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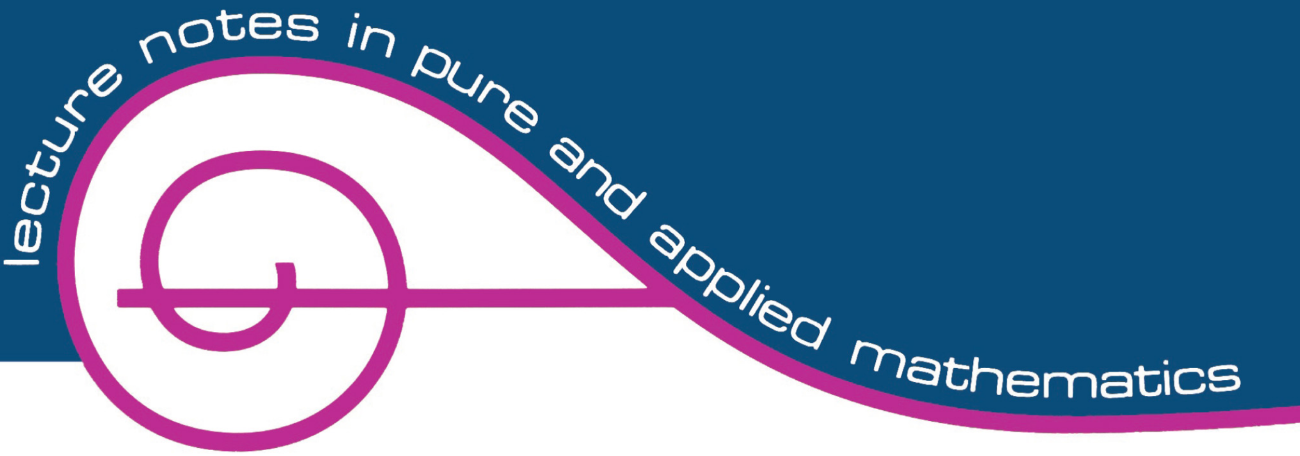

lecture notes in pure and applied mathematics

matrix-analytic methods in stochastic models

edited by
Srinivas R. Chakravarthy
Attahiru S. Alfa

matrix-analytic methods in stochastic models

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Preface

For a wide variety of stochastic models, the steady-state and occasionally the transient measures of the underlying process can be expressed in terms of a matrix R or G. That matrix is the minimal nonnegative solution to a nonlinear equation. Such matrix solutions to stochastic models were first proposed in the early 1970s by Marcel F. Neuts. Marcel, with his students, and several other researchers have since then provided much impetus to the mathematical development of this method.

Since their introduction these methods have found an astonishingly wide range of applications. Researchers in fields such as telecommunications, manufacturing, and computer engineering have fruitfully applied that methodology to models arising in their disciplines. Many practitioners in the applied sciences and engineering now use matrix-analytic methods to solve some important design problems.

Since the theoretical interest and the range of applications of matrix-analytic methods are now well established, we felt that it was timely to bring together researchers and practitioners in this area at an international conference. Researchers have seen rapid progress in the mathematical aspects and several new avenues toward applications have been opened, so our proposal was enthusiastically received by researchers in this area. Practitioners saw it as an opportunity to become acquainted with the latest algorithmic techniques to meet the challenges of application areas. Both groups felt that, despite their significance, the matrix-analytic developments are still not as widely known as they should be. That was in part attributed to the fact that a conference entirely devoted to these methods had hitherto not been held.

Therefore, the organizers of the conference sought to bring together researchers, practitioners, students, and others working on matrix-analytic methods in stochastic modeling, toward the following major goals:

(1) Through carefully designed tutorials, to impart a working knowledge of matrix-analytic methods to those new to the field
(2) To review the progress of research and to define the state-of-the-art, especially in large-scale applications
(3) To discuss the application of matrix-analytic methods to new technological problems
(4) To present contributed papers on recent, unpublished research
(5) To identify the directions of future growth of this subject

This conference on matrix-analytic methods in applied probability has confirmed the importance of regular, focused meetings of the specialists in an area. All participants agreed to meet again soon to exchange ideas, to stimulate interaction between various research groups, and to create new ideas. The second conference is tentatively planned to be held in Manitoba, Canada, during 1998.

We express our deep and sincere gratitude to the persons and organizations who have contributed to the great success of the conference. Marcel F. Neuts, the pioneer in matrix-analytic methods, was highly supportive of the organization of such a conference. His constant encouragement and participation were important to us during the long months of preparation.

The National Science Foundation provided funds for three student fellowships to participate in the conference, for secretarial help, and for some publication costs. GMI Engineering & Management Institute helped defray the costs of mailing the announcements and of printing the preliminary announcement of the proceedings. In addition, GMI hosted a dinner on the first day of the conference. Dr. Jim Luxon, Dean of Academic Affairs and Research, was especially supportive from beginning to end. His encouragement gave S. R. Chakravarthy the extra energy to get through the conference.

Last, but not least, we thank the members of the scientific advisory committee and the external referees for timely, informative reviews, and Ms. Corrine Anthony and Ms. Cheryl Cochran of GMI and Ms. Bev Dunlop of the University of Manitoba for their conscientious secretarial help.

Finally, S. R. Chakravarthy would like to dedicate this book to the memory of Dr. K. N. Venkataraman, who instilled in him a strong interest in probability when he was a graduate student at the University of Madras, Madras, India.

Srinivas R. Chakravarthy
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The Markovian Arrival Process: Some Future Directions

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Abstract

I informally discuss some questions on the *MAP* and its applications that appear to be of interest to current and future investigations. The discussion elaborates on remarks made during my presentation at the First International Conference on Matrix-Analytic Methods in Stochastic Models at Flint, Michigan.

1 Introduction

We use the standard notation for the Markovian arrival process (*MAP*) and assume familiarity with the classical properties and examples as treated in the surveys of Lucantoni (1991) and Neuts (1992). The discussion is organized around three major themes:

- The mathematical descriptors of qualitative behavior of the *MAP*.
- The computational problems of *MAP*s with many phases.
- Hailing a *MAP* when you see one.

The versatile behavior of *MAP*s, simply parameterized by two or more coefficient matrices, is also a major challenge to their practical use. Except for the simplest cases, the coefficient matrices by themselves do not convey much information about the path behavior of the induced point processes. That accounts, I believe, for the pervasive use of the simplest two-phase *MAP*s in studies of specific models. The point processes represented by these are only marginally more general than homogeneous Bernoulli or Poisson processes. Neuts (1992) discusses the coefficient matrices for a variety of *MAP*s with specified qualitative behavior.

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The exploration of *MAP*s with richer structure should go hand-in-hand with qualitative information gleaned from applications. If we know which features of an arrival or service process, beyond rate and second-order properties, significantly affect queueing behavior, we can build such features into the specifications of *MAP*s. Vice versa, the examination of important, non-traditional queueing models whose input is a *MAP* with recognized qualitative properties, can suggest which features of the input process affect queueing behavior in important ways. Instances of such investigations are the articles of Johnson (1993) and Ramaswami and Latouche (1989). Such comparative studies can be done by numerical methods or by carefully planned simulations. They would benefit from the availability of a portable software package with options to compute and display graphs of all known descriptors of *MAP*s. On behalf of all users of the *MAP*, I make a plea for the development of such software.

2 Descriptors of the *MAP*

The second-order properties, such as the dispersion curve, peakedness, and correlations, have been extensively studied. A summary with extensive references of these is found in Narayana and Neuts (1992). For coefficient matrices of moderately large sizes, the computation of second-order descriptors is straightforward. It mainly requires the evaluation of the exponential matrix $\exp(Dt)$ and of a number of vectors from explicit formulas.

The power spectral density of a random square wave with sign changes at the (single) events of a *MAP* is studied in Neuts (1990). Taken by themselves, I have not found graphs of that density to be very informative, yet they may still prove useful in making comparisons of *MAP*s with different, but related parameter matrices.

For some approaches to the burstiness of point processes, we have used the *MAP* as a benchmark. These approaches are based on *smoothing, thinning,* and *on data-analytic methods.* One smoothing of a point process is *local poissonification.* In it, the events in successive windows of length a are uniformly redistributed over those intervals. Even starting with a *MAP*, the resulting process is no longer a *MAP*, but various descriptors of a single poissonification of a *MAP* can still be expressed in explicit forms. See, Neuts, Liu, and Narayana (1992) and Neuts (1994). The idea underlying local poissonification is an analogue of replacing a function of a single variable by a piecewise linear interpolation; over successive windows of length a, the detailed information is replaced by the randomness of a Poisson process. That transformation is clearly *rate-preserving.* The resistance of the macroscopic features of a point process to such a smoothing may be an empirical measure of its burstiness. Several issues, some theoretical, some practical, related to local poissonification require further investigation.

The local poissonification of a Poisson process is again Poisson. I have conjectured that successive poissonifications of a stationary point process converge weakly to a Poisson process of the same rate. In the successive smoothings, the equidistant window grid must start at a random point in a window of the preceding grid. I believe that the different window sizes may be used for successive poissonifications, as long as their lengths a_j remain bounded away from zero. In spite of excellent suggestions from Sidney Resnick and Richard Serfozo to attempt proving Harris recurrence of a Markov process on the space of atomic measures, and of an alternative idea of mine to use entropy, that conjecture is not yet decided.

A practical exploration by simulation of the effect of local poissonifications with different window sizes of the input stream to a queue, also remains to be done. Such an experiment may show the time scales where macroscopic fluctuations in the arrivals dominate the values of the queue characteristics.

We stress in passing that all procedures for continuous *MAP*s have analogues for discrete-time *MAP*s. For discrete *MAP*s with single events, *local bernoullification,* is defined in the same manner. Should these smoothing procedures prove useful, their analogues for *MAP*s with group arrivals merit attention. What the useful analogues for group arrivals are is not immediately obvious.

The smoothing procedures are qualitatively related to *traffic shaping* discussed in *ATM* research. The dissertation of D. Liu (1993) treats traffic shaping for *MAP*s and contains extensive references on that general subject.

In the dissertation of D. Rauschenberg (1994), the *MAP* serves as a benchmark process for various data-analytic smoothing procedures of large data sets of the type arising in *ATM* applications. These procedures may prove useful in construction *MAP*s whose behavior mimics the salient features of such data sets. Jian-Min Li, my current doctoral student, is studying, among other things, some entropy-based descriptors and waiting times between long runs of events for the discrete *MAP*.

Neuts (1993) proposes random thinning procedures for decomposing a point process into processes of different grades of burstiness. For a *MAP*, many descriptors of the component processes are explicitly computable. The *MAP* can therefore again serve as a benchmark to assess the potential merit of such decompositions. For an illustration of that thinning in queueing context, see Liu and Neuts (1994).

One important characteristic of "bursty" arrival process is the prevalence of long strings of closely spaced events. A proposal for the quantitative characterization of closely spaced events in discrete-time *MAP*s is made in Johnson and Narayana (1995), and Johnson, Liu, and Narayana (1995). In brief, events are closely spaced if the number of time slots between them is less than a specified positive integer ν. Distributions of the lengths of runs of closely spaced events are derived, and the effect of such runs on queueing behavior is examined.

I should like to propose an alternative in the same spirit of their approach. Its advantages are that it serves equally well for continuous-time *MAP*s, and that it could lead to analytically tractable queueing models with two types of customers. At each event, start (independent) clocks with a geometrically distributed lifetime of success probability α (or with an exponentially distributed lifetime of parameter α.) Call the events *dense* for the value α if they occur before the clock of the immediately preceding event runs out, or if they lead a string of one or more events occurring before the clock runs out. All other events are *sparse* for the value α. In the discrete case, $\alpha = 1$ corresponds to the usual notion of a run. For α close to one, a string of dense events, an α-run, would still allow rare interruptions by empty slots (zeros). In that manner, we can study the process as a *MAP* with two types of arrivals, dense and sparse ones. The analysis of queueing models such as *MAP/G/1* (in its discrete or continuous versions) should remain reasonably tractable. The joint steady-state queue length and waiting time distributions for dense and sparse customers, computed for various values of α should be informative.

The current state-of-the-art in matrix-analytic methods immediately shows how to proceed when the clock times have *PH*-distributions. However, the increased order of the matrices then soon increases the computational effort to where super-computers are needed. During the conference presentation, I described a few queueing models requiring massive computation and I encouraged algorithmic studies of such models. Still, for the exploratory stage of descriptors of *MAP*s, I would suggest that we examine the simplest parameterizations first and move on to generalizations only if the descriptors prove to be informative for the simple, well-understood cases.

*MAP*s with two or more types of arrivals, mentioned in the preceding discussion, or the two-dimensional *MAP*-like processes described by V. Ramaswami in his presentation, offer many new challenging problems to queueing analysis. The study of informative descriptors of such generalized *MAP*s has barely started; it is sure to be a fruitful field of inquiry.

For the sake of a specific example of an investigation of descriptors, consider the splitting of a stationary point process (with single events) by subjecting successive events to independent Bernoulli trials with probabilities p and $q = 1 - p$. One of the many appealing properties of the Poisson process is, of course, that the resulting processes are independent Poisson processes. For Markov renewal processes, the Bernoulli switch generally produces *two dependent* Markov renewal processes. Measures of that dependence as functions of the original process and of p are, in general, very hard to come by. In contrast, for a *MAP*, there are several easily calculated functionals, such as the correlation coefficient $\rho(t;\ p)$ of the numbers $N(t,\ p)$ and $N(t,\ q)$ of events in each process during an interval $(0,\ t]$. Do graphs of $\rho(t;\ p)$ as functions of t, plotted for various values of p between 0 and 1/2 convey interesting information on the original *MAP*? For small values of p, the thinnest process should be approximately Poisson. Is $\rho(t;\ p)$ close to zero for small p; if so, uniformly in t? If it is not, what feature of the original *MAP* could account for that? We do not have complete answers yet, but current investigations in collaboration with several current students suggest that this line of inquiry is promising. It is also germane to the model of two queues receiving the component processes as input that was briefly mentioned in my presentation.

3 *MAP*s with Many Phases

The departure processes of queues such as *MAP/MAP/1*, the overflow processes of finite-capacity Markovian queues, and some superpositions or transformations of *MAP*s remain *MAP*s. That structural insight is important, but its utility is limited by size of the coefficient matrices. That raises many questions.

Developing increasingly efficient algorithms for the descriptors of *MAP*s with many phases is an extremely important task for the near future. Without offering specific suggestions, I can only hope for breakthroughs similar to those of recent years in the computation of the matrices R and G for queueing models of the *GI/M/1* and *M/G/1* types.

Equally promising and challenging are algorithms for a systematic state space reduction mentioned in V. Naoumov's presentation. Systematic state space reduction or any other form of approximation crucially depends on what we are willing to give up and what must be preserved in the approximating *MAP*. That is why I consider the discovery of new, informative descriptors so important. I think that, ultimately, approximation techniques will be cast as complex non-linear optimization problems with the various descriptors serving to specify their constraints. Some of Mary Johnson's work on fitting *PH*-distributions is in that spirit. However, we have long-established, well-tried descriptors for univariate probability distributions. Behaviorally, *MAP*s are significantly more complicated than *PH*-distributions.

4 Hailing a *MAP* When You See One

*MAP*s lurk in unsuspected places. Last year, my colleague Don Rawlings of California Polytechnic State University, a specialist in combinatorial number theory and special functions, discussed the number of increases in i.i.d sequences of random variables on $\{1, \ldots, n\}$. Once we noticed that the times of increases are the events in a *MAP* with highly structured coefficient matrices, various distributional and asymptotic properties immediately followed from general results on *MAP*s.

In my collection of algorithmic problems (Neuts, 1995), Problem 4.4.48 deals with the distribution of the number of *reversals* in an absorbing random walk on $\{0, 1, \ldots, K+1\}$ with general transition probabilities. Again, the times of reversals are the events in an appropriate, terminating *MAP*. Some special occurrences in the path behavior of finite queues can be similarly identified as the events in a suitable *MAP*.

In the two examples mentioned, the problems of interest are readily solved by scalar recurrence relations. However, the matrix formalism of the *MAP* is so much more elegant and the normalizing constants needed in limit theorems can be explicitly written in matrix formulas. Finding a *MAP* embedded in a general mathematical question remains serendipitous. It helps to be alert to that possibility. It could be useful to all who are interested in *MAP*s to communicate new examples that are unrelated to our usual concerns with queueing models. *MAP*s may well have other, as yet unsuspected applications.

5 Concluding Remarks

The *MAP* is arguably the most versatile, yet tractable generalization of the Poisson (or Bernoulli) process. That is clear from its use as an arrival or service process in many single service queues. It enables us to study diverse qualitative features of point processes explicitly. The related issues of finding new informative descriptors and of the approximation, either of other processes by *MAP*s, or of *MAP*s with large coefficient matrices by some with fewer phases, are likely to stimulate fruitful research in the near future.

Already for many years, I have stressed the need for portable software and, more significantly, for a substantial organization to perform the massive computations needed in our rapidly evolving branch of applied probability. We ably establish the mathematical qualities of the matrix-analytic methods and invest scarce time in demonstrating their computational feasibility for problems of moderate size. However, truly large scale implementations of our algorithms cannot be done by a few individuals working in relative isolation. Yet, some of the most interesting questions on the use of *MAP*s in applications, involve very large matrices. I believe that the complete fruition of our methods will require joint efforts of probabilists, numerical analysts, and advanced scientific programmers with ready access to a super-computer. Nevertheless, until somewhere an algorithmic center for probability modelling is established, the immediate mathematical questions generated by the matrix-analytic methods will keep us well occupied with interesting research.

References

[1] Johnson, M. A. An empirical study of queueing approximations based on phase-type distributions. *Communications in Statistics: Stochastic Models*, 9(4):531 - 561, 1993.

[2] Johnson, M. A., Liu, D., and Narayana, S. Burstiness descriptors of Markov renewal processes and Markovian arrival processes. Technical report, Dept. of Mechanical and Industrial Engineering, University of Illinois, Urbana, IL 61801, 1995.

[3] Johnson, M. A. and Narayana, S. Descriptors of arrival-process burstiness with application to the discrete Markovian arrival process. Technical report, Dept. of Mechanical and Industrial Engineering, University of Illinois, Urbana, IL 61801, September 1995.

[4] Liu, D. *Some Traffic Shaping Procedures for ATM Networks.* PhD thesis, The University of Arizona, Tucson, Arizona, 1993.

[5] Liu, D. and Neuts, M. F. Assessing the effect of bursts of arrivals on the characteristics of a queue. In J. Pérez Vilaplana and M. L. Puri, editors, *Recent Advances in Statistics and Probability*, pages 371–387. VSP, 1994.

[6] Lucantoni, D. M. New results on the single server queue with a batch Markovian arrival process. *Communications in Statistics: Stochastic Models*, 7(1):1–46, 1991.

[7] Narayana, Surya and Neuts, Marcel F. The first two moments matrices of the counts for the Markovian arrival process. *Communications in Statistics: Stochastic Models*, 8(3):459–477, 1992.

[8] Neuts, M. F. The square wave spectrum of a Markov renewal process. In M. J. Beckmann, M. N. Gopalan, and R. Subramanian, editors, *Stochastic Processes and Their Applications: Proceedings of the Symposium held in honour of Professor S. K. Srinivasan*, pages 1–10, Bombay, India, December 1990.

[9] Neuts, M. F. Models based on the Markovian arrival processes. *IEICE Transactions On Communications*, E75-B(12):1255–65, 1992.

[10] Neuts, M. F. *Algorithmic Probability: A Collection of Problems.* Chapman & Hall, New York, New York, 1995.

[11] Neuts, M. F., Liu, D. and Narayana, S. Local poissonification of the Markovian arrival process. *Communications in Satistics: Stochastic Models*, 8(1):87–129, 1992.

[12] Neuts, Marcel F. The burstiness of point processes. *Communications in Satistics: Stochastic Models*, 9(3):445–466, 1993.

[13] Neuts, Marcel F. The palm measure of a poissonified stationary point process. In Ramon Gutierrez and Mariano J. Valderrama, editors, *Selected Topics on Stochastic Modelling*, pages 26–40. Singapore: World Scientific, 1994.

[14] Ramaswami, V. and Latouche, G. An experimental evaluation of the matrix-geometric method for the GI/PH/1 queue. *Communications in Statistics: Stochastic Models*, 5(4):629 - 667, 1989.

[15] Rauschenberg, D. E. *Computer-Graphical Exploration of Large Data Sets from Teletraffic.* PhD thesis, The University of Arizona, Tucson, Arizona, 1994.

An Algorithm for the $P(n,t)$ Matrices of a Continuous BMAP

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Abstract

We propose an efficient algorithm, based on the uniformization method, to compute the $P(n,t)$-matrices for the counting process of a continuous batch Markovian arrival process (*BMAP*). The accuracy of the algorithm is guaranteed by many internal numerical checks based on known explicit results. By numerical examples, we graphically illustrate the burstiness of such arrival processes.

1 Introduction

We discuss an efficient algorithm for the matrices $P(n,t)$ of the counting process of a continuous-parameter batch Markovian arrival process (*BMAP*). We assume familiarity with the definition and the basic properties of that point process. These and many examples are found in Lucantoni (1991) and Neuts (1992). Only properties and notation essential to this discussion are repeated.

$N(t)$ is the number of events in $(0,\ t]$ in a continuous *BMAP* with parameter matrices D_0, D_1, $\cdots$, D_M. The integer $M \geq 1$ is the maximum batch size. By m, we denote the number of states of the Markov chain with generator $D = \sum_{j=0}^{M} D_j$. The random variable $J(t)$ is the phase of the process at time t. For $n \geq 0$ and $t \geq 0$, the (i,j)-th entry of the matrix $P(n,t)$ is defined by

$$P_{ij}(n,t) = P\{N(t) = n, J(t) = j | N(0) = 0, J(0) = i\}. \tag{1}$$

The matrices $P(n,t)$ satisfy the Chapman-Kolmogorov equations

$$P'(0,t) \quad = \quad P(0,t)D_0 \tag{2}$$

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$$P'(n,t) \;=\; \sum_{j=\max\{0,n-M\}}^{\min\{n,M\}} P(n-j,t)D_j, \tag{3}$$

with initial conditions $P(0,0) = I,\ P(n,0) = 0$, for $n \geq 1$.

In addition, for $t \geq 0$ and $s \geq 0$, we have that

$$P(n,\ t+s) = \sum_{r=0}^{n} P(r,t)P(n-r,s), \tag{4}$$

so that the sequence of matrices $\{P(n,t+s)\}$ is the *matrix convolution product* of the corresponding sequences for t and s.

For small values of t, the algorithm evaluates the $P(n,t)$ by cautiously integrating the system of differential equations (2) and (3) by the *uniformization method.* For larger t, we use the *semigroup* property in (4). In practice, a detailed computation of the $P(n,t)$ is meaningful only for moderately large values of t. For a *BMAP* of rate one, that means t at most up to a few thousand. Beyond that, the detailed variability of the counting process is rarely of interest and one applies the central limit theorem.

2 Preliminary Steps

In working with a specific *BMAP*, it is advisable to rescale the time variable and to obtain a rough idea of the behavior of the process prior to computing the matrices $P(n,t)$. To that end, we carry out the following preliminary steps:

1. **Some Sample Paths of the Process.** A few simulated sample paths of the process are quite informative and the simulation of a *BMAP* is very easily done.

2. **Rescaling the Time Variable.** The rate of the stationary *BMAP* is $\lambda^* = \boldsymbol{\theta} D_1^* \mathbf{e}$, where $\boldsymbol{\theta}$ is the stationary probability vector of the generator D, $D_1^* = \sum_{n=1}^{\infty} nD_n$, and $\mathbf{e}$ is a column vector of ones. The reciprocal $1/\lambda^*$ is the mean interarrival time in the stationary *BMAP*. We often choose the time unit as that quantity by rescaling the parameter matrices by setting

$$D_n \leftarrow \frac{1}{\lambda^*} D_n, \qquad \text{for } 0 \leq n \leq M.$$

In what follows, we tacitly assume that such a rescaling was done.

3. **The Palm function,** $H(t)$, the expected number of arrivals in an interval $(0,t]$ starting from an arbitrary arrival epoch 0, is given by

$$H(t) = \lambda^* t + \boldsymbol{\theta}_{arr}[I - \exp(Dt)]\mathbf{d},$$

where

$$\boldsymbol{\theta}_{arr} = \frac{-\boldsymbol{\theta} D_0}{-\boldsymbol{\theta} D_0 \mathbf{e}}$$

is the probability vector of the phase immediately following an arbitrary arrival epoch, and $\mathbf{d} = (\mathbf{e}\boldsymbol{\theta} - D)^{-1} D_1^* \mathbf{e}$. We examine graphs of $H(t)$ and of its derivative $H'(t)$, the *Palm density,*

$$H'(t) = \lambda^* + \boldsymbol{\theta}_{arr}[-\exp(Dt)D]\mathbf{d},$$

to identify the magnitude of and the time to significant bursts of arrivals.

4. **The variance** $V(t)$ **of** $N(t)$ for the stationary version of the process is expressed in terms of the moment matrices

$$D_1^* = \sum_{n=1}^{M} nD_n, \qquad D_2^* = \sum_{n=1}^{M} n^2 D_n,$$

and the vectors $\mathbf{c} = \boldsymbol{\theta} D_1^*(\mathbf{e}\boldsymbol{\theta} - D)^{-1}$ and $\mathbf{d}$. It is explicitly given by

$$V(t) = [\lambda_2 - 2(\lambda^*)^2 + 2\mathbf{c}D_1^*\mathbf{e}]\, t - 2\mathbf{c}[I - \exp(Dt)]\mathbf{d},$$

where $\lambda_2 = \boldsymbol{\theta} D_2^* \mathbf{e}$. As $t \to \infty$, the graph of $V(t)$ has a linear asymptote given by the equation

$$v(t) = [\lambda_2 - 2(\lambda^*)^2 + 2\mathbf{c}D_1^*\mathbf{e}]\, t + 2[(\lambda^*)^2 - \mathbf{cd}].$$

For derivations of these formulas, see e.g., Narayana and Neuts (1992). Much insight into the variability of the counting process is obtained from an examination of the graphs of the functions $\max\{0,\ \lambda^* t \pm k\, S_d(t)\}$ for $k = 0,\ 1,\ \cdots,\ 4$, where $S_d(t) = \sqrt{V(t)}$ is the dispersion function.

5. For a highly variable *BMAP* whose behavior strongly depends on its initial conditions, slightly more belabored preliminary computations are done to examine how the quantities in Step 4 depend on the phase at time 0. We compute the factorial moment matrices

$$M_1(t) = \sum_{n=0}^{\infty} nP(n,t), \qquad M_2(t) = \sum_{n=0}^{\infty} n(n-1)P(n,t),$$

by the procedure in Narayana and Neuts (1992). We set $\mu_j(t) = [M_1(t)\mathbf{e}]_j$, and

$$V_j(t) = [M_2(t)\mathbf{e}]_j + [M_1(t)\mathbf{e}]_j - \{[M_1(t)\mathbf{e}]_j\}^2,$$

and similarly plot graphs of $\max\{0,\ \mu_j(t) \pm k\sqrt{V_j(t)}\}$ for $k = 0,\ 1,\ \cdots,\ 4$ and $j = 1,\ \cdots,\ m$.

The graphs of the means plus several standard deviations are used to judge what is meant by a "small" value of t for the specific *BMAP* at hand. In computing the $P(n,t)$ over a long interval $(0,\ t^*)$, we would, for example, call an interval of length t "short" if it is extremely unlikely that more than 25 events occur in $(0,\ t)$. In the discussion of the general algorithm it will be obvious why an operational definition of short intervals is useful.

3 Computation of $P(n,t)$

3.1 Computation of $P(n,t)$ for a Fixed t

We first discuss the computation of the $P(n,t)$ for a fixed small value of t by an adaptation of the uniformization method. Consider a bivariate Markov process $\{N(t), J(t)\}$ on the state space $\{(i,j) : i \geq 0,\ 1 \leq j \leq m\}$ with the generator

$$Q = \begin{pmatrix} D_0 & D_1 & D_2 & D_3 & \cdots & D_M & \cdots \\ & D_0 & D_1 & D_2 & \cdots & D_{M-1} & \cdots \\ & & D_0 & D_1 & \cdots & D_{M-2} & \cdots \\ & & & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Clearly, $\exp(Qt)$ is then the infinite matrix

$$\begin{pmatrix} P(0,t) & P(1,t) & P(2,t) & P(3,t) & \cdots \\ & P(0,t) & P(1,t) & P(2,t) & \cdots \\ & & P(0,t) & P(1,t) & \cdots \\ & & & \cdots & \cdots \\ & & & & \cdots \end{pmatrix}.$$

In the *uniformization method* $\exp(Qt)$ is computed as a partial sum of the series in

$$\exp(Qt) = e^{-\lambda t}\exp[(I + \lambda^{-1}Q)\lambda t] = \sum_{r=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^r}{r!}K^r,$$

where λ is a constant such that $\lambda \geq \max_j\{-(D_0)_{jj}\}$ and K is the (stochastic) matrix $I + \lambda^{-1}Q$. By defining the matrices $C_0 = I + \lambda^{-1}D_0$, $C_j = \lambda^{-1}D_j$, for $1 \leq j \leq M$, it follows that

$$K = \begin{pmatrix} C_0 & C_1 & C_2 & C_3 & \cdots & C_M & \cdots \\ & C_0 & C_1 & C_2 & \cdots & C_{M-1} & \cdots \\ & & C_0 & C_1 & \cdots & C_{M-2} & \cdots \\ & & & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and K^r has the structure

$$K^r = \begin{pmatrix} V(0,r) & V(1,r) & V(2,r) & V(3,r) & \cdots \\ & V(0,r) & V(1,r) & V(2,r) & \cdots \\ & & V(0,r) & V(1,r) & \cdots \\ & & & \cdots & \cdots \\ & & & & \cdots \end{pmatrix},$$

for which the blocks $V(n,r)$ can be recursively computed. The $V(n,r)$ are substochastic matrices in terms of which

$$P(n,t) = \sum_{r=\lfloor n/M \rfloor M}^{\infty} e^{-\lambda t}\frac{(\lambda t)^r}{r!}V(n,r).$$

Proposition 3.1 *Let* $A_n(t) = \int_0^t P(n,t)dH(u)$, *for* $n \geq 0$, *where* $H(\cdot)$ *is a cdf on* $[0,\infty)$. *Then*

$$A_n(t) = \sum_{r=\lfloor n/M \rfloor M}^{\infty} a_r(t)V(n,r), \tag{5}$$

where $a_r(t) = \int_0^t e^{-\lambda t}\frac{(\lambda t)^r}{r!}dH(u)$.

Proof.

$$\begin{aligned} A_n(t) &= \sum_{r=\lfloor n/M \rfloor M}^{\infty} \int_0^t e^{-\lambda u}\frac{(\lambda u)^r}{r!}dH(u)V(n,r) \\ &= \sum_{r=\lfloor n/M \rfloor M}^{\infty} a_r(t)V(n,r). \end{aligned}$$

■

Proposition 3.1 generalizes a corollary in Lucantoni and Ramaswami (1985).

Next, we show how the matrices $V(n,r)$ may be computed efficiently and how, in the process, computed values of matrices $P(n,\ t)$ are readily accumulated. Let us write $\{b_r\}$ for the Poisson density

$$b_r = e^{-\lambda t}\frac{(\lambda t)^r}{r!}, \quad \text{for } r \geq 0$$

of mean λt. We first find the smallest index N for which $\sum_{n=N+1}^{\infty} b_r \leq \epsilon$, where ϵ is a small specified value. Thereafter, we proceed with the following algorithmic steps:

1. For $n = 0$,

$$V(0,0) = I, \quad P(0,t) \leftarrow V(0,0)b_0.$$

For $n \geq 1$, set

$$V(n,0) \leftarrow 0, \quad P(n,t) \leftarrow 0.$$

2. For $1 \leq n \leq N$, $0 \leq i \leq nM$,

$$\begin{aligned} V(i,n) &= \sum_{j=\max\{0,i-(n-1)M\}}^{\min\{i,M\}} V(i-j,n-1)C_j, \\ P(i,t) &\leftarrow P(i,t) + V(i,k)b_k. \end{aligned}$$

If $M = 1$, the recursion reduces to: For $n \geq 1$, $0 \leq i \leq n$,

$$\begin{aligned} V(i,n) &= \sum_{j=\max\{0,i-(n-1)\}}^{\min\{i,1\}} V(i-j,n-1)C_j, \\ P(i,t) &\leftarrow P(i,t) + V(i,k)b_k. \end{aligned}$$

Remarks: The computation only involves multiplications and additions of nonnegative numbers smaller than one. Its results are therefore insensitive to loss of significance. Also, by the choice of N we readily have that

$$\begin{aligned} P(n,t)\mathbf{e} - \sum_{r=0}^{N} b_r V(n,r)\mathbf{e} &= \sum_{r=N+1}^{\infty} b_r V(n,r)\mathbf{e} \\ &\leq \sum_{r=N+1}^{\infty} b_r \mathbf{e} \leq \epsilon \mathbf{e} \end{aligned}$$

since the $V(n,r)$ are substochastic.

There are several obvious, easily implemented accuracy checks.

1. To within a reasonable tolerance, say, $\epsilon = 10^{-15}$, the computed sum $\sum_{n=0}^{N} P(n,t)$ should be close to the matrix $\exp(Dt)$. For any fixed t, $\exp(Dt)$ is a stochastic matrix easily computed by the uniformization method. Specifically,

$$\exp(Dt) = e^{-ct}\exp[(I + c^{-1}D)ct] = \sum_{r=0}^{\infty} e^{-ct}\frac{(ct)^r}{r!}C^r,$$

where $c = \max\{-D_{ii}\}$, and C is the stochastic matrix $I + c^{-1}D$. We evaluate the Poisson density with parameter ct up to the index N_D for which the remaining probabilities are negligible, for example $N_D = 44$ for $ct = 10$, $\epsilon = 10^{-15}$, and we compute

$$\sum_{r=0}^{N_D} e^{-ct}\frac{(ct)^r}{r!}C^r$$

by Horner's rule.
To compute $\exp(Dt)$ for a large value of t, we implement the following procedure. Suppose that $ct = 2^L ct_0$, where L is the smallest integer for which $ct_0 \leq 10$. For $\epsilon = 10^{-15}$, at most 50 matrix multiplications are needed to compute $\exp(Dt_0)$ by Horner's rule. The matrix $\exp(Dt)$ is then readily obtained from

$$\exp(Dt) = [\exp(Dt_0)]^{2^L}.$$

Clearly, at most $50+L$ matrix multiplications are needed. For applications of common interest, L is quite small. For example, $L = 10$ for $ct = 10,000$; $L = 14$ for $ct = 100,000$.

2. The first two factorial moment matrices $M_1(t)$ and $M_2(t)$ of the computed sequence $\{P(n,t)\}$ should be in close numerical agreement with the corresponding matrices computed by the explicit procedures.

3.2 Computation of $P(n,t)$ for t large

For large values of t, we compute the $P(n,t)$ by using the matrix convolution in the semigroup property (4). As the array of computed matrices grows rapidly with repeated convolution, that array is trimmed as we proceed. Trimming, see Chapter 3 of Neuts (1995), is a cautious and systematic way of removing negligible terms in the heads and the tails of higher convolution products. Using the approach in Section 2, we first find, say, by successive halving of $(0,\ t]$, a value t_0 for which there are few arrivals in $(0,\ t_0]$. Next, we successively compute the arrays corresponding to the $\{P(n, 2^k t_0)\}$ until $\{P(n,t)\}$ is obtained. At each stage, the negligible terms in the head and the tail of the matrix sequence is deleted. For efficient use of the storage arrays, the terms are shifted downward before the convolution of the trimmed array is evaluated. While this requires some minor bookkeeping to keep track of the indices, it is important when the number m of phases is large.

3.3 Computation of $P(n,t)$ for $0 < t \leq t^*$

If it is important to know how the matrices $P(n,t)$ change over some interval $(0,t^*]$, we evaluate the $P(n,t)$ for $t = kh$, $0 \leq k \leq t^*/h$, and for an appropriate step size h. Matrix convolution with trimming is also used here.

To represent the $P(n,t)$ for $0 < t \leq t^*$ informatively, we plot a 3-D graph of $\boldsymbol{\alpha} P(n,t)\mathbf{e}$ against n and t. The initial probability vector $\boldsymbol{\alpha}$ is chosen as $\boldsymbol{\theta}$, $\boldsymbol{\theta}_{arr}$, or some other vector that reflects initial conditions of interest.

4 Examples

We have written computer code in FORTRAN to compute $P(n,t)$ for a fixed t and also for $0 < t \leq t^*$ for the MAP with single arrivals.

As an illustrative example we used an *MMPP* with ten phases. The underlying Markov chain with generator D visits the states in cyclic order. The first five phases have a common rate σ_1, the remaining ones a common rate σ_2. During the first five phases, arrivals occur at a high rate λ_1, and at a lower rate λ_2 during the remaining five phases. That MAP has the following parameter matrices:

$$D = \begin{pmatrix} -\sigma_1 & \sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sigma_1 & \sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sigma_1 & \sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sigma_1 & \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sigma_1 & \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sigma_2 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sigma_2 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma_2 & \sigma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma_2 & \sigma_2 \\ \sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma_2 \end{pmatrix},$$

and

$$D_1 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{pmatrix}.$$

The steady-state probabilities are $\theta(i) = \frac{1}{5}\frac{\sigma_2}{\sigma_1+\sigma_2}$ for the first five states, $\theta(i) = \frac{1}{5}\frac{\sigma_1}{\sigma_1+\sigma_2}$ for the remaining five. The stationary arrival rate is

$$\lambda^* = \lambda_2 + (\lambda_1 - \lambda_2)\frac{\sigma_2}{\sigma_1 + \sigma_2}.$$

That arrival process was used by Liu and Neuts (1994) to study the effect of bursts of arrivals to a queue.

Let κ denote the ratio of the expected duration of the periods with high arrival rate to the mean cycle time. Then

$$\kappa = \frac{\sigma_2}{\sigma_1 + \sigma_2}.$$

Also for notational convenience, we write $S_d(t)$ for $\sqrt{V(t)}$.

In Example 1, $\kappa = 50/100$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\lambda_1 = 2$, and $\lambda_2 = 0$. In Example 2 we reduce the durations of periods with high arrival rate by choosing $\kappa = 1/100$, $\sigma_1 = 9.9$, $\sigma_2 = 0.1, \lambda_1 = 100$, and $\lambda_2 = 0$. As is to be expected, the process in that second

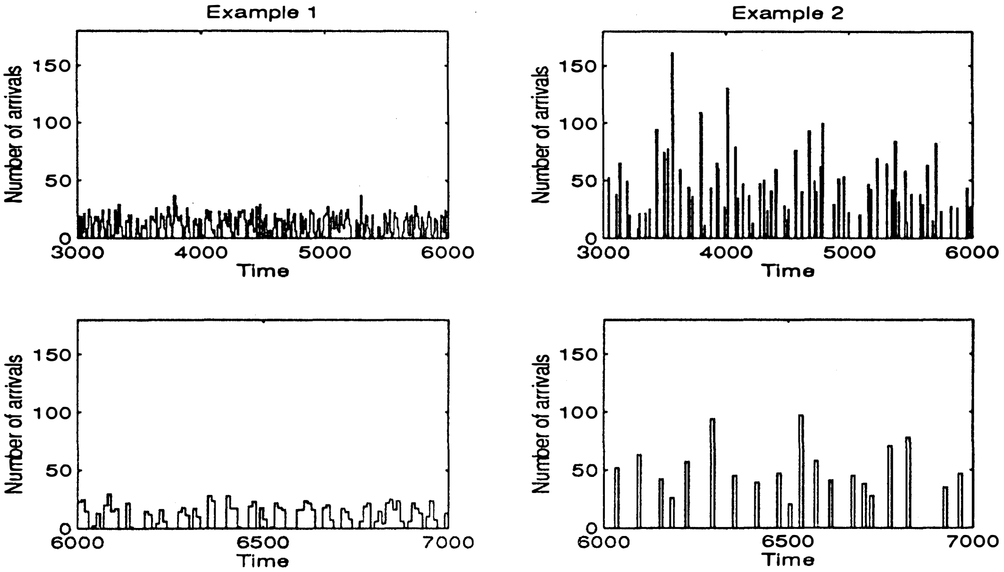

Figure 1: Sample paths of Examples 1 and 2

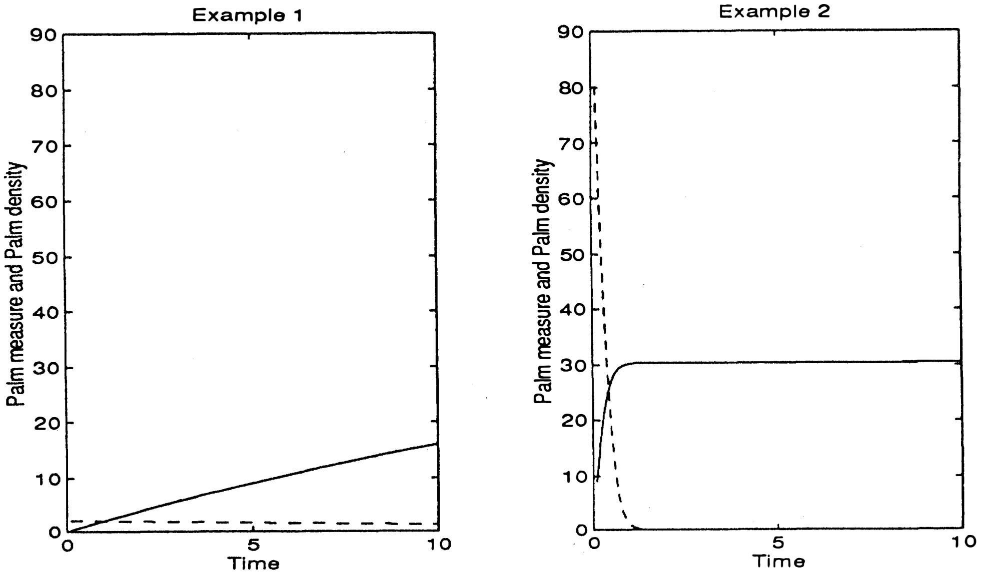

Figure 2: Plots of $H(t)$ and $H'(t)$ for Examples 1 and 2, with —— for $H(t)$, - - - for $H'(t)$

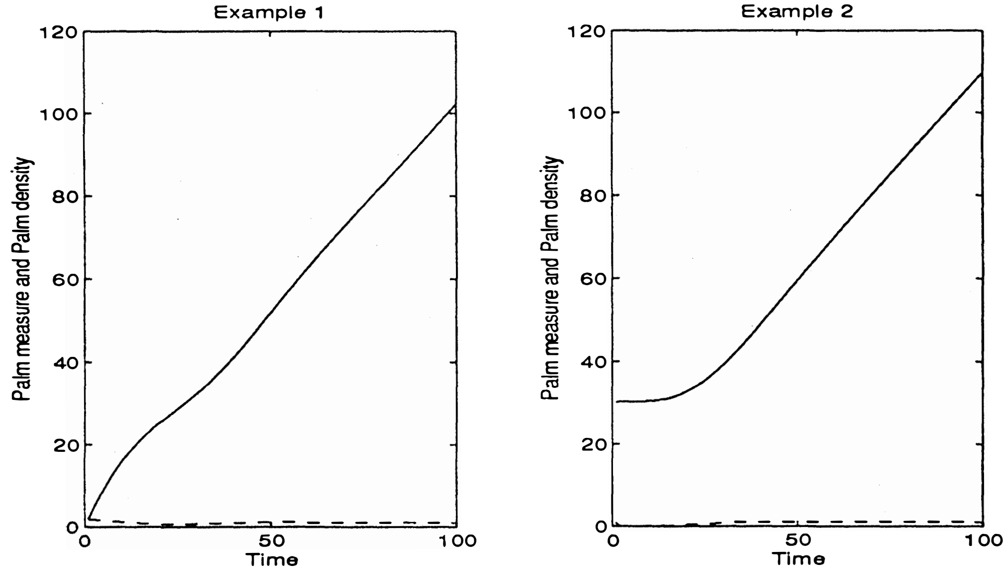

Figure 3: Plots of $H(t)$ and $H'(t)$ for Examples 1 and 2, with —— for $H(t)$, - - - for $H'(t)$

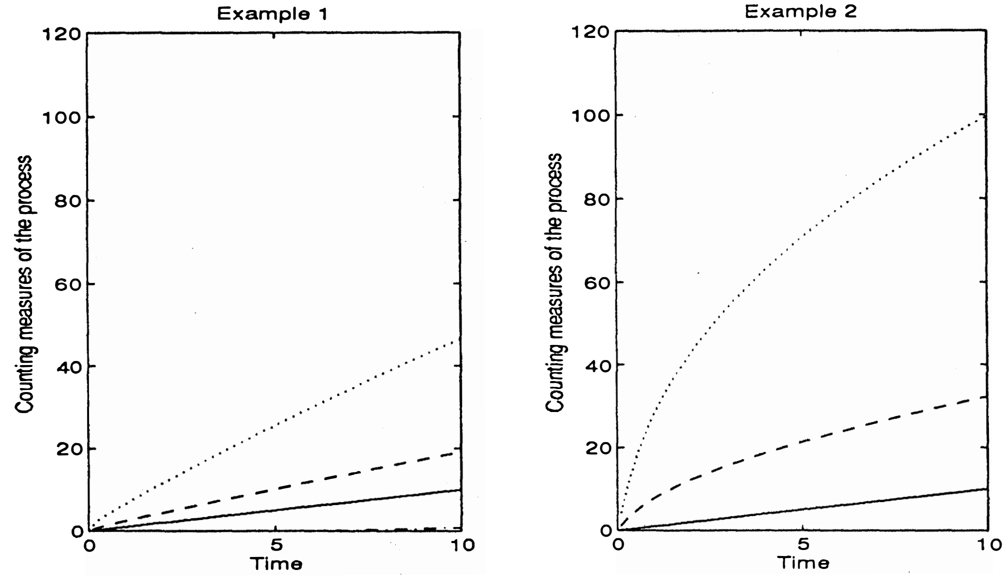

Figure 4: Plots of $\lambda^* t \pm k S_d(t)$ for Examples 1 and 2, with —— for $\lambda^* t$, - - - for $\lambda^* t + S_d(t)$, - · - · - · for $\lambda^* t - S_d(t)$, · · · · · · for $\lambda^* t \pm 4 S_d(t)$

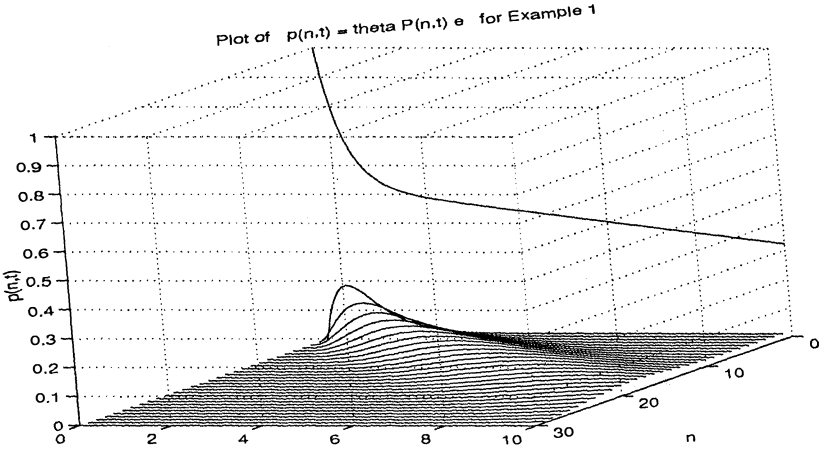

Figure 5: Plots for $\theta P(n,t)$e for Example 1

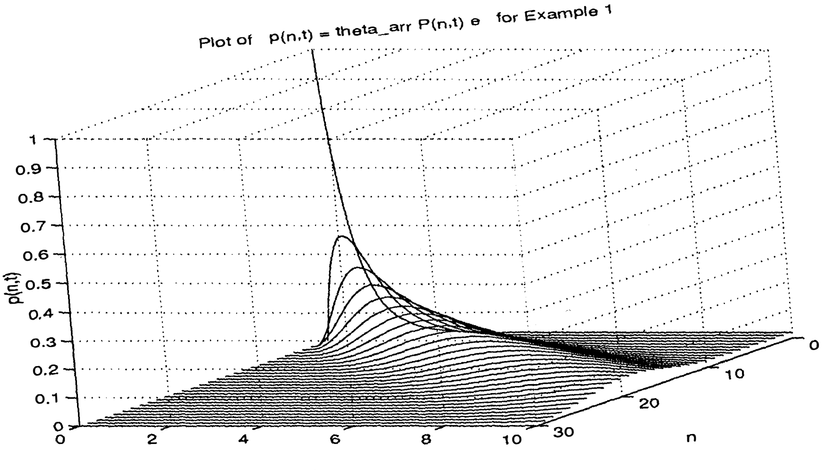

Figure 6: Plots for $\theta_{arr}P(n,t)$e for Example 1

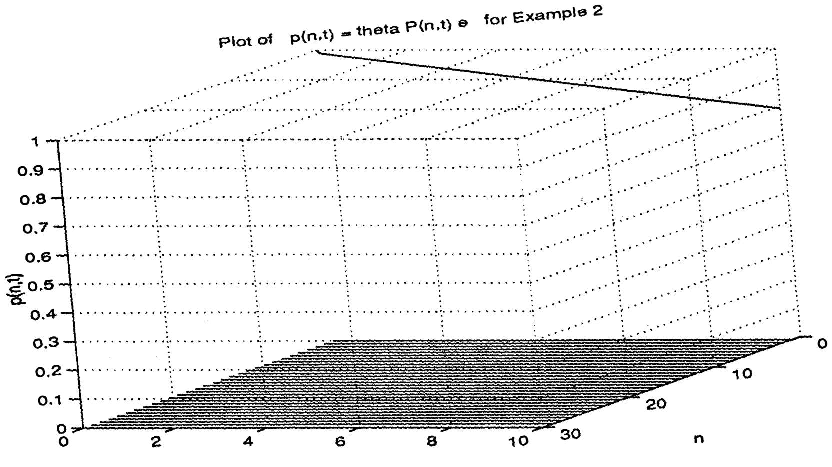

Figure 7: Plots for $\theta P(n,t)$e for Example 2

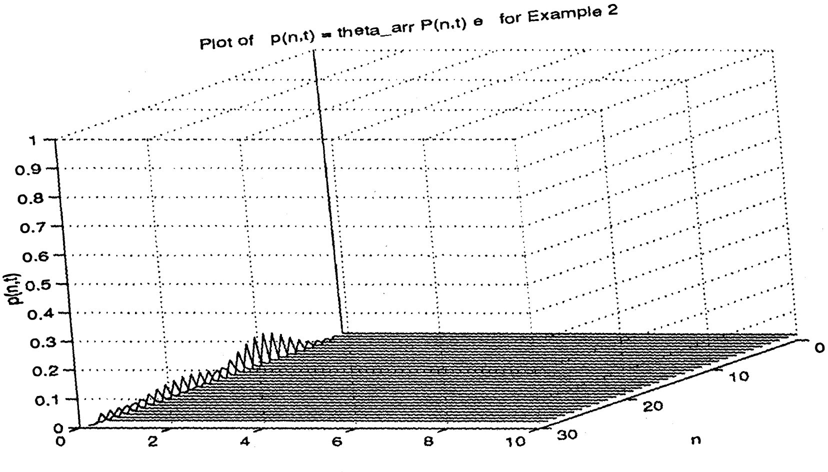

Figure 8: Plots for $\theta_{arr}P(n,t)$e for Example 2

example is much more bursty than that in Example 1. That is quite apparent in the following graphs.

Remarks: These figures suggest the following observations:

1. Figures 2 and 3 show that the Palm measure tends to $\lambda^* t$ and the Palm density $H'(t)$ tends to λ^* (here λ^* has been scaled to 1) for large t, as they should.

2. The differences between Figures 5 and 6 and between Figures 7 and 8 are striking. They show the high sensitivity to initial conditions of the probability distributions of similar descriptors of the *MAP*.

   For a bursty process, it is likely that an arbitrary arrival epoch is soon followed by another arrival. In contrast, in the stationary version, the waiting time for the first arrival will typically be much longer.

Appendix: Simulation of a Continuous *BMAP*

Consider an m-state continuous batch Markovian arrival process (*BMAP*) $\{(J(n),\ X(n)),\ n \geq 0\}$ in which each transition is associated with a number $Y(n)$ of arrivals. $J(n)$ is the state of the Markov chain immediately after the n-th transition. $X(n)$, with $X(0) = 0$, is the sojourn time between the $(n-1)$-th and the n-th transitions. Suppose that the *BMAP* has maximum batch size M, and is parameterized by matrices $D_0,\ D_1,\ \cdots,\ D_M$. The matrix $D = \sum_{k=0}^{M} D_k$ is irreducible. Let $S(n) = \sum_{k=0}^{n} Y(k)$ denote the number of arrivals up to and including the n-th transition. We want to generate a simulated sequence of random variables $\{J(n),\ X(n),\ Y(n)\}$ for $S(n)$ up to some value S_{max}.

The procedure is as follows:

1. Initialization at time $t = 0$.

   Select an initial probability vector $\boldsymbol{\alpha}$, generate the initial state $J(0)$ according to a multinomial trial with density $\boldsymbol{\alpha}$.

   Set $n = 0,\ t = 0,\ X(0) = 0,\ Y(0) = 0$.

2. While $S(n) < S_{max}$, do

   $\{$

   Suppose that $J(n) = j$.

   The sojourn time $X(n+1)$ is generated by an exponential variate with rate $-[D_0]_{jj}$.

   Generate an index i by a multinomial trial with density

$$\frac{1}{-[D_0]_{jj}} \{\ [D_0]_{j1}\ \cdots\ [D_0]_{j,j-1}\ \ 0\ \ [D_0]_{j,j+1}\ \cdots\ [D_0]_{jm}\ \ [D_1]_{j1}\ \cdots\ [D_1]_{jm}\ \cdots\ [D_M]_{j1}\ \cdots\ [D_M]_{jm}\ \}.$$

   The number $Y(n+1)$ of arrivals at $(n+1)$-th transition is determined by

$$Y(n+1) = \left\lfloor \frac{i}{m} \right\rfloor .$$

```
    Suppose that $Y(n+1) = k$.
    Choose the state $J(n+1)$ by $J(n+1) = i - km$. Set $S(n+1) = S(n) + Y(n+1)$.
    Update time $t \leftarrow t + X(n+1)$.
    Set $n \leftarrow n + 1$.
}
```

References

[1] Liu, D. and Neuts, M. F. Assessing the effect of bursts of arrivals on the characteristics of a queue. In J. Pérez Vilaplana and M. L. Puri, editors, *Recent Advances in Statistics and Probability*, pages 371–387. VSP, 1994.

[2] Lucantoni, D. M. New results on the single server queue with a batch Markovian arrival process. *Communications in Statistics: Stochastic Models*, 7(1):1–46, 1991.

[3] Lucantoni, David M. and Ramaswami, V. Efficient algorithms for solving the nonlinear matrix-equations arising in phase type queues. *Communications in Statistics: Stochastic Models*, 1(1):29–51, 1985.

[4] Narayana, Surya and Neuts, Marcel F. The first two moments matrices of the counts for the Markovian arrival process. *Communications in Statistics: Stochastic Models*, 8(3):459–477, 1992.

[5] Neuts, M. F. Models based on the Markovian arrival processes. *IEICE Transactions On Communications*, E75-B(12):1255–65, 1992.

[6] Neuts, M. F. *Algorithmic Probability: A Collection of Problems.* Chapman & Hall, London, 1995.