the solutions $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$ if the inverse exists.

4.2.5 Orthogonality

A matrix $\mathbf{A}$ is orthogonal if $\mathbf{AA}^t=\mathbf{I}$. If $\mathbf{A}$ is the matrix of a coordinate transformation $\mathbf{X}=\mathbf{AY}$ from variables y_i to variables x_i, then if $\mathbf{A}$ is orthogonal $\mathbf{X}^t\mathbf{X}=\mathbf{Y}^t\mathbf{Y}$, or

$$\sum_{i=1}^{n}x_i^2=\sum_{i=1}^{n}y_i^2$$

4.3 Eigenvalues and eigenvectors

The equation

$$\mathbf{Ax}=\lambda\mathbf{x}$$

where $\mathbf{A}$ is a square matrix, $\mathbf{x}$ a column vector and λ a number (in general complex) has at most n solutions $(\mathbf{x},\lambda)$. The values of λ are *eigenvalues* and those of $\mathbf{x}$ *eigenvectors* of the matrix $\mathbf{A}$. The relation may be written

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$$

so that if $\mathbf{x}\neq 0$, the equation $\mathbf{A}-\lambda\mathbf{I}=0$ gives the eigenvalues. If $\mathbf{A}$ is symmetric and real, the eigenvalues are real. If $\mathbf{A}$ is symmetric, the eigenvectors are orthogonal. If $\mathbf{A}$ is not symmetric, the eigenvalues are complex and the eigenvectors are not orthogonal.

4.4 Coordinate transformation

Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors related by the equation

$$\mathbf{y}=\mathbf{Ax}$$

when their components are expressed in one orthogonal system, and that a second orthogonal system has unit vectors $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n$ expressed in the first system. The components of $\mathbf{x}$ and $\mathbf{y}$ expressed in the new system will be $\mathbf{x}'$ and $\mathbf{y}'$, where

$$\mathbf{x}'=\mathbf{U}^t\mathbf{x},\qquad\mathbf{y}'=\mathbf{U}^t\mathbf{y}$$

and $\mathbf{U}^t$ is the orthogonal matrix whose rows are the unit vectors $\mathbf{u}_1^t$, $\mathbf{u}_2^t$, etc. Then

$$\mathbf{y}'=\mathbf{U}^t\mathbf{y}=\mathbf{U}^t\mathbf{Ax}=\mathbf{U}^t\mathbf{A}\dot{\mathbf{x}}=\mathbf{U}^t\mathbf{AUx}'$$

or

$$\mathbf{y}'=\mathbf{A}'\mathbf{x}'$$

where

$$\mathbf{A}'=\mathbf{U}^t\mathbf{AU}$$

Matrices $\mathbf{A}$ and $\mathbf{A}'$ are *congruent.*

4.5 Determinants

The determinant

$$D=\begin{vmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\vdots&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{vmatrix}$$

is defined as follows. The first suffix in a_{rs} refers to the row, the second to the column which contains a_{rs}. Denote by M_{rs} the determinant left by deleting the rth row and sth column from D, then

$$D=\sum_{k=1}^{n}(-1)^{k+1}a_{1k}M_{1k}$$

gives the value of D in terms of determinants of order $n-1$, hence by repeated application, of the determinant in terms of the elements a_{rs}.

4.6 Properties of determinants

If the rows of $|a_{rs}|$ are identical with the columns of $|b_{sr}|$, $a_{rs}=b_{sr}$ and

$$|a_{rs}|=|b_{sr}|$$

that is, the *transposed* determinant is equal to the original.

If two rows or two columns are interchanged, the numerical value of the determinant is unaltered, but the sign will be changed if the permutation of rows or columns is odd.

If two rows or two columns are identical, the determinant is zero.

If each element of one row or one column is multiplied by k, so is the value of the determinant.

If any row or column is zero, so is the determinant.

If each element of the pth row or column of the determinant c_{rs} is equal to the sum of the elements of the same row or column in determinants a_{rs} and b_{rs}, then

$$|c_{rs}|=|a_{rs}|+|b_{rs}|$$

The addition of any multiple of one row (or column) to another row (or column) does not alter the value of the determinant.

4.6.1 Minor

If row p and column q are deleted from $|a_{rs}|$, the remaining determinant M_{pq} is called the *minor* of a_{pq}.

4.6.2 Cofactor

The *cofactor* of a_{pq} is the minor of a_{pq} prefixed by the sign which the product $M_{pq}a_{pq}$ would have in the expansion of the determinant, and is denoted by A_{pq}:

$$A_{pq}=(-1)^{p+q}M_{pq}$$

A determinant a_{ij} in which $a_{ij}=a_{ji}$ for all i and j is called *symmetric*, whilst if $a_{ij}=-a_{ji}$ for all i and j, the determinant is *skew-symmetric*. It follows that $a_{ii}=0$ for all i in a skew-symmetric determinant.

4.7 Numerical solution of linear equations

Evaluation of a determinant by direct expansion in terms of elements and cofactors is disastrously slow, and other methods are available, usually programmed on any existing computer system.

4.7.1 Reduction of determinant or matrix to upper triangular or to diagonal form

The system of equations may be written

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{matrix} \\ x_1 \\ \vdots \\ \\ \end{matrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The variable x_1 is eliminated from the last $n-1$ equations by adding a multiple $-a_{i1}/a_{11}$ of the first row to the ith, obtaining

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a''_{nn} \end{bmatrix} \begin{matrix} \\ x_1 \\ \vdots \\ \\ \end{matrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_1 \\ \vdots \\ b''_n \end{bmatrix}$$

where primes indicate altered coefficients. This process may be continued by eliminating x_2 from rows 3 to n, and so on. Eventually the form will become

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a''_{nn} \end{bmatrix} \begin{matrix} \\ x_1 \\ \vdots \\ \\ \end{matrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ \vdots \\ b''_n \end{bmatrix}$$

x_n can now be found from the nth equation, substituted in the $(n-1)$th to obtain x_{n-1} and so on.

Alternatively the process may be applied to the system of equations in the form

$$\mathbf{Ax}=\mathbf{Ib}$$

where $\mathbf{I}$ is the unit matrix, and the same operations carried out upon $\mathbf{I}$ as upon $\mathbf{A}$. If the process is continued after reaching the upper triangular form, the matrix $\mathbf{A}$ can eventually be reduced to diagonal form. Finally, each equation is divided by the corresponding diagonal element of $\mathbf{A}$, thus reducing $\mathbf{A}$ to the unit matrix. The system is now in the form

$$\mathbf{Ix}=\mathbf{Bb}$$

and evidently $\mathbf{B}=\mathbf{A}^{-1}$. The total number of operations required is $O(n^3)$.

5 Electric Circuit Theory

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Contents

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5.1 Types of source

If we were to measure the terminal voltage of a source as an increasing current was drawn we should find the relationships shown in *Figure 5.1(a)* where A is a line of constant voltage V_0 obtained when the generator is perfect and without internal impedance, and where B shows the practical case in which there is internal impedance Z and $V=V_0-IZ$. The corresponding graphs for a constant current generator are shown in *Figure 5.1(b)* where A shows the perfect case and B the imperfect case. Apart

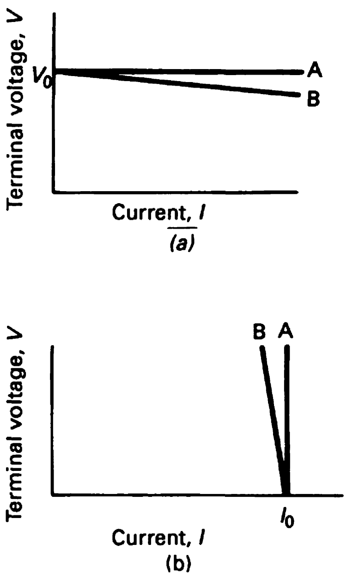

Figure 5.1 Types of sources: (a) constant voltage; (b) constant current

from the lack of familiarity the difficulty with a constant current generator is that the perfect case calls for an infinite impedance in parallel in any equivalent circuit and a practical case requires an impedance which is large but not infinite. This can best be seen by considering the equations which represent line B in *Figure 5.1(b)*, i.e. $I=I_0-VY$ or $I=I_0-V/Z$, where Y is the admittance of an element and equals $1/Z$ where Z is the impedance.

The equivalent circuits for the two types of generator are usually shown as in *Figures 5.2(a)* and *(b)*.

In practice the constant current generator is a useful aid towards the understanding of many transistors in which, crudely but often sufficiently accurately, the output is a current constant over a range of loads.

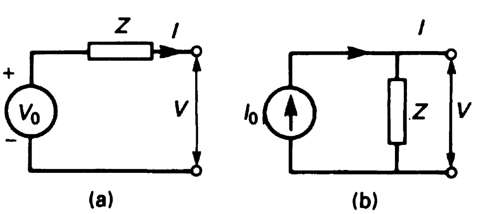

Figure 5.2 Equivalent circuit: (a) voltage source; (b) current source

5.2 Alternating current theory

Because it is possible to analyse all periodic functions of time into series of sinusoids and because of the mathematical properties of sinusoids we always consider currents and voltages varying in such a way that

$$\left.\begin{aligned} i &= I\sin\omega t = I\sin 2\pi ft \\ v &= V\sin\omega t = V\sin 2\pi ft \end{aligned}\right\} \tag{5.1}$$

or

$$\left.\begin{aligned} i &= I\cos\omega t = I\sin 2\pi ft \\ v &= V\cos\omega t = V\cos 2\pi ft \end{aligned}\right\} \tag{5.2}$$

where

i, v = instantaneous values of current, voltage
I, V = maximum or peak values of current, voltage
ω = angular frequency (in radians/s)
f = frequency (in cycles/s or hertz).

An alternative approach is to make use of complex number ideas based on de Moivre's theorem which states that

$$e^{j\theta}=\cos\theta+j\sin\theta \tag{5.3}$$

where $j=\sqrt{-1}$, so that we can write

$$v=V\,\mathrm{Re}(e^{j\omega t}) \tag{5.4}$$

for $V\cos\omega t$, or

$$v=V\,\mathrm{Im}(e^{j\omega t}) \tag{5.5}$$

for $V\sin\omega t$, where Re and Im stand for the real and imaginary parts of $e^{j\omega t}$. The convenient mathematical properties mentioned above are the simple forms of the differentials:

$$dv/dt=V\omega\cos\omega t \tag{5.6}$$

if $v=V\sin\omega t$, and

$$dv/dt=\mathrm{Re}\text{ or }\mathrm{Im}(j\omega V e^{j\omega t}) \tag{5.7}$$

dependent on whether v was assumed to be the real or the imaginary part of $Ve^{j\omega t}$.

Integration produces similar expressions:

$$\int v\,dt=-(V/\omega)\cos\omega t \tag{5.8}$$

for $v=V\sin\omega t$, and

$$\int v\,dt=\mathrm{Re}\text{ or }\mathrm{Im}\left(\frac{V}{j\omega}e^{j\omega t}\right)=\mathrm{Re}\text{ or }\mathrm{Im}\left(-\frac{jV}{\omega}e^{j\omega t}\right) \tag{5.9}$$

The nature of these variations is shown in *Figure 5.3*. Other properties of sinusoids and various trigonometrical relations will be developed in later sections.

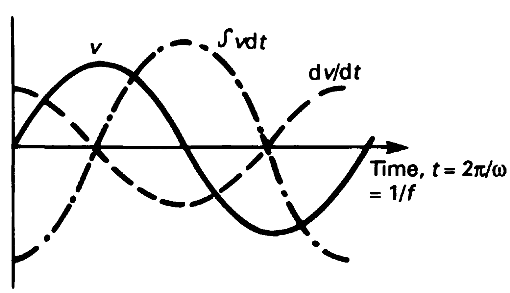

Figure 5.3 Sinusoidal variation

5.3 Resistance, inductance, capacitance and related quantities

Resistance, inductance and capacitance are the three main elements that either absorb or store electrical energy. They constitute the passive as opposed to the active parts of a circuit such as voltage and current sources. The relationship between voltage and current for each is of fundamental importance.

5.3.1 Resistance

Ohm's law states that

$$v = Ri \tag{5.10}$$

The unit of resistance is the ohm (symbol Ω).

5.3.2 Inductance

$$v = L\frac{di}{dt} \tag{5.11}$$

The unit of inductance is the henry (symbol H).

This equation is sometimes found with a minus sign, but then the 'v' is voltage induced in the inductor which has to be overcome by an applied voltage. Because our purpose is to study circuits we think in terms of the applied voltage.

5.3.3 Capacitance

The basic relation for a capacitor is

$$q = vC \tag{5.12}$$

where q is charge in coulombs (symbol C) and C is capacitance in farads (symbol F). Differentiation leads to the more usual expressions

$$\frac{dq}{dt} = i = C\frac{dv}{dt} \tag{5.13}$$

or

$$v = \frac{1}{C}\int i\,dt \tag{5.14}$$

In some circumstances it is convenient to use the reciprocal quantities, the main occasion being when many elements are in parallel, as in *Figure 5.4*. If we introduce the conductance G (equal

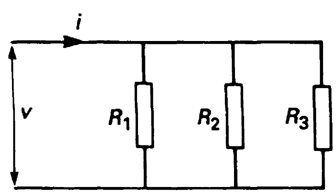

Figure 5.4 Parallel circuit to show advantage of using conductances

to $1/R$) then the conductance of the whole circuit becomes $G = G_1 + G_2 + G_3$ which is easier than solving

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

for R. Of course Ohm's law now takes the form $i = Gv$.

The unit of conductance is the siemen (symbol S). The phrase 'reciprocal ohms' (symbol ℧) is sometimes used.

Reciprocals for inductance and capacitance are not used in the same way but related quantities will be introduced when discussing a.c. circuits. Similarly the idea of impedance and its reciprocal, admittance, will be introduced later.

5.4 AC analysis of electric circuits

5.4.1 Mathematical approach

Consider a circuit such as that shown in *Figure 5.5* in which the

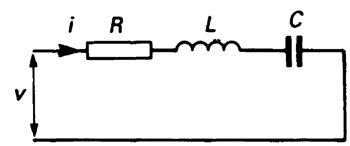

Figure 5.5 Alternating current series circuit

elements are connected in series. The applied voltage must be

$$v = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \tag{5.15}$$

If it is assumed that v and i vary sinusoidally i can be expressed as $I\exp(j\omega t)$ and Equation (5.15) becomes

$$\begin{aligned} v &= \left(RI + j\omega LI + \frac{1}{j\omega C}I\right)\exp(j\omega t) \\ &= \left(R + j\omega L - \frac{j}{\omega C}\right)I\exp(j\omega t) \\ &= \left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right]I\exp(j\omega t) \end{aligned} \tag{5.16}$$

The expression $R + j(\omega L - 1/\omega C)$ can be written

$$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\exp(j\varphi)$$

where

$$\varphi = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right)$$

The quantity $\sqrt{R^2 + (\omega L - 1/\omega C)^2}$ is known as the impedance, Z. Equation (5.16) can now be written

$$v = Z\exp(j\varphi)I\exp(j\omega t) = ZI\exp[j(\omega t + \varphi)] \tag{5.17}$$

The product ZI is the magnitude V of this applied voltage and angle φ is termed the phase angle.

5.4.2 Approach using phasor diagrams

Sinusoidal variations can be understood in terms of the projection onto a straight line of a point that moves round a circle, or of a radial line that sweeps round a circle as shown in *Figure 5.6*. If $\theta = \omega t$ then the lengths OA, OB represent $V\sin\omega t$ and $V\cos\omega t$ respectively. The line OP represents V and is known as a phasor. Such lines used to be called vectors but this is now regarded as misleading because vectors, as understood in

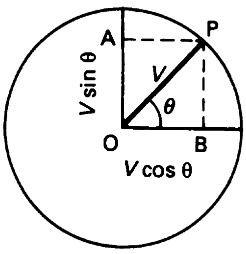

Figure 5.6 Phasor representation of sinusoidal variation

mechanics, represent *space*-dependent quantities such as force and momentum but the electrical quantities which we represent in a similar graphical manner are *time* dependent.

Figure 5.7 shows the phasor diagram for the circuit of *Figure 5.5*.

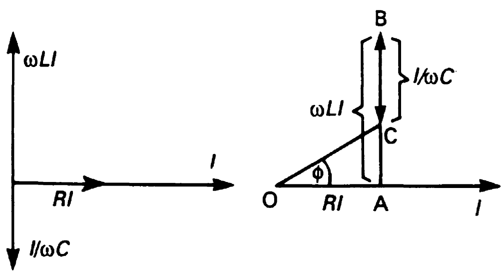

Figure 5.7 Phasor diagram for the series circuit of *Figure 5.5*

It will be realised that the triangle OAC is a graphical method of displaying and calculating the quantities Z and φ introduced in the mathematical approach. If $i = I \sin \omega t$ we can write $v = ZI \sin(\omega t + \varphi)$.

5.4.3 Summary

For a.c. circuits in which the frequency is ω the impedance to current flow of a simple series combination of R, L and C is denoted by $Z = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$ so that, confining our attention to magnitudes, $V = IZ$ or $I = V/Z$. There also exists a phase shift because the voltage and current will not, in general, be in phase, i.e. their maxima, minima, and zero values will not occur at the same instant of time.

Figure 5.8 shows how voltage and current vary with time. In this case the current lags behind the voltage or the voltage leads

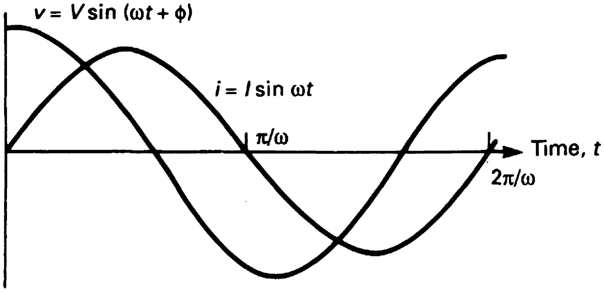

Figure 5.8 Voltage and current in a circuit in which current lags behind voltage

the current. If the effect of the capacitor were to exceed that of the inductor (i.e. if $I/\omega C > \omega LI$) the current would lead the voltage.

It should be appreciated that although diagrams could be drawn to scale and the answer for, say, the impedance could be measured, it is much more common to sketch the diagrams and use trigonometry to find the answer.

5.5 Impedance, reactance, admittance and susceptance

The impedance Z is best regarded as the quantity used in a.c. circuits analogous to resistance in d.c. circuits. In general it is a complex quantity consisting of two parts: the real part is resistance and represents energy dissipated in the circuit; while the imaginary part represents the energy stored and ultimately returned to the circuit. In this case, for an LCR series circuit,

$$Z = R + \mathrm{j}\left(\omega L - \frac{1}{\omega C}\right)$$

The terms representing the energy storage elements are $(\omega L - 1/\omega C)$ and we introduce the term reactance, X, to represent the energy storage element. Consequently $Z = R + \mathrm{j}X$. The units of Z and X must be the same as those of R, that is ohms. The inverse of impedance Z is the admittance Y and that of reactance X is susceptance B. As the reciprocal of resistance R is conductance G we can write $Y = G + \mathrm{j}B$.

5.6 Technique for a.c. circuits

Drawing on the example of the series LCR circuits we can form general rules. Considering a frequency ω we write $\mathrm{j}\omega L$ for every inductance, $1/\mathrm{j}\omega C$ for every capacitance and proceed as for a d.c. circuit containing only resistances. For example, consider the circuit in *Figure 5.9*, having written in $\mathrm{j}\omega L$ and $1/\mathrm{j}\omega C$ as necessary:

$$Z = \frac{R_2(R_1 + \mathrm{j}\omega L)}{R_2 + R_1 + \mathrm{j}\omega L} + \frac{1}{\mathrm{j}\omega C_1} + \frac{R_3/\mathrm{j}\omega C_2}{R_3 + 1/\mathrm{j}\omega C_2}$$

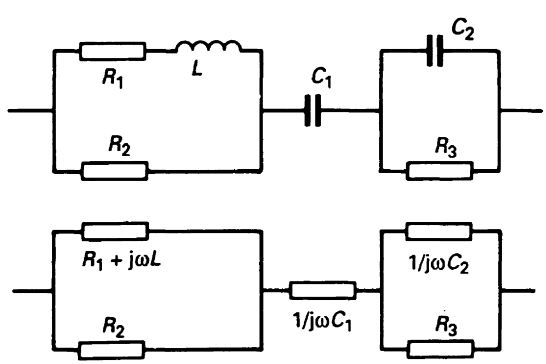

Figure 5.9 Circuit for analysis and first step in solution

This will reduce to the form $R + \mathrm{j}X$ enabling the relationship between overall voltage and current to be established. It is possible to achieve the same result using phasor diagrams, but when combinations of series and parallel elements are needed the work becomes tricky. The principles are that parts in series carry the same current whereas those in parallel experience the same voltage.

5.7 Average and r.m.s. values

So far we have dealt with the instantaneous values of voltage v and current i and the peak or maximum V and I. The true average of any quantity which is varying sinusoidally is zero. However, circumstances exist, such as full wave rectification, where we have waveforms like that shown in *Figure 5.10*. If the original voltage waveform was $v = V \sin \omega t$ the average of the fully rectified wave is $V_{av} = (2/\pi)V$.

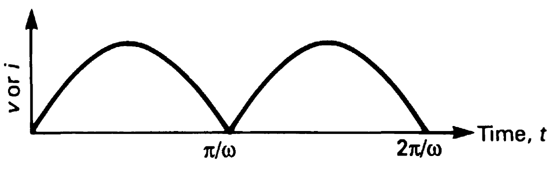

Figure 5.10 Full wave rectification

Another form of average is that associated with the current's ability to deliver power to a resistor R. If the current waveform is $i = I \sin \omega t$ then the power delivered is $i^2 R = (I \sin \omega t)^2 R$ averaged over a cycle. We introduce I_{rms} to represent this average such that $I_{rms} = (1/\sqrt{2})I$. The subscript stands for root mean square (r.m.s.) and those are the operations which, in the order square, mean and root, have been performed on the current.

Similarly we have $V_{rms} = (1/\sqrt{2})V$. It is the r.m.s. value of an a.c. quantity that is usually quoted and, unless stated to the contrary, it is to be assumed that a given voltage or current is the r.m.s. value.

5.8 Power, power factor

Instantaneously the power delivered to a circuit is given by $w = vi$ where w is the power and v and i the instantaneous voltage and current. The average power delivered is the mean of vi; therefore

$$W = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} vi \, dt$$

If $v = V \sin \omega t$ and $i = I \sin(\omega t + \varphi)$

$$W = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} VI \sin \omega t \sin(\omega t + \varphi) \, dt = \tfrac{1}{2} VI \cos \varphi \tag{5.18}$$

In this expression V and I are the peak values and φ is the phase angle. Introducing r.m.s. values we write

$$W = V_{rms} I_{rms} \cos \varphi \tag{5.19}$$

The quantity $\cos \varphi$ is known as the power factor and φ is sometimes known as the power factor angle. This form has the advantage that, for a circuit in which $\varphi = 0$, such as a pure resistance, the expression for power is simply $V_{rms} I_{rms}$ as would be expected from simple d.c. considerations.

Figure 5.11 shows how power varies through a cycle for different power factors. When current and voltage have the same polarity, either positive or negative, power is supplied to the circuit, and is shown shaded. When current and voltage have opposite polarities power is recovered from the circuit and is shown cross-hatched. The shaded areas represent power and it will be seen that power is always positive, i.e. being absorbed by the resistor in case I. In case II the shaded and cross-hatched areas are equal, indicating that energy is alternately stored and recovered with no net consumption. Case III is intermediate and it will be noticed that the shaded areas exceed the cross-hatched ones indicating a net consumption of power.

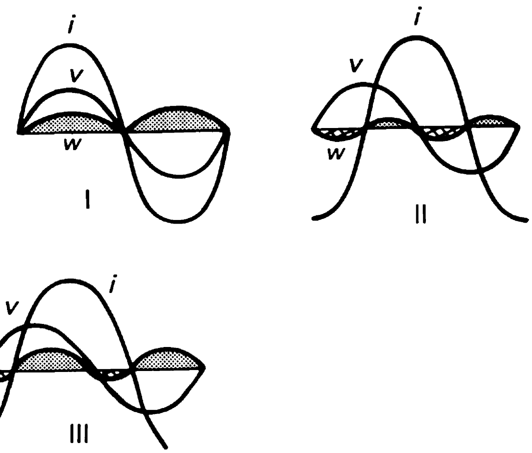

Figure 5.11 Power transfer in a.c. circuit. Case I: load entirely resistive, v and i in phase, maximum power w. Case II: load entirely inductive, i lags v by 90°, no net power. Case III load partially inductive, i lags v by less than 90°, some net power

5.9 Network laws and theorems: Kirchhoff's laws

The basis of systematic analysis of any circuit other than the most elementary is the two laws attributed to Kirchhoff. The first is the current law which states that the sum of all currents flowing into a node must be zero so in *Figure 5.12(a)*

$$i_1 + i_2 + i_3 + i_4 = 0$$

The second is the voltage law which states that there is no net change of voltage round a closed loop so in *Figure 5.12(b)* $V = iR_1 + iR_2$ when V is the rise of voltage from A to B and $iR_1 + iR_2$ is the fall of voltage from B through C and D to A.

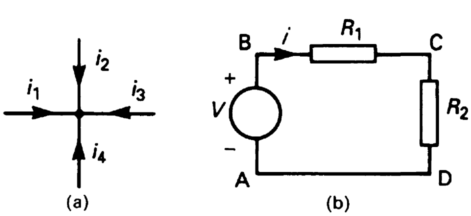

Figure 5.12 Kirchhoff's laws: (a) current law; (b) voltage law

The voltage law is usually used explicitly, for example a possible method of solving a circuit such as that shown in *Figure 5.13* would be to write in the currents i_1, i_2, $i_1 - i_2$, i_3 and $i_1 - i_2 - i_3$ in that order, implicitly making use of Kirchhoff's current law, and then to form as many equations as are necessary by the use of Kirchhoff's voltage law. Considering the loop comprising V, R_1, and R_3 we can write

$$V = i_1 R_1 + i_2 R_3$$

Considering R_2, R_3 and R_4

$$0 = (i_1 - i_2)R_2 + i_3 R_4 - i_2 R_3$$

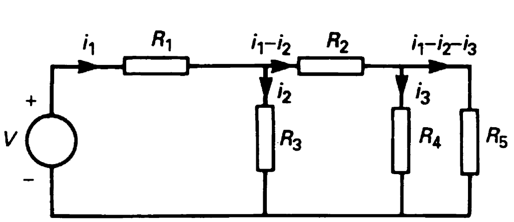

Figure 5.13 Illustration of Kirchhoff's laws

Considering R_4 and R_5

$0=(i_1-i_2-i_3)R_5-i_3R_4$

There are alternatives such as that obtained by considering V, R_1, R_2 and R_5 which would give

$V=i_1R_1+(i_1-i_2)R_2+(i_1-i_2-i_3)R_5$

However, this is not independent of the previous set of three equations as can be seen by adding these equations.

5.9.1 Loop or mesh currents

A device that is often helpful is to consider the current flowing round a loop rather than that actually in a wire. *Figure 5.14* shows the method. The current in any part is the sum of the loop currents which flow in the adjacent loops. If we apply the idea to the circuit of *Figure 5.13* we will consider loop currents i_4, i_5, i_6 as shown in *Figure 5.15*. In this case $i_1=i_4$, $i_2=i_4-i_5$, $i_3=i_5-i_6$.

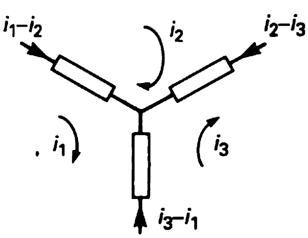

Figure 5.14 Loop currents

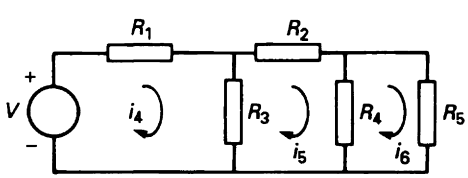

Figure 5.15 Illustration of loop currents

Writing the voltage loop equation we have the following equations. Considering loop i_4

$V=i_4R_1+(i_4-i_5)R_3$

Considering loop i_5

$0=i_5R_2+(i_5-i_6)R_4+(i_5-i_4)R_3$

Considering loop i_6

$0=i_6R_5+(i_6-i_5)R_4$

These can be arranged in a systematic manner which makes checking easy and leads to a matrix solution:

$$V=(R_1+R_3)i_4-R_3i_5$$
$$0=-R_3i_4 \quad +(R_2+R_3+R_4)i_5-R_4i_6$$
$$0=-R_4i_5 \quad +(R_4+R_5)i_6$$

The expression is symmetrical about the leading diagonal and the terms of the leading diagonal are the resistances as one traverses the appropriate loop.

5.9.2 Superposition

If a circuit is linear and contains more than one source the principle of superposition may be useful. Linearity means that currents are proportional to voltages. This means that diodes etc., which do not conduct equally for both directions of applied voltage, and devices (such as incandescent light bulbs) where the resistance changes, and so the current depends on the voltage to some power other than one, are excluded. Consider a circuit such as that shown in *Figure 5.16(a)*. Supposing we require the current I in R_2, the principle of superposition tells us that it is the sum of

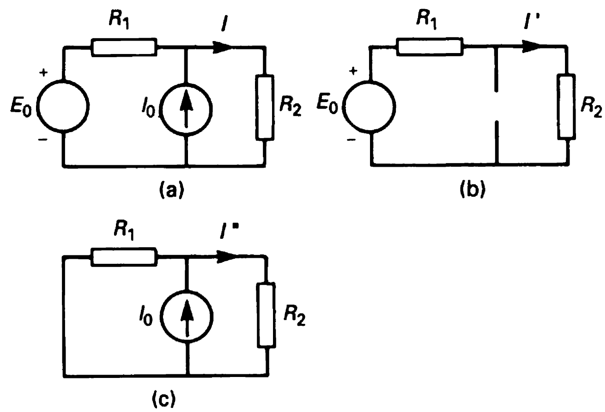

Figure 5.16 Illustration of superposition: (a) complete circuit; (b) current source deactivated or suppressed; (c) voltage source deactivated or suppressed

the currents I' and I'' where I' is the result of deactivating the current source and calculating the current due to the voltage source and I'' is the result of deactivating the voltage source and calculating the current due to the current source. A deactivated voltage source offers no impedance to the flow of current and so becomes a short circuit, whereas a deactivated current source offers an infinite impedance to the flow of current and so becomes an open circuit. Therefore *Figures 5.16(b)* and *(c)* show the two constituent parts and we can see that

$$I'=\frac{E_0}{R_1+R_2} \qquad I''=I_0\left(\frac{R_1}{R_1+R_2}\right)$$

and so

$$I=I'+I''=\frac{E_0+I_0R_1}{R_1+R_2}$$

5.9.3 Star–delta and delta–star transformations

These transformations may be useful in simplifying a circuit and are illustrated in *Figures 5.17* and *5.18*:

$$Y_{AB}=\frac{Y_A Y_B}{Y_A+Y_B+Y_C} \tag{5.20}$$

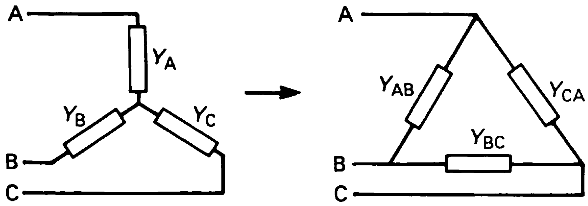

Figure 5.17 Star–delta transformation

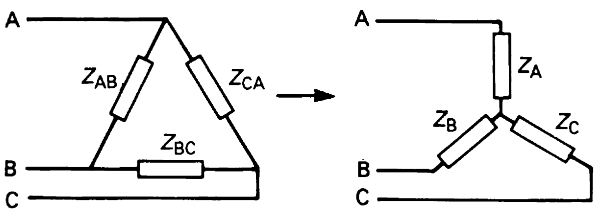

Figure 5.18 Delta–star transformation

Similarly for Y_{BC} and Y_{CA}.

$$Z_A = \frac{Z_{AB}Z_{CA}}{Z_{AB}+Z_{BC}+Z_{CA}} \tag{5.21}$$

Similarly for Z_B and Z_C.

5.10 Thevenin's theorem and Norton's theorem

Acceptance of these theorems enables one to analyse circuits consisting of one or more 'black boxes'. The so-called 'black box' approach encourages one to regard a piece of equipment in terms of what is observable at its terminals without regard to what is going on inside. Thevenin's and Norton's theorems give the quantities which must be used in any analysis. Previous sections have introduced current generators alongside voltage generators and the basic difference between Thevenin and Norton is that Thevenin expresses the circuit in terms of an equivalent voltage generator whereas Norton gives an equivalent current generator. Thevenin's theorem states that any two-terminal linear network can be represented by an ideal voltage source V_T in series with an impedance Z_T. The value of V_T is the voltage observed between the terminals when on open circuit and the value of Z_T is the impedance measured between the terminals with independent sources of voltage and current deactivated. The implications of deactivation were explained in Section 5.9.2. There is an alternative approach to the calculation of Z_T; it is that Z_T can be expressed as $Z_T = V_{oc}/I_{sc}$ when $V_{oc} = V_T$ and I_{sc} is the current that would flow in a short circuit placed across the terminals. Norton's theorem states that any two-terminal linear network can be represented by an ideal current generator I_N in parallel with an impedance. The value of the impedance is the same as that introduced with the Thevenin equivalent circuit but now it is in parallel with the source. The value of I_N equals I_{sc}, namely the short circuit current.

5.11 Resonance, 'Q' factor

If a circuit containing inductance and capacitance is subjected to a voltage of constant amplitude but varying frequency the current will have a maximum or minimum value for a particular frequency. Alternatively if the frequency is held fixed and the value of one of the components varied the same maximum or minimum will be observed. This phenomenon is known as resonance. To develop the important relations we will consider a simple circuit as shown in *Figure 5.19(a)* containing capacitance

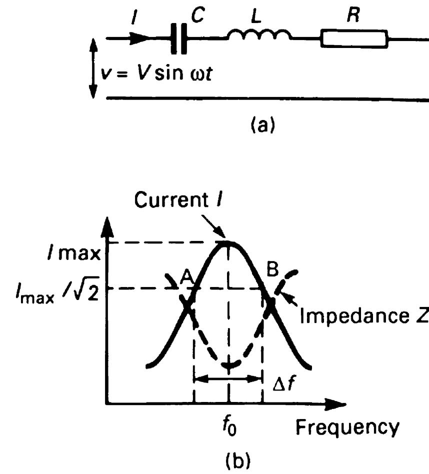

Figure 5.19 Resonance: (a) series circuit; (b) resonance curves

C, inductance L and resistance R. Using the techniques already developed for a.c. circuit analysis we write

$$v = \left(R + j\omega L + \frac{1}{j\omega C}\right) I \exp(j\omega t) \tag{5.22}$$

where the instantaneous current i is given by $I \exp(j\omega t)$. The instantaneous voltage v can be similarly represented by

$$v = V \exp[j(\omega t + \varphi)] \tag{5.23}$$

where φ is the angle by which V leads I. It follows that V, the peak value or modulus of the voltage, is related to I, the peak value or modulus of the current, by the relationship

$$V = \sqrt{R^2 + (\omega L - 1/\omega C)^2}\, I \tag{5.24}$$

and the phase angle φ is given by

$$\tan \varphi = \frac{\omega L - 1/\omega C}{R} \tag{5.25}$$

If we consider the usual elementary situation in which the circuit is supplied with a voltage of constant amplitude but varying frequency then the current I is given by

$$I = \frac{V}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \tag{5.26}$$

and a possible shape is sketched in *Figure 5.19(b)*.

The current has a maximum value at a frequency f_0 $(=\omega_0/2\pi)$ known as the resonant frequency. Its value is given by putting $\omega_0^2 = 1/LC$ in Equation (5.24). At this frequency it should be noticed that the phase angle φ is zero. If $\omega > \omega_0$ the phase angle is positive, that is, the voltage leads the current as is typical of an inductance; and if $\omega < \omega_0$ the phase angle is negative as is found with a capacitance. *Figure 5.19(b)* also shows the manner in which the impedance Z varies.

Resonance circuits, because of their frequency-dependent characteristics, are used for tuning purposes; that is, to pick out signals of one particular frequency from a range of signals, e.g. the tuning stages of a radio receiver. In these cases, of course, one of the circuit elements—normally the capacitor C—is varied until the resonant frequency equals that of the desired signal. For good selectivity the peak of the resonance curve needs to be sharp. The measure of the sharpness is the resonant frequency f_0 divided by the width Δf at some particular level. The level that is usually chosen is that at which the current has a value $1/\sqrt{2}$ of its maximum value. Bearing in mind that the power developed by a current in a resistor is I^2R it will be realised that if the current is $1/\sqrt{2}$ of its maximum then the power will be 1/2 of the maximum. These points are therefore known as the half power points (A and B in *Figure 5.19(b)*).

The symbol Q, defined as $f_0/\Delta f$ or $\omega_0/\Delta\omega$, is used for the sharpness as it measures the quality of the circuit when used as a tuning device. If we return to the simple series circuit shown in *Figure 5.19(a)* and if we draw the full phasor diagram as in *Figure 5.20* it will be seen that at resonance the voltages across both the

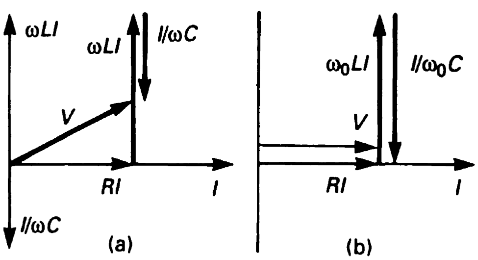

Figure 5.20 Phasor diagram for the circuit of *Figure 5.19(a)*: (a) circuit not in resonance; (b) circuit in resonance

inductor and the capacitor are equal. This leads to an alternative definition of Q, based on voltage magnification:

$$\frac{\text{voltage across inductor}}{\text{total voltage}} = \frac{\omega_0 LI}{RI} = \frac{\omega_0 L}{R} \tag{5.27}$$

$$\frac{\text{voltage across capacitor}}{\text{total voltage}} = \frac{I}{\omega_0 CRI} = \frac{1}{\omega_0 CR} \tag{5.28}$$

So we can define Q as $\omega_0 L/R$ or $1/\omega_0 CR$.

One should point out that, although voltage has been magnified, there has been no magnification of power. The voltages across both the inductor and the capacitor are in quadrature with the current so no power is available. Looked at from the energy point of view, relatively large amounts of energy are stored in the inductor and the capacitor but they are always returned to the source.

This section has, so far, only considered series circuits; the results for a parallel circuit are similar except that whereas the series circuit has a minimum impedance at resonance the parallel one has a maximum. The analysis is complicated by the fact that a true parallel circuit (R, L and C in parallel) is not a correct representation of a real inductor connected in parallel with a real capacitor. It will be realised that to obtain a high Q in a series circuit it is necessary to have a small R. A real good quality capacitor has negligible resistance but a real inductor, made of a coil of wire, must possess some resistance. However, it can be shown that the practical situation of a real capacitor in parallel with a real inductor complete with parasitic resistance, r (*Figure 5.21(a)*) is equivalent to a true parallel circuit (*Figure 5.21(b)*) provided that Q is large and we put $C = C'$, $L = L'$ and $R = Q^2 r$. In other words, the small parasitic r in series is equivalent to a large R in parallel.

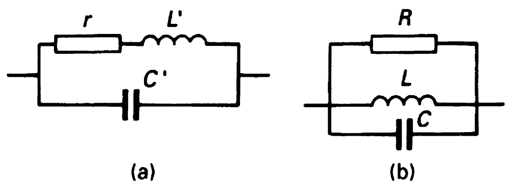

Figure 5.21 To show equivalence of real parallel circuit (a) with circuit (b) which is easier to analyse

5.12 Mutual inductance

This is the phenomenon whereby a changing current in one circuit produces a voltage in another. It is usually explained by appealing to the ideas of lines of magnetic flux which, when they change, produce an electromotive force in a circuit. Consider two coils as shown in *Figure 5.22*. If current in coil 1 is changing at a

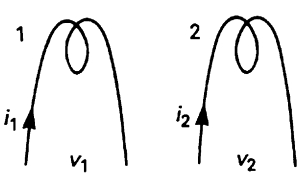

Figure 5.22 Illustration of mutual inductance (air cored)

rate di_1/dt then a voltage proportional to di_1/dt will be induced in coil 2. The coefficient of proportionality is known as the coefficient of mutual inductance M so that

$$v_2 = M\,di_1/dt \tag{5.29}$$

and

$$v_1 = M\,di_2/dt \tag{5.30}$$

The fact that the same value of M is used in both equations will probably be intuitively obvious; it can be proved by considering the energy stored in the coupled circuits when first one and then the other current is switched on. For two coils such as those shown in *Figure 5.22* the value of M will be much smaller than it would be if the coils were closer together to reduce leakage and if they were wound on an iron former. Much of the flux emanating from, say, coil 1 will not go through coil 2 and there is said to be a lot of leakage; this could be reduced by bringing the coils together. If, however, we were to wind the coils on an iron former the value of M would be much greater because more flux is produced due to ferromagnetism.

When analysing a circuit containing mutual inductance it is often helpful to redraw the mutual inductance adding generators as shown in *Figure 5.23*. The directions shown for positive current flow have the merit of symmetry but if one is considering

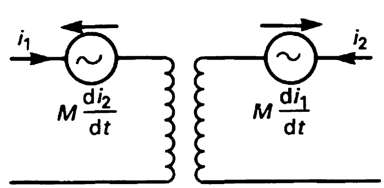

Figure 5.23 Circuit representation of mutual inductance

a transformer, the most commonly occurring example of mutual inductance, it is normal to think of an input current i, as shown, and of an output current, which would be $-i_2$ in *Figure 5.23*.

In line with the introduction to L and C and the analysis of a.c. circuits which has been presented earlier it should be pointed out that, if, as is often the case, we are dealing with sinusoidally varying quantitives, then $\mathrm{d}i/\mathrm{d}t$ becomes $\mathrm{j}\omega i$.

5.13 Differential equations and Laplace transforms

Equations containing integral and differential expressions are known as differential equations and many books have been devoted to their solution. To illustrate the process we will consider in more detail the equation developed in Section 5.4. The basic equation is (see Equation (5.15))

$$v=Ri+L\frac{\mathrm{d}i}{\mathrm{d}t}+\frac{1}{C}\int i\,\mathrm{d}t$$

An alternative form which may appear simpler, because it contains only differential coefficients and not a mixture of differential coefficients and integrals, is obtained by differentiating throughout and rearranging to give

$$L\frac{\mathrm{d}^2 i}{\mathrm{d}t^2}+R\frac{\mathrm{d}i}{\mathrm{d}t}+\frac{i}{C}=\frac{\mathrm{d}v}{\mathrm{d}t} \tag{5.31}$$

There are two parts to the solution of such an equation and the complete solution is the sum of both. The term on the right-hand side ($\mathrm{d}v/\mathrm{d}t$) is known as the forcing or driving function. In this case it is the voltage which is applied to the circuit even though in this analysis it is its rate of change that is used. Without a driving function nothing would happen unless, say, there was an initial charge on the capacitor in which case a solution is required to describe what happens as a result of this initial charge. The two parts of the solution referred to above are the complementary function and the particular integral.

(*1*) *The complementary function* is the solution to the equation with the independent variable, which is usually written on the right-hand side, put equal to zero. The complementary function describes the result of any initial charges or currents. In a stable system it decays to zero and represents the transient behaviour.

(2) *The particular integral* requires the inclusion of the independent variable, the driving voltage in the case we are considering. The particular integral is any solution to the full equation resulting from the inclusion of the forcing function. In our case it cannot be found until the amplitude and frequency of the driving voltage are known.

Traditionally, for this type of equation, the method of finding these two solutions goes as follows. For the complementary function we let $i=A\,\mathrm{e}^{mt}$ and substitute into Equation (5.31) with the right-hand side equal to zero, which produces values for m. For each m there will be a different value of A; these arbitrary constants depend on the initial conditions. The solution for the particular integral calls for an element of guesswork and intuition. If the driving function is sinusoidal it is a sensible guess that the response will also be sinusoidal and of the same frequency. It represents the steady-state, or long-term, solution.

All the guesswork can be taken out of the solution of these differential equations by the use of the Laplace transform. The effect of this is to transform the differential (and integral) expressions into straightforward algebraic equations. Once this has been done to the complete equation the transform of the dependent variable, in our case the current i, is expressed in terms of the other quantities and the use of the inverse transform produces the solution we require.

The Laplace transform of a function of time $f(t)$ is denoted by $F(s)$ and is defined as

$$F(s)=\int_0^{\infty} f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$$

The inverse transform is

$$f(t)=\frac{1}{2\pi\mathrm{j}}\int_{\sigma-\mathrm{j}\infty}^{\sigma+\mathrm{j}\infty} F(s)\,\mathrm{e}^{st}\,\mathrm{d}s$$

However, tables are frequently used rather than these formulae.

Use of the Laplace transform method quickly leads to the realisation that the equations can easily be set up by considering the impedance of an inductor to be sL and that of a capacitor to be $1/sC$. This is satisfactory as long as there are no initial currents or voltages respectively. The term generalised impedance is used for impedances expressed in this manner. The advantage of the method is that any type of input can be handled equally easily as long as the transform exists. In addition to sinusoids the other inputs that are encountered are step functions, repeated step functions or square waves and ramp functions. For sinusoids replace s with $\mathrm{j}\omega$ and we arrive back at the equation with which we started.

5.14 Transients and time constants

As the word transient implies, any phenomenon to which it is applied is short-lived and the time constant is a measure of its duration. A simple circuit that is often used to illustrate these ideas is that of a capacitor connected to a source of voltage through a switch and a resistor as shown in *Figure 5.24*(*a*).

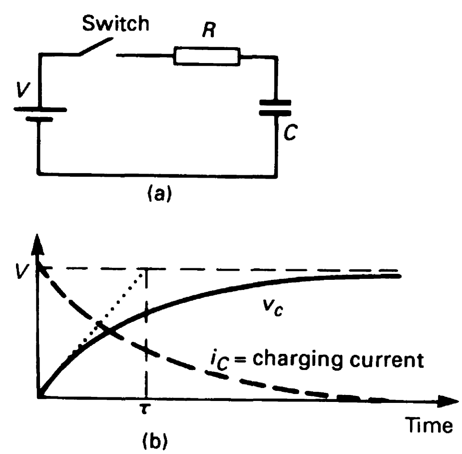

Figure 5.24 Illustration of transient phenomena: (a) circuit; (b) voltage and current graphs

If the capacitor is initially uncharged it can be shown that the voltage v_C across the capacitor at time t after closing the switch is given by

$$v_C=V[1-\exp(-t/RC)] \tag{5.32}$$

The current in the circuit decays according to the expression

$$i=(V/R)\exp(-t/RC) \tag{5.33}$$

The time constant τ can be regarded in two ways. Firstly it is defined, for this circuit, as being RC which means that it is the time it takes for the voltage to rise to $1-1/e=0.632$ of its final value. The second approach is to calculate how long it would take for v_C to reach V if the initial rate of change of v_C were to be maintained. The dotted line in *Figure 5.24(b)* is drawn tangential to the curve of v_C at the origin and illustrates this approach to the time constant.

If a circuit were to contain an inductor L instead of the capacitor the time constant would be L/R, and the current in a circuit containing an inductor and a resistor in series would be

$$i=\frac{V}{R}[1-\exp(-Rt/L)] \tag{5.34}$$

The final current must be V/R because there will be no voltage across the inductor in the ultimate steady state. If one is faced with a circuit containing both inductance and capacitance the results are more complicated and, in the absence of resistance, there will be no steady long-term solution. The solution can best be approached by setting up the differential equation and using Laplace transforms to solve it. The closing of a switch to apply a constant voltage is to apply a 'step function' and this can be handled easily using Laplace transforms because the transform of a step is $1/s$ times the size of the step.

5.14.1 Pulses and square waves

The response of a circuit to a pulse is often easily understood by considering the response to two step functions of opposite signs and one delayed relative to the other, as shown in *Figure 5.25(a)*. The shape depends on the relative sizes of the time constant τ and the delay interval T. *Figure 5.25(c)* shows the situation when $\tau \simeq T$.

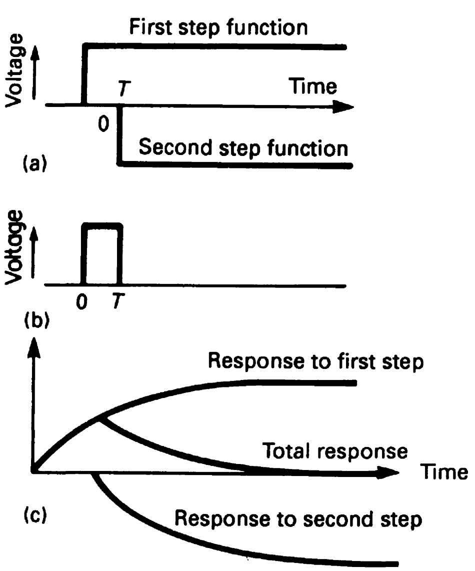

Figure 5.25 Effect of a pulse: (a) two-step representation; (b) pulse; (c) circuit response

If $\tau \ll T$ then the response would be much closer to the square wave input. If $\tau \gg T$ the output voltage would hardly change because the second and negative pulse would come so soon after the first and positive one.

Square waves are a series of such pulses and the response can be built up accordingly. The time gap between pulses and its relation to the time constant is the important quantity.

5.15 Three-phase circuits

Alternating current has many advantages over direct current when considering generation and transmission; no commutator is needed in the generator and the voltage can easily be changed to the level appropriate for economical transmission. However, single-phase a.c. systems are not as efficient as d.c. when considering the quantity of material required to transmit a given amount of energy. A further drawback to single-phase a.c. motors is that they are not inherently self-starting. A three-phase system overcomes both these disadvantages. Imagine a generator with three coils—R (red), Y (yellow) and B (blue)—on the stator and a magnet turning inside. The magnet could be a permanent one but is more likely to be an electromagnet fed from a d.c. source. If the pole faces are shaped so that the distribution of flux is sinusoidal then sinusoidal waves of voltage will be induced in the coils (see *Figure 5.26*).

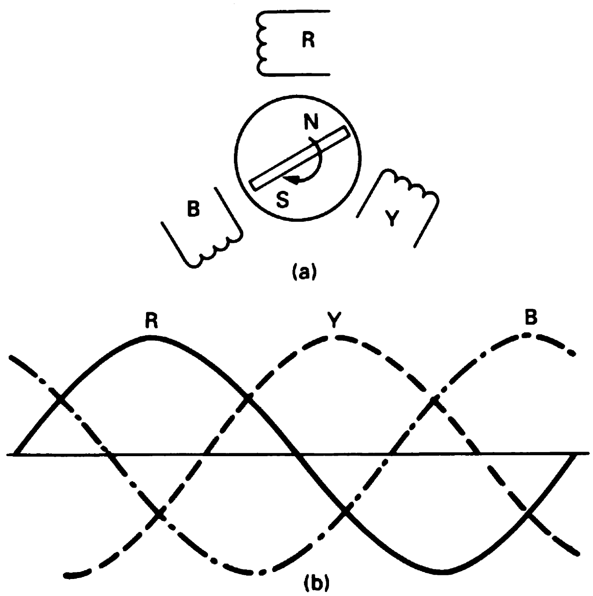

Figure 5.26 Elementary three-phase generator (a) and wavefofms (b)

5.15.1 Star or Y connection

It is not necessary to carry all six wires from the three coils to the load. The easiest way to connect them together is to join one end of each coil to a common point and call that point the star or neutral point. This is known as a star connection as shown in *Figure 5.27*.

There are two ways in which the voltage can be expressed, namely the phase value or the line value. The three-phase values V_R, V_Y and V_B are equal in magnitude (V_ϕ) but differ in phase by 120°; similarly the line values V_{RB}, V_{BY} and V_{YR} are also equal in magnitude (V_L), but differ in phase by 120°. The relationship between V_L and V_ϕ is

$$V_L=\sqrt{3}\,V_\phi \tag{5.35}$$