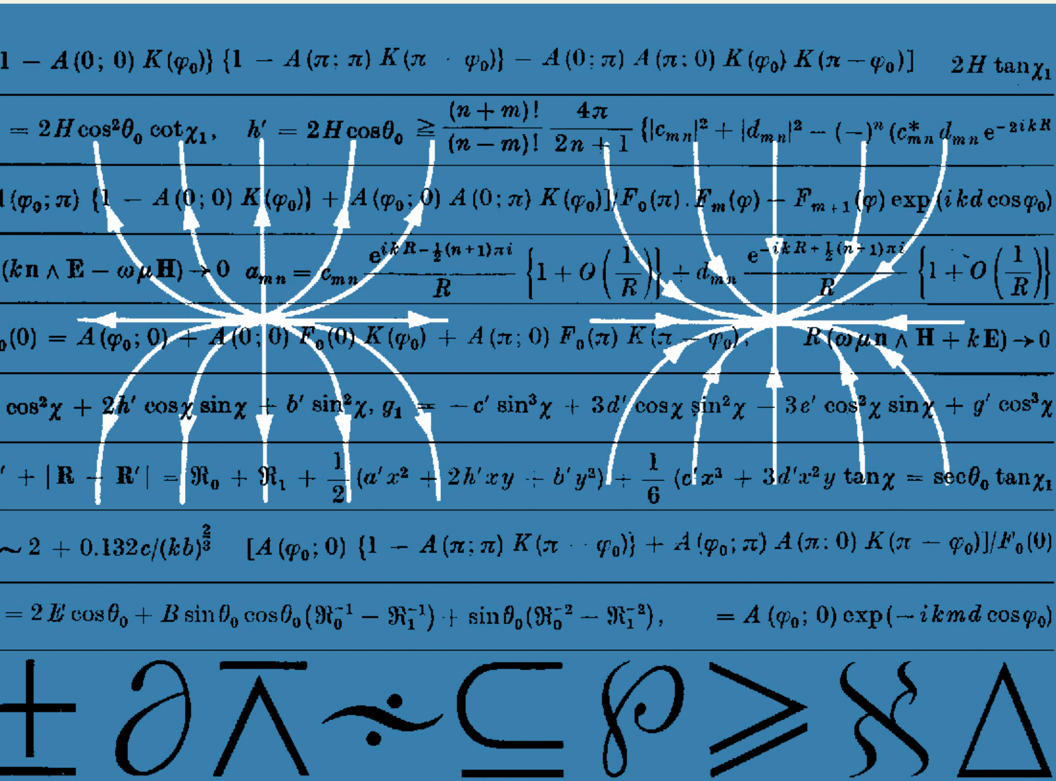
Regular Figures

L. Fejes Tóth



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REGULAR FIGURES

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REGULAR FIGURES

by

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PREFACE

ON BUILDINGS, machines and other products of our civilization, regularly arranged objects are often observed. The parquet blocks on the floor, the teeth on a cog-wheel, and the figures on a fancy cloth are all regularly arranged. Nature also produces a great variety of regular distributions, in the kingdom of the living and non-living as well. We recall the petals of flowers with various kinds of rotational symmetry, or the arrangement of atoms in crystals.

A discrete set of equal figures is said to be regularly arranged if each figure can be carried into any other one by a congruent transformation or *isometry* leaving the whole configuration unchanged. All such transformations together form a group, the *symmetry group* of the arrangement. The main aim of the classical theory of regular figures is to enumerate all possible symmetry groups in different spaces of constant curvature. This general problem is connected with a range of further problems: what kind of regular figures exists under certain restrictions? This question involves the theory of the regular polytopes, tessellations and lattices. Besides the enumeration of the various kinds of regular figures classical theory attempts to determine their metrical and topological properties. Thus it may be considered as the *systematology* and *morphology* of the regular figures.

This theory is one of the oldest branches of science, the foundations of which were laid by Greek and Egyptian artists. In the seventeenth century Kepler made essential contributions to the theory, but its golden age begins with the nineteenth century. This renascence of the regular figures of antiquity was due partly to the investigation of the inner structure of crystals, and partly to the discovery of the deep connection of regular figures with other branches of mathematics, especially with algebra, group theory, number theory and the theory of functions. The geometry of numbers is, today, one of the chief driving forces of the evolution of the theory.

Besides this classical theory, regular figures may be approached in another way, starting from the observation that extremum postulates often involve regularity. Classical theory starts with a more or less arbitrary *definition* of regularity. Here, in turn, regular arrangements are generated from unarranged, chaotic sets by the ordering effect of an *economy principle*, in the widest sense of the word. This theory may be called the *genetics* of regular figures.

Systematology plays a central part in directing the researches of genetics. On the other hand, the different extremum properties of the regular figures may be considered as precious contributions to the classical theory. This organic connection of the two approaches makes it reasonable to expound them in one book.

This book is divided into two parts: systematology and genetics of the regular figures. (Aspects of morphology are incorporated partly in the first, partly in the second part, where metrical properties of regular figures appear in various inequalities as the cases of equality.) In both parts we shall be content to present some typical, simple and interesting results and methods. For a more detailed discussion of the classical theory we refer to the excellent modern monographs of Coxeter (1948), Burckhardt (1947) and Coxeter and Moser (1957). Concerning recent theory the reader may consult the author's book (1953a) and the great number of original works on the subject quoted in the bibliography.

"C'est la dissymétrie qui crée le phénomène" — writes Pierre Curie, expressing by these words the frequently observed tendency towards symmetry in fundamental physical structures. It is always an extremum postulate which lies at the bottom of this tendency. Thus we seem to be on the right track towards the wider aim of throwing some light on the causes of their origin, besides describing and systematizing the regular figures occurring in nature.

But, in writing this book, we have a much narrower aim in mind which may be best expressed by echoing the words of

Clebsch written in his memoir on Plücker: "Es ist die Freude an der Gestalt in einem höheren Sinne, die den Geometer ausmacht". I should like to awaken this noble joy in the reader, showing that we are all, in the sense of Clebsch, geometers.

*

I wish to express my sincere gratitude to Professor H. S. M. Coxeter for having encouraged me to write this book, for reading the whole manuscript and making many valuable suggestions. I offer also my friendly thanks to A. Heppes for conscientiously criticizing the manuscript, to J. Molnár for the numerous expressive drawings and to I. Pál for the beautiful anaglyphs. Last but not least I remember with grateful affection the enjoyable winter semester (1960/61) at the University of Freiburg in Breisgau where I completed this book amid an inspiring circle of colleagues.

Budapest,
July, 1963

L. FEJES TÓTH

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PART ONE

SYSTEMATOLOGY OF THE REGULAR FIGURES

THE first part of our book contains some chapters on the classical theory of the regular figures. We start with the Euclidean plane, the "space" most familiar to us, giving a complete enumeration of its discontinuous groups of isometries. We proceed with the analogous problem in spherical geometry (which is the first step towards the far more laborious task concerning Euclidean 3-space). Then we make a journey into the 2-dimensional hyperbolic space and into the general spherical and Euclidean spaces, considering, first of all, their regular divisions.

CHAPTER I
PLANE ORNAMENTS

ONE of the most interesting instances of a deep connection between art and mathematics is provided by the surface ornaments, raised to such an admirable degree of perfection by ancient artists. The task of the artist is to find for a certain type of ornamental symmetry an elementary figure whose repetitions intertwine to give a harmonious whole. The mathematician, in turn, is interested only in the symmetry operations occurring in an ornament. Chapter I deals with the mathematical theory of plane ornaments. In addition, it provides a vivid introduction to one of the most fundamental notions of modern mathematics, the concept of a group.

1. Isometries

An isometry which leaves a figure invariant is called a *symmetry operation*. In order to classify the ornaments according to their symmetry operations we have to investigate the various isometries of the plane.

In the plane, an isometry, i.e. a distance-preserving mapping, is uniquely determined by its effect on a rectangular Cartesian co-ordinate frame. It is said to be *direct* or *opposite* according as it preserves or reverses the sense of the frame. A direct isometry can be achieved by a rigid motion of the plane in itself. Therefore it is often called a *proper motion*. On the other hand, an opposite isometry requires besides a proper motion a reflection in a line. Executing this reflection by a half-turn about the line as axis we obtain, as a final result, a rigid motion in which, however, we must come out of the plane. Therefore an opposite isometry is also called an *improper motion*.

The simplest direct isometries are the *translations*, in which every point of the plane moves through the same distance in the same direction. A translation is uniquely determined by a directed line-segment AB, called a *vector*, leading from a point A to its image B. We shall often denote this translation by $A \to B$. Another type of direct isometry is a *rotation* of the plane through a given angle about a given point. We shall show that no other proper motions exist in the plane.

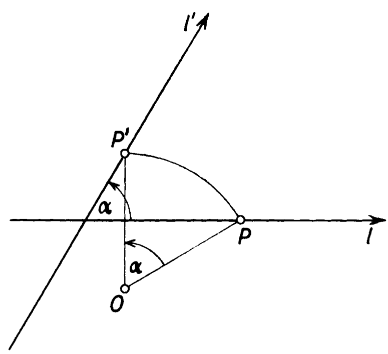

Fig. 1/1

We shall find it convenient to regard translations as rotations through a zero angle about an infinitely distant point. Then our statement reads as follows: *every proper motion of the plane is a rotation*. To make this evident, we note that a proper motion is uniquely determined by a point P, an oriented line l through P and the images P' and l' of P and l. Since the cases where P and P' or the directions of l and l' coincide are trivial, we may suppose that P and P' differ and l and l' include an angle a $(0 < a < 2\pi)$. Let O be a point equidistant from P and P', such that the rotation about O transforming P into P' has an angle equal to a. This rotation transforms l into l' (Fig. 1/1).

The improper motions can also be reduced to a simple type of isometry, called *glide-reflections*. A glide-reflection is the resultant of a reflection in a line and a translation in the direction of this line. Considering reflections as special cases of glide-reflections, we may assert that *every improper motion of the plane is a glide-reflection.*

In order to show this we notice that an improper motion is determined by the transforms P' and l' of a point P and an oriented line l through it. Consider the line parallel to the bisector of the angle between l and l' passing through the

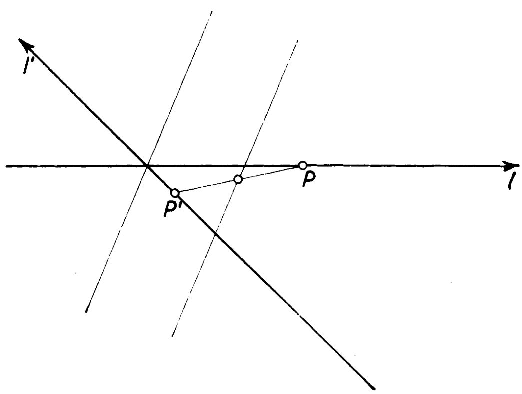

Fig. 1/2

midpoint of the segment PP'. The glide-reflection in this line which carries P into P', transfers l into l' (Fig. 1/2).

Now we make some remarks concerning the composition of isometries. A certain analogy exists between the composition of isometries and the multiplication of numbers. Therefore we denote the transformation arising by performing first the transformation U, then the transformation V, by UV. We call this resultant transformation the *product* of U and V. This kind of multiplication is *associative* :

$$(UV)W = U(VW),$$

so that either side may be denoted by UVW. But it is generally *not commutative* : $UV \neq VU$. If $UV = VU$, (as, for example,