volume 211

lecture notes in pure and applied mathematics

nonassociative algebra and its applications

edited by

Roberto Costa
Alexander Grishkov
Henrique Guzzo, Jr.
Luiz A. Peresi

CRC Press
Taylor & Francis Group

nonassociative algebra
and its applications

PURE AND APPLIED MATHEMATICS

A Program of Monographs, Textbooks, and Lecture Notes

LECTURE NOTES IN PURE AND APPLIED MATHEMATICS

1. *N. Jacobson,* Exceptional Lie Algebras
2. *L.-Å. Lindahl and F. Poulsen,* Thin Sets in Harmonic Analysis
3. *I. Satake,* Classification Theory of Semi-Simple Algebraic Groups
4. *F. Hirzebruch et al., Differentiable Manifolds and Quadratic Forms*
5. *I. Chavel,* Riemannian Symmetric Spaces of Rank One
6. *R. B. Burckel,* Characterization of C(X) Among Its Subalgebras
7. *B. R. McDonald et al.,* Ring Theory
8. *Y.-T. Siu,* Techniques of Extension on Analytic Objects
9. *S. R. Caradus et al.,* Calkin Algebras and Algebras of Operators on Banach Spaces
10. *E. O. Roxin et al.,* Differential Games and Control Theory
11. *M. Orzech and C. Small,* The Brauer Group of Commutative Rings
12. *S. Thomier,* Topology and Its Applications
13. *J. M. Lopez and K. A. Ross,* Sidon Sets
14. *W. W. Comfort and S. Negrepontis,* Continuous Pseudometrics
15. *K. McKennon and J. M. Robertson,* Locally Convex Spaces
16. *M. Carmeli and S. Malin,* Representations of the Rotation and Lorentz Groups
17. *G. B. Seligman,* Rational Methods in Lie Algebras
18. *D. G. de Figueiredo,* Functional Analysis
19. *L. Cesari et al.,* Nonlinear Functional Analysis and Differential Equations
20. *J. J. Schäffer,* Geometry of Spheres in Normed Spaces
21. *K. Yano and M. Kon,* Anti-Invariant Submanifolds
22. *W. V. Vasconcelos,* The Rings of Dimension Two
23. *R. E. Chandler,* Hausdorff Compactifications
24. *S. P. Franklin and B. V. S. Thomas,* Topology
25. *S. K. Jain,* Ring Theory
26. *B. R. McDonald and R. A. Morris,* Ring Theory II
27. *R. B. Mura and A. Rhemtulla,* Orderable Groups
28. *J. R. Graef,* Stability of Dynamical Systems
29. *H.-C. Wang,* Homogeneous Branch Algebras
30. *E. O. Roxin et al.,* Differential Games and Control Theory II
31. *R. D. Porter,* Introduction to Fibre Bundles
32. *M. Altman,* Contractors and Contractor Directions Theory and Applications
33. *J. S. Golan,* Decomposition and Dimension in Module Categories
34. *G. Fairweather,* Finite Element Galerkin Methods for Differential Equations
35. *J. D. Sally,* Numbers of Generators of Ideals in Local Rings
36. *S. S. Miller,* Complex Analysis
37. *R. Gordon,* Representation Theory of Algebras
38. *M. Goto and F. D. Grosshans,* Semisimple Lie Algebras
39. *A. I. Arruda et al.,* Mathematical Logic
40. *F. Van Oystaeyen,* Ring Theory
41. *F. Van Oystaeyen and A. Verschoren,* Reflectors and Localization
42. *M. Satyanarayana,* Positively Ordered Semigroups
43. *D. L Russell,* Mathematics of Finite-Dimensional Control Systems
44. *P.-T. Liu and E. Roxin,* Differential Games and Control Theory III
45. *A. Geramita and J. Seberry,* Orthogonal Designs
46. *J. Cigler, V. Losert, and P. Michor,* Banach Modules and Functors on Categories of Banach Spaces
47. *P.-T. Liu and J. G. Sutinen,* Control Theory in Mathematical Economics
48. *C. Byrnes,* Partial Differential Equations and Geometry
49. *G. Klambauer,* Problems and Propositions in Analysis
50. *J. Knopfmacher,* Analytic Arithmetic of Algebraic Function Fields
51. *F. Van Oystaeyen,* Ring Theory
52. *B. Kadem,* Binary Time Series
53. *J. Barros-Neto and R. A. Artino,* Hypoelliptic Boundary-Value Problems
54. *R. L. Sternberg et al.,* Nonlinear Partial Differential Equations in Engineering and Applied Science
55. *B. R. McDonald,* Ring Theory and Algebra III
56. *J. S. Golan,* Structure Sheaves Over a Noncommutative Ring
57. *T. V. Narayana et al.,* Combinatorics, Representation Theory and Statistical Methods in Groups
58. *T. A. Burton,* Modeling and Differential Equations in Biology
59. *K. H. Kim and F. W. Roush,* Introduction to Mathematical Consensus Theory

60. *J. Banas and K. Goebel,* Measures of Noncompactness in Banach Spaces
61. *O. A. Nielson,* Direct Integral Theory
62. *J. E. Smith et al.,* Ordered Groups
63. *J. Cronin,* Mathematics of Cell Electrophysiology
64. *J. W. Brewer,* Power Series Over Commutative Rings
65. *P. K. Kamthan and M. Gupta,* Sequence Spaces and Series
66. *T. G. McLaughlin,* Regressive Sets and the Theory of Isols
67. *T. L. Herdman et al.,* Integral and Functional Differential Equations
68. *R. Draper,* Commutative Algebra
69. *W. G. McKay and J. Patera,* Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras
70. *R. L. Devaney and Z. H. Nitecki,* Classical Mechanics and Dynamical Systems
71. *J. Van Geel,* Places and Valuations in Noncommutative Ring Theory
72. *C. Faith,* Injective Modules and Injective Quotient Rings
73. *A. Fiacco,* Mathematical Programming with Data Perturbations I
74. *P. Schultz et al.,* Algebraic Structures and Applications
75. *L Bican et al.,* Rings, Modules, and Preradicals
76. *D. C. Kay and M. Breen,* Convexity and Related Combinatorial Geometry
77. *P. Fletcher and W. F. Lindgren,* Quasi-Uniform Spaces
78. *C.-C. Yang,* Factorization Theory of Meromorphic Functions
79. *O. Taussky,* Ternary Quadratic Forms and Norms
80. *S. P. Singh and J. H. Burry,* Nonlinear Analysis and Applications
81. *K. B. Hannsgen et al.,* Volterra and Functional Differential Equations
82. *N. L. Johnson et al.,* Finite Geometries
83. *G. I. Zapata,* Functional Analysis, Holomorphy, and Approximation Theory
84. *S. Greco and G. Valla,* Commutative Algebra
85. *A. V. Fiacco,* Mathematical Programming with Data Perturbations II
86. *J.-B. Hiriart-Urruty et al.,* Optimization
87. *A. Figa Talamanca and M. A. Picardello,* Harmonic Analysis on Free Groups
88. *M. Harada,* Factor Categories with Applications to Direct Decomposition of Modules
89. *V. I. Istrăţescu,* Strict Convexity and Complex Strict Convexity
90. *V. Lakshmikantham,* Trends in Theory and Practice of Nonlinear Differential Equations
91. *H. L. Manocha and J. B. Srivastava,* Algebra and Its Applications
92. *D. V. Chudnovsky and G. V. Chudnovsky,* Classical and Quantum Models and Arithmetic Problems
93. *J. W. Longley,* Least Squares Computations Using Orthogonalization Methods
94. *L. P. de Alcantara,* Mathematical Logic and Formal Systems
95. *C. E. Aull,* Rings of Continuous Functions
96. *R. Chuaqui,* Analysis, Geometry, and Probability
97. *L. Fuchs and L. Salce,* Modules Over Valuation Domains
98. *P. Fischer and W. R. Smith,* Chaos, Fractals, and Dynamics
99. *W. B. Powell and C. Tsinakis,* Ordered Algebraic Structures
100. *G. M. Rassias and T. M. Rassias,* Differential Geometry, Calculus of Variations, and Their Applications
101. *R.-E. Hoffmann and K. H. Hofmann,* Continuous Lattices and Their Applications
102. *J. H. Lightbourne III and S. M. Rankin III,* Physical Mathematics and Nonlinear Partial Differential Equations
103. *C. A. Baker and L. M. Batten,* Finite Geometrics
104. *J. W. Brewer et al.,* Linear Systems Over Commutative Rings
105. *C. McCrory and T. Shifrin,* Geometry and Topology
106. *D. W. Kueke et al.,* Mathematical Logic and Theoretical Computer Science
107. *B.-L. Lin and S. Simons,* Nonlinear and Convex Analysis
108. *S. J. Lee,* Operator Methods for Optimal Control Problems
109. *V. Lakshmikantham,* Nonlinear Analysis and Applications
110. *S. F. McCormick,* Multigrid Methods
111. *M. C. Tangora,* Computers in Algebra
112. *D. V. Chudnovsky and G. V. Chudnovsky,* Search Theory
113. *D. V. Chudnovsky and R. D. Jenks,* Computer Algebra
114. *M. C. Tangora,* Computers in Geometry and Topology
115. *P. Nelson et al.,* Transport Theory, Invariant Imbedding, and Integral Equations
116. *P. Clément et al.,* Semigroup Theory and Applications
117. *J. Vinuesa,* Orthogonal Polynomials and Their Applications
118. *C. M. Dafermos et al.,* Differential Equations
119. *E. O. Roxin,* Modern Optimal Control
120. *J. C. Díaz,* Mathematics for Large Scale Computing

121. *P. S. Milojević,* Nonlinear Functional Analysis
122. *C. Sadosky,* Analysis and Partial Differential Equations
123. *R. M. Shortt,* General Topology and Applications
124. *R. Wong,* Asymptotic and Computational Analysis
125. *D. V. Chudnovsky and R. D. Jenks,* Computers in Mathematics
126. *W. D. Wallis et al.,* Combinatorial Designs and Applications
127. *S. Elaydi,* Differential Equations
128. *G. Chen et al.,* Distributed Parameter Control Systems
129. *W. N. Everitt,* Inequalities
130. *H. G. Kaper and M. Garbey,* Asymptotic Analysis and the Numerical Solution of Partial Differential Equations
131. *O. Arino et al.,* Mathematical Population Dynamics
132. *S. Coen,* Geometry and Complex Variables
133. *J. A. Goldstein et al.,* Differential Equations with Applications in Biology, Physics, and Engineering
134. *S. J. Andima et al.,* General Topology and Applications
135. *P Clément et al.,* Semigroup Theory and Evolution Equations
136. *K. Jarosz,* Function Spaces
137. *J. M. Bayod et al.,* *p*-adic Functional Analysis
138. *G. A. Anastassiou,* Approximation Theory
139. *R. S. Rees,* Graphs, Matrices, and Designs
140. *G. Abrams et al.,* Methods in Module Theory
141. *G. L. Mullen and P. J.-S. Shiue,* Finite Fields, Coding Theory, and Advances in Communications and Computing
142. *M. C. Joshi and A. V. Balakrishnan,* Mathematical Theory of Control
143. *G. Komatsu and Y. Sakane,* Complex Geometry
144. *I. J. Bakelman,* Geometric Analysis and Nonlinear Partial Differential Equations
145. *T. Mabuchi and S. Mukai,* Einstein Metrics and Yang–Mills Connections
146. *L. Fuchs and R. Göbel,* Abelian Groups
147. *A. D. Pollington and W. Moran,* Number Theory with an Emphasis on the Markoff Spectrum
148. *G. Dore et al.,* Differential Equations in Banach Spaces
149. *T. West,* Continuum Theory and Dynamical Systems
150. *K. D. Bierstedt et al.,* Functional Analysis
151. *K. G. Fischer et al.,* Computational Algebra
152. *K. D. Elworthy et al.,* Differential Equations, Dynamical Systems, and Control Science
153. *P.-J. Cahen, et al.,* Commutative Ring Theory
154. *S. C. Cooper and W. J. Thron,* Continued Fractions and Orthogonal Functions
155. *P. Clément and G. Lumer,* Evolution Equations, Control Theory, and Biomathematics
156. *M. Gyllenberg and L. Persson,* Analysis, Algebra, and Computers in Mathematical Research
157. *W. O. Bray et al.,* Fourier Analysis
158. *J. Bergen and S. Montgomery,* Advances in Hopf Algebras
159. *A. R. Magid,* Rings, Extensions, and Cohomology
160. *N. H. Pavel,* Optimal Control of Differential Equations
161. *M. Ikawa,* Spectral and Scattering Theory
162. *X. Liu and D. Siegel,* Comparison Methods and Stability Theory
163. *J.-P. Zolésio,* Boundary Control and Variation
164. *M. Křížek et al.,* Finite Element Methods
165. *G. Da Prato and L. Tubaro,* Control of Partial Differential Equations
166. *E. Ballico,* Projective Geometry with Applications
167. *M. Costabel et al.,* Boundary Value Problems and Integral Equations in Nonsmooth Domains
168. *G. Ferreyra, G. R. Goldstein, and F. Neubrander,* Evolution Equations
169. *S. Huggett,* Twistor Theory
170. *H. Cook et al.,* Continua
171. *D. F. Anderson and D. E. Dobbs,* Zero-Dimensional Commutative Rings
172. *K. Jarosz,* Function Spaces
173. *V. Ancona et al.,* Complex Analysis and Geometry
174. *E. Casas,* Control of Partial Differential Equations and Applications
175. *N. Kalton et al.,* Interaction Between Functional Analysis, Harmonic Analysis, and Probability
176. *Z. Deng et al.,* Differential Equations and Control Theory
177. *P. Marcellini et al.* Partial Differential Equations and Applications
178. *A. Kartsatos,* Theory and Applications of Nonlinear Operators of Accretive and Monotone Type
179. *M. Maruyama,* Moduli of Vector Bundles
180. *A. Ursini and P. Aglianò,* Logic and Algebra
181. *X. H. Cao et al.,* Rings, Groups, and Algebras
182. *D. Arnold and R. M. Rangaswamy,* Abelian Groups and Modules
183. *S. R. Chakravarthy and A. S. Alfa,* Matrix-Analytic Methods in Stochastic Models

184. *J. E. Andersen et al.*, Geometry and Physics
185. *P.-J. Cahen et al.*, Commutative Ring Theory
186. *J. A. Goldstein et al.*, Stochastic Processes and Functional Analysis
187. *A. Sorbi*, Complexity, Logic, and Recursion Theory
188. *G. Da Prato and J.-P. Zolésio*, Partial Differential Equation Methods in Control and Shape Analysis
189. *D. D. Anderson*, Factorization in Integral Domains
190. *N. L. Johnson*, Mostly Finite Geometries
191. *D. Hinton and P. W. Schaefer*, Spectral Theory and Computational Methods of Sturm–Liouville Problems
192. *W. H. Schikhof et al.*, *p*-adic Functional Analysis
193. *S. Sertöz*, Algebraic Geometry
194. *G. Caristi and E. Mitidieri*, Reaction Diffusion Systems
195. *A. V. Fiacco*, Mathematical Programming with Data Perturbations
196. *M. Křížek et al.*, Finite Element Methods: Superconvergence, Post-Processing, and A Posteriori Estimates
197. *S. Caenepeel and A. Verschoren*, Rings, Hopf Algebras, and Brauer Groups
198. *V. Drensky et al.*, Methods in Ring Theory
199. *W. B. Jones and A. Sri Ranga*, Orthogonal Functions, Moment Theory, and Continued Fractions
200. *P. E. Newstead*, Algebraic Geometry
201. *D. Dikranjan and L. Salce*, Abelian Groups, Module Theory, and Topology
202. *Z. Chen et al.*, Advances in Computational Mathematics
203. *X. Caicedo and C. H. Montenegro*, Models, Algebras, and Proofs
204. *C. Y. Yıldırım and S. A. Stepanov*, Number Theory and Its Applications
205. *D. E. Dobbs et al.*, Advances in Commutative Ring Theory
206. *F. Van Oystaeyen*, Commutative Algebra and Algebraic Geometry
207. *J. Kakol et al.*, *p*-adic Functional Analysis
208. *M. Boulagouaz and J.-P. Tignol*, Algebra and Number Theory
209. *S. Caenepeel and F. Van Oystaeyen*, Hopf Algebras and Quantum Groups
210. *F. Van Oystaeyen and M. Saorin*, Interactions Between Ring Theory and Representations of Algebras
211. *R. Costa et al.*, Nonassociative Algebra and Its Applications

Additional Volumes in Preparation

nonassociative algebra and its applications

the fourth international conference

edited by

Roberto Costa
Alexander Grishkov
Henrique Guzzo, Jr.
Luiz A. Peresi
University of São Paulo
São Paulo, Brazil

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

First published 2000 by Marcel Dekker, Inc.

Published 2018 by CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

ISBN-13: 978-0-8247-0406-3 (pbk)

Library of Congress Cataloging-in-Publication Data

Nonassociative algebra and its applications : the fourth international conference / edited by Roberto Costa ... [et al.].
p. cm. — (Lecture notes in pure and applied mathematics ; v. 211)
ISBN 0-8247-0406-1 (alk. paper)
1. Nonassociative algebras—Congresses. I. Costa, Roberto. II. Series.

QA252 .N66 2000
512'.24—dc21 00-026394

Preface

The Fourth International Conference on Nonassociative Algebra and Its Applications was held in São Paulo (Brazil). The three previous Conferences were in: Novosibirsk (Russia, 1988), Tashkent (Russia, 1990), and Oviedo (Spain, 1993).

The Organising Committee members were Roberto Costa, Alexander Grishkov, Henrique Guzzo, Jr., and Luiz Antonio Peresi, from the University of São Paulo. The Scientific Committee was made up of the following members:

Benkart, G. (University of Wisconsin, U.S.)
Costa, R. (USP, Brazil)
Elduque, A. (Universidad de La Rioja, Spain)
Ferreira, L. A. (UNESP, Brazil)
Forger, M. (USP, Brazil)
González, S. (Universidad de Oviedo, Spain)
Hentzel, I. R. (Iowa State University, U.S.)
Kamiya, N. (Shimane University, Japan)
Kaup, W. (Universität Tubingen, Germany)
Kleinfeld, E. (University of Iowa, U.S.)
Kostrikin, A. I. (Moscow State University, Russia)
Kuzmin, E. N. (Institute of Mathematics, Novosibirsk, Russia)
Lyubich, Y. (Technion University, Israel)
Mallol, C. (Universidad de La Frontera, Chile)
Martínez, C. (Universidad de Oviedo, Spain)
McCrimmon, K. (University of Virginia, U.S.)
Medvedev, Y. (York University, Canada)
Meyberg, K. (Technische Universität München, Germany)
Moody, R. (University of Alberta, Canada)
Myung, H. C. (KAIST, Taejon, South Korea)
Okubo, S. (University of Rochester, U.S.)
Osborn, J. M. (University of Wisconsin, U.S.)
Pchelintsev, S. (Lenin Moscow Pedagogical Institute, Russia)
Petersson, H. (Fern-Universität, Germany)
Polcino Milies, F. C. (USP, Brazil)
Racine, M. (University of Ottawa, Canada)
Schafer, R. D. (Massachusetts Institute of Technology, U.S.)
Shestakov, I. P.(Institute of Mathematics, Novosibirsk, Russia)
Slinko, A. (University of Auckland, New Zealand)
Taft, E. (Rutgers University, U.S.)
Zelmanov, E. (Yale University, U.S.)

Nearly 100 people came to the conference. Several topics in nonassociative algebra were presented, including applications to population genetics theory and physics.

During the five days of the conference, 14 plenary talks were delivered by members of the Scientific Committee and also some invited speakers. Twelve conferences on some special topics were presented, as well as 35 short communications.

The editors would like to thank Professor L. A. Ferreira, who was in charge of the conferences in the field of physics. We regret the death of Professor M. V. Saveliev some months after the meeting.

The editors want also to express thanks to the following institutions for their financial support: The Institute of Mathematics and Statistics of the University of São Paulo, Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Sociedade Brasileira de Matemática (SBM), CAPES, and CCInt-USP.

The editors also thank the referees for their invaluable work of revision of the written version of the conferences and communications.

Roberto Costa
Alexander Grishkov
Henrique Guzzo, Jr.
Luiz A. Peresi

Contents

Contributors

Helena Albuquerque University of Coimbra, Coimbra, Portugal

Y. Bahturin Moscow State University, Moscow, Russia

H. Belich Centro Brasileiro de Pesquisas Fisicas, Rio de Janeiro, Brazil

Pilar Benito University of La Rioja, Logroño, Spain

A. Catalan University of La Frontera, Temuco, Chile

Roberto Costa Institute of Mathematics and Statistics, University of São Paulo, São Paulo, Brazil

João Carlos da Motta Ferreira Federal University of Mato Grosso do Sul, Campo Grande, Brazil

Cristina Draper University of La Rioja, Logroño, Spain

Alberto Elduque University of Zaragoza, Zaragoza, Spain

Mohammed El Houari Institute of Mathematics and Statistics, University of São Paulo, São Paulo, Brazil

Luiz A. Ferreira Institute of Theoretical Physics –IFT/UNESP, São Paulo-SP, Brazil

Alice Fialowski Eötvös Loránd University, Budapest, Hungary

M. A. Garcia-Muñiz University of Oviedo, Oviedo, Spain

J. F. Gomes Institute of Theoretical Physics – UNESP, São Paulo-SP, Brazil

Carlos Gomez-Ambrosi University of Zaragoza, Zaragoza, Spain

S. González University of Oviedo, Oviedo, Spain

Edgar G. Goodaire Memorial University of Newfoundland, St. John's, Newfoundland, Canada

Alexander Grishkov Institute of Mathematics and Statistics, University of São Paulo, São Paulo, Brazil

Henrique Guzzo, Jr. Institute of Mathematics and Statistics, University of São Paulo, São Paulo, Brazil

Irvin Roy Hentzel Iowa State University, Ames, Iowa

Noriaki Kamiya Shimane University, Japan

Bakhadyr Khoussainov University of Auckland, Auckland, New Zealand

Erwin Kleinfeld University of Iowa, Iowa City, Iowa

Margaret Kleinfeld University of Iowa, Iowa City, Iowa

Jesus Laliena University of La Rioja, Logroño, Spain

C. López-Díaz University of Oviedo, Oviedo, Spain

Yu. Lyubich Technion, Haifa, Israel

Shahn Majid University of Cambridge, Cambridge, England

C. Mallol University of La Frontera, Temuco, Chile

C. Martínez University of Oviedo, Oviedo, Spain

J. Martínez Moreno University of Granada, Granada, Spain

Martin Mathieu The Queen's University of Belfast, Belfast, Northern Ireland

F. E. Mendonça da Silveira Institute of Theoretical Physics – UNESP, São Paulo, Brazil

M. L. Merlini Giuliani Federal University of Santa Maria, Santa Maria, Brazil

Artibano Micali University of Montpellier II, Montpellier, France

Hyo Chul Myung Korea Institute for Advanced Study and KAIST, School of Mathematics, Seoul, Korea

Alexander I. Nesterov CUCEI, University of Guadalajara, Guadalajara, Jalisco, Mexico

Susumu Okubo University of Rochester, Rochester, New York

J. Marshall Osborn University of Wisconsin, Madison, Wisconsin

R. Paunov Centro Brasileiro de Pesquisas Fisicas, Rio de Janeiro, Brazil

Luiz Antonio Peresi University of São Paulo, São Paulo, Brazil

J. M. Perez-Izquierdo University of La Rioja, Logroño, Spain,

César Polcino Milies Institute of Mathematics and Statistics, University of São Paulo, São Paulo SP, Brazil

A. V. Razumov Institute for High Energy Physics, Moscow region, Russia

L. V. Sabinin University of Quintana Roo, Chetumal, Quintana Roo, Mexico

Liudmila Sabinina Faculty of Sciences, UAEM., Morelos, Mexico

M. V. Saveliev Institute for High Energy Physics, Moscow region, Russia

L. V. Sbitneva University of Quintana Roo, Chetumal, Quintana Roo, Mexico

J. Setó University of Oviedo, Oviedo, Spain

Ivan P. Shestakov Sobolev Institute of Mathematics, Novosibirsk, Russia

Ivan P. Shestakov University of São Paulo, São Paulo-SP, Brazil

Pavel Shumyatsky University of Brasilia, Brasilia – DF, Brazil

Arkadii Slinko University of Auckland, Auckland, New Zealand

G. M. Sotkov Departamento de Campos e Partículas – CBPF, Rio de Janeiro – RJ, Brazil

R. Varro University of Montpellier III, Montpellier, France

P. Vicente University of León, León, Spain

G. P. Wene The University of Texas at San Antonio, San Antonio, Texas

Richard Wiegandt Mathematical Institute, HAS, Budapest, Hungary

M. Zaicev Moscow State University, Moscow, Russia

Kaiming Zhao Institute of Systems Science, Academia Sinica, Beijing, China

A. H. Zimerman Institute of Theoretical Physics – UNESP, São Paulo–SP, Brazil

New approach to octonions and Cayley algebras

HELENA ALBUQUERQUE[1] Departamento de Matematica-Faculdade de Ciencias e Tecnologia, Universidade de Coimbra, Apartado 3008, 3000 Coimbra, Portugal

SHAHN MAJID[2] Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, UK

ABSTRACT We announce a new approach to the octonions as *quasiassociative* algebras. We strip out the categorical and quasi-quantum group considerations in our longer paper and present here (without proof) some of the more algebraic conclusions.

1 INTRODUCTION

Usually one recognises the nonassociativity of the octonions by saying that they are instead alternative algebras. While this is true, the property of being alternative has a much weaker character than associativity and, as a result, many standard ideas and constructions for associative algebras do not go through in the alternative case. In our paper [1], to which this note is a short introduction, we have introduced a full solution to this problem based on modern ideas from category theory and Drinfeld's theory of quasiquantum groups.

Without going into any details (see [1]), the new formulation is that the octonions and other Cayley algebras live naturally as objects in a monoidal category[2][3]. For any three objects V, W, Z in such a category there is an associator isomorphism $\Phi_{V,W,Z} : (V \otimes W) \otimes Z \to V \otimes (W \otimes Z)$ which performs the rebracketting. Mac Lane's pentagon condition on Φ ensures that *we can do all constructions as if there*

[1] Supported by CMUC-JNICT and by Praxis 2/2.1/Mat7458/94
[2] Royal Society University Research Fellow and Fellow of Pembroke College, Cambridge

are no brackets (i.e. as if $\otimes$ is strictly associative). After writing any desired constructions as the composition of various maps, we simply insert Φ as needed for the compositions to make sense, and all different ways to do this will give the same result (this is Mac Lane's coherence theorem). So working in such a category is no harder than usual associative linear algebra. For example, an algebra A in such a category means

$$\bullet \circ (\bullet \otimes \mathrm{id}) = \bullet \circ (\mathrm{id} \otimes \bullet) \circ \Phi_{A,A,A}$$

for the product $\bullet$, where Φ is inserted for the bracketting to make sense. So, recognizing the octonions as such a *quasiassociative algebra* (or *quasialgebra* for short) makes them as good as associative in the precise sense explained above.

In [1] we introduce and study a class of such quasialgebras that contain composition algebras and more general algebras obtained by a generalised Cayley-Dickson process. All algebras are considered over a field k of characteristic different from 2. The required class of quasialgebras arises naturally by a certain 'Drinfeld twisting'[4][3] or deformation of classical group algebras, as follows.

2 QUASIALGEBRAS $k_F G$

First of all, we know that if we consider the set of complex numbers, the quaternions or the octonions, all these algebras have something in common: If we choose a suitable basis and remove the $\pm$ signs from the multiplication tables of these algebras, we have the tables of the additive groups $G = \mathbb{Z}_2$ (for complex numbers), $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (for quaternions) and $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (for octonions)[5]. We view the signs in the multiplication tables as an invertible 2-cochain $F : G \times G \to k$ (a nowhere-zero function which is 1 when either argument is the group identity $e \in G$). Writing $F(x, y) = (-1)^{f(x,y)}$, one has explicitly[1],

$$G = \mathbb{Z}_2, \quad f(x, y) = xy, \quad x, y \in \mathbb{Z}_2 \qquad \text{(Complex numbers)},$$

$$G = (\mathbb{Z}_2)^2, \quad f(\vec{x}, \vec{y}) = x_1 y_1 + (x_1 + x_2) y_2 \qquad \text{(Quaternions)},$$

where $\vec{x} = (x_1, x_2) \in G$ is a vector notation and the components x_1, x_2 are viewed in the field $\mathbb{Z}_2$.

$$G = (\mathbb{Z}_2)^3, \quad f(\vec{x}, \vec{y}) = \sum_{i \le j} x_i y_j + y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3 \quad \text{(Octonions)},$$

where $\vec{x} = (x_1, x_2, x_3) \in G$ is a vector notation. Similarly for higher Cayley algebras where $G = (\mathbb{Z}_2)^n$.

From the group G and the cochain F we recover the complex, quaternion and octonion algebras as the 'deformation' $k_F G$ of the group algebra of the appropriate G. This is the vector space with basis labeled by G and the product

$$x \bullet y = F(x, y) xy$$

for $x, y \in G$, where xy is the group product in G. So this is a kind of deformation of the usual group algebra of G. In quantum groups we do the deformation by introducing a parameter q such that when q tends to 1 we have the original algebras. Here we do the deformation by introducing a cochain F.

PROPOSITION 1 *[1] Let G be a group and F any invertible 2-cochain. Then $k_F G$ is a G-graded quasialgebra with associator Φ determined by the coboundary ϕ of F. Explicitly, it is quasiassociative in the sense*

$$(x \bullet y) \bullet z = \phi(x, y, z) x \bullet (y \bullet z)$$

for all $x, y, z \in G$, and

$$\phi(x, y, z) = \frac{F(x, y)F(xy, z)}{F(y, z)F(x, yz)}.$$

To explain the setting here, for G a group and $\phi : G \times G \times G \to k$ an invertible group 3-cocycle, the category of G-graded vector spaces becomes monoidal with the associator Φ determined by the 3-cocycle ϕ and the grading. A quasialgebra with this form of Φ is called a *G-graded quasialgebra*. It consists of an algebra A, a G-grading which respects the product and unit (so the degree of $1 \in A$ is $e \in G$, the group identity), and the quasiassociativity law

$$(a \bullet b) \bullet c = \phi(|a|, |b|, |c|) a \bullet (b \bullet c)$$

for all elements $a, b, c \in A$ of degree $|a|, |b|, |c|$. In our case, $k_F G$ is such a G-graded quasialgebra with ϕ built from F and with $|x| = x$ for $x \in G$.

For the complex number and the quaternion algebras, F is closed, i.e. ϕ is trivial and the algebras happen to be strictly associative. This is because f in these cases is quadratic. As soon as we introduce a cubic or higher 'interaction' term in f, as in the case of the octonions, ϕ typically becomes nontrivial and the algebra $k_F G$ nonassociative. In the case of the Octonions it is

$$\phi(\vec{x}, \vec{y}, \vec{z}) = (-1)^{(\vec{x} \times \vec{y}) \cdot \vec{z}}$$

(the vector cross product and vector dot product in the exponent, i.e. the determinant $|\vec{x}\ \vec{y}\ \vec{z}|$). On the other hand, if we simply drop the cubic or higher terms in the above family, we clearly obtain the corresponding Clifford algebra with negative signature[3] (the relations are immediate and the dimensions match) i.e. these are obtained as $k_F G$ with

$$G = (Z_2)^n, \quad f(\vec{x}, \vec{y}) = \sum_{i \leq j} x_i y_j \qquad \text{(Clifford algebras)},$$

as the associative version of the octonion or Cayley algebra, which is another way to see the close relationship between these and Clifford algebras. The positive signature algebras are obtained similarly with $i < j$ in f.

[3] We would like to thank Tony Smith for asking us to clarify this point

Also, we are mainly interested in G Abelian and specialise to this case from now on. For ϕ of the coboundary form, the category of G-graded spaces is symmetric[2], i.e. for any two objects V, W there is a generalised transposition isomorphism $\Psi_{V,W}$: $V \otimes W \to W \otimes V$. A quasialgebra A is *quasicommutative* if $\bullet = \bullet \circ \Psi_{A,A}$. This is the case for all $k_F G$ with Ψ determined by a function $\mathcal{R}(x,y) = F(x,y)/F(y,x)$. Explicitly,

$$x \bullet y = \mathcal{R}(x,y) y \bullet x,$$

for all $x, y \in G$. For complex numbers, quaternions and octonions, etc., the function $\mathcal{R}$ has the simple form

$$\mathcal{R}(x,y) = \begin{cases} 1 & \text{if } x = e \text{ or } y = e \text{ or } x = y \\ -1 & \text{otherwise.} \end{cases}$$

We call this important case *altercommutative.* By contrast, for the Clifford algebras one has

$$\mathcal{R}(\vec{x}, \vec{y}) = (-1)^{\sum_{i \neq j} x_i y_j}$$

which is not in general altercommutative (for $n > 2$).

3 ALGEBRAIC PROPERTIES OF $k_F G$

We have just seen that the functions $\phi, \mathcal{R}$ built from F allow for the categorical setting of the algebras $k_F G$ as quasiassociative and quasicommutative. In particular, the algebra is associative iff $\phi = 1$ and commutative iff $\mathcal{R} = 1$. We now summarise less obvious results expressing the more conventional algebraic properties of $k_F G$ in terms of these functions $\phi, \mathcal{R}$. We refer to [1] for proofs and details.

PROPOSITION 2 *[1] $k_F G$ is an alternative algebra* iff

$$\phi^{-1}(y,x,z) + \mathcal{R}(x,y)\phi^{-1}(x,y,z) = 1 + \mathcal{R}(x,y)$$

$$\phi(x,y,z) + \mathcal{R}(z,y)\phi(x,z,y) = 1 + \mathcal{R}(z,y)$$

for all $x, y, z \in G$. In this case,

$$\phi(x,x,y) = \phi(x,y,y) = \phi(x,y,x) = 1$$

for all $x, y \in G$.

Next we consider involutions. Since we have a special basis of $k_F G$ it is natural to consider involutions diagonal in this basis.

PROPOSITION 3 *[1] $k_F G$ admits an involution which is diagonal in the basis G* iff $\mathcal{R}(x,y) = \frac{s(x)s(y)}{s(xy)}$ *for some 1-cochain $s : G \to k$ (a nowhere-zero function with $s(e) = 1$) obeying $s^2 = 1$. In this case, one has $\mathcal{R}(x,y) = \mathcal{R}(y,x)$ and $\phi(x,y,z) = \phi(z,y,x)^{-1}$ for all $x, y, z \in G$.*

The corresponding involution here is $\sigma(x) = s(x)x$ for all $x \in G$. Let A be a finite dimensional algebra with identity element 1 and let σ be an involution in A. We say that σ is a strong involution if $a + \sigma(a)$, $a \bullet \sigma(a) \in k1$ for all $a \in A$.

PROPOSITION 4 *[1] $k_F G$ admits a diagonal strong involution σ* iff
i) $G \simeq (Z_2)^n$ for some n,
ii) $\sigma(e) = e$, $\sigma(x) = -x$ for all $x \neq e$,
iii) $k_F G$ is altercommutative.

Given $k_F G$ we have a natural function $s(x) = F(x,x)$ and consider now the possibility of defining a strong involution using this. For all statements of simplicity in the following we assume $|G| > 2$.

PROPOSITION 5 *[1] If $\sigma(x) = F(x,x)x$ for all $x \in G$ defines a strong involution, then the algebra $k_F G$ is simple and the following are equivalent*
i) $k_F G$ is an alternative algebra,
ii) $k_F G$ is a composition algebra.

Finally, it is possible to characterize a natural class of $k_F G$ algebras that are composition algebras,

PROPOSITION 6 *[1] Let $k_F G$ admit a strong diagonal involution $\sigma(x) = s(x)x$. Then $q(x) = x \bullet \sigma(x)$ makes $k_F G$ a composition algebra* iff
i) $s(xy)F(x,y)^2 F(xy,xy) = s(x)s(y)F(x,x)F(y,y)$, for all $x, y \in G$.
ii) $F(x,xz)F(y,yz)F(z,z)s(z) + F(x,yz)F(y,xz)F(xyz,xyz)s(xyz) = 0$, for all $x, y \in G$ with $x \neq y$.

An important corollary of the last result is:

COROLLARY 7 *[1] If $G \cong (Z_2)^n$ then the Euclidean norm quadratic function defined by $q(x) = 1$ for all $x \in G$ makes $k_F G$ a composition algebra* iff
i) $F^2(x,y) = 1$ for all $x, y \in G$
ii) $F(x,xz)F(y,yz) + F(x,yz)F(y,xz) = 0$, for all $x, y, z \in G$ with $x \neq y$.

In this case $\sigma(x) = F(x,x)x$ for all $x \in G$ is a strong involution and $k_F G$ is simple and alternative.

4 CAYLEY-DICKSON PROCESS FOR $k_F G$

We have a generalisation of the Cayley-Dickson process as follows. Again, details are in [1].

DEFINITION 8 *Let G be an Abelian group F a 2-cochain on it. For any 1-cochain $s : G \to k$ and $\alpha \neq 0$ we define $\bar{G} = G \times Z_2$ and on it the 2-cochain $\bar{F}$ and 1-cochain $\bar{s}$,*

$$\bar{F}(x,y) = F(x,y), \quad \bar{F}(x,vy) = s(x)F(x,y), \quad \bar{F}(vx,y) = F(y,x),$$

$$\bar{F}(vx,vy)=\alpha s(x)F(y,x),\quad \bar{s}(x)=s(x),\quad \bar{s}(vx)=-1$$

for all $x,y\in G$. *Here* $x\equiv(x,e)$ *and* $vx\equiv(x,\nu)$ *denote elements of* $\bar{G}$, *where* $Z_2=\{e,\nu\}$ *with product* $\nu^2=e$.

We say that $k_{\bar{F}}\bar{G}$ is the *generalised Cayley-Dickson extension* of $k_F G$ associated to s,α. The motivation is that if $\sigma(x)=s(x)x$ is a strong involution, then $k_{\bar{F}}\bar{G}$ is the usual Cayley-Dickson extension of $k_F G$ associated to σ,α. Note that since all unital composition algebras over k are obtained by repeated Cayley-Dickson extension[6], they are all of the form of a quasialgebra $k_F G$ in the last proposition of the preceding section with G a power of Z_2.

The natural application for our generalised Cayley-Dickson process is when $k_F G$ admits a diagonal involution $\sigma(x)=s(x)x$ (but not necessarily strong). In this case we have:

PROPOSITION 9 *[1] The 3-cocycle* $\bar{\phi}$ *of* $k_{\bar{F}}\bar{G}$ *is given by*

$$\bar{\phi}(x,y,z)=\phi(x,y,z),\quad \bar{\phi}(vx,y,z)=\mathcal{R}(y,z)\phi(x,y,z),$$

$$\bar{\phi}(x,vy,z)=\mathcal{R}(y,z)\mathcal{R}(xy,z)\phi(x,y,z),\quad \bar{\phi}(x,y,vz)=\mathcal{R}(x,y)\phi(x,y,z),$$

$$\bar{\phi}(vx,vy,z)=\mathcal{R}(xy,z)\phi(x,y,z),\quad \bar{\phi}(vx,y,vz)=\mathcal{R}(y,z)\mathcal{R}(x,y)\phi(x,y,z),$$

$$\bar{\phi}(x,vy,vz)=\mathcal{R}(x,yz)\phi(x,y,z),\quad \bar{\phi}(vx,vy,vz)=\mathcal{R}(xy,z)\mathcal{R}(x,y)\phi(x,y,z),$$

for $x,y,z\in G$.

Using this calculation, one may show under the same assumptions:

PROPOSITION 10 *[1]*

i) $k_{\bar{F}}\bar{G}$ *is associative* iff $k_F G$ *is associative and commutative.*

ii) If $k_F G$ *has trivial centre then* $k_{\bar{F}}\bar{G}$ *is alternative* iff $k_F G$ *is associative and* $s(x)=-1$ *for all* $x\in G$ *and* $x\neq e$.

iii) If s *defines a strong diagonal involution then* $k_{\bar{F}}\bar{G}$ *is alternative* iff $k_F G$ *is associative.*

This extends to more general $k_F G$ some well-known considerations for octonions and higher Cayley algebras.

5 CONCLUDING REMARKS

We conclude with some remarks about other classes if quasialgebras. In fact, the input data for our $k_F G$ construction is clearly very general. If we denote the elements of the finite group G by $\{x_1=e,x_2,\ldots,x_n\}$ then we can represent the cochain F by an $n\times n$ matrix with entries $F_{ij}=F(x_i,x_j)$. It has 1 in the first row and column and all entries non-zero. Conversely, any such matrix will do for a cochain and yield a quasialgebra. Therefore it is a wide-open question what other groups and cochains might be interesting; here we list just a few natural classes.

First of all, motivated by composition algebras for the Euclidean norm (isomorphic to complex, quaternions or octonions), where the cochain is represented by certain normalized Hadamard matrices, a natural more general class of examples is $k_F G$ where F is a general normalised Hadamard matrix. Some results for (and low-dimensional examples of) this kind of $k_F G$ are given in [1]. Hadamard matrices have even dimension but odd dimensional examples of $k_F G$ can be obtained by taking F with first column and row 1 and the rest an Hadamard matrix.

Another general choice of F, overlapping with the extended Hadamard case, is (for G any group),

$$F(x,y) = \begin{cases} 1 & \text{if } x = e \text{ or } y = e \text{ or } x \neq y \\ -1 & \text{otherwise} \end{cases},$$

where one may show that $k_F G$ is simple, commutative and in general nonassociative.

Other interesting examples of $k_F G$ quasialgebras come from the theory of finite fields, and include the generalisations of octonions based on Galois sequences in [5].

Finally, we would also like to recall that our approach answers such questions as what means a representation of the octonions. Following the method explained in the introduction, a representation of the quasialgebra $k_F G$ means a G-graded vector space V and a degree-preserving action $\triangleright$ obeying

$$(x \bullet y)\triangleright v = \phi(x, y, |v|)x\triangleright(y\triangleright v)$$

for all $v \in V$. This is explained in [1], where it is also shown that a representation is equivalent to an algebra map from $k_F G$ to a certain quasialgebra $\mathrm{End}_\phi(V)$ of *quasimatrices* associated to any V.

REFERENCES

[1] H.Albuquerque and S. Majid. Quasialgebra structure of the octonions. Preprint Damtp/97-138 and math.QA/9802116.

[2] S. Mac Lane. Categories for the Working Mathematician. Springer Verlag, 1974.

[3] S.Majid. Foundations of Quantum Group Theory. Cambridge University Press, 1995.

[4] V.G. Drinfeld. QuasiHopf algebras. Leningrad Math. J. 1:1419-1457, 1990.

[5] G.Dixon. Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics. Kluwer, 1994.

[6] K.A. Zhevlakov, A.M. Slin'ko, I.P. Shestakov, and A.I.Shirshov. Rings That are Nearly Associative. Academic Press, 1982.

Identities of graded alternative algebras

Y. BAHTURIN and M. ZAICEV Department of Algebra,Faculty of Mechanics and Mathematics, Moscow State University, 119899 Moscow, RUSSIA e-mail: bahturin@mech.math.msu.su and zaicev@nw.math.msu.su

Dedicated to Alexei Ivanovich KOSTRIKIN on his 70th birthday

1 INTRODUCTION

In this paper we study identities of alternative algebras over a field.
The main results are Theorems 1, 2, 3. In Theorem 1 we prove that for any constant $a > 0$ and any alternative PI-algebra A the growth of identities of A is asymptotically less $\frac{n!}{a^n}$. The statement of this result is similar to a theorem about Lie algebras due to A. Grishkov [7], but we leave open the question about the possibility of bounding the growth from above by a function of the form b^n for a suitable b. In the class of associative algebras this estimate had been obtained more than 20 years ago (see [10]).
Our second result treats the identities of *graded* alternative algebras. It is well-known [6] that if $A = \oplus_{g\in G} A_g$ is an associative algebra graded by a finite semigroup G and any subalgebra A_e, where e is an idempotent element in G, satisfies a non-trivial identity then also A is a PI-algebra. It is known also that the degree of identity guaranteed for A does not depend on the algebra itself and is completely determined by the degrees of identities satisfied by A_e's and the order of G [2]. Later similar results had been obtained for Lie algebras, special Jordan algebras and some other types of algebras (see [4], [5]). In Theorem 2 we prove that alternative algebras graded also enjoy a similar property (G commutative).

This work was partially supported by Russian Foundation for Fundamental Research Grants # 6-01-00146 and 96-15-96050.

Our third main result, Theorem 3, shows that the matrix algebra over an alternative PI-algebra is a non-associative algebra with a non-trivial identity of a very specific form. This theorem can be viewed as a generalization of the classical result from [8] saying that any matrix algebra over an associative PI-algebra is a PI-algebra itself.

2 DEFINITIONS AND PRELIMINARY RESULTS

We recall some basic notions. Let A be an algebra over a field Φ. It is called *alternative* if the following identities hold:

$$(xx)y = x(xy) \quad , \quad y(xx) = (yx)x \quad . \tag{1}$$

It follows from (1) that a sufficient condition for an algebra to be alternative is the associativity of any 2-generator subalgebra. By a well-known Artin's Theorem (see, for example, [13]) this condition is also necessary

We introduce the following notation: $x \circ y = xy + yx$. The linearization procedure applied to the second of identities (1) after a change of notation yields a relation of the form:

$$a(x \circ y) = (ax)y + (ay)x, \tag{2}$$

which will be repeatedly used in what follows. Following [13] we write also $\langle a_1 a_2 \ldots a_k \rangle$ for the left - normed product of the elements $a_1, \cdots, a_k$ of an alternative algebra. Specifically, $\langle a_1 a_2 \rangle = a_1 a_2$ and $\langle a_1 \cdots a_k \rangle = \langle a_1 \cdots a_{k-1} \rangle a_k$. We will also omit the brackets in the left - normed Jordan product and write $a_1 \circ a_2 \circ \ldots \circ a_k = (a_1 \circ a_2 \circ \ldots \circ a_{k-1}) \circ a_k$.

According to a conventional definition (see [13]) an algebra A is called an alternative *PI-algebra*, if it satisfies an *essential* identity

$$f(x_1, \ldots, x_n) \equiv 0.$$

A non-associative polynomial f from a free alternative algebra is called essential if after forgetting all brackets it becomes a non-trivial associative polynomial.

We need a result about the identities in alternative algebras. If char $F \neq 2$ this easily follows from Lemma 5.14 of [13].

LEMMA 1. *Any alternative PI - algebra satisfies an identity of the form*

$$\langle x_0 x_1 \ldots x_m \rangle \equiv \sum_{1 \neq \sigma \in S_n} \alpha_\sigma \langle x_0 x_{\sigma(1)} \ldots x_{\sigma(m)} \rangle. \tag{3}$$

Proof. Let A be an alternative PI-algebra and $V = \text{var } A$ the variety generated by A. We denote by B the relatively free 2-generator algebra from V. Then B is an the PI-algebra. Since the growth of identities of any associative variety is at most exponential B satisfies an identity of the form

$$f = \sum_{\sigma \in S_n} \beta_\sigma y \circ x_{\sigma(1)} \circ \ldots \circ x_{\sigma(n)} \equiv 0. \tag{4}$$

One can assume that the coefficient of the Jordan monomial $y \circ x_1 \circ \ldots \circ x_n$ in (4) is non-zero. We perform in (4) a substitution $\varphi(y) = x \circ y^{(t_0)}, \varphi(x_1) = x \circ y^{(t_1)}, \ldots, \varphi(x_n) = x \circ y^{(t_n)}$, where $t_0 < t_1 < \ldots < t_n$, and $x \circ y^{(k)}$ is a Jordan monomial $x \circ y \circ \ldots \circ y$ of degree k in y. If we introduce the lexicographic order on the monomials in x, y in the free associative algebra by setting $x > y$ then the leading monomial of the element $(x \circ y^{(t_0)}) \circ (x \circ y^{(t_1)}) \circ \ldots \circ (x \circ y^{(t_n)})$, written associatively, is equal to $a = xy^{t_0}xy^{t_1} \ldots xy^{t_n}$ since $t_0 < t_1, \ldots, t_n$. On the other hand, the leading term of any $(x \circ y^{(t_0)}) \circ (x \circ y^{(t_{\sigma(1)})}) \circ \ldots \circ (x \circ y^{(t_{\sigma(n)})})$ is strictly less than a, as soon as $\sigma \neq 1$. This means, that $\varphi(f) \equiv 0$ is a non-trivial identity in B, while $\varphi(f)$ is a Jordan polynomial in two variables. Hence, in A we have an identity $\varphi(f) \equiv 0$, and the also $z\varphi(f) \equiv 0$. After a linearization of this identity by x, y in the free alternative algebra and making a change of variables, considering (2) we obtain a relation

$$\sum_{\sigma \in S_m} \alpha_\sigma \langle x_0 x_{\sigma(1)} \ldots x_{\sigma(m)} \rangle \equiv 0. \tag{5}$$

The non-triviality of the coefficients in (5) follows since the associative polynomial

$$\sum_{\sigma \in S_m} \alpha_\sigma x_{\sigma(1)} \ldots x_{\sigma(m)}$$

is a linearization in the free associative algebra of a Jordan polynomial $\varphi(f)$. Since not all coefficients in (5) are zero, it easily implies (3), and Lemma 1 has been proven. □

3 MAIN RESULTS

We denote by $\mathrm{Alt}(Y)$ the free alternative algebra with the free generating set $Y = \{y_1, y_2, \ldots\}$. As before, let A be an alternative PI-algebra, I the ideal of identities of A in $\mathrm{Alt}(Y)$. We consider in $\mathrm{Alt}(Y)$ the subspace P_n of multilinear elements in $y_1, \ldots, y_n$.

THEOREM 1. *Let A satisfy an identity (3) of degree m. Then for any $c > 0$ there exists n, depending only on m and c, such that*

$$\dim \frac{P_n}{P_n \cap I} < \frac{n!}{c^n}.$$

Proof. We first consider the subspace $Q_n \subset P_{n+1}$ spanned by all left - normed monomials $\langle y_0 y_{i_1} \ldots y_{i_n} \rangle$. A monomial $\langle y_0 y_{i_1} \ldots y_{i_n} \rangle$ from Q_n is called *k-decomposable*, if there exist indices $1 \leq j_1 < \ldots < j_k \leq n$, such that $i_{j_s} > i_{1+j_s} > i_{2+j_s} > \ldots > i_{j_k}$ for all $s = 1, \ldots, k$. In other words, in our monomial we can find k variables, each of which greater than all the variables between it and the last one, and also greater then the last, k-th variable. All the rest monomials in Q_n we will call *k-indecomposable*. For the proof of Theorem 1 we need the following.

LEMMA 2. *Let A satisfy an identity of the form (3). Then modulo I the subspace Q_n coincides with the linear span V of m-indecomposable monomials.*

Proof. We order the variables $y_0, y_1, \dots, y_n$ by setting $y_0 < y_1 < \dots < y_n$ and extend this order to the lexicographic order on the monomials from Q_n, comparing the letters from the left to the right. Then there exists a counterexample which is minimal in the sense of this order, that is, an m-decomposable left-normed monomial a which is not in $I+V$. Let $a = \langle y_0 y_{i_1} \dots y_{i_n}\rangle$ and $j_1 < \dots < j_m$ be a set of indices determining an m-decomposition on a. We introduce the following notation. If $j_1 > 1$ then we set $a_0 = \langle y_0 y_{i_1} \dots y_l\rangle$ where $l = i_{j_1 - 1}$. If $j_1 = 1$ then $a_0 = y_0$. Further, we denote by $b_r, 1 \le r \le m-1$ the left-normed Jordan product of the variables y_s with the indices running from i_{j_r} to $i_{j_{r+1}-1}$. For $r = m$ we set $b_m = y_l$, $l = i_{j_m}$. By a_{m+1} we will for the convenience denote the multiplication from the right by the remaining variables y_{i_s} with the indices $s = j_m + 1, \dots, n$, if $j_m < n$. From (2) it follows that for any $c, z_1, \dots, z_r$ from $\mathrm{Alt}(Y)$ we have an equation

$$c\langle z_1 \circ z_2 \circ \dots \circ z_r\rangle - \langle c z_1 \dots z_r\rangle = \sum_{1 \ne \sigma \in S_n} \langle c z_{\sigma(1)} \dots z_{\sigma(r)}\rangle. \quad (6)$$

By the minimality of the counterexample selected, it follows from (6) that we have

$$a \equiv \langle a_0 b_1 \dots b_m a_{m+1}\rangle (\mathrm{mod}(I+V)).$$

Now (3) implies:

$$a \equiv \sum_{1 \ne \sigma \in S_n} \alpha_\sigma \langle a_0 b_{\sigma(1)} \dots b_{\sigma(m)} a_{m+1}\rangle (\mathrm{mod}(I+V)). \quad (7)$$

If we apply (2) to each summand on the right hand side of (7) we will obtain an expression for a modulo $I+V$ in the form of a linear combination of left-normed products $\langle y_0 y_{k_1} \dots y_{k_n}\rangle$, each of which is lexicographically less than a and thus belongs to $I+V$. The contradiction thus obtained completes the proof of Lemma 2. □

We continue the proof of Theorem 1. By a Theorem of A. I. Shirshov (see [13]) P_n is the linear span of the products $\langle y_1 \dots y_k\rangle$, where each y_i in its turn is a left-normed product of some of $x_1, \dots, x_n$, $y_i = x_{\alpha_1} \dots x_{\alpha_s}$, with $\alpha_1 < \dots < \alpha_s$. From here it follows, that if we denote by $S(n,k)$ the Stirling number of the second kind, that is, the number of decompositions of an n-element set into the disjoint union of k subsets then

$$\dim \frac{P_n}{P_n \cap I} \le \sum_{k=0}^{n} S(n,k) n \dim \frac{Q_{n-1}}{Q_{n-1} \cap I}. \quad (8)$$

Let $q_n = \dim \frac{Q_{n-1}}{Q_{n-1} \cap I}$. From Lemma 2 it follows that q_n does not exceed the number of m-indecomposable monomials in Q_{n-1}.It has been shown in [3] (see also [1]) that for any $a > 0$ there exists n_0, depending only on m and a, such that

$$q_n < \frac{n!}{a^n}$$

for all $n > n_0$. In order to obtain the upper bound for the sum on the right side of (8), we first consider the sum

$$\sum_{k=0}^{n} S(n,k)\frac{k!}{a^k}, \tag{9}$$

in which $a = 6c$, and c is the constant from the proof of our Theorem. It is well-known (see [12], Ch.1), that for any natural $t \geq n$ we have

$$t^n = \sum_{k=0}^{n} S(n,k)(t)_k,$$

in which $(t)_k = t(t-1)(t-2)\dots(t-k+1)$. From here one can obtain an estimate for the value (9):

$$\sum_{k=0}^{n} S(n,k)\frac{k!}{a^k} < \frac{1}{a^n}\sum_{k=0}^{n} S(n,k)k! < \frac{1}{a^n}\sum_{k=0}^{n} S(n,k)(n)_k = \frac{n^n}{a^n}. \tag{10}$$

From Stirling's formula for $n!$ it is easy to derive $n^n < 3^n n!$ hence, from (10), it follows inequality

$$\sum_{k=0}^{n} S(n,k)\frac{k!}{a^k} < \frac{n!}{b^n}, \tag{11}$$

where $b = \frac{a}{3} = 2c$.

We now split the sum on the right side of (8) into two components

$$A = \sum_{k=0}^{n_0} S(n,k)nq_k; \quad B = \sum_{k=n_0+1}^{n} S(n,k)nq_k.$$

Then, as it follows from (11), $B < n\frac{n!}{b^n}$. In order to estimate A we notice, that the values $S(n,k)$ satisfy the inequalities: $S(n,2) \leq 2^n, S(n,k) \leq 2^n S(n,k-1)$. It follows then that $S(n,k) \leq t^n$, where $t = 2^{n_0}$, for all $k \leq n_0$. Considering an obvious inequality $q_k \leq (k-1)!$, we obtain

$$A \leq n(n_0)!n_0t^n = Dnt^n, \tag{12}$$

where D and t are some constants depending only on c and m. Now from (8), (11) and (12) we have the bound

$$\dim\frac{P_n}{P_n \cap I} \leq Dnt^n + B < Dnt^n + n\frac{n!}{b^n} = Dnt^n + n\frac{n!}{2^n c^n}.$$

For given c, D and t one can select $n_1 = n_1(D,t,c)$, such that $Dnt^n < \frac{1}{2}\frac{n!}{c^n}$ for all $n \geq n_1$. Since $\frac{n}{2^n} \leq \frac{1}{2}$ for all natural n,

$$\dim\frac{P_n}{P_n \cap I} < \frac{1}{2}\frac{n!}{c^n} + \frac{1}{2}\frac{n!}{c^n} = \frac{n!}{c^n},$$

and Theorem 1 has been proven. □

Let now G be a commutative semigroup and

$$A = \sum_{g \in G} A_g$$

a G-graded alternative algebra, that is, $A_g A_h \subset A_{gh}$. We will say that A is an algebra with a finite grading, if $A_g \neq 0$ only for a finite number of elements $g \in G$.

THEOREM 2. *Let $A = \sum_{g \in G} A_g$ be a alternative algebra over a field Φ with a finite grading. If for any idempotent $e \in G$ the subalgebra A_e is a PI-algebra then also A is an alternative PI-algebra. Moreover, the degree of the nontrivial identity in A is bounded from above by a function only on the degrees of non-trivial identities in A_e and on the number of elements $g \in G$, for which $A_g \neq 0$.*

Proof. We introduce some necessary notation. Let $H = \{g_1, \dots, g_t\}$ be a subset of all elements from G, for which $A_g \neq 0$. We consider a G-graded set of the variables $X = \cup_{g \in G} X_g$, where $X_g = \{x_1^g, x_2^g, \dots\}$, and generate by this the set a free alternative algebra $\mathrm{Alt}(X)$. By $\overline{\mathrm{Alt}}(Y)$ we denote the alternative subalgebra in $\mathrm{Alt}(X)$ generated by the elements $\{y_1, y_2, \dots\}$, where $y_i = x_i^{g_1} + \dots + x_i^{g_t}$. It is obvious that $\overline{\mathrm{Alt}}(Y)$ is a free alternative algebra and $\mathrm{Alt}(X)$ is endowed by a natural G-grading,

$$\mathrm{Alt}(X) = \bigoplus_{g \in G} Alt_g(X).$$

We denote by Ψ the set all homomorphisms $\psi : \overline{\mathrm{Alt}}(Y) \to A$, and by Ψ_0 the set all homomorphisms $\psi_0 : \mathrm{Alt}(X) \to A$, for which $\psi_0(x_i^g) \in A_g$. We set

$$I = \bigcap_{\psi \in \Psi} \mathrm{Ker}\ \psi \ \triangleleft\ \overline{\mathrm{Alt}}(Y), \quad J = \bigcap_{\psi_0 \in \Psi_0} \mathrm{Ker}\ \psi_0 \ \triangleleft\ \mathrm{Alt}(X).$$

The elements from I (resp., J) are the *non-graded* (resp., *graded*) identities of algebra A.

Let e be an idempotent of G. By a hypothesis of Theorem A_e is PI-algebra. Repeating verbatim the argument of Lemma 1, one can observe, that for any $g \in G$ and $x_0 \in A_g$ the elements $x_1, \dots, x_m \in A_e$ satisfy a left-normed identity of the form (3), and, moreover, from the proof of Lemma it follows, that the degree m of this identity depends only on the degree of the initial identity of A_e. Since A is a algebra with finite grading one can assume that for all e and g there exists m such that we have a graded identity

$$\langle x_0^g x_1^e \dots x_m^e \rangle \equiv \sum_{1 \neq \sigma \in S_n} \alpha_\sigma^{e,g} \langle x_0^g x_{\sigma(1)}^e \cdots x_{\sigma(m)}^e \rangle,$$

that is, we have an inclusion

$$\langle x_0^g x_1^e \dots x_m^e \rangle - \sum_{1 \neq \sigma \in S_n} \alpha_\sigma^{e,g} \langle x_0^g x_{\sigma(1)}^e \cdots x_{\sigma(m)}^e \rangle \in J. \tag{13}$$

We denote now by P_n subspace in $\overline{\text{Alt}}(Y)$ of multilinear polynomials in $y_1, \dots, y_n$, and by $P_{n,q}$ a subspace of multilinear polynomials in $x_1^{q_1}, \dots, x_n^{q_n}$ in $\text{Alt}(X)$, where $q = (q_1, \dots q_n) \in G^n$. In order to prove the theorem it is sufficient for some n, bounded by a function of m and t, to obtain the inequality

$$\dim \frac{P_n}{P_n \cap I} < n!\,, \tag{14}$$

since if (14) holds the ideal I cannot be zero. .

We show first, that $P_n \cap I = P_n \cap J$ in $\text{Alt}(X)$. Let $f = f(y_1, \dots, y_n) \in I$, ψ_0 is a homomorphism $\text{Alt}(X) \to A$. We consider a homomorphism $\psi : \overline{\text{Alt}}(Y) \to A$, for which $\psi(y_i) = \psi(x_i^{g_1} + \ldots + x_i^{g_t}) = \psi_0(x_i^{g_1}) + \ldots + \psi_0(x_i^{g_t}) = \psi_0(y_i)$. As $f \in I$ we have $\psi_0(f) = \psi(f) = 0$, so that $f \in \text{Ker}\ \psi_0$. Hence, we have proved the inclusion $I \subset J$. Let now $f = f(y_1, \dots, y_n)$ be a multilinear polynomial in $y_1, \dots, y_n$, an element in J, and ψ is an arbitrary homomorphism from $\overline{\text{Alt}}(Y)$ into A. Then $\psi(y_i) = b_i^{g_1} + \ldots + b_i^{g_t}$, where $b_i^{g_j} \in A_{g_j}$

$$\psi(f) = \sum_{\alpha, \beta, \dots, \gamma} f(b_1^{g_\alpha}, b_2^{g_\beta}, \dots, b_n^{g_\gamma}).$$

But all summands on the right side equal zero hence $f \in I$ and then $P_n \cap I = P_n \cap J$ has been verified.

Now, considering the inclusion $P_n \subset \bigoplus_{q \in G^n} P_{n,q}$, we obtain the inequality

$$\dim \frac{P_n}{P_n \cap I} = \dim \frac{P_n}{P_n \cap J} = \dim \frac{P_n + J}{J} \leq \sum_{q \in G^n} \dim \frac{P_{n,q} + J}{J} =$$

$$\sum_{q \in G^n} \dim \frac{P_{n,q}}{P_{n,q} \cap J} = \sum_{q \in H^n} \dim \frac{P_{n,q}}{P_{n,q} \cap J}. \tag{15}$$

The inequality (14) will follow from (15) if we prove, that for some n, depending only on m and t, for any $q \in G^n$ we have

$$\dim \frac{P_{n,q}}{P_{n,q} \cap J} < \frac{n!}{t^n}, \tag{16}$$

since the number of the summands on the right side (15) is equal to $|H| = t^n$.

The proof of (16) is similar to the proof of Theorem 1. Change n by $n + 1$, fix a tuple $q = (q_1, \dots, q_{n+1})$ and write, for convenience,

$$x_1^{q_1} = x_0, \dots, x_{n+1}^{q_{n+1}} = x_n.$$

We introduce on $x_0, \dots, x_n$ an order by setting $x_0 < x_1 < \ldots < x_n$. As before, we denote by $Q_{n,q}$ subspace in $P_{n,q}$, spanned by the monomials $\langle x_0 x_{i_1} \dots x_{i_n} \rangle$ and consider k-indecomposable monomials, where the number k will be chosen later. We prove now an analogue of Lemma 2 for the graded case.

LEMMA 3. *Modulo J the subspace $Q_{n,q}$ coincides with the linear span V of all k - indecomposable monomials, where $k = \frac{t^N-1}{t-1}$, and $N = t(m-1)+1$.*

Proof. As in Lemma 2, consider the lexicographically minimal counterexample

$$a = \langle x_0 x_{i_1} \dots x_{i_n} \rangle,$$

that exists if Lemma is wrong. Let $j_1 < \dots < j_k$ be a set of indices defining a k-decomposition for a. We set $a_0 = \langle x_0 x_{i_1} \dots x_l \rangle$, where $l = i_{j_1 - 1}$, We denote by $b_r, 1 \le r \le k-1$ a left-normed Jordan product of the variables x_s where $s = i_{j_r}, \dots, i_{j_{r+1}-1}$. If $r = k$ we will assume $b_k = x_{i_s}$ where $s = j_k$. As before, for brevity by a_{k+1} we denote the right multiplication by all remaining variables x_{i_s}. As in Lemma 2,

$$a \equiv \langle a_0 b_1 \dots b_k a_{k+1} \rangle (\mathrm{mod}(J+V)).$$

By construction, the elements $b_1, \dots, b_k$ are homogeneous relative the G - grading. Let $B_i \in Alt_{p_i}(X), i = 1, \dots, k$. We consider the product $p_1 p_2 \dots p_k$ in G. By Lemma 1 from [5] for k selected by us there exist $1 \le t_0 < \dots < t_m \le k$ such that

$$p_{t_0} p_{t_0+1} \cdots p_{t_1-1} = p_{t_1} p_{t_1+1} \cdots p_{t_2-1} = \dots = p_{t_{m-1}} p_{t_{m-1}+1} \cdots p_{t_m} = e$$

$e^2 = e$ in G. Now we set $c_0 = \langle a_0 b_1 \dots b_{t_0-1} \rangle, c_1 = b_{t_0} \circ b_{t_0+1} \circ \dots \circ b_{t_1-1}, \dots, c_m = b_{t_{m-1}} \circ b_{t_{m-1}+1} \circ \dots \circ b_{t_m}$, and by c_{m+1} we denote the right multiplication by all remaining b_i and by the operator a_{k+1}. As before, by (2), the lexicographically leading term of $\langle c_0 c_1 \dots c_{m+1} \rangle$ is equal to the leading term of $\langle a_0 b_1 \dots b_k a_{k+1} \rangle$ and leading term of a hence

$$a \equiv \langle c_0 c_1 \dots c_{m+1} \rangle (\mathrm{mod}(J+V))$$

by the minimality of the counterexample a. But according to (13)

$$\langle c_0 c_1 \dots c_{m+1} \rangle \equiv \sum_{1 \ne \sigma \in S_n} \beta_\sigma \langle c_0 c_{\sigma(1)} \dots c_{\sigma(m)} c_{m+1} \rangle (\mathrm{mod}(J+V)).$$

The leading term of any product $\langle c_0 c_{\sigma(1)} \dots c_{\sigma(m)} c_{m+1} \rangle$ strictly less than a as soon as $\sigma \ne 1$, from so that we have $a \in J + V$, contradicting the choice of a. Lemma 3 has been proven. □

The remaining portion of the proof of Theorem 2 completely repeats the proof of Theorem 1 with a reference to Lemma 3 in place of Lemma 2. □

4 SOME COROLLARIES

Now we list a number of consequences from the results obtained.

COROLLARY 1. *The growth of identities of any alternative PI - algebra is asymptotically less than $\frac{n!}{a^n}$ for of any constant a.*

This result follows from of Theorem 1 and of Lemma 1, since by definition the growth of identities of an algebra A is the growth of the sequence

$$c_n(A) = \dim \frac{P_n}{P_n \cap I},$$

where I is the ideal of identities of A in the free alternative algebra.

COROLLARY 2. *Let G be a finite solvable group of the automorphisms of an alternative algebra A over an arbitrary field such that* $\operatorname{char} F$ *is not a divisor of the order $|G|$. If the subalgebra of G-fixed points A^G is a PI-algebra with a non-trivial identity of degree d then A itself satisfies a non-trivial identity whose degree depends only on d and the order $|G|$.*

We omit the proof of this corollary since it is standard in all similar situations for various classes of algebras (see., for example, [4]). It should be remarked that A.P.Semenov [11] proved a similar result, with G arbitrary finite but without the connection between the degrees of identities in A^G and A.
The techniques developed above enable us also to consider the question about the existence of identities in matrices over alternative algebras. We prove first a more general result, of independent interest.

LEMMA 4. *Let A be a non-associative algebra, the growth of identities of which is asymptotically less than $\frac{n!}{a^n}$ for of any constants $a \rangle 0$. Let also B be a finite - dimensional algebra. Then the tensor product $A \otimes B$ satisfies a non-trivial identity of the form*

$$\sum_{\sigma \in S_n} \alpha_\sigma \langle z_{\sigma(1)} \dots z_{\sigma(n)} \rangle \equiv 0, \tag{17}$$

where $\langle z_1 \dots z_n \rangle$ stands for the left-normed product of the elements $z_1, \dots, z_n$ of a non-associative algebra.

Proof. Let $e_1, \dots, e_m$ be a basis of the algebra B, and $\mathcal{A}(X)$ a relatively free algebra of countable rank in the variety var A, where $X = \{x_i^j; i, j = 1, 2, \dots\}$. We notice first that if on the elements $x_1^1 \otimes e_1, \dots, x_1^m \otimes e_m, \dots, x_n^1 \otimes e_1, \dots, x_n^m \otimes e_m$ of the tensor product $\mathcal{A}(X) \otimes B$ a relation

$$g(x_1^1 \otimes e_1, \dots, x_1^m \otimes e_m, \dots, x_n^1 \otimes e_1, \dots, x_n^m \otimes e_m) = 0, \tag{18}$$

holds then also for any $a_i^j \in A$ we have a similar relation

$$g(a_1^1 \otimes e_1, \dots, a_1^m \otimes e_m, \dots, a_n^1 \otimes e_1, \dots, a_n^m \otimes e_m) = 0. \tag{19}$$

The reason is that for any algebras P, Q, R, S and any homomorphisms of algebras $\varphi : P \to R$, $\psi : Q \to S$ the natural linear map $\varphi \otimes \psi : P \otimes Q \to R \otimes S$ defined by

$(\varphi \otimes \psi)(p \otimes q) = \varphi(p) \otimes \psi(q)$, is a homomorphism of algebras :

$$(\varphi \otimes \psi)((p \otimes q)(p' \otimes q')) = (\varphi \otimes \psi)(pp' \otimes qq')) = \\ \varphi(pp') \otimes \psi(qq') = \varphi(p)\varphi(p') \otimes \psi(q)\psi(q') = \\ (\varphi(p) \otimes \psi(q))(\varphi(p') \otimes \psi(q')) = ((\varphi \otimes \psi)(p \otimes q))((\varphi \otimes \psi)(p' \otimes q')).$$

Since $\mathcal{A}(X)$ is a free algebra of the variety $\operatorname{var} A$, the map $x_j^i \mapsto a_j^i$ extends to a homomorphism $\varphi : \mathcal{A}(X) \to A$. Taking ψ equal to the identity map $B \to B$ we obtain a homomorphism of algebras $(\varphi \otimes \psi) : \mathcal{A}(X) \otimes B \to A \otimes B$.
We consider now in $\mathcal{A}(X)$ the elements

$$z_i = \sum_{j=1}^{m} x_i^j \otimes e_j, \quad j = 1, 2, \ldots .$$

If

$$f(z_1, \ldots, z_n) = \\ f(x_1^1 \otimes e_1 + \ldots + x_1^m \otimes e_m, \ldots, x_n^1 \otimes e_1 + \ldots + x_n^m \otimes e_m) \\ = 0,$$

then, as noticed above,

$$f(a_1^1 \otimes e_1 + \ldots + a_1^m \otimes e_m, \ldots, a_n^1 \otimes e_1 + \ldots + a_n^m \otimes e_m) = 0$$

for any $a_1^1, \ldots, a_n^m \in A$. Since any element from $A \otimes B$ is a sum of the form $c_1 \otimes e_1 + \cdots + c_m \otimes e_m$ then f is identically equal to zero on $A \otimes B$. In other words, it is sufficient to prove, that a relation of the form (17) holds for the elements selected by us: $z_1, z_2, \ldots$.
We consider a multilinear non-associative polynomial $f = f(y_1, \ldots, y_n)$. If f is a monomial then its value

$$f(x_1^1 \otimes e_1 + \ldots + x_1^m \otimes e_m, \ldots, x_n^1 \otimes e_1 + \ldots + x_n^m \otimes e_m)$$

can be expressed linearly through

$$f(x_1^{i_1}, \ldots, x_n^{i_n}) \otimes e_k$$

where $1 \leq i_1, \ldots, i_n \leq m$. Hence the value of any multilinear polynomial f in $z_1, \ldots, z_n$ can be linearly expressed through the tensors of the form

$$p(x_1^{i_1}, \ldots, x_n^{i_n}) \otimes e_k, \quad 1 \leq k, i_1, \ldots, i_n \leq m,$$

where $p(x_1^{i_1}, \ldots, x_n^{i_n})$ is a monomial in $x_1^{i_1}, \ldots, x_n^{i_n}$. We denote by $P_{n,q}$ the linear span of all monomials of $\mathcal{A}(X)$ in $x_1^{i_1}, \ldots, x_n^{i_n}$, if $q = (i_1, \ldots, i_n)$. By the hypothesis of Lemma, starting with some n the following inequality holds

$$\dim P_{n,q} < \frac{n!}{(2m)^n}.$$

But then

$$\dim \sum_{k=1}^{n} \sum_{q} P_{n,q} \otimes e_k \leq n! m^n \dim P_n < \frac{(n+1)!}{2^n} < n!.$$

This means that the left - normed products $\langle z_{\sigma(1)} \ldots z_{\sigma(n)} \rangle$ are linearly dependent, that is, satisfy the relation (17), and Lemma 4 has been proven.
An immediate consequence of Theorem 1 and Lemma 4 is the following.

THEOREM 3. *Let A be a alternative PI-algebra. Then the matrix algebra $M_t(A)$ satisfies an identity of the form*

$$\sum_{\sigma \in S_n} \alpha_\sigma \langle x_{\sigma(1)} \ldots x_{\sigma(n)} \rangle \equiv 0,$$

where $\langle x_1 \ldots x_n \rangle$ is a left - normed non - associative monomial in $x_1, \ldots, x_n$.

REFERENCES

[1] Bahturin, Yuri. Identities of algebras with actions of Hopf algebras. Lecture Notes in Pure Appl. Math. 198 (1998), 1 - 36.

[2] Bahturin, Yu.; Giambruno, A.; Riley, D., Group - graded algebras with polynomial identity, Israel J. Math., 104 (1998), 145 - 156.

[3] Bahturin, Yu.; Giambruno, A.; Zaicev, M., G - identities of associative algebras, Proc. Amer. Math. Soc., 127 (1998), pp. 63-69.

[4] Bahturin, Yu.; Zaicev, M., Identities of graded algebras, J. Algebra, 205 (1998), 1 - 12

[5] Bahturin, Yu.; Zaicev, M., Identities of graded special Jordan algebras, Vestnik MSU. Math., Mech., 1997, No. 6, 18 - 20..

[6] Bergen G., Cohen M., Actions of commutative Hopf algebras. Bull. London Math. Soc., 18(1986), 159 - 164.

[7] Grishkov, A.N., On the growth of varieties of Lie algebras , Mat. Zametki, 4 (1988), 51 - 54.

[8] Procesi, Claudio; Small, Lance, Endomorphism rings of modules over PI - algebras, Math. Z., 106 (1968), 178 - 180.

[9] Razmyslov, Yu. P., Identities of algebras and their representations, Nauka, Moscow, 1985.

[10] Regev A. Existence of identities in $A \otimes B$, Israel J. Math., 11(1972), 131-152.

[11] Semenov A.P., Subrings of invariants of a finite group of automorphisms of a Jordan ring, Siberian Math. J., 32 (1991), 203 - 211.

[12] Stanley, R.P., Enumerative Combinatorics, v.1, Wadsworth & Brooks/Cole, Adv. Books & Software, Monterey, CA, 1986.

[13] Zhevlakov, K.A.; Shestakov, I.P.; Shirshov, A.I.; Slin'ko, A.M., Rings that are nearly associative, Nauka, Moscow., 1978.

Invariant connections on symmetric spaces

PILAR BENITO* and CRISTINA DRAPER** Departamento de Matemáticas y Computación, Universidad de La Rioja, 26004 Logroño, Spain.

ALBERTO ELDUQUE*** Departamento de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain.

ABSTRACT: A classical problem in Differential Geometry, the determination of the invariant affine connections in the simply connected irreducible symmetric spaces, is equivalent to the algebraic problem of computing the set $\mathrm{Hom}_S(T \otimes_{\mathbb{R}} T, T)$ for any $\mathbb{Z}_2$-graded simple Lie algebra $L = S \oplus T$. The algebraic problem is solved using known information about the Lie triple system structure on T, because the simple $\mathbb{Z}_2$-graded Lie algebra $L = S \oplus T$ is just the embedding for the simple Lie triple system T. It turns out that the set of homomorphisms contains non trivial elements if and only if T is related to a simple Jordan algebra. Now it is possible to come back to the geometric context to describe the affine connections and express the holonomy and torsion and curvature tensors in algebraic terms.

1. INTRODUCTION.

In another paper of these proceedings [1], it is explained how the geometrical problem of describing the invariant affine connections on a reductive homogeneous space $M = G/H$ is equivalent to the algebraic problem of describing the multiplications $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ with $(\mathrm{Ad}\, h)\alpha(x, y) = \alpha\big((\mathrm{Ad}\, h)(x), (\mathrm{Ad}\, h)(y)\big)$ for any $h \in H$ and any $x, y \in \mathfrak{m}$, where $\mathfrak{m}$ is a vector subspace of the Lie algebra $\mathfrak{g}$ of G such that

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$$

(where $\mathfrak{h}$ is the Lie subalgebra of $\mathfrak{g}$ which corresponds to the closed subgroup H

* Supported by the Spanish DGICYT (Pb 94-1311-C03-02) and by the Universidad de La Rioja (API-98/B15).

** Supported by a grant from the "Programa de Formación de Profesorado Universitario" (MEC), by the Spanish DGICYT (Pb 94-1311-C03-02) and by the Universidad de La Rioja (API-98/B15).

*** Supported by the Spanish DGICYT (Pb 94-1311-C03-03) and by the Universidad de La Rioja (API-98/B15).

of G) and $AdH(\mathfrak{m}) \subseteq \mathfrak{m}$, a condition equivalent to $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ if H is connected. Such decomposition is called a *reductive decomposition*. This interplay between Geometry and Algebra was proved by Nomizu in [2].

The aim of this paper is to study the connection algebras $(\mathfrak{m}, \alpha)$ that appear in the irreducible symmetric spaces, a special type of reductive homogeneous spaces.

If M is a manifold and ∇ an affine connection, (M, ∇) is a *symmetric space* if for each point $p \in M$ there exists a central symmetry S_p with center at p. Any symmetric space is isomorphic to a space of the form G/H, where the closed subgroup $H \subseteq G$ is such that ${G_0}^\sigma \subseteq H \subseteq G^\sigma$, with σ an involutive automorphism of the Lie group G, G^σ the subgroup of fixed points under σ and ${G_0}^\sigma$ the connected component of the identity in G^σ.

For any symmetric space G/H, there exists a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ satisfying $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$, because $d\sigma$ provides a $\mathbb{Z}_2$-gradation ($\mathfrak{h}$ and $\mathfrak{g}$, as before, are the Lie algebras of H and G respectively). This kind of decomposition is called *a symmetric decomposition*. The converse is true.

For the simply connected irreducible symmetric spaces $M = G/H$, the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is graded-simple. Therefore the geometric problem of computing invariant affine connections in a simply connected irreducible symmetric space $M = G/H$ is equivalent, after applying Nomizu's result, to the algebraic problem of computing the set $\mathrm{Hom}_{\mathfrak{h}}(\mathfrak{m} \otimes_{\mathbb{R}} \mathfrak{m}, \mathfrak{m})$, where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a graded-simple $\mathbb{Z}_2$-graded Lie algebra, because each homomorphism of $\mathfrak{h}$-modules $\tilde{\alpha} \in \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{m} \otimes_{\mathbb{R}} \mathfrak{m}, \mathfrak{m})$ determines an $\mathbb{R}$-bilinear multiplication $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ such that $\mathrm{ad}\, \mathfrak{h}|_{\mathfrak{m}} \subseteq Der(\mathfrak{m}, \alpha)$ and, since H is connected, this is equivalent to $(\mathrm{Ad}\, h)\alpha(x, y) = \alpha\big((\mathrm{Ad}\, h)(x), (\mathrm{Ad}\, h)(y)\big)$ for any $h \in H$ and any $x, y \in \mathfrak{m}$; that is, it determines a connection algebra and so an invariant affine connection.

In the next section, we will present briefly the solution of the general problem of determining the set $\mathrm{Hom}_S(T \otimes_F T, T)$ for any simple finite-dimensional $\mathbb{Z}_2$-graded Lie algebra $L = S \oplus T$ over an arbitrary field F of characteristic zero. In particular, in the real case we will have obtained an algebraic description of all invariant affine connections in irreducible symmetric spaces (this was done previously in the Riemannian case, using different methods, by Laquer [3,4]). This description will be used in section 3 to determine the holonomy algebra and give explicit formulas for the torsion and curvature tensors associated to each connection.

2. THE SOLUTION OF THE ALGEBRAIC PROBLEM.

The determination of the set $\mathrm{Hom}_S(T \otimes_F T, T)$ for any $\mathbb{Z}_2$-graded simple Lie algebra $L = S \oplus T$ has been done in [5], the purpose of this section is to present the main result in [5, Theorem 4.3] explaining briefly how the structure of a Lie triple system can be effectively used to compute those sets without decomposing $T \otimes_F T$.

From now on all the algebras and systems considered will be assumed to be finite-dimensional over a field F of characteristic 0.

2.1. Basic definitions.

A Lie triple system (L.t.s) is a vector space T endowed with a trilinear product $[x, y, z]$ satisfying:

i) $[x, x, y] = 0$,

ii) $[x, y, z] + [y, z, x] + [z, x, y] = 0$,

iii) $[x, y, -] \in \mathrm{Der}\, T$,

for any $x, y, z \in T$. The set $S = \mathrm{span}\,\langle [x, y, -] \mid x, y \in T\rangle$ is a Lie subalgebra of linear mappings on T and the $\mathbb{Z}_2$-graded Lie algebra $L(T) = S \oplus T$, whose bracket operation is given by:

$$[s_1, s_2] := s_1 s_2 \in S \text{ (the multiplication in } S)$$
$$[s, t] := s(t) \in T \text{ (the natural action)}$$
$$[t_1, t_2] := [t_1, t_2, -] \in S$$

for any $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$, is called the *standard embedding* of T. Note that S and T are the even and odd part of the embedding and T is a faithful S-module. Thus any Lie triple system is nothing else but the odd part of a $\mathbb{Z}_2$-graded Lie algebra with product $[x, y, z] = [[x, y], z]$.

Among the Lie triple systems we are interested in the so called simple ones (without nontrivial ideals). They are just the odd part of simple $\mathbb{Z}_2$-graded Lie algebras (see [6]). Consequently the problem now is to compute the set $\mathrm{Hom}_S(T \otimes_F T, T)$ for any simple Lie triple system T with standard embedding $L(T) = S \oplus T$.

2.2. Examples.

It is not difficult to get elementary examples of simple L.t.s. We point out three of them which are specially relevant for our purposes. First we note that if T is a (simple) L.t.s. and $\mu \in F \setminus \{0\} = F^*$, the vector space T with the product $[x, y, z]^\mu = \mu[x, y, z]$ is a (simple) L.t.s. (in case $\mu \in F^2$ both systems are isomorphic).

Example 1. Any simple Lie algebra A is a simple L.t.s under the trilinear product $[x, y, z] = [[x, y], z]$ ($[\,,\,]$ the Lie bracket in A). In this case $S = \mathrm{ad}\, A = \mathrm{Der}\, A \simeq A$ and the standard embedding is $L(A) = \mathrm{Der}\, A \oplus A \simeq A \oplus A$. Moreover

A is the adjoint module for S and therefore the Lie bracket in A is a nonzero skew-symmetric element in $\mathrm{Hom}_S(A\otimes A, A)$. For any $\mu \in F^*$, we shall refer to the system $(A, [\,,\,,\,]^\mu)$ as a L.t.s *of adjoint type*.

Over algebraically closed fields, it is known ([7,8]) that

$$\dim \mathrm{Hom}_A(A\otimes A, A) = \begin{cases} 2 & \text{if } A \text{ is of type } A_n \ (n \geq 2), \\ 1 & \text{otherwise.} \end{cases}$$

Consequently, $\mathrm{Hom}_S(A\otimes A, A)$ is spanned by the "Lie bracket in A" for types other than A_n $(n \geq 2)$. But, what happens in the remaining cases? The answer can be found in the following example.

Example 2. For any simple Jordan algebra J of degree $n \geq 2$ with generic trace t, the set J_0 of trace zero elements in J becomes a simple L.t.s via $[x,y,z] = (xz)y - x(zy)$ (the associator of x, z and y). Since $[x,y,-] = [R_y, R_x]$ $(R_x u = ux)$, it follows that $S = [R_J, R_J] = \mathrm{Der}\, J$ and the standard embedding for this system is $L(J_0) = \mathrm{Der}\, J \oplus J_0 \simeq \mathrm{Der}\, J \oplus R_{J_0}$. In this case we can obtain a symmetric element in $\mathrm{Hom}_S(J_0 \otimes J_0, J_0)$ by means of

$$\begin{aligned} J_0 \times J_0 &\longrightarrow J_0 \\ x\otimes y &\mapsto x\cdot y = xy - \frac{1}{n}t(xy)1 \end{aligned}$$

(that is, the projection of the element xy onto J_0) and this product is nontrivial if and only if $n \geq 3$. The systems $(J_0, [\,,\,,\,]^\mu)$ will be said to be of *Jordan type*.

In case F is algebraically closed, if J is a simple Jordan algebra of degree $n \geq 3$ over F, then either:

i) J is the algebra of symmetric $n\times n$ matrices. In this case $\mathrm{Der}\, J$ is isomorphic to the set of skew-symmetric matrices ([9, Theorem VI.9]) and $L(J_0) \simeq sl(n,F)$.

ii) J is the algebra of symmetric $2n \times 2n$ matrices with respect to the standard symplectic involution. Again in this case $\mathrm{Der}\, J$ is isomorphic to the skew-symmetric matrices with respect to this involution [9, Theorem VI.9] and $L(J_0) \simeq sl(2n, F)$.

iii) J is the exceptional simple Jordan algebra, $\mathrm{Der}\, J$ is a Lie algebra of type F_4 and $L(J_0)$ is a Lie algebra of type E_6 [9, Section IV.11].

In all these cases, $\dim \mathrm{Hom}_S(J_0\otimes J_0, J_0) \geq 1$, and the main Theorem will assert that, in fact, it is 1.

iv) $J = Mat_n(F)^+$. In this case $J_0 = sl(n,F)$ and $\mathrm{Der}\, J = \mathrm{ad}\, J_0$. Then the triple system J_0 is both of adjoint type (A_{n-1}) and of Jordan type.
Therefore, $\mathrm{Hom}_S(T \otimes_F T, T)$ is spanned by the "Lie bracket in S" and the "Jordan product $\cdot$" for the adjoint type A_n $(n \geq 2)$. This is the answer to the question in Example 1.

The situation in iv) can be settled in a more general context over arbitrary fields of characteristic 0, as the following and final example ([9, Section V.7]) shows:

Example 3. Let (A, j) be a central simple associative involutorial algebra of degree $n \geq 3$ with j an involution of second kind and center $P = F[q]$, a quadratic extension of the base field or isomorphic to $F \times F$. Then the Jordan algebra $J = H(A,j)^+$ of j-symmetric elements of A is central simple of type A and degree n (that is, after extension by scalars it becomes a total $n \times n$ matrix algebra) and the derived algebra $S(A,j)_0 = [S(A,j), S(A,j)]$ of the Lie algebra $S(A,j)^-$ of j-skew elements of A is a central simple Lie algebra of type A_{n-1} [10, Theorem X.8] (Note that if $P = F \oplus F$, $A = B \oplus B^{op}$, with B a central simple associative algebra and j the exchange involution: $j(b_1, b_2) = (b_2, b_1)$. So we can identify $H(A,j)^+ \simeq B^+$ and $S(A,j)^- \simeq B^-$). The converse is also true: any central simple Jordan algebra of type A and degree $n \geq 3$ can be obtained in this manner.

If we consider the L.t.s. of Jordan type $T = J_0 = H(A,j)_0 = \{x \in H(A,j) \mid t(x) = 0\}$ (t the generic trace of $H(A,j)$), we have that S=Der $J = \mathrm{ad}\, S(A,j)_0|_{H(A,j)_0}$ [9].

As the linear map $x \mapsto qx$ of J_0 onto $S(A,j)_0$ is an isomorphism of $S(A,j)_0$-modules, our L.t.s. is both of adjoint type and of Jordan type, and the bracket in $S(A,j)_0$ under this isomorphism provides the product

$$x * y = q[x, y],$$

which is a nonzero skew-symmetric element in $\mathrm{Hom}_{\mathrm{Der}\, J}(J_0 \otimes J_0, J_0)$. Consequently the dimension of this space is at least 2. We shall refer to the systems $(H(A,j)_0, [\,,\,,\,]^\mu)$ as L.t.s. of *adjoint-Jordan type*.

2.3. The centroid of a Lie triple system.

A natural and useful tool in the study of simple algebraic structures is the centroid. Given any L.t.s. T we define the centroid of T in the natural way as $\Gamma = \{\alpha \in End_F(T) \mid \alpha([x,y,z]) = [\alpha(x), y, z] = [x, \alpha(y), z] = [x, y, \alpha(z)] \quad \forall x, y, z \in T\}$. If T is simple, Γ is a field contained in the centroid of $L(T)$, and T over Γ remains simple by scalars extensions.

A technical result in [5, Lemma 4.1] shows that for any simple Lie triple system T with centroid Γ, in order to determine $\mathrm{Hom}_S(T \otimes_F T, T)$ it is enough to determine $\mathrm{Hom}_S(T \otimes_\Gamma T, T)$.

In this way, our problem is reduced to the special case of central simple L.t.s. for which we can extend scalars up to an algebraically closed field. Then, the problem becomes to compute the set of homomorphisms for simple L.t.s. over algebraically closed fields.

Now we shall sketch how to solve this situation.

2.4. Solution in the algebraically closed case.

In this paragraph let T be a simple L.t.s. over an algebraically closed field F and $L(T) = S \oplus T$ be its standard embedding.

The complete classification of such systems was obtained by Lister in 1952 [6]. In 1980 Faulkner [11] gave a method based on the use of the affine Dynkin diagrams (see [12, Ch.8]) that provides an useful and quicker classification. This one will be used here to deduce our results.

From [6,11] the algebra $L(T)$ is a simple ungraded algebra unless T is of adjoint type, and S is a reductive algebra with $\dim Z(S) \leq 1$, consequently the derived subalgebra $[S, S]$ of S is semisimple. In case $Z(S) = 0$, T is an irreducible selfdual module for S; otherwise $T = T_1 \oplus T_2$, T_i being irreducible dual S-modules.

The central idea in Faulkner's classification is to associate a diagram to each simple L.t.s. T, starting with the Dynkin diagram of the semisimple algebra $[S, S]$ and adding a node for each $[S, S]$-irreducible component of T (which represents its minimum weight). All the nodes in the diagrams are equipped by numerical labels representing the linear dependence of the roots of $[S, S]$ and the weights of T involved in the diagrams. These weighted diagrams contain the whole information on the Lie algebra S and the S-module T. They can be grouped into four types.

i) Diagrams representing simple L.t.s. of adjoint type. For these ones, the problem was already solved.

ii) Diagrams representing non irreducible simple L.t.s.. Given such a system T, the center of S is one-dimensional and $T = T_1 \oplus T_2$ with T_i S-irreducible. In this case, there exists a nonzero central element $z \in S$ with $\operatorname{ad} z|_{T_1} = 1$ and $\operatorname{ad} z|_{T_2} = -1$. Therefore $T \otimes T$ is the sum of the eigenspaces 2, 0, -2 and so $\operatorname{Hom}_S(T \otimes T, T) = 0$.

iii) Diagrams representing irreducible simple L.t.s with trivial 0-weight space, and

iv) Diagrams representing irreducible simple L.t.s. with nontrivial 0-weight space.

In the two latter cases the corresponding L.t.s. are selfdual modules and the determination of the set $\operatorname{Hom}_S(T \otimes T, T)$ can be reduced to the problem of bounding the dimension of a certain subspace in the 0-weight space of T (which is 0 for systems of type iii)) because of the following result [5, Lemma 3.3] and [8]:

LEMMA *Let S be a semisimple Lie algebra, H a Cartan subalgebra of S, $T = V(\lambda)$ an irreducible selfdual module with highest weight λ relative to H, v_λ and $v_{-\lambda}$ nonzero weight vectors for λ and $-\lambda$. For any root α relative to H let S_α be the corresponding root space. If T_0 is the 0-weight space in T, the linear map*

$$\begin{array}{rcl} \phi : \operatorname{Hom}_S(T \otimes T, T) & \longrightarrow & T_0^\lambda = \{v \in T_0 \mid S_\alpha \cdot v = 0 \quad \forall \alpha \perp \lambda\} \\ \varphi & \mapsto & \varphi(v_\lambda \otimes v_{-\lambda}) \end{array}$$

is well defined and one-to-one. That is, $\dim \mathrm{Hom}_S(T \otimes T, T) \leq \dim T_0^\lambda$.

So, given a simple L.t.s. T of type iii), from the previous Lemma we get that $\mathrm{Hom}_S(T \otimes T, T) = 0$; and given a T of type iv), by using the technical Lemma in [5, Lemma 3.4] and the information provided by the affine diagrams of types i), iii) and iv) it is possible to obtain that the space $\mathrm{Hom}_S(T \otimes T, T)$ is trivial or one-dimensional, turning out that $\mathrm{Hom}_S(T \otimes T, T) = 0$ unless T is a simple L.t.s. of Jordan type (the L.t.s. i), ii), iii) described in Example 2), in this case $\mathrm{Hom}_S(T \otimes T, T)$ is spanned by the product $\cdot$ (J is not of type A).

2.5. The main Theorem.

Now, taking into account the considerations above, we can establish the main result in [5, Theorem 4.3]:

THEOREM *Let T be a simple Lie triple system over a field F of characteristic zero with centroid Γ and let $L = S \oplus T$ be its standard embedding. Then $\mathrm{Hom}_S(T \otimes_F T, T) = 0$ unless either:*

a) *T is of adjoint type with S being a central simple Lie algebra over Γ of type different from A_n ($n \geq 2$). In this case $\mathrm{Hom}_S(T \otimes_F T, T) \simeq \mathrm{Hom}_S(S \otimes_F S, S)$, which is spanned over Γ by the Lie multiplication in S, or*

b) *there exists a central simple Jordan algebra J of degree $n \geq 3$ over Γ and a nonzero scalar $\mu \in \Gamma$ (which can be taken modulo Γ^2) such that T is the Lie triple system $(J_0, [\,,\,,\,]^\mu)$. In this case either:*

    i) *J is not of type A over Γ, then $\mathrm{Hom}_S(T \otimes_F T, T) = \mathrm{Hom}_{\mathrm{Der}\,J}(J_0 \otimes_F J_0, J_0)$ is spanned over Γ by the product $x \cdot y$,*

    ii) *$J = H(A, j)$ for some central simple associative involutorial algebra (A, j) of second kind over Γ. Let $P = \Gamma[q]$ ($0 \neq q^2 \in \Gamma$) be the center of A. Then $\mathrm{Hom}_S(T \otimes_F T, T) = \mathrm{Hom}_{\mathrm{Der}\,J}(J_0 \otimes_F J_0, J_0) = \mathrm{Hom}_{S(A,j)_0}(H(A, j)_0 \otimes_F H(A, j)_0, H(A, j)_0)$ is spanned over Γ by the products $\cdot$ and $*$.*

From the point of view of nonassociative algebras, this result points out the central role played by the multiplication $\cdot$, defined in the set J_0 of the trace zero elements in a Jordan algebra J, in the computation of the sets $\mathrm{Hom}_S(T \otimes T, T)$.

The restriction of the Theorem to the real field gives us the invariant affine connections on the irreducible symmetric spaces. So, in a sense, we can say that the Jordan algebras are responsible for the existence of noncanonical connections on the symmetric spaces.

3. APPLICATIONS TO DIFFERENTIAL GEOMETRY.

In this section we are going to compute some important geometric objects in a symmetric space, namely, the torsion and curvature tensors and the holonomy algebra associated to the invariant affine connections on it.

First of all, we will get expressions for these objects in any reductive homogeneous space in terms of the multiplication in the associated connection algebra. Afterwards we will specialize the previous computations to the symmetric spaces, because for these spaces we know the concrete connection algebras (thanks to the central Theorem in Section 2).

The torsion and curvature tensors are the most important tensors associated to an affine connection ∇, and they are given by the following formulas:

- Torsion tensor: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$
- Curvature tensor: $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

Now, if $M = G/H$ is a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and ∇ is a G-invariant affine connection, let $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ be the associated multiplication by Nomizu's result. In this case the torsion and curvature tensors can be expressed in terms of the multiplication α by means of:

$$T(X,Y) = \alpha(X,Y) - \alpha(Y,X) - [X,Y]_{\mathfrak{m}}$$
$$R(X,Y)Z = \alpha(X,\alpha(Y,Z)) - \alpha(Y,\alpha(X,Z)) - \alpha([X,Y]_{\mathfrak{m}}, Z) - [[X,Y]_{\mathfrak{h}}, Z]$$

for any $X, Y, Z \in \mathfrak{m}$, where $Z_{\mathfrak{m}}$ and $Z_{\mathfrak{h}}$ denote the projections of $Z \in \mathfrak{g}$ onto $\mathfrak{m}$ and $\mathfrak{h}$ respectively, and we are identifying $\mathfrak{m}$ with the tangent space $T_{\bar{e}}M$.

If M is symmetric, the projection over $\mathfrak{m}$ of the bracket of two elements in $\mathfrak{m}$ is zero, hence:

$$T(X,Y) = \alpha(X,Y) - \alpha(Y,X)$$
$$R(X,Y)Z = \alpha(X,\alpha(Y,Z)) - \alpha(Y,\alpha(X,Z)) - [[X,Y]_{\mathfrak{h}}, Z].$$

On the other hand, the holonomy algebra $\operatorname{Hol}\nabla$ is the smallest Lie subalgebra of $\operatorname{End}_{\mathbb{R}}(\mathfrak{m})$ containing $R(X,Y)$ for all $X, Y \in \mathfrak{m}$ and closed under commutation by ∇_X for any $X \in \mathfrak{m}$.

The previous facts lead us to introduce the following definitions:

Given a Lie triple system T with standard embedding $L(T) = S \oplus T$ and given $\alpha \in \operatorname{Hom}_S(T \otimes_F T, T)$, we define:

i) Torsion tensor: $T^\alpha(x,y) = \alpha(x,y) - \alpha(y,x)$

ii) Curvature tensor: $R^\alpha(x,y)z = \alpha(x,\alpha(y,z)) - \alpha(y,\alpha(x,z)) - [x,y,z]$

iii) Holonomy algebra: $\operatorname{Hol}^\alpha(T)$ = the smallest subalgebra of $\operatorname{End}_F T$ containing $R^\alpha(x,y)$ for all $x, y \in T$ and closed under commutation by the operators $\alpha(x,-)$.

We are going to compute all these concepts for any simple Lie triple system T over an arbitrary field F of characteristic zero. In the particular case $F = \mathbb{R}$, each $\alpha \in \mathrm{Hom}_S(T \otimes_{\mathbb{R}} T, T)$ corresponds to an invariant affine connection in the symmetric space so that T^α, R^α and Hol^α are the true torsion tensor, curvature tensor and holonomy algebra associated to the connection.

According to the main theorem in Section 2 we can assume that T is a central simple Lie triple system over F with standard embedding $L(T) = S \oplus T$ and the pair (T, α) has one of the following forms:

I) **Trivial case**: $\alpha = 0$

Clearly we have for any $x, y \in T$:

$$\begin{cases} T^\alpha(x,y) = 0 \\ R^\alpha(x,y) = -[x,y,-] \\ \mathrm{Hol}^\alpha(T) = \mathrm{alg}\langle [x,y,-] \mid x,y \in T\rangle = S \end{cases}$$

II) **Adjoint type**: there exists a central simple Lie algebra S and $\mu, \eta \in F^* = F \setminus \{0\}$ such that $T = S$ and for any $x, y \in T$:

$$\begin{cases} [x,y,-] = \mu\, \mathrm{ad}[x,y] \\ \alpha(x,y) = \eta[x,y] \end{cases}$$

where $[\,,\,]$ is the product in S.

Immediately we obtain in this case

$$\begin{cases} T^\alpha(x,y) = 2\eta[x,y] \\ R^\alpha(x,y) = (\eta^2 - \mu)\,\mathrm{ad}[x,y] \\ \mathrm{Hol}^\alpha(T) = \begin{cases} 0 & \text{if } \mu = \eta^2 \\ \mathrm{ad}\, S \simeq S & \text{if } \mu \neq \eta^2 \end{cases} \end{cases}$$

III) **Jordan type**: there exists a central simple Jordan algebra J of degree $n \geq 3$ with generic trace t and $\mu, \eta \in F^*$ such that $T = J_0 = \{x \in J \mid t(x) = 0\}$ is the set of trace zero elements and for any $x, y \in J_0$:

$$\begin{cases} [x,y,-] = \mu[R_x, R_y]|_{J_0} \\ \alpha(x,y) = \eta\, x \cdot y \end{cases}$$

where $x \cdot y = xy - \frac{1}{n}t(xy)1$ and xy is the product in J.

In order to compute the holonomy we need a previous result:

LEMMA *Let J be a central simple Jordan algebra of degree $n \geq 3$ with generic trace t and $J_0 = \{x \in J \mid t(x) = 0\}$ in which we consider the product $x \cdot y = xy - \frac{1}{n}t(xy)1$ and the trace form $t(x,y) = t(xy)$.*

Then the derivation algebra and the Lie multiplication algebra of $(J_0, \cdot)$ are the Lie algebras given by:

$$\mathrm{Der}(J_0, \cdot) = \{d|_{J_0} : d \in \mathrm{Der}\, J\} \quad (\simeq \mathrm{Der}\, J)$$

$$\mathrm{Lie}(J_0, \cdot) = sl(J_0)$$

Furthermore, $(J_0, \cdot)$ *is simple.*

Proof: If $(a,b,c) = (ab)c - a(bc) = [R_c, R_a](b)$ is the associator in J and $(a,b,c)^{\cdot} = (a \cdot b) \cdot c - a \cdot (b \cdot c)$ the associator in $(J_0, \cdot)$, it is straightforward to check that

$$(a,b,c)^{\cdot} = (a,b,c) - \frac{1}{n}(t(a,b)c - t(b,c)a) \qquad (*)$$

for any $a, b, c \in J_0$. Therefore for any $x, y \in J_0$

$$(x \cdot x, y, x)^{\cdot} = \frac{1}{n}(t(x,y)x \cdot x - t(x \cdot x, y)x).$$

For any $d \in \mathrm{Der}(J_0, \cdot)$, its action on this equality gives

$$(t(dx,y) + t(x,dy))\, x \cdot x - (t(d(x \cdot x), y) + t(x \cdot x, dy))\, x = 0$$

for any $x, y \in J_0$. Since the degree of J is ≥ 3, the set of elements $x \in J_0$ such that x and $x \cdot x$ are linearly independent is a dense subset in the Zariski topology, we obtain from the above that

$$t(dx,y) + t(x,dy) = 0$$

for any $x, y \in J_0$. That is, $\mathrm{Der}(J_0, \cdot)$ is contained in the orthogonal Lie algebra related to t: $so(J_0, t)$. Now, extending d to J by means of $d1 = 0$, we get immediately that $d \in \mathrm{Der}\, J$. Conversely, any derivation of J restricts to a derivation of $(J_0, \cdot)$, so that

$$\mathrm{Der}(J_0, \cdot) = \{d|_{J_0} : d \in \mathrm{Der}\, J\} \simeq \mathrm{Der}\, J\,.$$

It is known that J_0 is an irreducible $\mathrm{Der}\, J$-module, therefore J_0 is an irreducible $\mathrm{Der}(J_0, \cdot)$-module, which implies that $(J_0, \cdot)$ is simple and their derivations are inner ([13, Theorem 3.4] and [14, Lemma 1]), that is, $\mathrm{Der}(J_0, \cdot) \subseteq \mathrm{Lie}(J_0, \cdot)$.

We denote by $R^{\cdot}_x$ the multiplication operator in $(J_0, \cdot)$ given by $R^{\cdot}_x(y) = x \cdot y$. Since $trace(R_x)$ is a multiple of $t(x)$, we have $trace(R^{\cdot}_x) = trace(R_x) = 0$ if $x \in J_0$, hence $\mathrm{Lie}(J_0, \cdot) \subseteq sl(J_0)$.

In order to prove the converse, we note that (*) is equivalent to:

$$[R_x, R_y] = [R^{\cdot}_x, R^{\cdot}_y] - \frac{1}{n}\big(t(x,-)y - t(y,-)x\big)$$

for any $x, y \in J_0$. Hence

$$\begin{aligned}\frac{1}{n}\big(t(x,-)y - t(y,-)x\big) &= \\ &= [R^{\cdot}_x, R^{\cdot}_y] - [R_x, R_y] \in \mathrm{Lie}(J_0, \cdot) + \mathrm{Der}(J_0, \cdot) \subseteq \mathrm{Lie}(J_0, \cdot)\end{aligned}$$

since $[R_x, R_y]$ is always a derivation of J. But the linear maps on the left hand side above span the orthogonal Lie algebra $so(J_0, t)$, so that $so(J_0, t) \subseteq \mathrm{Lie}(J_0, \cdot)$. Now