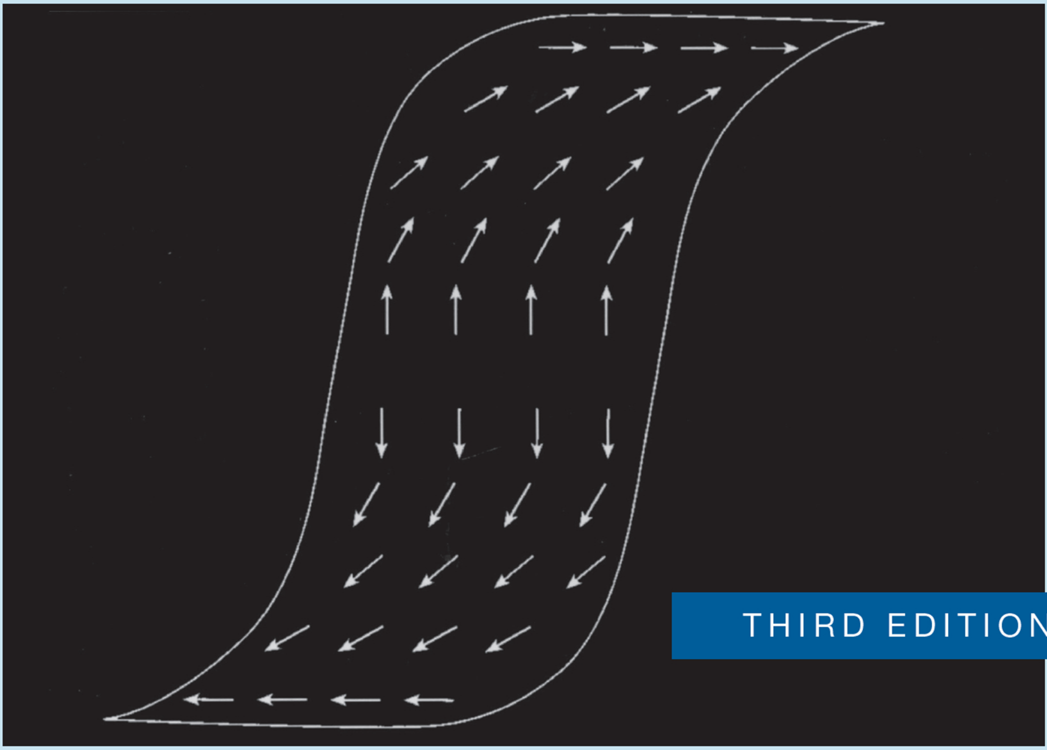
Introduction to MAGNETISM and MAGNETIC MATERIALS

DAVID JILES



THIRD EDITION

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Introduction to Magnetism and Magnetic Materials

Third Edition

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Third Edition

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Few subjects in science are more difficult to understand than magnetism.

—Encylopaedia Britannica, 1989

This book is dedicated to Helen, Sarah, Elizabeth, Andrew and Richard

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Preface

A new edition of *Introduction to Magnetism and Magnetic Materials* is long overdue, the project having been held up first by the need for a second edition of *Introduction to the Electronic Materials*, and then by the new book *Introduction to the Principles of Materials Evaluation*. This third edition of *Introduction to Magnetism and Magnetic Materials* has been completely revised. The basic science of magnetism evolves slowly, so that most of the content of Chapters 1 through 11 is as relevant today as it was at the time of the second edition. Nevertheless, I have taken the opportunity to add a few more exercises to these chapters with once again complete worked solutions to all exercises at the end of the book. It is my belief that the best way to learn this subject is to read through the chapters, try the exercises that test your knowledge on the subject area of the chapter, and then check your solution against the model answers in the back of the book. That is why I continue to provide complete worked solutions rather than just a numerical answer, because in the latter case if you don't get the answer right, it can be difficult to see where you went wrong.

The book also serves a secondary role as a reference text with a large number of references to key works in magnetism and magnetic materials. In Chapters 12 through 15, which deal with applications of magnetism and magnetic materials, there has been enormous progress in the time since the previous edition—particularly in magnetic recording. In preparing the new edition that is where most of the time and effort had to be devoted. So in these later chapters, you will find new results and many new references—often references to web pages, where a lot of the new results now appear. However, the classic works still provide information on the important milestones in the development of our subject. So I have interspersed references to some of the classic works with newer references in these chapters to provide a more complete treatment of the subject matter.

I would like to acknowledge the assistance of many friends, colleagues and students in preparing this third edition. In particular I would like to thank: Lawrence Crowther, Ravi Hadimani, Orfeas Kypris, Ikenna Nlebedim, Zhen Zhang, Helena Khazdozian, Yan (Michelle) Ni, Neelam Prabhu Gaunkar and Priyam Rastogi.

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Glossary of Symbols

A	Magnetic vector potential
A	Area
	Helmholz energy
	Exchange stiffness coefficient (= E_{ex}/az)
a	Distance
	Lattice spacing
	Radius of coil or solenoid
α	Mean field constant
$\alpha_1, \alpha_2, \alpha_3$	Direction cosines of magnetic vector with respect to the applied field
$\boldsymbol{B}$	Magnetic induction
$\boldsymbol{B}_g$	Magnetic induction in air gap
$\boldsymbol{B}_j(x)$	Brillouin function of x
$\boldsymbol{B}_R$	Remanent magnetic induction
$\boldsymbol{B}_s$	Saturation magnetic induction
$\beta_1, \beta_2, \beta_3$	Direction cosines of direction of measurement with respect to the applied field
C	Capacitance
	Curie constant
c	Velocity of light
χ	Susceptibility
$\boldsymbol{D}$	Electric displacement
D	Diameter of solenoid
d	Diameter
δ	Domain wall thickness
$\boldsymbol{E}$	Electric field strength
E	Energy
e	Electronic charge spontaneous strain within a domain
E_a	Anisotropy energy
E_{ex}	Exchange energy per atom
$E_{ex,vol}$	Exchange energy per unit volume (= NE_{ex})
$E_{ex,nn}$	Exchange energy per nearest neighbor spin (= E_{ex}/Z)
E_f	Fermi energy
E_H	Magnetic field energy (Zeeman energy)
E_{Hall}	Hall field
E_{loss}	Energy loss
E_p	Potential energy
E_σ	Stress energy
ϵ	Permittivity
ϵ_0	Permittivity of free space
ϵ_{pin}	Domain-wall pinning energy
η	Magnetomotive force
F	Field factor
$\boldsymbol{F}$	Force

(*Continued*)

f	Frequency
	Current factor
G	Gibbs free energy
g	Spectroscopic splitting factor
	Lande splitting factor
γ	Gyromagnetic ratio
	Domain-wall energy per unit area
$\boldsymbol{H}$	Magnetic field strength
h	Planck's constant
$\boldsymbol{H}_{\mathrm{c}}$	Coercivity
	Critical field
$\boldsymbol{H}_{\mathrm{cr}}$	Remanent coercivity
$\boldsymbol{H}_{\mathrm{d}}$	Demagnetizing field
$\boldsymbol{H}_{e}$	Weiss mean field
$\boldsymbol{H}_{\mathrm{eff}}$	Effective magnetic field
$\boldsymbol{H}_{\mathrm{g}}$	Magnetic field in air gap
$\boldsymbol{I}$	Magnetic polarization
	Intensity of magnetization
I	Current
$\boldsymbol{J}$	Current density
	Total atomic angular momentum quantum number
J	Exchange constant
J	Total electronic angular momentum quantum number
$\mathscr{J}$	Coupling between nearest-neighbor magnetic moments
J_{atom}	Exchange integral for an electron on an atom with electrons on several nearest neighbors
J_{ex}	Exchange integral; exchange interaction between two electrons
K	Anisotropy constant
	Kundt's constant
k	Pinning or loss coefficient in hysteresis equation
k_{B}	Boltzmann's constant
K_{ul}	First anisotropy constant for uniaxial system
K_{u2}	Second anisotropy constant for uniaxial system
K_1	First anisotropy constant for cubic system
K_2	Second anisotropy constant for cubic system
$\boldsymbol{L}$	Inductance
	Length
	Length of solenoid
	Atomic orbital angular momentum
	Electronic orbit length
L	Length
	Orbital angular momentum quantum number
$\mathscr{L}(x)$	Langevin function of x
λ	Wavelength
	Magnetostriction
	Filling factor for solenoid

(*Continued*)

λ_t	Transverse magnetostriction
λ_s	Saturation magnetostriction
λ_0	Spontaneous bulk magnetostriction
$\boldsymbol{M}$	Magnetization
$\boldsymbol{m}$	Magnetic moment
m	Mass
	Momentum
$\boldsymbol{M}_{an}$	Anhysteretic magnetization
m_e	Electronic mass
m_1	Orbital magnetic quantum number
m_o	Orbital magnetic moment of electron
$\boldsymbol{M}_R$	Remanent magnetization
$\boldsymbol{M}_0$	Saturation magnetization (spontaneous magnetization at 0 K)
$\boldsymbol{M}_s$	Spontaneous magnetization within a domain
$\boldsymbol{m}_s$	Spin magnetic moment of electron
m_s	Spin magnetic quantum number
$\boldsymbol{m}_{tot}$	Total magnetic moment of atom
μ	Permeability
μ,ν	Rayleigh coefficients
μ_B	Bohr magneton
μ_0	Permeability of free space
N	Number of turns of solenoid
	Number of atoms per unit volume
n	Number of turns per unit length on solenoid
	Principal quantum number
N_d	Demagnetizing factor
N_0	Avogadro's number
n_s	Number density of paired electrons
υ	Frequency
ω	Angular frequency
P	Pressure
P	Magnetic pole strength
	Angular momentum operator
P	Density
	Resistivity
$\boldsymbol{P}_0$	Orbital angular momentum of electron
$\boldsymbol{P}$S	Spin angular momentum of electron
$\boldsymbol{P}_{tot}$	Total angular momentum of electron
Φ	Magnetic flux
ϕ	Angle
	Spin wave function
Ψ	Total wave function
ψ	Electron wave function
Q	Electric charge
R	Resistance

(*Continued*)

r	Radius vector
	Radius
	Electronic orbit radius
R_m	Magnetic reluctance
$\boldsymbol{S}$	Atomic spin angular momentum
S	Entropy
s	Electronic spin angular momentum quantum number
σ	Conductivity
	Stress
T	Temperature
t	Time
	Thickness
T_c	Curie temperature
	Critical temperature
t_0	Orbital period of electron
θ	Angle
τ	Torque
	Orbital period
τ_{max}	Maximum torque
U	Internal energy
$\mathbf{u}$	Unit vector
V	Potential difference
	Volume
	Verdet's constant
υ	Velocity
W	Power
W_a	Atomic weight
W_H	Hysteresis loss
x	Distance along x-axis
y	Distance along y-axis
Z	Impedance
	Atomic number
z	Distance along z-axis
	Number of nearest-neighbor atoms

SI Units, Symbols, and Dimensions

Quantity	Unit		MKSA	
	Symbol	Name	Base Units	Dimensions
Length	m	meter	m	L
Mass	kg	kilogram	kg	M
Time	s	second	s	T
Frequency	Hz	hertz	s^{-1}	T^{-1}
Force	N	newton	kg m s^{-2}	MLT^{-2}
Pressure	Pa	pascal	kg m^{-1} s^{-2}	ML^{-1} T^{-2}
Energy	J	joule	kg m^2 s^{-2}	ML^2 T^{-2}
Power	W	watt	kg m^2 s^{-3}	ML^2 T^{-3}
Electric charge	C	coulomb	A s	CT
Electric current	A	ampere	A	C
Electric potential	V	volt	kg m^2 A^{-1} s^{-3}	ML^2 C^{-1} T^{-3}
Resistance	Ω	ohm	kg m^2 A^{-2} s^{-3}	ML^2 C^{-2} T^{-3}
Resistivity	Ωm	ohm meter	kg m^3 A^{-2} s^{-3}	ML^3 C^{-2} T^{-3}
Conductance	S	siemens	A^2 s^3 kg^{-1} m^{-2}	M^{-1} L^{-2} C^2 T^3
Conductivity	S m^{-1}	siemens $meter^{-1}$	A^2 s^3 kg^{-1} m^{-3}	M^{-1} L^{-3} C^2 T^3
Capacitance	F	farad	A^2 s^4 kg^{-1} m^{-2}	M^{-1} L^{-2} C^2 T^4
Electric displacement (flux density)	C m^{-2}	coulomb $meter^{-2}$	A s m^{-2}	CL^{-2} T
Electric field	V m^{-1}	volt $meter^{-1}$	kg m A^{-1} s^{-3}	MLC^{-1} T^{-3}
Electric polarization	C m^{-2}	coulomb $meter^{-2}$	A s m^{-2}	CL^{-2} T
Permittivity	F m^{-1}	farad $meter^{-1}$	A^2 s^4 kg^{-1} m^{-3}	M^{-1} L^{-3} C^2 T^4
Inductance	H	henry	kg m^2 A^{-2} s^{-2}	ML^2 C^{-2} T^{-2}
Magnetic flux	Wb	weber	kg m^2 A^{-1} s^{-2}	ML^2 C^{-1} T^{-2}
Magnetic induction (flux density)	T	Tesla	kg A^{-1} s^{-2}	MC^{-1} T^{-2}
Magnetic field	A m^{-1}	ampere $meter^{-1}$	A m^{-1}	CL^{-1}
Magnetization	A m^{-1}	ampere $meter^{-1}$	A m^{-1}	CL^{-1}
Permeability	H m^{-1}	henry $meter^{-1}$	kg m A^{-2} s^{-2}	MLC^{-2} T^{-2}

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Values of Selected Physical Constants

Avogadro's Number	$N_0 = 6.022 \times 10^{26}$ atoms kg mole^{-1}
Boltzmann's constant	$k_B = 1.381 \times 10^{-23}$ J K^{-1}
Gas constant	$R = 8.314$ J mole^{-1} K^{-1}
Planck's constant	$h = 6.626 \times 10^{-34}$ J s
	$h/2\pi = 1.054 \times 10^{-34}$ J s
Velocity of light in empty space	$c = 2.998 \times 10^{8}$ m s^{-1}
Permittivity of empty space	$\epsilon_0 = 8.854 \times 10^{-12}$ F m^{-1}
Permeability of empty space	$\mu_0 = 1.257 \times 10^{-6}$ H m^{-1}
Atomic mass unit	a.m.u. = 1.661×10^{-27} kg
Properties of electrons	
Electronic charge	$e = -1.602 \times 10^{-19}$ C
Electronic rest mass	$m_e = 9.109 \times 10^{-31}$ kg
Charge to mass ratio	$e/m_e = 1.759 \times 10^{11}$ C kg^{-1}
Electron volt	eV = 1.602×10^{-19} J
Properties of protons	
Proton charge	$e_p = 1.602 \times 10^{-19}$ C
Rest mass	$m_p = 1.673 \times 10^{-27}$ kg
Gyromagnetic ratio of proton	$\gamma_p = 2.675 \times 10^{8}$ Hz T^{-1}
Magnetic constants	
Bohr magneton	$\mu_B = 9.274 \times 10^{-24}$ A m^2 (= J T^{-1})
	$= 1.165 \times 10^{-29}$ J m A^{-1}
Nuclear magneton	$\mu_N = 5.051 \times 10^{-27}$ A m^2 (= J T^{-1})
Magnetic flux quantum	$\Phi_0 = 2.067 \times 10^{-15}$ Wb (= V s)

Introduction

As you study the intricate subject of magnetism in this book, you will find that the journey begins at a familiar level, with electric currents passing through wires, compass needles rotating in magnetic fields, and bar magnets attracting or repelling each other. As the journey progresses, though, in order to understand our observations, we must soon peel back the surface and begin to delve into the materials, to use ever-increasing magnification levels and look in greater and greater detail to explain what is happening. This process takes us from bulk magnets (10^{23}–10^{26} atoms) down to the domain scale (10^{12}–10^{18} atoms) and then down to the scale of a domain wall (10^{3}–10^{2} atoms). In critical phenomena, one is often concerned with the behavior of even smaller numbers (10 atoms or less) in a localized array. Then comes the question of how the magnetic moment of a single atom arises. We must go inside the atom to find the answer by looking at the behavior of a single electron orbiting a nucleus. The next question is, *Why are the magnetic moments of neighboring atoms aligned?* In order to answer this, we must go even further and consider the quantum mechanical exchange interaction between two electrons on neighboring atoms. This then marks the limit of our journey into the fundamentals of our subject. Subsequently, we must ask how this knowledge can be used to our benefit. In Chapters 12 through 15, we look at the most significant applications of magnetism. It is no surprise that these applications deal with ferromagnetism. Ferromagnetism is easily the most important technological branch of magnetism; most scientific studies, even of other forms of magnetism, are ultimately designed to help further our understanding of ferromagnetism, so that we may both fabricate new magnetic materials with improved properties and make better use of existing material.

Finally, I have adopted an unusual format for the book, in which each major heading within a chapter is introduced by a question which the subsequent discussion attempts to answer. Many have said they found this feature useful in focusing attention on the subject matter at hand since it is then clear what the objective of each section is. I have decided therefore to retain this format from my original notes, realizing that though it is unusual for a textbook it would prove helpful to the reader.

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Section I

Electromagnetism

Magnetic Phenomena on the Macroscopic Scale

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1 Magnetic Fields

In this chapter, we will clarify ideas about what is meant by a *magnetic field* and then show that it is always the result of electrical charge in motion. This will be followed by a discussion of the concept of *magnetic induction*, also known as *magnetic flux density*, and its relation to the magnetic field. We will look at the various unit conventions currently in use in magnetism and finally discuss methods for calculating magnetic fields.

1.1 MAGNETIC FIELD

What do we mean by magnetic field*?*

One of the most fundamental ideas in magnetism is the concept of the magnetic field. When a field is generated in a volume of space, it means that there is a force produced, which can be detected by the acceleration of an electric charge moving in the field, by the force on a current-carrying conductor, by the torque on a magnetic dipole such as a bar magnet, or even by a reorientation of spins on electrons within certain types of atoms. The torque on a compass needle, which is an example of a magnetic dipole, is probably the most widely recognized property of a magnetic field.

1.1.1 Generation of a Magnetic Field

How are magnetic fields produced?

A magnetic field is produced whenever there is an electrical charge in motion. This can be due to an electrical current flowing in a conductor, for example, as was first discovered by Oersted in 1819 [1]. A magnetic field is also produced by a permanent magnet. In this case, there is no conventional electric current, but there are the orbital motions and spins of electrons (the so-called Ampèrian currents) within the magnetic material that lead to a magnetization within the material and a magnetic field outside. The magnetic field exerts a force on both current-carrying conductors and permanent magnets.

In order to explain why a moving charge causes a magnetic field, we need to ask ourselves what forces exist between two electrical charges when they are at rest and when they are in motion, relative to whoever or whatever is measuring the forces between them. In the former case, when the charges are *at rest*, the only force between the charges is the electrostatic Coulomb force. However, when the charges are in motion, there is an additional force on the charges, which we commonly call the *magnetic force* or the *magnetic field*. The existence of this magnetic field can be shown by application of the relativistic Lorentz transformation to the Coulomb force between the charges, which is explained in detail in Appendix A. Therefore, the magnetic field emerges as an extension or correction to the electrostatic Coulomb force.

1.1.2 Biot-Savart Law

Is there any way we can calculate the magnetic field strength generated by an electric current?

The Biot-Savart law, which enables us to calculate the magnetic field $\boldsymbol{H}$ generated by an electrical current, is one of the fundamental laws of electromagnetism. It is a statement of experimental observation rather than a theoretical prediction. In its usual form, the law gives the field contribution generated by a current flowing in an elementary length of conductor

$$\delta \boldsymbol{H} = \frac{1}{4\pi r^2} i\,\delta \boldsymbol{l} \times \mathbf{u} \tag{1.1}$$

where:

i is the current flowing in an elemental length δl of a conductor
r is the radial distance
$\mathbf{u}$ is a unit vector along the radial direction
$\delta \boldsymbol{H}$ is the contribution to the magnetic field at r due to the current element $i\delta l$

The magnetic field decreases with the square of the distance from the current element that produces it.

This form is known as the *Biot-Savart law*, although it was also discovered independently in a different form by Ampère in the same year. For steady currents, it is equivalent to Ampère's circuital law. It is not really capable of direct proof, but is justified by experimental measurements. Notice in particular that it is an inverse square law.

1.1.3 Magnetic Field Due to a Circular Coil

What is the field strength produced by a single-turn circular coil?

The Biot-Savart law can be used to determine the magnetic field $\boldsymbol{H}$ at the center of a circular coil of one turn with a radius of a meters, carrying a current of i amperes. We divide the coil into elements of arc length δl, each of which contributes $\delta \boldsymbol{H}$ to the field at the center of the coil. Since by the Biot-Savart law,

$$\boldsymbol{H} = \sum \frac{1}{4\pi r^2} i\delta l \sin\theta \tag{1.2}$$

and $\sum \delta l = 2\pi a$, while dl is perpendicular to $\mathbf{u}$, so $\theta = 90°$, and hence $\sin\theta = 1$, and $r = a$

$$\boldsymbol{H} = \frac{i}{2a} \quad \mathrm{A\,m^{-1}} \tag{1.3}$$

1.1.4 Definition of Magnetic Field Strength, *H*

What is the unit of magnetic field strength?

There are a number of ways in which the ampere per meter, the unit of magnetic field strength $\boldsymbol{H}$, can be defined. In accordance with the ideas developed here, we wish to emphasize the connection between the magnetic field $\boldsymbol{H}$ and the electric current. We shall therefore define the unit of magnetic field strength, the ampere per meter, in terms of the generating current. The simplest definition is as follows:

The ampere per meter: A magnetic field strength of 1 A m^{-1} is produced at the center of a single circular coil of conductor of diameter 1 m when it carries a current of 1 A.

For the time being, we will take the viewpoint that the magnetic field $\boldsymbol{H}$ is solely determined by the size and distribution of currents producing it and is independent of the material medium. This will allow us to draw a distinction between magnetic field and magnetic induction. However, we shall see in Section 2.4.3 that this assumption needs to be modified under certain circumstances, particularly when demagnetizing fields are encountered in magnetic materials.

1.1.5 Magnetic Field Generated by a Long Current-Carrying Conductor

What is the field strength generated at a fixed distance from a long current-carrying conductor?

We shall determine the magnetic field $\boldsymbol{H}$ at some point P that is at a distance of a meters from an infinitely long conductor carrying a current of i amperes and calculate the field at a distance of 0.1 m from the conductor when it carries a current of 0.1 A (Figure 1.1).

Using the Biot-Savart law, the contribution $\delta\boldsymbol{H}$ to the field at the point P, as shown in Figure 1.2, due to current element $i\delta l$ at an angle α is given by

$$\delta\boldsymbol{H} = \frac{1}{4\pi r^2} i\delta\boldsymbol{l} \sin(90-\alpha) \times \mathbf{u} \tag{1.4}$$

We can write $\delta l = r\delta\alpha/\cos\alpha = a\delta\alpha/\cos^2\alpha$.

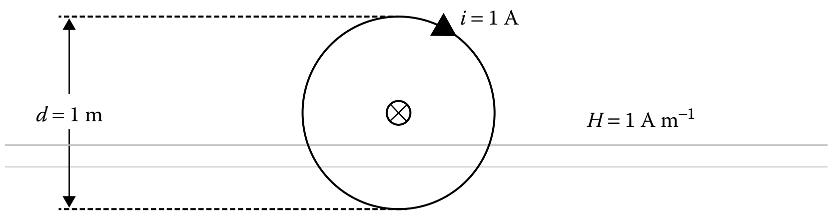

FIGURE 1.1 The magnetic field $\boldsymbol{H}$ at the center of a single circular coil of diameter 1 m carrying a current of 1 A.

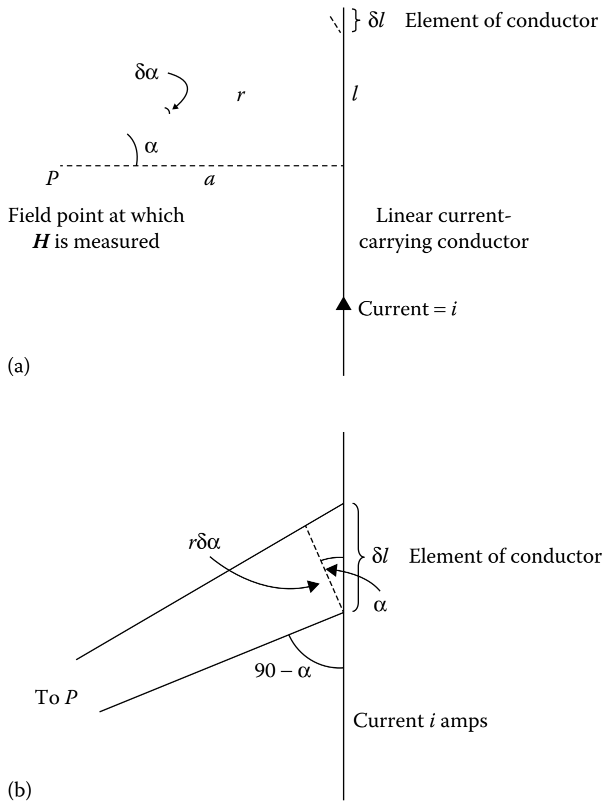

FIGURE 1.2 Magnetic field $\boldsymbol{H}$ due to a long straight conductor carrying an electric current *i*.

$$\delta \boldsymbol{H} = \frac{i \cos\alpha \, \delta\alpha}{4\pi a} \tag{1.5}$$

Now integrating the expression from $\alpha = -\pi/2$ to $\alpha = \pi/2$ to obtain the total $\boldsymbol{H}$

$$\boldsymbol{H} = \int_{-\pi/2}^{\pi/2} \frac{i}{4\pi a} \cos\alpha \, d\alpha \tag{1.6}$$

$$\boldsymbol{H} = \frac{i}{2\pi a} \ \text{A m}^{-1} \tag{1.7}$$

Therefore, if $a = 0.1$ m and $i = 0.1$ A, the field is $1/2\pi$ A m^{-1}, or $\boldsymbol{H} = 0.159$ A m^{-1}.

The direction of this magnetic field is such that it circulates the conductor obeying the right-hand rule. That is, if we look along the conductor in the direction of the conventional current, the magnetic field circulates in a clockwise direction.

1.1.6 Field Patterns around Current-Carrying Conductors

What do these fields look like?

The magnetic field patterns, detected by the magnetic powder, around a bar magnet (magnetic dipole), a straight conductor, a single circular loop, and a solenoid are shown in Figure 1.3a through e. The field circulates around a single current-carrying

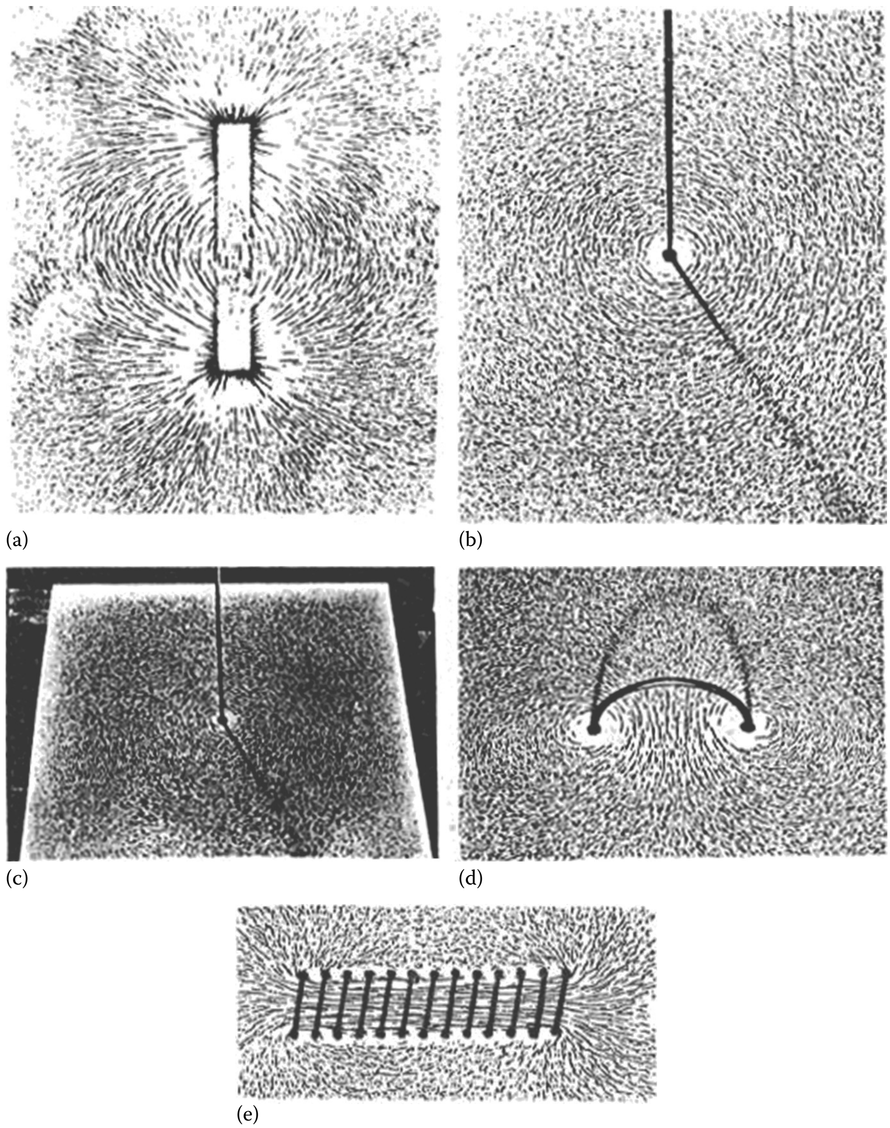

FIGURE 1.3 Magnetic field patterns in various situations obtained using iron filings: (a) a bar magnet, (b) a straight conductor carrying an electric current, (c) a perspective view of (b), (d) a single circular loop of conductor carrying a current, and (e) a solenoid with an air core.

conductor in a direction given by the right-hand corkscrew rule. The fields around a single current loop and a solenoid are similar to those around a bar magnet.

In a bar magnet, the field emerges from one end of the magnet, conventionally known as the *north pole* and passes through the air making a return path to the other end of the bar magnet, known conventionally as the *south pole*. We can think of the *north pole* of a magnet as a source of magnetic field $\boldsymbol{H}$ while the *south pole* behaves as a field sink. Whether such poles have any real existence is debatable. At present, the convention is to assume that such poles are fictitious, although the concept of the magnetic pole is very useful to those working with magnetic materials. The matter is discussed again in Section 2.1.2.

Notice that the magnetic field produced by a bar magnet is not identical to that of a solenoid. In particular, the magnetic field lines within the bar magnet run in the opposite direction to the field lines within the solenoid. We shall look at this again in Sections 2.3.1 and 2.3.2. It can be explained because the bar magnet has a magnetization $\boldsymbol{M}$ while the solenoid does not, and this magnetization leads to the generation of a magnetic dipole, which acts as a source and sink for the magnetic field.

1.1.7 Ampère's Circuital Law

How can we calculate the strength of a magnetic field generated by an electrical current?

Ampère deduced that a magnetic field is produced by an electrical charge in motion when he read of Oersted's discovery of the effect of an electric current on a compass needle. This was a rather remarkable conclusion considering that until then magnetic fields were known to be generated only by permanent magnets and the Earth and in neither case was the presence of electrical charge in motion obvious.

According to Ampère, the magnetic field generated by an electrical circuit depended on the shape of the circuit (i.e., the conduction path) and the current carried. By assuming that each circuit is made up of an infinite number of current elements, each contributing to the field, and by summing or integrating these contributions at a point to determine the field, Ampère arrived at the following result [2]:

$$Ni = \int_{\text{closed path}} \boldsymbol{H} \cdot d\mathbf{l} \tag{1.8}$$

where N is the number of current-carrying conductors, each carrying a current i amperes. This is the source of the magnetic field $\boldsymbol{H}$, and $\mathbf{l}$ is simply a line vector. The total current Ni equals the line integral of $\boldsymbol{H}$ around a closed path containing the current. We should note that this equation is only true for steady currents.

Ampère's law (Equation 1.8) and the Biot-Savart law can be shown to be equivalent. Consider the field due to a steady current flowing in a long current-carrying conductor. By the Biot-Savart rule, the field at a radial distance r from the conductor is

$$H = \frac{i}{2\pi r} \tag{1.9}$$

while from Ampère's circuital theorem

$$\int H \cdot d\mathbf{l} = i \tag{1.10}$$

and integrating along a closed path around the conductor at a radial distance r leads to

$$\int H \cdot d\mathbf{l} = 2\pi r H = i \tag{1.11}$$

$$H = \frac{i}{2\pi r} \tag{1.12}$$

Furthermore, Ampère's law, which we have really used to define $\boldsymbol{H}$ above, can be shown to be equivalent to one of Maxwell's equations of electromagnetism, specifically $\nabla \times \boldsymbol{H} = \boldsymbol{J}_f$, where $\boldsymbol{J}_f$ is the current density of conventional electrical currents.

1.1.8 Orders of Magnitudes of Magnetic Fields in Various Situations

How large or how small are magnetic fields in different situations?

The magnitudes of the magnetic field strengths in a variety of situations are shown in Table 1.1.

TABLE 1.1
Magnetic Field Strengths ($A\ m^{-1}$) in a Variety of Situations, Showing a Range of 19 Orders of Magnitude

10^{14}	Surface of neutron stars
10^{8}	Implosive magnets (microsecond duration)
$2–5 \times 10^{7}$	Pulsed electromagnets (microsecond duration)
$1–3 \times 10^{7}$	High field electromagnets
$1–1.5 \times 10^{7}$	Superconducting magnets
$1–2 \times 10^{6}$	Laboratory electromagnets
1×10^{6}	Strongest permanent magnets
10^{2}	Earth's magnetic field on the surface
10	Stray fields from electrical machinery
1	Urban magnetic noise level
5×10^{-2}	Contours for geomagnetic anomaly maps
10^{-4}	Magnetocardiograms
10^{-5}	Fetal heartbeat
10^{-6}	Magnetic field from human brain
10^{-8}	Limits of detection for superconducting quantum interference devices

1.2 MAGNETIC INDUCTION

How does a medium respond to a magnetic field?

When a magnetic field $\boldsymbol{H}$ has been generated in a medium by a current, in accordance with Ampère's law, the response of the medium is its magnetic induction $\boldsymbol{B}$, also sometimes called the *flux density*. All media will respond with some induction and, as we shall see, the relation between magnetic induction and magnetic field is a property called the *permeability of the medium*. For our purposes, we shall also consider free space to be a medium since a magnetic induction is produced by the presence of a magnetic field in free space.

1.2.1 Magnetic Flux

If a loop of conductor of cross-sectional area A *carries a current* i, *what is the energy associated with it?*

When current passes around a loop of conductor or a solenoid, there is a certain amount of energy associated with this current loop. The magnetic energy is given by

$$E = \frac{1}{2} i\Phi = \frac{1}{2} i^2 L \tag{1.13}$$

where:
- i is the current passing around the loop
- Φ is the amount of magnetic flux generated by the current
- L is the electrical inductance, which has a value $L = \mu_o \mu_r N^2 A/l$ for an air-cored solenoid with N turns, cross-sectional area A, and length l

This result links the energy with a magnetic quantity, flux, and as we shall see later, this will be useful in characterizing the *strength* of a magnetic dipole, such as a single current loop, a solenoid, or a bar magnet, in terms of the quantity Φ, the total magnetic flux that it generates, which can then be related directly to the energy stored in the magnetic dipole.

Whenever a magnetic field is present, there will be a magnetic flux Φ. This magnetic flux is measured in units of webers and its rate of change with time can be measured since it generates a voltage in a closed circuit of conductor through which the flux passes. Small magnetic particles such as iron filings align themselves along the direction of the magnetic flux as shown in Figure 1.3. We can consider the magnetic flux to be caused by the presence of a magnetic field in a medium. We shall see in Chapter 2 that the amount of flux generated by a given field strength depends on the properties of the medium and varies from one medium to another.

The weber: The weber is the amount of magnetic flux that when reduced uniformly to zero in 1 s produces an electromotive force of 1 V in a one-turn coil of conductor through which the flux passes.

1.2.2 Definition of Magnetic Induction

What is the unit of magnetic induction?

The magnetic induction $\boldsymbol{B}$ in webers per square meter is also known as the magnetic flux density and consequently a magnetic induction of 1 Wb m^{-2} is identical to a magnetic induction of 1 T. The magnetic induction is most usefully described in terms of the force on a moving electric charge or an electric current. If the induction is constant, then we can define the tesla as follows.

The tesla: A magnetic induction $\boldsymbol{B}$ of 1 T generates a force of 1 N m^{-1} on a conductor carrying a current of 1 A perpendicular to the direction of the induction.

This definition can be shown to be equivalent to the older definition of the tesla as the couple exerted in newtons per meter on a small current loop when its axis is normal to the field, divided by the product of the loop current and surface area. We shall see in Chapter 2 that there are two contributions to the magnetic induction, one from the magnetic field $\boldsymbol{H}$ and the other from the magnetization $\boldsymbol{M}$ of the medium.

There is often some confusion between the concept of the magnetic field $\boldsymbol{H}$ and the magnetic induction **B**, and since a clear idea of the important difference between these two is essential to the development of the subject presented here, a discussion of the difference is called for. In many media, $\boldsymbol{B}$ is a linear function of $\boldsymbol{H}$.

In particular, in free space, we can write

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} \tag{1.14}$$

where μ_0 is the permeability of free space, which is a universal constant. In the unit convention adopted in this book, $\boldsymbol{H}$ is measured in amperes per meter and $\boldsymbol{B}$ is measured in tesla (= V s m^{-2}), the units of μ_0 are therefore (volt second)/(ampere per meter), also known as *henries per meter*, and its value is $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}. If the value of $\boldsymbol{B}$ in free space is known, then $\boldsymbol{H}$ in free space is immediately known from this relationship.

However, in another media, particularly ferromagnets and ferrimagnets, $\boldsymbol{B}$ is no longer a linear function of $\boldsymbol{H}$, nor is it even a single-valued function of $\boldsymbol{H}$. In these materials, the distinction becomes readily apparent and important. A simple measurement of the $\boldsymbol{BH}$ loop of a ferromagnet should be all that is necessary to convince anyone of this. Finally, $\boldsymbol{H}$ and $\boldsymbol{B}$ are still related by the permeability of the medium μ through the following equation:

$$\boldsymbol{B} = \mu \boldsymbol{H} \tag{1.15}$$

but now of course μ is not necessarily a constant. We shall see shortly that in paramagnets and diamagnets μ is constant over a considerable range of values of $\boldsymbol{H}$. However, in ferromagnets, μ varies rapidly with $\boldsymbol{H}$.

All of this means that a field $\boldsymbol{H}$ in amperes per meter gives rise to a magnetic induction or flux density $\boldsymbol{B}$ in tesla in a medium with permeability μ measured in henries per meter.

1.2.3 Force per Unit Length on a Current-Carrying Conductor in a Magnetic Field: Ampère's Force Law

How does the presence of a magnetic induction affect the passage of an electric current?

The unit of magnetic induction has been defined in terms of the force exerted on a current-carrying conductor. This will now be generalized to obtain the force $\boldsymbol{F}$ on a current-carrying conductor in a magnetic induction $\boldsymbol{B}$. The force per meter on a conductor carrying a current i in the direction of the unit vector $\mathbf{l}$ caused by a magnetic induction $\boldsymbol{B}$ is given by an equation known as *Ampère's force law*

$$\boldsymbol{F} = i\mathbf{l} \times \boldsymbol{B} \tag{1.16}$$

and hence in free space,

$$\boldsymbol{F} = \mu_0 i\mathbf{l} \times \boldsymbol{H} \tag{1.17}$$

Therefore, if two long wires are arranged parallel at a distance of a meters apart and carrying currents of i_1 and i_2 amperes, the force per meter exerted by one wire on the other is as follows:

$$\boldsymbol{F} = \frac{\mu_0}{2\pi a} i_1 i_2 \tag{1.18}$$

1.2.4 Lines of Magnetic Induction

How can we visualize the magnetic induction?

The lines of magnetic induction are a geometrical abstraction, which help us to visualize the direction and strength of a magnetic field. The direction of the induction can be examined using a small compass needle (magnetic dipole) or a fine magnetic powder such as iron filings. These show that the magnetic induction around a single linear current-carrying conductor are coaxial with the conductor and follow the right-hand, or corkscrew, rule. In a solenoid, the lines are uniform within the solenoid but form a closed return path outside the solenoid. The lines of induction around a bar magnet are very similar to those around a solenoid since both act as magnetic dipoles.

The lines of magnetic induction always form a closed path since we have no direct evidence that isolated magnetic poles exist that would act as sources and sinks of magnetic flux. This means that through any closed surface, the amount of flux entering is equal to the amount of flux leaving. That is the divergence of $\boldsymbol{B}$ is always zero. This is Gauss's law.

$$\int_{\text{closed surface}} \boldsymbol{B} \cdot d\boldsymbol{A} = 0 \tag{1.19}$$

We sometimes say that $\boldsymbol{B}$ is *solenoidal*, which is the same as saying that the lines of $\boldsymbol{B}$ form closed paths—Equation 1.19 is equivalent to another of Maxwell's equations of electromagnetism, specifically $\nabla \cdot \boldsymbol{B} = 0$.

1.2.5 Electromagnetic Induction

Can the magnetic field generate an electrical current or voltage in return?

When the magnetic flux linking an electric circuit changes, an electromotive force is induced and this phenomenon is called *electromagnetic induction*. Faraday and Lenz were two of the early investigators of this effect, and from their work, we have the two laws of induction.

Faraday's law: The voltage induced in an electrical circuit is proportional to the rate of change of magnetic flux linking the circuit.
Lenz's law: The induced voltage is in a direction that opposes the flux change producing it.

The phenomenon of electromagnetic induction can be used to determine the magnetic flux Φ. The unit of magnetic flux is the weber, which has been chosen so that the rate of change of flux linking a circuit is equal to the induced electromotive force in volts.

$$V = -N\frac{d\Phi}{dt} \tag{1.20}$$

where:
Φ is the magnetic flux passing through a coil of N turns
$d\Phi/dt$ the rate of change of flux

Since the magnetic induction is the flux density,

$$\boldsymbol{B} = \frac{\Phi}{A} \tag{1.21}$$

we can rewrite the law of electromagnetic induction as

$$V = -NA\frac{d\boldsymbol{B}}{dt} \tag{1.22}$$

This is an important result since it tells us that an electrical voltage can be generated by a time-dependent magnetic induction. As an example, we can consider the voltage induced in a 50-turn coil of area 1 cm^2 when the magnetic induction linking it changes uniformly from 3 T to zero in 0.01 s.

$$\begin{aligned} V &= -NA\frac{d\boldsymbol{B}}{dt} \\ &= -\frac{(50)\left(1\times10^{-4}\right)(3)}{0.01} \\ &= 1.5\ \text{V} \end{aligned} \tag{1.23}$$

1.2.6 Magnetic Dipole

What is the most elementary entity in magnetism?

As shown above in Ampère's theorem, a current in an electrical circuit generates a field. A circular loop of a conductor carrying an electric current is the simplest circuit that can generate a magnetic field. The most elementary units of the equivalent current and pole models are the current loop and the dipole, as shown in Figure 1.4.

In each case, there is a magnetic moment $\boldsymbol{m}$ associated with the elementary unit of magnetism. In the current loop, the magnetic moment is equal to the product of the current i and the area of the loop A. In the magnetic dipole, the magnetic moment is the product of the pole strength p and the separation between the poles l.

The torque on a magnetic dipole of moment $\boldsymbol{m}$ in magnetic induction $\boldsymbol{B}$ is then simply

$$\tau = \boldsymbol{m} \times \boldsymbol{B} \tag{1.24}$$

This is a variant on Ampère's force law and can be derived directly using Equation 1.6. In free space,

$$\tau = \mu_0 \boldsymbol{m} \times \boldsymbol{H} \tag{1.25}$$

This means that the magnetic induction $\boldsymbol{B}$ tries to align the dipole so that the moment $\boldsymbol{m}$ lies parallel to the induction. Alternatively, we can consider that $\boldsymbol{B}$ tries to align the current loop such that the field produced by the current loop is parallel to it.

If no frictional forces are operating, the work done by the turning force will be conserved. This gives rise to the following expression for the energy of the dipole moment $\boldsymbol{m}$ in the presence of magnetic induction $\boldsymbol{B}$:

$$E = -\boldsymbol{m} \cdot \boldsymbol{B} \tag{1.26}$$

and hence in free space

$$E = -\mu_0 \boldsymbol{m} \cdot \boldsymbol{H} \tag{1.27}$$

The current loop is known as a *magnetic dipole* for historical reasons, since at large distances from the loop, the field produced by such a loop is identical in form to the field produced by calculation from two hypothetical magnetic poles of strength p separated by a distance l, the dipole moment of such an arrangement being

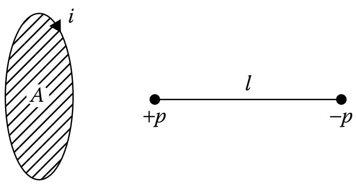

FIGURE 1.4 Configurations of the most basic mathematical entities: the current loop and the linear dipole.

$$\boldsymbol{m} = pl \tag{1.28}$$

We will see in Chapter 2 how important the concept of magnetic dipole moment is in the case of magnetic materials. For in that case, the electrical *current* is caused by the motion of electrons within the solid, particularly the spins of unpaired electrons, which generate a magnetic moment even in the absence of a conventional current.

1.2.7 Unit Systems in Magnetism

What unit systems are currently used to measure the various magnetic quantities?

There are currently three systems of units in widespread use in magnetism and several other systems of units, which are variants of these. The three unit systems are the Gaussian or CGS system and two MKS unit systems, the Sommerfeld and the Kennelly conventions (Table 1.2). Each of these unit systems has certain advantages and disadvantages. The CGS and SI systems of magnetic units have different philosophies. The CGS system took an approach based on magnetostatics and the concept of the *magnetic pole*, while the SI system takes an electrodynamic approach to magnetism based on electric currents. The SI system of units was adopted at the 11th General Congress on Weights and Measures (1960). The Sommerfeld convention was subsequently the one accepted for magnetic measurements by the International Union for Pure and Applied Physics (IUPAP), and therefore, this system has slowly been adopted by the magnetism community. This is the system of units used in this book.

TABLE 1.2
Principal Unit Systems Currently Used in Magnetism

Quantity		SI (Sommerfeld)	SI (Kennelly)	EMU (Gaussian)
Field	$\boldsymbol{H}$	A m^{-1}	A m^{-1}	oersteds
Induction	$\boldsymbol{B}$	tesla	tesla	gauss
Magnetization	$\boldsymbol{M}$	A m^{-1}	–	emu/cc
Intensity of magnetization	$\boldsymbol{I}$	–	tesla	–
Flux	Φ	weber	weber	maxwell
Moment	$\boldsymbol{m}$	A m^2	weber meter	emu
Pole strength	$\boldsymbol{p}$	A m	weber	emu cm^{-1}
Field equation		$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$	$\boldsymbol{B} = \mu_0\boldsymbol{H} + \boldsymbol{I}$	$\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M}$
Energy of moment (in free space)		$E = -\mu_0\boldsymbol{m}\cdot\boldsymbol{H}$	$E = -\boldsymbol{m}\cdot\boldsymbol{H}$	$E = -\boldsymbol{m}\cdot\boldsymbol{H}$
Torque on moment (in free space)		$\tau = \mu_0\boldsymbol{m} \times \boldsymbol{H}$	$\tau = \boldsymbol{m} \times \boldsymbol{H}$	$\tau = \boldsymbol{m} \times \boldsymbol{H}$

Note: The intensity of magnetization $\boldsymbol{I}$ used in the Kennelly system of units is merely an alternative measure of the magnetization $\boldsymbol{M}$, in which tesla is used instead of A m^{-1}. Under all circumstances, therefore, $\boldsymbol{I} = \mu_0\boldsymbol{M}$.

Conversion factors

$$1\ \text{Oe} = \left(\frac{1000}{4\pi}\right) \text{A m}^{-1} = 79.58\ \text{A m}^{-1} \quad (1.29)$$

$$1\ \text{gauss} = 10^{-4}\ \text{T} \quad (1.30)$$

$$1\ \text{emu cm}^{-3} = 1000\ \text{A m}^{-1} \quad (1.31)$$

To give some idea of the sizes of these units, the Earth's magnetic field is typically $\boldsymbol{H} = 56$ A m^{-1} (0.7 Oe), $\boldsymbol{B} = 0.7 \times 10^{-4}$ T. The saturation magnetization of iron is $\boldsymbol{M}_0 = 1.7 \times 10^6$ A m^{-1}. Remanence of iron is typically 0.8×10^6 A m^{-1}. The magnetic field generated by a large laboratory electromagnet is $\boldsymbol{H} = 1.6 \times 10^6$ A m^{-1}, $\boldsymbol{B} = 2$ T.

1.2.8 Maxwell's Equations of the Electromagnetic Field

Is there a more general and perhaps interconnected set of equations that can be used to describe electromagnetic fields?

In previous sections, we have alluded to a more general form of the Ampère/Biot-Savart law and Gauss's law. In fact, at the classical macroscopic level, electromagnetic fields can be described by four differential equations formulated by Maxwell. These are as follows:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \quad \text{(Ampère's law)} \quad (1.32)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \text{(Faraday's law)} \quad (1.33)$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad \text{(Gauss's law for magnetic flux density)} \quad (1.34)$$

$$\nabla \cdot \boldsymbol{D} = \rho \quad \text{(Gauss's law for electric flux density)} \quad (1.35)$$

where:

$\boldsymbol{E}$ is the electric field
$\boldsymbol{D}$ is the electric flux density
$\boldsymbol{J}$ is the current density
ρ is the charge density

The derivation of these equations is given in Appendix B. At this point in the discussion, we can treat them as simply experimental facts, since each was obtained as a result of observation. Only the first three are of concern to us in magnetism. Notice from these equations that there appears to be some correspondence between the magnetic field strength $\boldsymbol{H}$ and the electric field strength $\boldsymbol{E}$, and between the magnetic flux density **B** and the electric flux density **D**, and that there is one equation for each of **H**, **B**, **E**, and **D**.

1.2.9 Alternating or Time-Dependent Magnetic Fields

What happens if the currents that cause the magnetic field are not steady?

Maxwell's equations form the basis of the description of the electromagnetic field. They apply whether the fields are steady or time dependent, and therefore, the solution of the above equations provides a general result. Two special cases arise, however. First, if the frequency is low enough (typically $<10^{14}$ Hz), then the displacement current term $\partial D/\partial t$ is small, and so the first Maxwell equation reduces to the steady-state Ampère's law. This means that in most practical cases, $\nabla \times \boldsymbol{H}$ is determined by the free current density **J**. Second, if magnetic materials are present, the properties of the materials can significantly alter the penetration of magnetic fields into the material. The most familiar example of this is the penetration of a time-dependent field into an electrically conducting material. In this case, eddy currents are induced in the material by Maxwell's second equation (Faraday's law of induction) and this results in the alteration of the field amplitude with depth into the material. As the frequency of the exciting field increases, the rate of attenuation increases, resulting in a *penetration depth* or *skin depth*, which decreases with increasing frequency. However, this requires the presence of materials that we have not yet addressed in this book, so further discussion of this effect will be deferred.

Before proceeding further, we should derive the equation of a time-dependent magnetic field beginning from Maxwell's equations.

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{1.36}$$

and taking the curl of this

$$\nabla \times \nabla \times \boldsymbol{H} = \nabla \times \boldsymbol{J} + \nabla \times \frac{\partial \boldsymbol{D}}{\partial t} \tag{1.37}$$

now the current density **J** is related to the electric field **E** via the conductivity σ, $\boldsymbol{J} = \sigma \boldsymbol{E}$, and the electric flux density **D** is related to the electric field through the permittivity ε. Therefore, substituting for these and assuming that **E** is a well-behaved function, so that the $\partial D/\partial t$ and ∇ operators commute, we obtain

$$\nabla \times \nabla \times \boldsymbol{H} = \sigma \nabla \times \boldsymbol{E} + \in \frac{\partial}{\partial t}(\nabla \times \boldsymbol{E}) \tag{1.38}$$

and from Maxwell's equation for $\nabla \times \boldsymbol{E}$, we can substitute $-\partial B/\partial t$. Furthermore, $\nabla \times \nabla \times \boldsymbol{H} = \nabla\nabla \cdot \boldsymbol{H} - \nabla^2 \boldsymbol{H}$ for any vector quantity.

$$\nabla^2 \boldsymbol{H} - \nabla\nabla \cdot \boldsymbol{H} = \sigma \frac{\partial \boldsymbol{B}}{\partial t} + \in \frac{\partial^2 \boldsymbol{B}}{\partial t^2} \tag{1.39}$$

Now considering the special case of fields in free space $\boldsymbol{B} = \mu_0 \boldsymbol{H}$ and $\nabla \cdot \boldsymbol{B} = 0$, implying $\nabla \cdot \boldsymbol{H} = 0$ in free space. Therefore, the $\nabla\nabla \cdot \boldsymbol{H}$ term must be identically zero in free space:

$$\nabla^2 \boldsymbol{H} - \sigma\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} - \in \mu_0 \frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0 \tag{1.40}$$

This is a wave equation for the magnetic field $\boldsymbol{H}$, which is undamped when $\sigma = 0$, but is damped in any electrically conducting medium.

1.3 MAGNETIC FIELD CALCULATIONS

How are magnetic fields of known strength usually produced?

Magnetic fields are usually produced by solenoids or electromagnets. A solenoid is made by winding a large number of turns of insulated copper wire, or a similar electrical conductor, in a helical fashion on an insulated tube known as a *former*. Solenoids are often cylindrical in shape. An electromagnet is made in a similar way except that the windings are made on a soft ferromagnetic material, such as soft iron. The ferromagnetic core of an electromagnet generates a higher magnetic induction $\boldsymbol{B}$ than a solenoid for the same magnetic field $\boldsymbol{H}$.

In view of the widespread use of solenoid of various forms to produce magnetic fields, we shall take some time to examine the field strengths produced by a number of different configurations.

1.3.1 Field at the Center of a Long Thin Solenoid

What is the simplest way to produce a uniform magnetic field?

The simplest way to produce a uniform magnetic field is in a long, thin solenoid. If the solenoid has N turns wound on a former of length L and carries a current i amperes, the field inside it will be

$$\boldsymbol{H} = \frac{Ni}{L} = ni \tag{1.41}$$

where n is defined as the number of turns per unit length.

The magnetic field lines in and around a solenoid are shown in Figure 1.5. A practical method of making an *infinite* solenoid is to make the solenoid toroidal in shape. This ensures that the field is uniform throughout the length of the solenoid. The magnetic field is then

$$\boldsymbol{H} = \frac{N}{2\pi r} i \tag{1.42}$$

where:

N is the total number of turns
r is the radius of the toroid
i is the current flowing in the windings in amperes

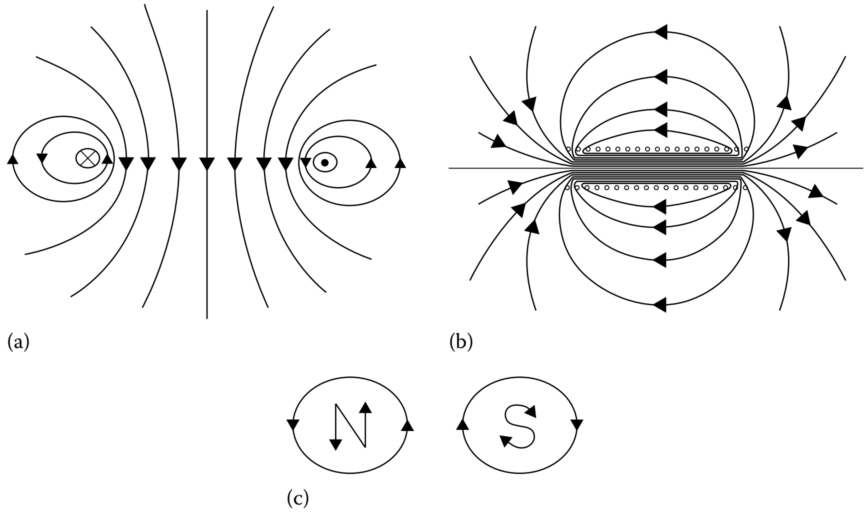

FIGURE 1.5 Magnetic field lines: (a) around a single loop of current carrying a conductor, (b) around a solenoid, and (c) convention for finding which end of a solenoid acts as the north pole (field source) and the south pole (field sink).

1.3.2 Magnetic Field along the Axis of a Circular Coil

What is the field strength along the axis of a single coil of wire carrying a current of i amperes?

The previous calculation of the field at the center of a circular coil can be generalized to obtain the exact expression for the magnetic field on the axis of a circular coil. Using the situation depicted in Figure 1.6, and applying the Biot-Savart rule, the field at the general point P is

$$d\boldsymbol{H} = \frac{1}{4\pi r^2} i\, d\mathbf{l} \times \mathbf{u} \tag{1.43}$$

where $\mathbf{u}$ is a unit vector along the r direction. We can make the substitution

$$r = \frac{a}{\sin\alpha} \tag{1.44}$$

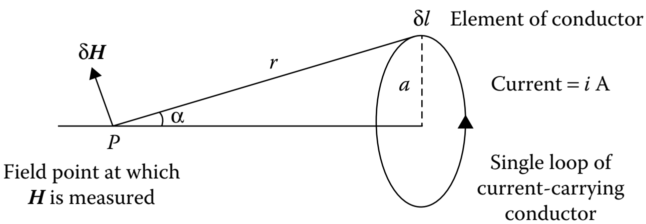

FIGURE 1.6 The magnetic field $\boldsymbol{H}$ due to a single circular coil carrying an electric current i.

which gives

$$dH = \frac{1}{4\pi a^2}\left(\sin^2\alpha\right) i\, d\mathbf{l} \times \mathbf{u} \tag{1.45}$$

The component of the field along the axis, which by symmetry will be the only resultant, is $d\boldsymbol{H}_{\text{axial}} = d\boldsymbol{H} \sin\alpha$

$$d\boldsymbol{H}_{\text{axial}} = \frac{1}{4\pi a^2}\left(\sin^3\alpha\right) i\, d\mathbf{l} \times \mathbf{u} \tag{1.46}$$

Integrating round the coil, $\int d\mathbf{l} = 2\pi a$ and remembering $d\mathbf{l}$ is perpendicular to $\mathbf{u}$

$$\boldsymbol{H} = \frac{i}{2a}\sin^3\alpha \tag{1.47}$$

or, equivalently,

$$\boldsymbol{H} = \frac{ia^2}{2\left(a^2 + x^2\right)^{3/2}} \tag{1.48}$$

where x is the distance along the axis of the coil from its center.

This can be expressed in the form of a series in x and by symmetry all terms of odd order must have zero coefficients so the form of the dipole field becomes

$$\boldsymbol{H} = \boldsymbol{H}_0\left(1 + c_2 x^2 + c_4 x^4 + c_6 x^6 + \cdots\right)$$

where $\boldsymbol{H}_0 = i/2a$ is the field at the center of the coil, and the coefficients have the values $c_2 = -3/2a^2$, $c_4 = 15/8a^4$, and $c_6 = -105/48a^6$.

As an example, we can consider a coil of 100 turns and diameter 0.1 m carrying a current of 0.1 A, and calculate the magnetic field at a distance of 50 cm along the axis of the coil.

When $i = 0.1$ A, $a = 5$ cm, $x = 50$ cm and the coil has 100 turns

$$\boldsymbol{H} = \frac{(100)(0.05)^2(0.1)}{2\left[(0.05)^2 + (0.5)^2\right]^{3/2}} = 0.098\,\text{A}\,\text{m}^{-1} \tag{1.49}$$

Off-axis field of a circular coil

As shown in the above derivations, a simple analytical expression can be obtained for the magnetic field along the axis of a single loop of conductor carrying a current using the Biot-Savart law. However, in the vast majority of cases, there is no closed-form analytic solution for the field generated by a current-carrying conductor. Those that can produce closed-form analytic solutions are only the very simplest types of situation.

To give an example, there is no closed-form analytic solution for the off-axis field of a single circular loop of conductor carrying a steady current, except for the far-field dipole field, which varies with $1/r^3$. This comes at first as somewhat of a surprise given the extreme simplicity of the situation. However, the example does show how very limited the situations are that yield analytical solutions.

In the case of the off-axis field of the single circular loop, the analysis leads to an elliptic integral that has no exact solution. From the Biot-Savart law, the magnetic field contribution at any point $d\boldsymbol{H}$ due to a current element $i\,d\mathbf{l}$ is

$$d\boldsymbol{H} = \frac{i d\mathbf{l} \times \mathbf{u}}{4\pi r^2} \tag{1.50}$$

where r is the distance from the coil.

$$d\boldsymbol{H} = \frac{i\,d\mathrm{l}\sin\theta}{4\pi\left(x^2 + a^2\right)} \tag{1.51}$$

where now a is also a function of θ instead of being a constant. In the case of the off-axis field, the field strength can be calculated from Equation 1.51 by using numerical techniques.

1.3.3 Field Due to Two Coaxial Coils

Which simple coil configurations produce: (i) a constant magnetic field, or (ii) a constant field gradient?

In superposition

Often, when it is necessary to produce a uniform field over a large volume of space, a pair of Helmholtz coils is used. This consists of two flat coaxial coils, each containing N turns, with the current flowing in the same sense in each coil as shown in Figure 1.7. The separation d of the coils in a Helmholtz pair is equal to their common radius a.

The axial component of the magnetic field on the axis of the two coils can be calculated from the Biot-Savart law. The field on the axis of a single coil of N turns and radius a carrying a current i at a distance x from the plane of the coil is

$$\boldsymbol{H} = \left(\frac{Ni}{2a}\right)\left(1 + \frac{x^2}{a^2}\right)^{-1.5} \tag{1.52}$$

If we define one coil at location $x = 0$ and the other at location $x = a$, the field at the center of two such coils wound in superposition is

$$\boldsymbol{H} = \left(\frac{Ni}{2a}\right)\left\{\left[1 + \frac{x^2}{a^2}\right]^{-1.5} + \left[1 + \frac{(a+x)^2}{a^2}\right]^{-1.5}\right\} \tag{1.53}$$

and since for the Helmholtz coils $x - a/2$ at the point on the axis midway between the coils, this gives the axial component of the magnetic field at the midpoint as

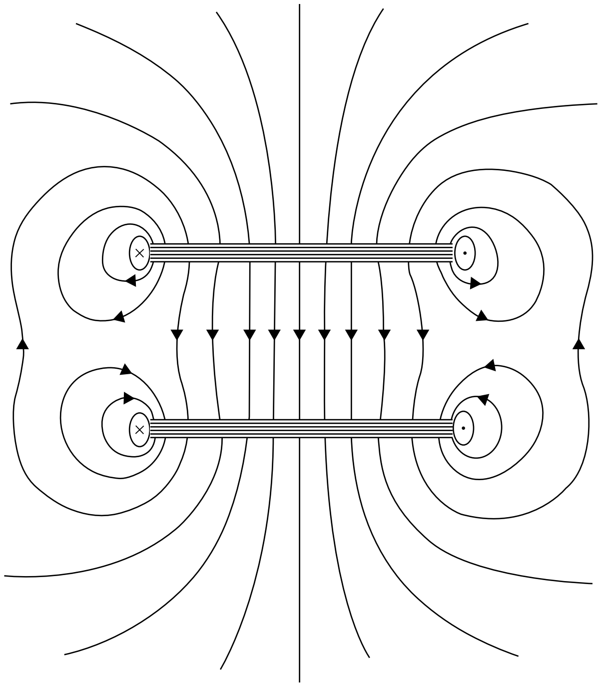

FIGURE 1.7 Two coaxial coils configured as a Helmholtz pair with the separation distance between the coils equal to their common radius.

$$\boldsymbol{H} = \left(\frac{Ni}{2a}\right)\left[(1.25)^{-1.5} + (1.25)^{-1.5}\right] = \frac{0.7155Ni}{a} \tag{1.54}$$

and by symmetry the radial component on the axis must be zero.

In fact, if a series expansion is made for the axial component of $\boldsymbol{H}$ in terms of the distance x along the axis from the center of the coils, as was given for the single coil in Section 1.3.2, it is found that the term in x^2 disappears when the coil separation d equals the coil radius a, so that the fourth-order correction term becomes the most significant. The series expansion for the field in terms of x is then

$$\boldsymbol{H} = \boldsymbol{H}_0\left(1 + c_4x^4 + c_6x^6 + \cdots\right)$$

This results in a small value of $d\boldsymbol{H}/dx$ at the center of the coils, and consequently a very uniform field along the axis as x is varied close to zero, which is shown in Figure 1.8 for three different values of coil radius a. In addition, the axial component of the magnetic field close to the center of a pair of Helmholtz coils is only very weakly dependent on the radial distance z from the axis. This means that the magnetic field strength $\boldsymbol{H}$ is maintained fairly constant over a large volume of space between the Helmholtz coils.

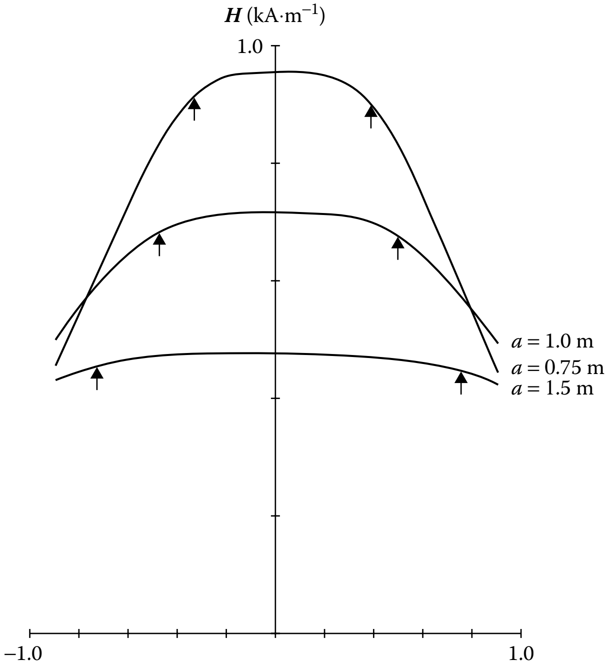

FIGURE 1.8 Axial component of the magnetic field $\boldsymbol{H}$ as a function of position along the axis of a pair of Helmholtz coils for various coil radii. The calculation is for $N = 100$ turns, with the coil carrying a current $i = 10$ A, with coil radii $a = 0.75$, 1, and 1.5 m. The arrows mark the location of the coils in each case.

The useful region of uniform field between a Helmholtz pair can be increased by making the coil spacing slightly larger than $a/2$, although this leads to a slight reduction in field strength over this region.

In opposition

If the current in one of the two coaxial coils described above is reversed, then the magnetic fields generated by the two coils will be in opposition. This is a specific example of a quadrupole field, so called because the form of the field obtained is similar to that obtained from two magnetic dipoles aligned coaxially and anti-parallel.

Under these conditions, the magnetic field along the axis of the pair of coils is given by

$$\boldsymbol{H} = \left(\frac{Ni}{2a}\right)\left[\left(1+\frac{x^2}{a^2}\right)^{-1.5} - \left(1+\frac{(a-x)^2}{a^2}\right)^{-1.5}\right] \tag{1.55}$$

Such a configuration generates a uniform field gradient, which can be useful for applying a constant force to a sample. See, for example, Section 3.2.2.

1.3.4 Field Due to a Thin Solenoid of Finite Length

What field strength is produced in the more practical case of a solenoid of limited length?

So far the field of an infinite solenoid has been considered. Now solenoids of finite length will be considered. A thin solenoid is one in which the inner and outer diameters of the coil windings are equal. So, for example, a solenoid consisting of one layer of windings would be considered as a thin solenoid.

The field of a long thin solenoid has already been calculated in Section 1.3.1. The field on the axis of a thin solenoid of finite length has an analytical solution. If L is the length of the solenoid, D the diameter, i the current in the windings, and x the distance from the center of the solenoid, then the field at x is given by

$$\boldsymbol{H} = \left(\frac{Ni}{L}\right)\left\{\frac{(L+2x)}{2\left[D^2+(L+2x)^2\right]^{1/2}} + \frac{(L-2x)}{2\left[D^2+(L-2x)^2\right]^{1/2}}\right\} \tag{1.56}$$

At the center of the solenoid, $x = 0$ and hence

$$\boldsymbol{H} = \left(\frac{Ni}{L}\right)\left[\frac{L}{\left(L^2+D^2\right)^{1/2}}\right] \tag{1.57}$$

Finally, for a long solenoid, $L >> D$ and $(L^2 + D^2)^{1/2} = L$, so that the result from Section 1.3.1 is a limiting case:

$$\boldsymbol{H} = \frac{Ni}{L} = ni \tag{1.58}$$

The fields generated by solenoids are of course dipole fields.

The field calculations for thin solenoids, that is, solenoids with $L >> D$, at least along the axis, are relatively straightforward and yield analytical solutions as shown. A useful result to remember is that the field at the end of a solenoid is half the value of the field at the center. The field in the middle 50% of a solenoid is also known to be very uniform.

1.3.5 General Formula for the Field of a Solenoid

How can the field strength generated by a solenoid be determined in more general cases?

Not surprisingly, there is no general analytical formula for the magnetic field from a solenoid at a general point in space. However, there are some methods of calculation available. The most obvious method is by the straightforward procedure of using either Ampère's law or the Biot-Savart law, as in Section 1.1.3. This leads to a solution containing an elliptic integral, which can then be solved numerically.