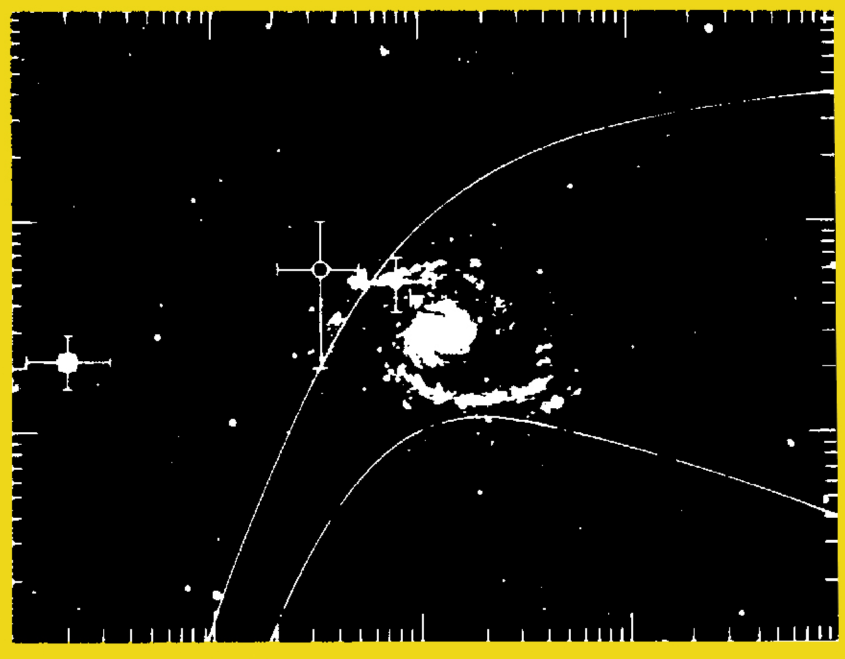
Essays in Theoretical Physics

In Honour of Dirk ter Haar

Edited by W. E. Parry
Fellow of Oriel College, Oxford



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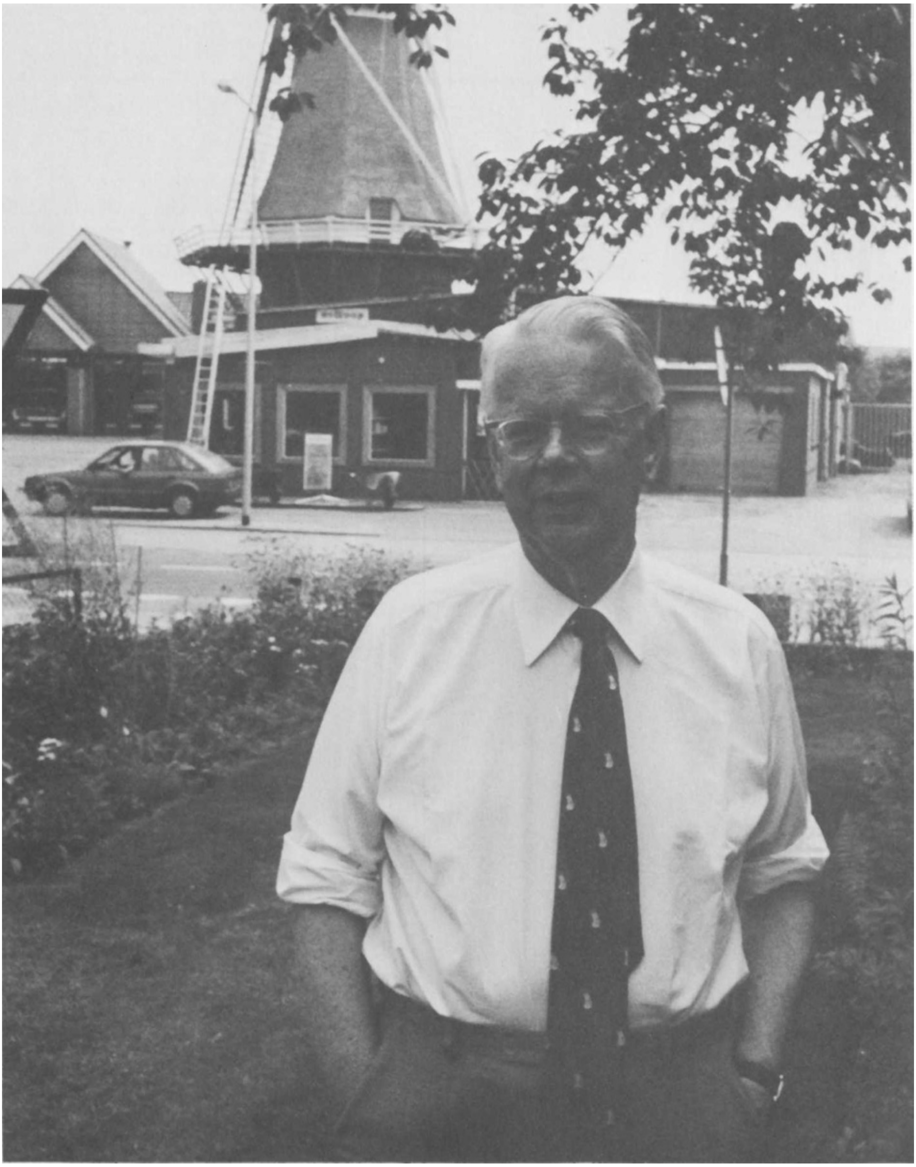

DIRK TER HAAR

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Dedication

Dirk ter Haar was born on 22nd April, 1919. He was a student of Leiden University. He has been a research fellow at Bohr's Institute in Copenhagen, a professor at Purdue, and a lecturer at St. Andrew's. Since 1956 he has been at Oxford where he is a fellow of Magdalen College and University Reader in Theoretical Physics. He has been an editor of Physics Letters since 1952.

He has published many papers, chiefly in astrophysics and the many-body problem, and has written books on Statistical Mechanics and Hamiltonian Mechanics. He has edited the collected papers of Landau and Kapitza, and is the general editor of Pergamon's Natural Philosophy Series.

In 1947 he married Christine Janet Lound; they have two sons and one daughter.

The remarkable breadth of Dirk's interest in physics is to some extent reflected in this volume, which is a product of the affection of his pupils and colleagues, and of the high regard in which they hold his own research, his inspiration to others in their research, and his work for the scientific community.

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Stellar Dynamics with Non-classical Integrals

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ABSTRACT

Effective integrals of motion that do not derive from a manifest Lagrangian symmetry appear to play a vital role in structuring galaxies. Recent work suggests that these effective integrals may sometimes by understood in terms of potentials that lead to separable Hamilton-Jacobi equations. However, studies of the action integrals of numerically integrated orbits in given rotating potentials suggest that many effective integrals of astronomical interest cannot be understood in this way. A hybrid approach to stellar dynamics in which the angle-action representations of orbits in model potentials are obtained by numerical integration and are then analyzed by the techniques of Hamiltonian perturbation theory may lead to a more satisfactory understanding of the relationship between potentials and the orbits that they support.

1. Introduction

From Newton through Laplace, Gauss and Jacobi to Poincaré it may be safely asserted that dynamics was driven by astronomy. In recent years dynamical problems have attracted the attention of scientists from a wider range of backgrounds and astrophysicists can no longer claim to be in undisputed possession of the frontier of dynamical theory. However, several important branches of astrophysics still turn on problems in dynamics and the interest of astrophysicists in developing this theory is as keen as ever. In this article I review some recent work in one such branch of astrophysics — the dynamics of large groupings of point masses, or 'stellar dynamics' as it is usually rather confusingly called. My aim in this article has been to tell physicists what astrophysicists have been up to recently, and to describe two new developments which promise to bring mathematical concepts that were developed by Jacobi and others in connection with the dynamics of the solar system, into contact with the empirical-numerical approach to stellar dynamics that has grown up since electronic computers became widely available. I have tried to make the mathematical development in this article as self-contained as possible, but I have assumed a familiarity with the basic ideas of Hamiltonian mechanics at the level of Berry's excellent review article (Berry, 1978).

This article is organized as follows: § 2 introduces the mean-field approximation that underlies all work in stellar dynamics. §3 provides a very brief summary of the classical 'third integral' problem in galactic dynamics. §4 reviews recent work on the nature of orbits in triaxial gravitational potentials, with particular emphasis on the classification of orbits into families and the natural explanation which these families find in the context of certain separable potentials. §5 introduces the spectral approach to stellar dynamics and describes its application to studies of the action integrals of orbits that have been obtained by numerical integration of the equations of motion. These studies throw an interesting light on how widely separable potentials may be applicable to problems in stellar dynamics. §6 sums up and indicates some possible directions of future work.

2. The Mean-Field Approximation

The central idea of all stellar dynamics is the use of a mean-field approximation for the gravitational field in which individual stars move: we calculate the orbits of individual stars as they move through the potential of an idealized model of the system under study. In this model mass is smoothly distributed rather than concentrated into individual stars. Clearly an orbit that is calculated using this model potential will not be exactly the same as the true orbit of the star. The relaxation time $t_{\rm rel}$ is a measure of the length of time during which orbits in the model potential remain a reasonable approximation to the true orbits. Orbits in our model potential are entirely untrustworthy after a time $t \gg t_{\rm rel}$ and are a good approximation to the truth for $t \ll t_{\rm rel}$.

The true orbit deviates from the model orbit for two main reasons:

1) Stars sometimes pass close to each other and experience short-lived bursts of relatively intense acceleration toward one another. No such bursts

of acceleration occur along a model orbit.

2) Since there are only a finite number of stars in even a large volume of the stellar system, the number of stars in such a volume will fluctuate more-or-less randomly about the mean value on which the mean-field model is based. Large-scale density fluctuations of this nature generate large-scale fluctuations in the gravitational potential of the system and these deflect the true orbit from the path calculated within the context of the mean-field model.

These processes that cause true orbits to deviate from model orbits are generally referred to as 'encounters'. Process (1) above is that of 'close encounters', while process (2) is rather misleadingly described as that of 'distant encounters'. Since Jeans' pioneering investigations (Jeans, 1913) many authors (Chandrasekhar 1941, Cohen, Spitzer and Routly 1950) have calculated the relaxation time t_{rel} to which encounters lead. The relative importance of close and distant encounters still generates a surprising amount of controversy. Fortunately the order of magnitude of t_{rel} is not controversial and one may safely adopt Spitzer's (Spitzer and Hart 1958) estimate

$$t_{\mathrm{rel}} = \frac{(2/3)^{3/2} v^3}{2\pi G^2 \rho m \log_e (0.4 N)} . \tag{1}$$

Here G is Newton's gravitational constant, ρ is the mass density of (identical) stars of mass m, and v is the rms speed of the N stars in the system. If the system has a characteristic radius r, one has $v^2 \simeq GNm/r \simeq 4\pi G\rho r^2/3$ by the virial theorem, so

$$t_{\mathrm{rel}} = \frac{(2/3)^{5/2}\, rN}{\log_e(0.4\, N)\, v} \simeq \frac{0.36\, N}{\log_e (N)}\, t_{\mathrm{cross}} \tag{2}$$

where $t_{\mathrm{cross}} \equiv r/v$ is the characteristic dynamical or 'crossing' time of the system.

In some smaller stellar systems, for example globular clusters (which have $N \simeq 10^5$ and $t_{\mathrm{cross}} \simeq 10^6\, y$), t_{rel} is less than the age of the Universe, $t_{\mathrm{Hubble}} \simeq 10^{10}\, y$, and one cannot ignore the effects of encounters. However, whenever $N \geqslant 100$, equation (2) yields $t_{\mathrm{rel}} \gg t_{\mathrm{cross}}$, so the mean-field approximation gives a description of the dynamics of the system that is valid over many dynamical times. Thus it is expedient to treat encounters among cluster stars as perturbations within a mean-field model; these perturbations cause the mean-field model to evolve on the long time-scale t_{rel}.

A typical giant galaxy like our own has $N \geqslant 10^{11}$ and $t_{\mathrm{cross}} \geqslant 10^8\, y$, so $t_{\mathrm{rel}} \simeq 10^{18}\, y \gg t_{\mathrm{Hubble}}$. Hence the mean-field approximation should provide an entirely sound basis for understanding the structure of galaxies.

The first step towards constructing a self-consistent mean field model of a stellar system is to understand the structure of orbits in the mean-field potential. Once these orbits have been conveniently catalogued, we can try to populate them in such a way that the net density due to all the stars in the system is at each point equal to the known density of the system.

It should be emphasized that we are interested in the structure of orbits over only a few hundred to some thousands of characteristic dynamical times. This greatly simplifies the task of cataloguing stellar orbits as it implies that we do not have to worry about subtle effects that might become apparent if we were to integrate the equations of motion for millions or even thousands of millions of orbital times. This situation contrasts with that obtaining in celestial mechanics and plasma physics. In those fields one does have to worry about the long run. In stellar dynamics we can safely argue that in the long run we are all dead.

3. Orbits in Axisymmetric Potentials

What sort of potentials do galaxies have ? While the detailed distribution of gravitating matter in even our own Galaxy is still highly controvertial. it is certain that few if any galaxies have spherically symmetric potentials. Therefore the highest symmetry that we need consider is that of axisymmetry. Studies of motion in flattened axisymmetric potentials became popular in the mid 1960's, when electronic computers became widely available, and we now have a pretty clear picture of the types of orbits that these potentials support.

If (R,ϕ,z) is a system of suitably oriented cylindrical polar coordinates, motion in an axisymmetric potential, $\Phi(R,z)$ can be reduced to the two-dimensional problem of motion in the 'comoving meridional plane' (R,z) by replacing $\dot{\phi}$ in the R-equation of motion by $\dot{\phi}=L_z/R^2$ where L_z is the star's component of angular momentum about the symmetry axis of the potential. The equations governing the two-dimensional motion are

$$\ddot{R}=-\partial\Phi_{\mathbf{eff}}/\partial R\ ,\quad \ddot{z}=-\partial\Phi_{\mathbf{eff}}/\partial z$$

where

$$\Phi_{\mathbf{eff}} = \Phi + L_z/2R^2$$

is the sum of the gravitational and 'centrifugal' energies. $\Phi_{\mathbf{eff}}$ has a minimum at $z=0$ and $R=R_c$ where R_c is the radius of the circular orbit with angular momentum L_z. Expanding $\Phi_{\mathbf{eff}}$ in a Taylor series

$$\Phi_{\mathbf{eff}}(R,z) = \Phi_0+\tfrac{1}{2}\kappa^2(R-R_c)^2+\tfrac{1}{2}\omega_z^2 z^2+\ \ldots$$

about this minimum, one can regard the star's motion in the (R,z) plane as that of an harmonic oscillator that is perturbed by the cubic and higher-order terms in the excursions $R-R_c$ and z from the point of equilibrium.

If the deviation of $\Phi_{\mathbf{eff}}$ from harmonicity is not very great, we would expect the motion to admit two integrals

$$E_{\mathbf{R}} \equiv \tfrac{1}{2}(\dot{R}^2 + \kappa^2 R^2) + \ldots \tag{3a}$$

$$E_{\mathbf{z}} = \tfrac{1}{2}(\dot{z}^2 + \omega_{\mathbf{z}}^2 z^2) + \ldots \tag{3b}$$

that are in lowest order κ times the radial action for motion in the plane $z = 0$, and the 'z-energy' that Oort and Lindblad had already used in the 1930's to interpret observations of the motions through our Galaxy of stars that pass near the Sun. It is often convenient to regard $E_{\mathbf{R}}$ as dependent on $E_{\mathbf{z}}$, $L_{\mathbf{z}}$ and the total energy E; $E_{\mathbf{R}} = E - E_{\mathbf{z}} - \Phi_0$. $E_{\mathbf{z}}$ is then called the 'third integral'.

In 1960 Contopoulos showed that one can use standard Hamiltonian perturbation theory to obtain higher-order terms in the expression (3b) for $E_{\mathbf{z}}$ (Contopoulos 1960, 1963). As Henon and Heiles (1964) subsequently demonstrated by writing down a special polynomial approximation to $\Phi_{\mathbf{eff}}$, this approach to the problem of motion in an axisymmetric potential can be misleading; the Henon and Heiles potential exhibits a wealth of stochastic orbits which are not constrained by a third integral. However, numerous numerical studies (Ollongren 1965, Martinet and Mayer 1975) have shown that most orbits in realistic axisymmetric potentials are in fact constrained by third integrals of the type discussed by Contopoulos.

4. Orbits in Triaxial Potentials

In recent years evidence has been accumulating that the majority of galaxies do not have axisymmetric potentials, but may have potentials that are of more-or-less ellipsoidal form [see Binney (1982) for a review]. Consequently it is important to understand the sorts of orbit that are supported by 'triaxial' potentials of this type.

The figure of a triaxial galaxy is likely to rotate with respect to inertial space, so it is necessary to allow the possibility that the frame of reference in which the potential is stationary (assuming such a frame exists) rotates with some angular speed ω. However, it is helpful to consider first the important special case $\omega = 0$.

4.1 *Schwarzschild's Potential*

Schwarzschild (1979) presented the first systematic investigation of orbits in the potential of a body that might resemble a triaxial elliptical galaxy. The particular potential studied by Schwarzschild is that generated by a body whose isodensity surfaces are roughly elliptical with axis lengths in the ratio 1:0.625:0.5. If we orient the body with its longest axis along the x_1 axis, and its shortest axis along the x_3 axis, Schwarzschild's distribution is

$$\rho(x) = \rho_0\left[1 + \sum_{i=1}^{3} \left(x_i/a_i\right)^2\right]^{-3/2}. \tag{4}$$

If light were emitted from each volume of this body in proportion to the mass in that volume, then in projection the body would closely resemble an elliptical galaxy.

Over the timescales of astronomical interest (a few hundred orbital times) Schwarzschild found by direct numerical integration of the equations of motion that the great majority of orbits are confined to three-dimensional regions of phase space. This result suggests that in addition to the classical Hamiltonian integral $H(\boldsymbol{x},\boldsymbol{p})$, two further isolating integrals (or possibly quasi-integrals) exist at most points in phase space. Since these additional integrals do not arise via Noether's theorem from a manifest symmetry of the Lagrangian for motion in the given potential, Schwarzschild called them 'non-classical integrals'.

Superficially Schwarzschild's demonstration that most orbits in a particular ellipsoidal potential admit three effective isolating integrals may appear no more than a minor extension of the earlier work on motion in axysymmetric potentials. However, if we exclude a small and physically unreal region near the centre of Schwarzschild's model, the potential of the model cannot be regarded as that of a perturbed harmonic oscillator. The potential generated by the spherical density distribution $\rho = \rho_0\left[1 + (r/a)^2\right]^{-3/2}$ is

$$\Phi(r) = -4\pi G\rho_0 a^3 \frac{1}{r} \log_e\left[\frac{r}{a} + \left(\frac{r^2}{a^2} + 1\right)^{\frac{1}{2}}\right] \tag{5}$$

and if we expand this in powers of (r/a) around the point of equilibrium at $r = 0$, we obtain a series which diverges beyond $r = a$. Many of Schwarzschild's orbits reach $r = 20a$. Thus Schwarzschild's result cannot be easily understood in terms of perturbations of a three-dimensional harmonic oscillator.

Of course orbits at very large binding energies do not move to $r \gg a$, and one can attempt to understand these orbits in terms of the motions of a perturbed harmonic oscillator. De Zeeuw and Merritt (1983) have taken this approach with a certain measure of success, but recently de Zeeuw (1983) and de Zeeuw and Lynden-Bell (1983) have shown that there is a much better way of understanding Schwarzschild's numerical results in analytic terms. The approach taken by de Zeeuw and Lynden-Bell is to go back to the classical work of Stäckel (1893) on separable potentials. This development is sufficiently interesting to merit a digression into the nature of motion in Stäckel's potentials.

4.2 *Stäckel Potentials*

Stäckel's potentials are most conveniently expressed in terms of ellipsoidal coordinates. An ellipsoidal coordinate system is defined by choosing three numbers $\alpha_1 < \alpha_2 < \alpha_3$: the ellipsoidal coordinates of the point $\boldsymbol{x}$ are then the roots $q_1(\boldsymbol{x}) > q_2(\boldsymbol{x}) > q_3(\boldsymbol{x})$ of the cubic equation for q

$$\sum_{i=1}^{3} \frac{x_i^2}{q+\alpha_i} = 1 \quad (\alpha_1 < \alpha_2 < \alpha_3 \text{ arbitrary constants}). \tag{6a}$$

The ranges of variation of the q_i are given by

$$-\alpha_3 \leqslant q_3 \leqslant -\alpha_2 \leqslant q_2 \leqslant -\alpha_1 \leqslant q_1 . \tag{6b}$$

The coordinate system provided by the q_i proves to be orthogonal. That is

$$\sum_{i=1}^{3} dx_i^2 = ds^2 = \sum_{i=1}^{3} g_i \, dq_i^2 \tag{7a}$$

where $g_i(q_1, q_2, q_3)$. A straightforward but tedius calculation shows that

$$1/g_i = -\frac{2}{\Delta} \sum_{j,k=1}^{3} \varepsilon_{ijk}(q_j - q_k) \prod_{\ell=1}^{3} (q_i + \alpha_\ell). \tag{7b}$$

where

$$\Delta = \tfrac{1}{6} \sum_{ijk} \varepsilon_{ijk}(q_i - q_j)(q_j - q_k)(q_k - q_i) = -\tfrac{1}{2} \sum_{ijk} \varepsilon_{ijk} q_i^2 (q_j - q_k) . \tag{7c}$$

Stäckel considered potentials of the form

$$\Phi(\boldsymbol{x}) = \frac{V_1(q_1)}{g_1} + \frac{V_2(q_2)}{g_2} + \frac{V_3(q_3)}{g_3} \tag{8}$$

where the V_i are three arbitrary functions of one variable. All such potentials lead to a separable Hamilton-Jacobi equation as the following analysis shows.

If S is Hamilton's principal function, the Hamiltonian for motion in a potential Φ may be written

$$E = \frac{1}{2m} (\nabla S \cdot \nabla S) + \Phi$$
$$= \frac{1}{2m} \sum_{ij} g^{ij} \frac{\partial S}{\partial q_i} \frac{\partial S}{\partial g_j} + \Phi = \frac{1}{2m} \sum_i \frac{1}{g_i} \left(\frac{\partial S}{\partial q_i}\right)^2 + \Phi \tag{9}$$

where $g^{ij} = (1/g_i)\,\delta_{ij}$ is the inverse of the covariant metric tensor $g_{ij} = g_i \, \delta_{ij}$. With (8) the Hamilton-Jacobi equation becomes

$$\sum_i \frac{1}{g_i} \left[\frac{1}{2m} \left(\frac{\partial S}{\partial q_i}\right)^2 + V_i(q_i) \right] = E . \tag{10}$$

When one substitutes from equations (7) for the g_i, this becomes