MOHR CIRCLES, STRESS PATHS AND GEOTECHNICS

R. H. G. PARRY

Mohr Circles, Stress Paths and Geotechnics

Second Edition

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Preface

On turning the pages of the many textbooks which already exist on soil mechanics and rock mechanics, the important roles of Mohr circles and stress paths in geotechnics becomes readily apparent. They are used for representing and interpreting data, for the analysis of geotechnical problems and for predicting soil and rock behaviour. In the present book, Mohr circles and stress paths are explained in detail – including the link between Mohr stress circles and stress paths – and soil and rock strength and deformation behaviour are viewed from the vantage points of these graphical techniques. Their various applications are drawn together in this volume to provide a unifying link to diverse aspects of soil and rock behaviour. The reader can judge if the book succeeds in this.

Past and present members of the Cambridge Soil Mechanics Group will see much in the book which is familiar to them, as I have drawn, where appropriate, on the accumulated Cambridge corpus of geotechnical material in the form of reports, handouts and examples. Thus, a number of people have influenced the contents of the book, and I must take this opportunity to express my gratitude to them. I am particularly grateful to Ian Johnston and Malcolm Bolton for looking through the first drafts and coming up with many useful suggestions. Similarly my thanks must go to the anonymous reviewers to whom the publishers sent the first draft for comments, and who also came up with some very useful suggestions. The shortcomings of the book are entirely of my own making.

Inevitably, there has been much typing and retyping, and I owe a special debt of gratitude to Stephanie Saunders, Reveria Wells and Amy Cobb for their contributions in producing the original manuscript. I am also especially indebted to Ulrich Smoltczyk, who provided me with a copy of Mohr's 1882 paper and an excellent photograph of Mohr, and to Stille Olthoff for translating the paper. Markus Caprez kindly assisted me in my efforts to locate a copy of Culmann's early work.

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Cambridge

Historical note: Karl Culmann (1821–1881) and Christian Otto Mohr (1835–1918)

Although the stress circle is invariably attributed to Mohr, it was in fact Culmann who first conceived this graphical means of representing stress. Mohr's contribution lay in making an extended study of its usage for both two-dimensional and three-dimensional stresses, and in developing a strength criterion based on the stress circle at a time when most engineers accepted Saint-Venant's maximum strain theory as a valid failure criterion. Anyone wishing to pursue the relative contributions of Culmann and Mohr is recommended to read the excellent accounts in *History of Strength of Materials* by Timoshenko (McGraw-Hill, 1953).

Born in Bergzabern, Rheinpfalz, in 1821, Karl Culmann graduated from the Karlsruhe Polytechnikum in 1841 and immediately started work at Hof on the Bavarian railroads. In 1849 the Railways Commission sent him to England and the United States for a period of two years to study bridge construction in those countries. The excellent engineering education which he had received enabled him to view, from a theoretical standpoint, the work of his English and American counterparts, whose expertise was based largely on experience. The outcome was a report by Culmann published in 1852 which strongly influenced the theory of structures and bridge engineering in Germany. His appointment as Professor of Theory of Structures at the Zurich Polytechnikum in 1855 gave him the opportunity to develop and teach his ideas on the use of graphical methods of analysis for engineering structures, culminating in his book *Die Graphische Statik*, published by Verlag von Meyer and Zeller in 1866. The many areas of graphical statics dealt with in the book include the application of the polygon of forces and the funicular polygon, construction of the bending moment diagram, the graphical solution for continuous beams (later simplified by Mohr) and the use of the method of Sections for analysing trusses. He concluded this book with Sections on calculating the pressures on retaining walls and tunnels.

Culmann introduced his stress circle in considering longitudinal and vertical stresses in horizontal beams during bending. Isolating a small element of the beam and using rectangular coordinates, he drew a circle with its centre on the (horizontal) zero shear stress axis, passing through the two stress points represented by the normal and conjugate shear stresses on the vertical and horizontal faces of the element. He took the normal stress on the horizontal faces to be zero. In making

Figure 1 Karl Culmann.

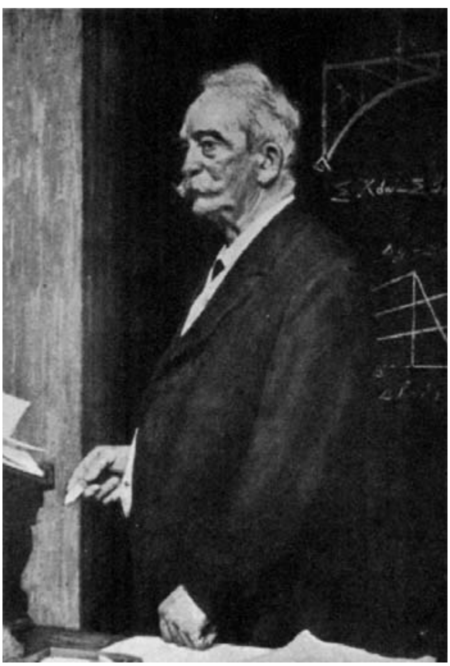

Figure 2 Otto Mohr.

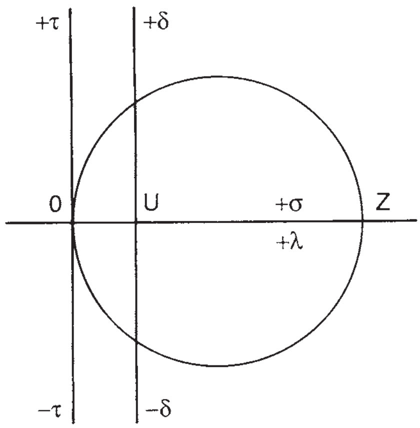

Figure 3 A figure from Mohr's 1882 paper showing his use of a single circle to illustrate both stress and strain circles for uniaxial tension.

this construction Culmann established a point on the circle, now known as the pole point, and showed that the stresses on a plane at any specified inclination could be found by a line through this point drawn parallel to the plane. Such a line met the circle again at the required stress point. Extensive use is made of the pole point in the present text. Culmann went on to plot trajectories of principal stresses for a beam, obtained directly from the stress circles.

Christian Otto Mohr was born in 1835 in Wesselburen, on the inhospitable North Sea coast of Schleswig-Holstein. After graduating from the Hannover Polytechnical Institute he first worked, like Culmann, as a railway engineer before taking up, at the age of 32, the post of Professor of Engineering Mechanics at the Stuttgart Polytechnikum. In 1873 he moved to the Dresden Polytechnikum, where he continued to pursue his interests in both strength of materials and the theory of structures. Pioneering contributions which he made to the theory of structures included the use of influence lines to calculate the deflections of continuous beams, a graphical solution to the three-moments equations, and the concept of virtual work to calculate displacements at truss joints. His work on the stress circle included both two-dimensional and three-dimensional applications and, in addition, he formulated the trigonometrical expressions for an elastic material, relating stresses and strains, as well as the expression relating direct and shear strain moduli. As with stress, he showed that shear strains and direct strains could be represented graphically by circles in a rectangular coordinate system.

Believing, as Coulomb had done a hundred years before, that shear stresses caused failure in engineering materials, Mohr proposed a failure criterion based on envelopes tangential to stress circles at fracture in tension and compression. He then assumed that any stress conditions represented by a circle touching these envelopes would initiate failure. This failure criterion was found to give better agreement with experiment than the maximum strain theory of Saint-Venant, which was widely accepted at that time.

Mohr first published his work on stress and strain circles in 1882 in *Civil-ingenieur* and it was repeated in *Abhandlungen aus dem Gebiete der Technischen Mechanik* (2nd edn), a collection of his works published by Wilhelm Ernst & Sohn, Berlin, 1914.

Chapter 1

Stresses, strains and Mohr circles

1.1 The concept of stress

The concept of stress, defined as force per unit area, was introduced into the theory of elasticity by Cauchy in about 1822. It has become universally used as an expedient in engineering design and analysis, despite the fact that it cannot be measured directly and gives no indication of how forces are transmitted through a stressed material. Clearly the manner of transfer in a solid crystalline material, such as a metal or hard rock, is different from the point-to-point contacts in a particulate material, such as a soil. Nevertheless, in both cases it is convenient to visualize an imaginary plane within the material and calculate the stress across it by simply dividing the force across the plane by the total area of the plane.

1.2 Simple axial stress

A simple illustration of stress is given by considering a cylindrical test specimen, with uniform Section of radius r, subjected to an axial compressive force F as shown in Figure 1.1(a). Assuming the force acts uniformly across the Section of the specimen, the stress σ_{n0} on a plane PQ perpendicular to the direction of the force, as shown in Figure 1.1(a), is given by

$$\sigma_{n0} = \frac{F}{A} \tag{1.1}$$

where A is the cross-sectional area of the specimen. As this is the only stress acting across the plane, and it is perpendicular to the plane, σ_{n0} is a principal stress.

Consider now a plane such as PR in Figure 1.1(b), inclined at an angle θ to the radial planes on which σ_{n0} acts. The force F has components N acting normal (perpendicular) to the plane and T acting along the plane, in the direction of maximum inclination θ. Thus

$$N = F\cos\theta \tag{1.2a}$$

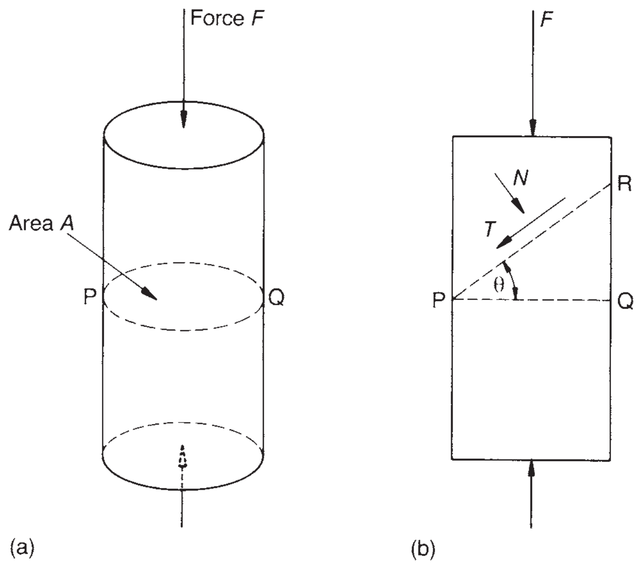

Figure 1.1 Cylindrical test specimen subjected to axial force *F*.

$$T = F \sin \theta \tag{1.2b}$$

As the inclined plane is an ellipse with area $A/\cos\,\theta$, the direct stress $\sigma_{n\theta}$ normal to the plane and shear stress τ_θ along the plane, in the direction of maximum inclination, are given by:

$$\sigma_{n\theta} = \frac{N\cos\theta}{A} = \frac{F}{A}\cos^2\theta \tag{1.3a}$$

$$\tau_\theta = \frac{T\cos\theta}{A} = \frac{F}{2A}\sin 2\theta \tag{1.3b}$$

It is obvious by inspection that the maximum normal stress, equal to F/A, acts on radial planes. The magnitude and direction of the maximum value of τ_θ can be found by differentiating equation 1.3b:

$$\frac{\mathrm{d}\tau_\theta}{\mathrm{d}\theta} = \frac{F}{A}\cos 2\theta$$

The maximum value of τ_θ is found by putting $\mathrm{d}\tau_\theta/\mathrm{d}\theta = 0$, thus:

$$\cos 2\theta = 0$$
$$\theta = 45°(\text{or } 135°) \qquad (1.4)$$
$$\tau_{\theta\max} = \frac{F}{2A}$$

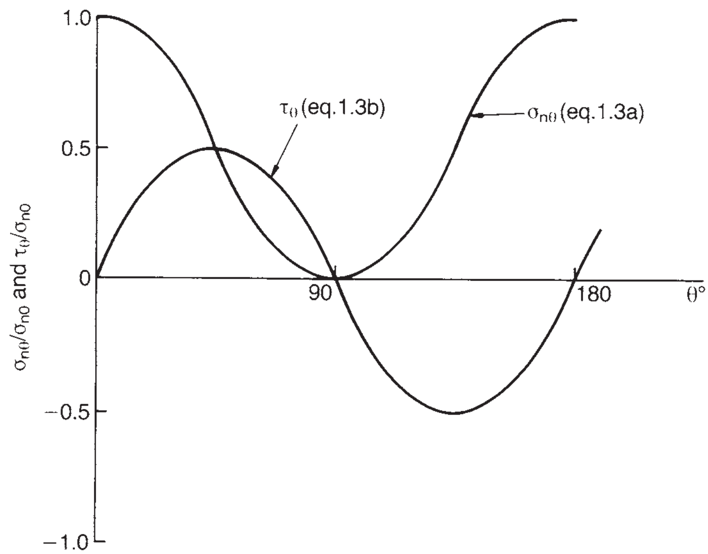

Figure 1.2 Variation of normal stress $\sigma_{n\theta}$ and shear stress τ_θ with angle of plane θ in cylindrical test specimen.

The variations of $\sigma_{n\theta}$ and τ_θ with θ, given by equations 1.3a and l.3b, are shown in Figure 1.2. It can be seen that $\tau_{\theta\max}$ occurs on a plane with $\theta = 45°$ and $\sigma_{n\theta\max}$ on a plane with $\theta = 0°$.

EXAMPLE 1.1 AXIAL PRINCIPAL STRESSES

A cylindrical specimen of rock, 50 mm in diameter and 100 mm long is subjected to an axial compressive force of 5 kN. Find:

1. the normal stress $\sigma_{n\theta}$ and shear stress τ_θ on a plane inclined at 30° to the radial direction;

2. the maximum value of shear stress;
3. the inclination of planes on which the shear stress τ_θ is equal to one-half $\tau_{\theta\max}$.

Solution

1. Area $A = \pi r^2 = \pi \times 0.025^2\ \text{m}^2 = 1.96 \times 10^{-3}\ \text{m}^2$

$$\text{Equation 1.3a:}\ \sigma_{n\theta} = \frac{5}{1.96 \times 10^{-3}} \times \cos^2 30° = 1913\ \text{kPa}$$

$$\text{Equation 1.3b:}\ \tau_\theta = \frac{5}{2 \times 1.96 \times 10^{-3}} \times \sin 60° = 1105\ \text{kPa}$$

2. Equation 1.4: $$\tau_{\theta\max} = \frac{F}{2A} = \frac{5}{2 \times 1.96 \times 10^{-3}}\ \text{kPa} = 1275\ \text{kPa}$$

3. Equation 1.3b: $$\tfrac{1}{2}\tau_{\theta\max} = \tau_{\theta\max} \sin 2\theta$$
$$\therefore\ \sin 2\theta = \tfrac{1}{2}$$
$$\therefore \theta = 15° \text{ or } 75°$$

1.3 Biaxial stress

Although in most stressed bodies the stresses acting at any point are fully three-dimensional, it is useful for the sake of clarity to consider stresses in two dimensions only before considering the full three-dimensional stress state.

1.3.1 Simple biaxial stress system

A simple biaxial stress system is shown in Figure 1.3(a), which represents a rectangular plate of unit thickness with stresses σ_1, σ_2 acting normally to the squared edges of the plate. As the shear stresses along the edges are assumed to be zero, σ_1 and σ_2 are principal stresses.

A small square element of the plate is shown in the two-dimensional diagram in Figure 1.3(b). The stresses $\sigma_{n\theta}$, τ_θ acting on a plane inclined at an angle θ to the direction of the plane on which σ_1 acts can be found by considering the forces acting on the triangular element in Figure 1.3(c).

If length CD = l, then for a plate of unit thickness:

$$F_1 = \sigma_1 l \qquad (1.5a)$$

$$N_1 = \sigma_1 l \cos\theta \tag{1.5b}$$

$$T_1 = \sigma_1 l \sin\theta \tag{1.5c}$$

$$F_2 = \sigma_2 l \tan\theta \tag{1.5d}$$

$$N_2 = \sigma_2 l \tan\theta \sin\theta \tag{1.5e}$$

$$T_2 = \sigma_2 l \tan\theta \cos\theta \tag{1.5f}$$

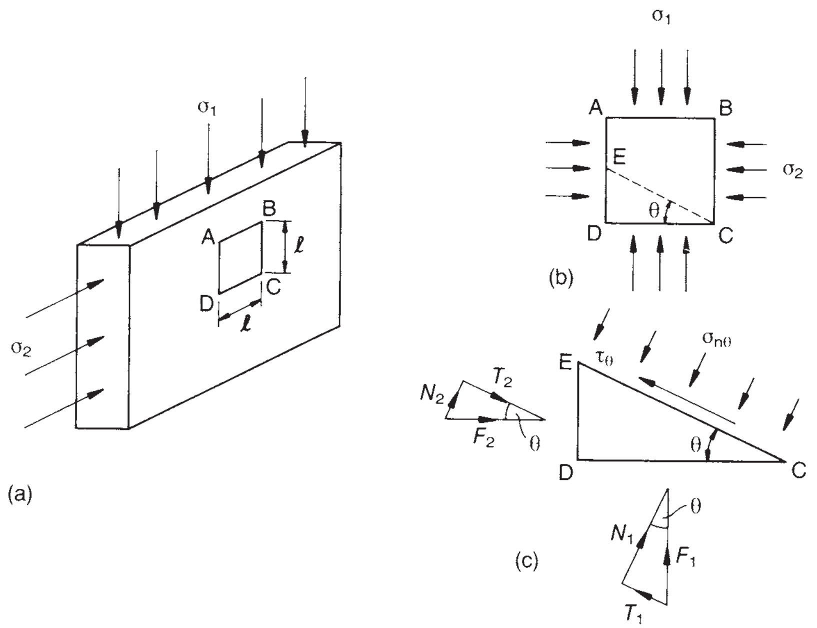

Figure 1.3 Biaxial stress system in a rectangular plate: (a) boundary stresses; (b) stresses on element ABCD; (c) determination of stresses $\sigma_{n\theta}$, τ_θ on plane inclined at angle θ.

Resolving forces in the direction of action of $\sigma_{n\theta}$:

$$\sigma_{n\theta} l \sec\theta = N_1 + N_2 \tag{1.6}$$

Substituting equations 1.5b, 1.5e into equation 1.6:

$$\sigma_{n\theta} = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta \tag{1.7}$$

Resolving forces in the direction of τ_θ:

$$\tau_\theta / \sec\theta = T_1 + T_2 \tag{1.8}$$

Substituting equations 1.5c, 1.5f into equation 1.8:

$$\tau_\theta = \tfrac{1}{2}\left(\sigma_1 - \sigma_2\right)\sin 2\theta \tag{1.9}$$

Comparing equation 1.9 with equation 1.3b it is seen that the maximum value of τ_θ acts on a plane with $\theta = 45°$, for which

$$\tau_{\theta\max} = \tfrac{1}{2}\left(\sigma_1 - \sigma_2\right) \tag{1.10}$$

This is not the maximum value of shear stress in the plate. As the third principal stress is zero, the maximum value of τ in the plate acts on a plane at 45° to both σ_1 and σ_2 and has the value

$$\tau_{\max} = \frac{\sigma_1}{2} \text{ if } \sigma_1 > \sigma_2$$

EXAMPLE 1.2 BIAXIAL PRINCIPAL STRESSES

A flat piece of slate with uniform thickness 20 mm is cut into the shape of a square with 100 mm long squared edges. A test is devised which allows uniform compressive stress σ_1 to be applied along two opposite edges and uniform tensile stress σ_2 along the other two opposite edges, as shown in Figure 1.4(a). The stresses σ_1 and σ_2 act normally to the edges of the test specimen. The test is performed by increasing the magnitudes of σ_1 and σ_2 simultaneously, but keeping the magnitude of σ_1 always four times the magnitude of σ_2. If failure of the slate occurs when the shear stress on any plane exceeds 1 MPa, what would be the values of σ_1 and σ_2 at the moment of failure?

Would the values of σ_1 and σ_2 at failure be changed if:

1. the rock had a tensile strength of 0.5 MPa?
2. a planar weakness running through the test specimen as shown in Figure 1.4(b), inclined at 60° to the direction of σ_2, would rupture if the shear stress on it exceeds 0.8 MPa?

Solution

The maximum shear stress $\tau_{\theta\max}$ occurs on a plane with $\theta = 45°$.

Equation 1.10: $$\tau_{\theta\max} = \tfrac{1}{2}(\sigma_1 - \sigma_2)$$

As $\tau_{\theta\max}$ = 1 MPa and $\sigma_1 = -4\sigma_2$, then at failure:

$$\sigma_1 = 1.6 \text{ MPa} \qquad \sigma_2 = -0.4 \text{ MPa}$$

1. As σ_2 at failure is less in magnitude than 0.5 MPa, the tensile strength of the slate does not influence failure.
2. As the planar weakness acts at 30° to the direction of σ_1, the normal stress on this weakness $\sigma_{n\theta}$ acts in a direction of 60° to σ_1. Thus the stress τ_θ on the plane of weakness is found by putting $\theta = 60°$.

Equation 1.9:

$$\tau_\theta = \tfrac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta$$

If θ =60°,

$$\tau_\theta = 0.866 \text{ MPa}$$

As rupture occurs on the plane of weakness when τ_θ = 0.8 MPa, the weakness would influence failure. The stresses at rupture are found by putting τ = 0.8 MPa and $\sigma_1 = -4\sigma_2$ into equation 1.9, giving

$$\sigma_1 = 1.48 \text{ MPa} \qquad \sigma_2 = -0.37 \text{ MPa}$$

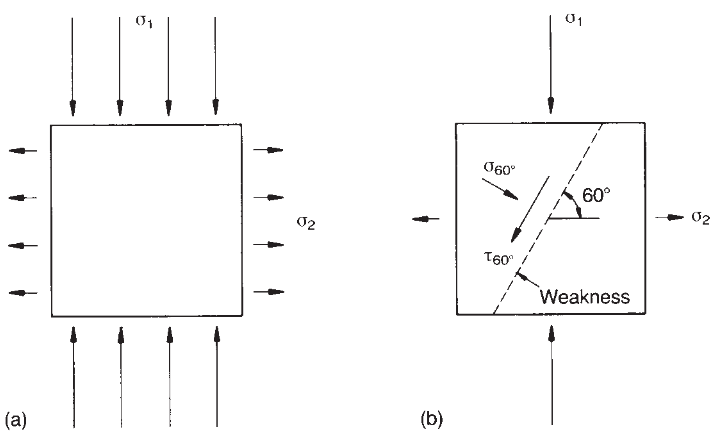

Figure 1.4 Example 1.2.

1.3.2 Generalized biaxial stress system

In Section 1.3.1 the special case is considered of an element ABCD with principal stresses only acting along its edges, but it has been seen that shear stresses are generated along all other planes which are not parallel to the normally loaded edges of the plate. Thus, an element with edges oriented in directions z, y different from the principal stress directions will have both normal and direct stresses acting along its edges, as shown in Figure 1.5(a).

It is appropriate here to consider the conventions of stress representation usually adopted in geotechnics. Referring to Figure 1.5(a) we have the following.

1. Shear stress τ_{zy} acts tangentially along the edge or face normal to the direction z and in the direction y. The converse applies for τ_{yz}.
2. Compressive normal stresses are positive and tensile normal stresses are negative.
3. Anticlockwise shear stresses are positive and clockwise shear stresses are negative. Thus, in Figure 1.5(a):

$$\tau_{zy} \text{ is +ve}$$

$$\tau_{yz} \text{ is –ve}$$

If moments are taken about a point such as M in Figure 1.5(a), static equilibrium of the element JKLM can be maintained if the conjugate shear stresses τ_{zy} and τ_{yz} are equal in magnitude, i.e.

$$\tau_{zy} = -\tau_{yz} \tag{1.11}$$

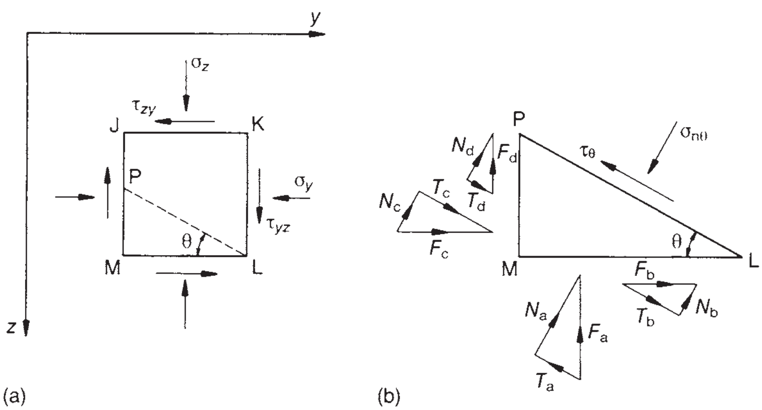

Figure 1.5 Generalized biaxial stress system: (a) stresses on element JKLM; (b) determination of stresses $\sigma_{n\theta}$, τ_θ on plane inclined at angle θ.

This is known as the principle of complementary shear.

The stresses on a plane inclined at angle θ to the direction of the plane on which σ_1 acts can be obtained by considering the equilibrium of the triangular portion LMP of element JKLM shown in Figure 1.5(b). The forces acting on edges LM and MP can be determined from the force diagrams:

$$
\begin{array}{lll}
F_a = \sigma_z l & N_a = F_a \cos\theta & T_a = F_a \sin\theta \\
F_b = \tau_{zy} l & N_b = F_b \sin\theta & T_b = F_b \cos\theta \\
F_c = \sigma_y l \tan\theta & N_c = F_c \sin\theta & T_c = F_c \cos\theta \\
F_d = -\tau_{yz} l \tan\theta & N_d = F_d \cos\theta & T_d = F_d \sin\theta
\end{array}
$$

But, noting that $-\tau_{yz} = \tau_{zy}$

$$F_d = -\tau_{yz} l \tan\theta = \tau_{zy} l \tan\theta$$

Resolving forces in the direction of $\sigma_{n\theta}$:

$$\sigma_{n\theta} l \sec\theta = N_a + N_b + N_c + N_d$$

which becomes, on substitution of the force diagram values:

$$\sigma_{n\theta} = \sigma_z \cos^2\theta + \sigma_y \sin^2\theta + \tau_{zy} \sin 2\theta \qquad (1.12)$$

Resolving forces in the direction of τ_θ:

$$\tau_\theta l \sec\theta = -T_a + T_b + T_c + T_d$$

which becomes, on substitution of the force diagram values:

$$\tau_\theta = \tfrac{1}{2}\left(\sigma_y - \sigma_z\right)\sin 2\theta + \tau_{zy}\cos 2\theta \qquad (1.13)$$

(a) Planes on which $\tau_\theta = 0$

The directions of planes on which $\tau_\theta = 0$ can be found by putting $\tau_\theta = 0$ in equation 1.13, from which

$$\tan 2\theta = \frac{2\tau_{zy}}{\sigma_z - \sigma_y} \qquad (1.14)$$

Equation 1.14 gives two sets of orthogonal planes. As the shear stress is zero on these planes, these are the planes on which the principal stresses act.

It is possible to evaluate the principal stresses on these planes by substituting equation 1.14 into equation 1.12, noting that equation 1.14 gives

$$\sin 2\theta = \frac{2\tau_{zy}}{\left[\left(\sigma_z - \sigma_y\right)^2 + 4\tau_{zy}^2\right]^{1/2}}$$

$$\cos 2\theta = \frac{\sigma_z - \sigma_y}{\left[\left(\sigma_z - \sigma_y\right)^2 + 4\tau_{zy}^2\right]^{1/2}}$$

and using the trigonometrical relationships

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

Substitution of these relationships into equation 1.12 gives

$$\sigma_{\mathrm{n}\theta} = \tfrac{1}{2}\left(\sigma_z + \sigma_y\right) \pm \tfrac{1}{2}\left[\left(\sigma_z - \sigma_y\right)^2 + 4\tau_{zy}^2\right]^{1/2} \tag{1.15}$$

As the expression under the root sign yields both positive and negative values, two values of $\sigma_{\mathrm{n}\theta}$ are obtained, the larger of which is the major and the smaller the minor of the two principal stresses.

(b) Planes on which maximum τ_θ acts

The directions of planes on which the maximum values of τ_θ act can be found by differentiating equation 1.13 with respect to θ and equating to zero:

$$\tan 2\theta = \frac{\sigma_y - \sigma_z}{2\tau_{zy}} \tag{1.16}$$

It is possible to evaluate the maximum shear stress $\tau_{\theta\max}$ by substituting equation 1.16 into equation 1.13:

$$\tau_{\theta\max} = \tfrac{1}{2}\left[\left(\sigma_y - \sigma_z\right)^2 + 4\tau_{zy}^2\right]^{1/2} \tag{1.17}$$

In the mathematical sense $\tau_{\theta\max}$ is the positive root of equation 1.17, while the negative root gives the minimum value of τ_θ, which has the same magnitude as, but is opposite in sense to, $\tau_{\theta\max}$. In the physical sense the minimum value of τ_θ is zero.

1.4 Mohr stress circle

A graphical means of representing the foregoing stress relationships was discovered by Culmann (1866) and developed in detail by Mohr (1882), after whom the graphical method is now named.

By using the relationships

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

it is possible to rewrite equation 1.12 in the form

$$\sigma_{n\theta} - \tfrac{1}{2}(\sigma_z + \sigma_y) = \tfrac{1}{2}(\sigma_z - \sigma_y)\cos 2\theta + \tau_{zy}\sin 2\theta \qquad (1.18)$$

If equations 1.13 and 1.18 are squared and added, after some manipulation (remembering that $\sin^2 2\theta + \cos^2 2\theta = 1$) the following expression results:

$$\left[\sigma_{n\theta} - \tfrac{1}{2}(\sigma_z + \sigma_y)\right]^2 + \tau_\theta^2 = \left[\tfrac{1}{2}(\sigma_z - \sigma_y)\right]^2 + \tau_{zy}^2 \qquad (1.19)$$

Putting

$$s = \tfrac{1}{2}(\sigma_z + \sigma_y)$$
$$r^2 = \left[\tfrac{1}{2}(\sigma_z - \sigma_y)\right]^2 + \tau_{zy}^2$$

equation 1.19 becomes

$$(\sigma_{n\theta} - s)^2 + \tau_\theta^2 = r^2$$

which is the equation of a circle with radius r and with a centre, on a τ–σ plot, at

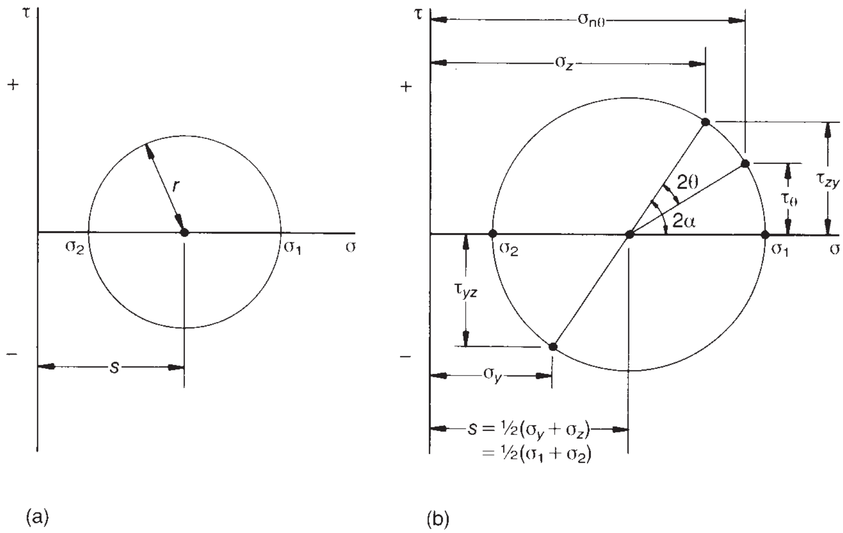

Figure 1.6 Mohr stress circle: (a) geometry; (b) stress representation.

the point $\sigma = s$, $\tau = 0$, as shown in Figure 1.6(a). This is the Mohr circle of stress. The complete state of two-dimensional stress is represented by points on this circle. The principal stresses σ_1, σ_2 are given by the points where the circle crosses the $\tau = 0$ axis.

In Figure 1.6(b) the boundary stresses σ_z, σ_y, τ_{zy}, τ_{yz}, on the element shown in Figure 1.5 are plotted in τ–σ space and a circle drawn through them. The Mohr circle represents completely the two-dimensional stresses acting within the element, and σ_1, σ_2 are the major and minor principal stresses respectively. The stresses $\sigma_{n\theta}$, τ_θ acting on a plane at an angle θ clockwise to the plane on which σ_z acts, as shown in Figure 1.5, can be found by travelling clockwise around the circle from stress point σ_z, τ_{zy} a distance subtending an angle 2θ at the centre of the circle. Thus, the major principal stress σ_1 acts on a plane inclined at an angle α to the plane on which σ_z acts.

The stresses $\sigma_{n\theta}$, τ_θ can be evaluated from the known boundary stresses σ_z, σ_y, τ_{zy} by observing from Figure 1.6(b) that

$$\sigma_{n\theta} = \tfrac{1}{2}\left(\sigma_z + \sigma_y\right) + r\cos\left(2\alpha - 2\theta\right) \tag{1.20}$$

Substituting

$$r = \frac{\tau_{zy}}{\sin 2\alpha} \tag{1.21a}$$

and

$$\tan 2\alpha = \frac{2\tau_{zy}}{\sigma_z - \sigma_y} \tag{1.21b}$$

obtained from Figure 1.6(b) into equation 1.20 leads to the expression

$$\sigma_{n\theta} = \tfrac{1}{2}\left(\sigma_z + \sigma_y\right) + \tfrac{1}{2}\left(\sigma_z - \sigma_y\right)\cos 2\theta + \tau_{zy}\sin 2\theta$$

This expression is identical to equation 1.18 as required.

Similarly, from Figure 1.6(b):

$$\tau_\theta = r\sin\left(2\alpha - 2\theta\right) \tag{1.22}$$

Substituting equations 1.21a and 1.21b into equation 1.22 leads to the expression

$$\tau_\theta = \tfrac{1}{2}\left(\sigma_y - \sigma_z\right)\sin 2\theta + \tau_{zy}\cos 2\theta$$

This expression is identical to equation 1.13 as required.

It can also be seen in Figure 1.6(b) that

$$s = \tfrac{1}{2}\left(\sigma_y + \sigma_z\right) = \tfrac{1}{2}\left(\sigma_1 + \sigma_2\right)$$

1.5 Mohr circles for simple two-dimensional stress systems

Examples of Mohr circles for simple two-dimensional stress systems are shown in Figure 1.7. As stresses in only two dimensions are considered, the diagrams are incomplete. The complete three-dimensional stress diagrams are discussed in Section 1.6, but it is instructive to consider firstly stresses in two dimensions only.

1. **Biaxial compression**. The biaxial stresses are represented by a circle which plots in positive σ space, passing through stress points σ_1, σ_2, on the $\tau = 0$ axis. The centre of the circle is located on the $\tau = 0$ axis at stress point ½($\sigma_1 + \sigma_2$). The radius of the circle has the magnitude ½($\sigma_1 - \sigma_2$), which is equal to τ_{max}.
2. **Biaxial compression/tension**. In this case the stress circle extends into both positive and negative σ space. The centre of the circle is located on the $\tau = 0$ axis at stress point ½($\sigma_1 + \sigma_2$) and has radius ½($\sigma_1 - \sigma_2$). This is also the