SETS, FUNCTIONS, AND LOGIC

AN INTRODUCTION TO ABSTRACT MATHEMATICS

THIRD EDITION

KEITH DEVLIN

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Sets, Functions, and Logic

An Introduction to Abstract Mathematics

Third Edition

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Preface

This book is designed to help the university student make the difficult transition from calculus to university-level pure mathematics. A quick glance at the contents page will indicate to any instructor what the book sets out to do.

I first starting teaching this material in the late 1970s at the University of Lancaster in England. I wrote the first version of this book, published in 1981, to accompany a 6-week course I developed there. Teaching the same kind of material a decade later in the U.S., I looked at the original version, found it wanting, and completely rewrote it. I changed almost everything: I reordered the topics, changed the treatment of each topic, and increased the number of exercises. That second edition was published in 1992.

One decision I faced when I wrote the second edition was the size of the book. The original edition was deliberately written as a "little book" that would be both unintimidating and cheap. At the time, there were no competing books on the market (at least in the U.K.), and as a result the book did quite well. By the time I came to write the second edition, there was a whole slew of books (in the U.S.) targeted at the same students. All of these books were far more substantial than mine and, in keeping with the trend in textbook publishing, getting larger all the time in a furious struggle to be all things to all students and to include everything that appeared in any competing book. I looked at those books and decided firmly — I had to be firm because my publisher had other ideas — to stick with my original plan: Despite the many changes that I felt were necessary, the second edition would remain a "little book".

Now another decade has passed and the time has come to rewrite the book once again. Not because the core material has changed; that is the same now as it was when I myself was a student in the 1960s. What has changed is the background that the beginning student brings to his or her study and the environment in which the study is carried out. This new version of the book tries to take those changes into account. But I still believe that for this material, small is better.

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Students Start Here

If you are a mathematics instructor, this book will tell you absolutely nothing you do not already know. Assuming you have read the Preface (which was written for instructors, under the assumption that students never read prefaces but instructors usually do), you really do not need to read any further in this book than this paragraph.

Those two sentences are the last ones in the entire book directed at instructors. Everything else (including this paragraph) is written for the beginning student of post-calculus university-level mathematics ("you", from now on).

You have completed courses on differential and integral calculus. Maybe you aced those courses; or maybe you got through only after a struggle. Either way, now you are trying to make the transition to what comes next. Chances are, regardless of your performance in mathematics up until now, you are going to find the next step unfamiliar and challenging. This book won't make it easy. No book can do that. Anyone who claims otherwise is trying to sell you a book — probably a thick, expensive one at that.[1] But after guiding many generations of young people through precisely this difficult transition,[2] I think I can help.

The main distinction between most high-school mathematics and post-calculus university math lies in the degree of rigor and abstraction required at the university level. In general, you (a student embarking on post-calculus university mathematics) will have had little or no prior experience of wholly rigorous definitions and proofs. The result is that, although you may be competent to handle quite difficult problems in calculus, you are likely to find yourself totally lost when presented with a rigorous definition of limits and derivatives — the fundamental mathematical ideas that lie behind calculus.

In effect, what you need in order to progress further in mathematics is to acquire mastery of what is virtually an entire new language ("the language of mathematics") and to adopt an entirely new mode of thinking ("mathematical thinking"). Throughout my many years teaching university mathematics,[3] I have met very few students who came through this process without a great deal of difficulty. (I certainly didn't when I was a student.) This book is intended to assist you in making this transition.

Chances are that almost everything you find in this book will be new to you and will probably seem very strange. Indeed, you may feel that it is not "mathematics" at all. Be patient. Given time and a fair amount of hard work, this stage will pass.

[1] See the Preface.

[2] See the Preface again.

[3] See the Preface yet again.

Do not try to rush through any part of the book, even if at first glance a particular section looks easy. This entire book consists of *basic* material required elsewhere (indeed practically *everywhere*) in post-calculus mathematics. Everything you will find in this book is included because it generally causes problems for the beginner. (Trust me on this. Hey, I got you to look at the Preface, didn't I, and how often have you done *that* with a math textbook?)

Part of the reason you are likely to find this material difficult is that it will seem unmotivated. It is bound to: the sole motivation is to provide you with the foundation on which to build the mathematics that comes later — mathematics that you do not yet know about!

So take it steadily, and try to *understand* the new concepts as you meet them. There is little in the way of new facts to learn, but a great deal to comprehend! (The actual facts contained in this book could be listed on three or four pages of notes.) And try the exercises — as many of them as possible. They are included for a purpose: to aid your understanding.

Discuss any difficulties that arise with your colleagues and with your instructor. Do not give up. Students all around the world managed it last year. Likewise the previous year, and the year before that. So did I. So will you!

Keith Devlin
Stanford University

Chapter 1

What Is Mathematics and What Does It Do for Us?

You might think these are odd questions to begin with. After all, won't you, my intended reader, already know what mathematics is? And doesn't everyone know that it's important to be able to "do math" — which presumably implies that they think it is useful stuff?

Maybe yes. But in my experience, given the way mathematics is often taught in schools — and in some universities, come to that — many students are never given a "big picture" view of the subject. I've long lost count of the number of adults I've met, many years after they have graduated with degrees in such mathematically rich subjects as engineering, physics, computer science, or even mathematics itself, who have told me that they went through their entire education without ever gaining a good overview of what constitutes modern mathematics. Only later in life do they start to catch a glimpse of the true nature of mathematics and the extent of its pervasive influence in modern-day society.

It's not hard to understand why this is the case. Most of the mathematics that underpins present-day science and technology is no more than 300 or 400 years old, in many cases less than a century old. Yet the typical high school curriculum covers mathematics that is at least 500 years old — in fact, much of what is taught is over 2,000 years old. It's as if our literature courses gave students Homer and Chaucer but never mentioned Shakespeare, Dickens, or Proust.

1.1 It's Not Just Numbers

Most people think that mathematics is the study of numbers. That description of mathematics ceased to be accurate about 2,500 years ago! Anyone who has that view of mathematics is unlikely to appreciate that research in mathematics is a thriving, worldwide activity, or to accept a suggestion that mathematics permeates, often to a considerable extent, most walks of present-day life and society. Nor are they likely to know which organization in the U.S. employs the greatest number of Ph.D.s in mathematics. (The answer is the National Security Agency. Exactly what the 30 or so new mathematics Ph.D.s who are hired there each year — the exact number is an official secret — do to earn their paychecks is never made public, but it is generally assumed that the majority of them work on code breaking, to enable the agency to read encrypted messages that are intercepted by monitoring systems.)

In fact, the answer to the question "What is mathematics?" has changed several times during the course of history.

Up to 500 B.C. or thereabouts, mathematics was indeed the study of number. This was the period of Egyptian and Babylonian mathematics.[1] In those civilizations, mathematics consisted almost solely of arithmetic. It was largely utilitarian, and very much of a "cookbook" variety. ("Do such and such to a number and you will get the answer.")

The period from around 500 B.C. to 300 A.D. was the era of Greek mathematics. The mathematicians of ancient Greece were primarily concerned with geometry. Indeed, they regarded numbers in a geometric fashion, as measurements of length, and when they discovered that there were lengths to which their numbers did not correspond (the discovery of irrational lengths), their study of number largely came to a halt.[2] For the Greeks, with their emphasis on geometry, mathematics was the study of number and shape.

In fact, it was only with the Greeks that mathematics came into being as an area of study, and ceased being merely a collection of techniques for measuring, counting, and accounting. Greek interest in mathematics was not just utilitarian; they regarded mathematics as an intellectual pursuit having both aesthetic and religious elements. Around 500 B.C., Thales of Miletus (now part of Turkey) introduced the idea that the precisely stated assertions of mathematics could be logically proved by a formal argument. This innovation marked the birth of the theorem, now the bedrock of mathematics. For the Greeks, this approach culminated in the publication of Euclid's *Elements*, reputedly the most widely circulated book of all time after the Bible.

There was no major change in the overall nature of mathematics, and hardly any significant advances within the subject, until the middle of the 17th century, when Isaac Newton (in England) and Gottfried Leibniz (in Germany) independently invented calculus. In essence, calculus is the study of continuous motion and change. Previous mathematics had been largely restricted to the static issues of counting, measuring, and describing shape. With the introduction of techniques to handle motion and change, mathematicians were able to study the motion of the planets and of falling bodies on earth, the workings of machinery, the flow of liquids, the expansion of gases, physical forces such as magnetism and electricity, flight, the growth of plants and animals, the spread of epidemics, the fluctuation of profits, and so on.

[1]Other civilizations also developed mathematics, for example, the Chinese and the Japanese. But the mathematics of those cultures did not appear to have a direct influence on the development of modern Western mathematics, so in this book I will ignore them.

[2]There is an oft repeated story that the young Greek mathematician who made this discovery was taken out to sea and drowned, lest the awful news of what he had stumbled upon should leak out. As far as I know, there is no evidence whatsoever to support this fanciful tale. Pity. It's a great story.

After Newton and Leibniz, mathematics became the study of number, shape, motion, change, and space.

Most of the initial work involving calculus was directed toward the study of physics; indeed, many of the great mathematicians of the period are also regarded as physicists. But from about the middle of the 18th century there was an increasing interest in mathematics itself, not just its applications, as mathematicians sought to understand what lay behind the enormous power that calculus gave to humankind. Here the old Greek tradition of formal proof came back into ascendancy, as a large part of present-day pure mathematics was developed. By the end of the 19th century, mathematics had become the study of number, shape, motion, change, and space, and of the mathematical tools that are used in this study.

The explosion of mathematical activity that has taken place over the past 100 years or so has been dramatic. The growth has not just been a further development of previous mathematics; many quite new branches have sprung up. At the start of the 20th century, mathematics could reasonably be regarded as consisting of about 12 distinct subjects: arithmetic, geometry, calculus, and so on. Today, between 60 and 70 distinct categories would be a reasonable figure. Some subjects, such as algebra or topology, have split into various subfields; others, such as complexity theory or dynamical systems theory, are completely new areas of study.

Given this tremendous growth in mathematical activity, for a while it seemed as though the only simple answer to the question "What is mathematics?" was to say, somewhat fatuously, "It is what mathematicians do for a living." A particular study was classified as mathematics not so much because of what was studied but because of how it was studied — that is, the methodology used. It was only in the 1980s that a definition of mathematics emerged on which most mathematicians now agree: *mathematics is the science of patterns*. What the mathematician does is examine abstract patterns — numerical patterns, patterns of shape, patterns of motion, patterns of behavior, voting patterns in a population, patterns of repeating chance events, and so on. Those patterns can be either real or imagined, visual or mental, static or dynamic, qualitative or quantitative, purely utilitarian or of little more than recreational interest. They can arise from the world around us, from the depths of space and time, or from the inner workings of the human mind. Different kinds of patterns give rise to different branches of mathematics. For example:

- Arithmetic and number theory study the patterns of numbers and counting.
- Geometry studies the patterns of shape.
- Calculus allows us to handle patterns of motion.
- Logic studies patterns of reasoning.
- Probability theory deals with patterns of chance.
- Topology studies patterns of closeness and position.

and so forth.

1.2 Mathematical Notation

One aspect of modern mathematics that is obvious to even the casual observer is the use of abstract notations: algebraic expressions, complicated-looking formulas, and geometric diagrams. The mathematician's reliance on abstract notation is a reflection of the abstract nature of the patterns she studies.

Different aspects of reality require different forms of description. For example, the most appropriate way to study the lay of the land or to describe to someone how to find their way around a strange town is to draw a map. Text is far less appropriate. Analogously, line drawings in the form of blueprints are the appropriate way to specify the construction of a building. And musical notation is the most appropriate medium to convey music, apart from, perhaps, actually playing the piece.

In the case of various kinds of abstract, formal patterns and abstract structures, the most appropriate means of description and analysis is mathematics, using mathematical notations, concepts, and procedures. For instance, the symbolic notation of algebra is the most appropriate means of describing and analyzing general behavioral properties of addition and multiplication.

For example, the commutative law for addition could be written in English as:

When two numbers are added, their order is not important.

However, it is usually written in the symbolic form

$$m + n = n + m$$

Such is the complexity and the degree of abstraction of the majority of mathematical patterns, that to use anything other than symbolic notation would be prohibitively cumbersome. And so the development of mathematics has involved a steady increase in the use of abstract notations.

The first systematic use of a recognizably algebraic notation in mathematics seems to have been made by Diophantus, who lived in Alexandria some time around 250 A.D. His treatise *Arithmetica*, of which only six of the original thirteen volumes have been preserved, is generally regarded as the first algebra textbook. In particular, Diophantus used special symbols to denote the unknown in an equation and to denote powers of the unknown, and he employed symbols for subtraction and for equality.

These days, mathematics books tend to be awash with symbols, but mathematical notation no more is mathematics than musical notation is music. A page of sheet music *represents* a piece of music; the music itself is what you get when the notes on the page are sung or performed on a musical instrument. It is in its performance that the music comes alive and becomes part of our experience; the music exists not on the printed page but in our minds. The same is true for mathematics; the symbols on a page are just a *representation* of the mathematics. When read by a competent performer (in this case, someone trained in mathematics), the symbols on the printed page come alive — the mathematics lives and breathes in the mind of the reader like some abstract symphony.

Given the strong similarity between mathematics and music, both of which have their own highly abstract notations and are governed by their own structural rules, it is hardly surprising that many (perhaps most) mathematicians also have some musical talent. Although some commentators make more of this connection in terms of mental ability than I believe is warranted, it is true that for most of the 2,500 years of Western civilization, starting with the ancient Greeks, mathematics and music were regarded as two sides of the same coin. It was only with the rise of the scientific method in the 17th century that the two started to go their separate ways.

For all the historical connections, however, there was, until recently, one very obvious difference between mathematics and music. Although only someone well trained in music can read a musical score and hear the music in her head, if that same piece of music is performed by a competent musician, anyone with a sense of hearing can appreciate the result. It requires no musical training to experience and enjoy music when it is performed.

For most of its history, however, the only way to appreciate mathematics was to learn how to "sight-read" the symbols. Although the structures and patterns of mathematics reflect the structure of, and resonate in, the human mind every bit as much as do the structures and patterns of music, human beings have developed no mathematical equivalent to a pair of ears. Mathematics can only be "seen" with the "eyes of the mind". It is as if we had no sense of hearing, so that only someone able to sight-read musical notation would be able to appreciate the patterns and harmonies of music.

In recent years, however, the development of computer and video technologies has to some extent made mathematics accessible to the untrained spectator. In the hands of a skilled user, the computer can be used to "perform" mathematics, and the result can be displayed in a visual form on the screen for all to see. Although only a relatively small part of mathematics lends itself to such visual "performance", it is now possible to convey to the layperson at least something of the beauty and the harmony that the mathematician "sees" and experiences when she does mathematics.

Sometimes, the use of computer graphics can be of significant use to the mathematician as well as providing the layperson with a glimpse of the inner world of mathematics. For instance, the study of complex dynamical systems was begun in the 1920s by the French mathematicians Pierre Fatou and Gaston Julia, but it was not until the late 1970s and early 1980s that the rapidly developing technology of computer graphics enabled Benoit Mandelbrot and other mathematicians to see some of the structures Fatou and Julia had been working with. The strikingly beautiful pictures that emerged from this study have since become something of an art form in their own right.

Without its algebraic symbols, large parts of mathematics simply would not exist. Indeed, the issue is a deep one having to do with human cognitive abilities. The recognition of abstract concepts and the development of an appropriate language are really two sides of the same coin.

The use of a symbol such as a letter, a word, or a picture to denote an abstract entity goes hand in hand with the recognition of that entity as an entity. The use of the numeral "7" to denote the number 7 requires that the number 7 be recognized as