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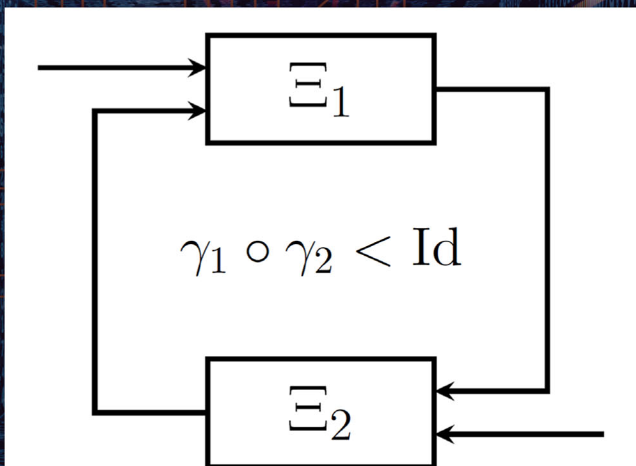

Tengfei Liu • Zhong-Ping Jiang
David J. Hill

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Nonlinear Control of Dynamic Networks

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20140224

International Standard Book Number-13: 978-1-4665-8460-0 (eBook - PDF)

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Dedication

This work is dedicated to
Lina and Debbie (TFL)
Xiaoming, Jenny, and Jack (ZPJ)
Gloria (DJH)

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Preface

The rapid development of computing, communications, and sensing technologies has been enabling new potential applications of advanced control of complex systems like smart power grids, biological processes, distributed computing networks, transportation systems, and robotic networks. Significant problems are to integrally deal with the fundamental system characteristics such as nonlinearity, dimensionality, uncertainty, and information constraints, and diverse kinds of networked behaviors, which may arise from quantization, data sampling, and impulsive events.

Physical systems are inherently nonlinear and interconnected in nature. Significant progress has been made on nonlinear control systems in the past three decades. However, new system analysis and design tools that are capable of addressing more communication and networking issues are still highly desired to handle the emerging theoretical challenges underlying the new engineering problems. As an example, small quantization errors may cause the performance of a "well-designed" nonlinear control system to deteriorate. The need for new tools motivates this book, the purpose of which is to present a set of novel analysis and design tools to address the newly arising theoretical problems from the viewpoint of dynamic networks. The results are intended to help solve real-world nonlinear control problems, including quantized control and distributed control aspects.

In this book, dynamic networks are regarded as systems composed of structurally interconnected subsystems. Such systems often display complex dynamic behaviors. The control problem of such a complex system could be simplified with the notion of a dynamic network if the subsystems have some common characteristic which, together with the structural feature of the dynamic network, can guarantee the achievement of the control objective. For the research in this book, one such characteristic is Sontag's input-to-state stability (ISS), based on which, refined small-gain theorems are extremely useful in solving control problems of complex systems by taking advantage of the structural feature.

By bridging the gap between the stability concepts defined in the input–output and the state–space contexts, the notion of ISS has proved to be extremely useful in analysis and control design of nonlinear systems with the influence of external inputs represented by nonlinear gains. Its essential relationship with robust stability provides an effective approach to robust control by means of input-to-state stabilization. For a dynamic network of ISS subsystems, the small-gain theorem is capable of testing the overall ISS by directly checking compositions of the ISS gains of the subsystems. Based on the ISS small-gain theorem, complex systems can be input-to-state stabilized by appropriately designing the subsystems.

This book is based on the authors' recent research results on nonlinear control of dynamic networks. In particular, it contains refined small-gain results for dynamic networks and their applications in solving the control problems of nonlinear uncertain systems subject to sensor noise, quantization error, and information exchange constraints. The widely known Lyapunov functions approach is mainly used for proofs and discussions. The relationship between the new tools and the existing nonlinear control methods is highlighted. In this way, not only control researchers but also students interested in related topics may understand and use the tools for control designs.

The organization of the book is as follows. To make the book self-contained, Chapter 1 provides some prerequisite knowledge on useful characteristics of Lyapunov stability and ISS. Chapter 2 presents ISS small-gain results for interconnected systems composed of two subsystems. Both trajectory-based and Lyapunov-based formulations of the ISS small-gain theorem are reviewed with proofs. For dynamic networks that may contain more than two subsystems, Chapter 3 introduces more readily usable cyclic-small-gain methods to reduce the complexity of analysis and control design problems for more general dynamic networks. Detailed proofs of some of the background theorems in Chapters 1–3, which need a higher level of mathematical sophistication and are available in the literature, are not provided. However, the basic ideas are highlighted.

The applications of the cyclic-small-gain theorem to nonlinear control designs are studied in Chapters 4–6. Specifically, Chapter 4 investigates the important measurement feedback control problem for uncertain nonlinear systems with disturbed measurements. In Chapter 5, the quantized nonlinear control problem is studied. Chapter 6 discusses the distributed control problem for coordination of groups of nonlinear systems under information exchange constraints. The control problems are transformed into input-to-state stabilization problems of dynamic networks, and the influence of the uncertain sources, i.e., sensor noise, quantization, and information exchange constraints, are explicitly evaluated and attenuated by new cyclic-small-gain designs.

Certainly, most of the results presented in this book can be extended for more general systems. Some of the easier extensions mentioned in the book are not thoroughly discussed and may be used as exercises for interested readers. Several future challenges in this research direction are outlined in Chapter 7. The Appendix gives supplementary materials on graph theory and discontinuous systems, and the proofs of the technical lemmas which seem too mathematical to be placed in the main chapters. Finally, historical discussion will be confined to brief notes at appropriate points in the text.

TFL wishes to express his sincere gratitude to his coauthors, Professor Zhong-Ping Jiang and Professor David Hill, who are also TFL's postdoctoral adviser and PhD supervisor, respectively. They introduced TFL to nonlinear control, drew his attention to ISS and small-gain, and offered him precious opportunities for working on the frontier research subjects in the field. Their persistent support, expert guidance, and willingness to share wisdom have been invaluable for TFL's academic career. TFL would also like to give thanks to his previous and current labmates in Canberra and New York. He has benefited a lot from the discussions/debates with them. TFL would need more than a lifetime to thank his wife, Lina Zhang, for her understanding and patience and their daughter, Debbie Liu, for a lot of happiness.

ZPJ would like to thank, from the bottom of his heart, all his coauthors and friends for sharing their passion for nonlinear control. The ideas and methods presented in this book truly reflect their wisdom and vision for the nonlinear control of dynamic networks. Special thanks go to Iven Mareels, Laurent Praly (his former PhD adviser), Andy Teel, and Yuan Wang for collaborations on the very first, nonlinear ISS small-gain theorems, and to Hiroshi Ito, Iasson Karafyllis, Pierdomenico Pepe, and again Yuan Wang for recent joint work on various extensions of the small-gain theorem for dynamic networks. Finally, it is only under the strong and constant support and love of his family that ZPJ can discover the beauty of nonlinear feedback and control, while having fun doing research.

DJH firstly thanks his coauthors for their hard work and collaboration throughout the research leading to this book. It has been a pleasure to see the ideas of state-space-based small-gain theorems progress through all our PhD theses, as well as work with colleagues—with special mention of Iven Mareels, and now into this book. (As a memorial note, his thesis was over 30 years ago, following the seminal work on dissipative systems by Jan Willems, who sadly passed away during our writing.) Personally, DJH would like to thank his wife, Gloria Sunnie Wright, whose positive supportive approach and excitement for life are a perfect match for an academic who (as she often hears) has "got to run" for deadlines.

The authors are grateful to the series editors, Frank Lewis and Sam Ge, for the opportunity to publish the book. The authors would also like to thank the editorial staff, in particular, Nora Konopka, Michele Smith, Amber Donley, Michael Davidson, John Gandour, and Shashi Kumar, of Taylor & Francis for their efforts in publishing the book.

The research presented in this book was supported partly by the NYU-Poly Faculty Fellowship provided to the first author during his visit at the Polytechnic Institute of New York University, partly by the U.S. National Science Foundation and by the Australian Research Council.

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A Fellow of both the IEEE and IFAC, Dr. Jiang is an editor for the *International Journal of Robust and Nonlinear Control* and has served as an associate editor for several journals including *Mathematics of Control, Signals and Systems* (MCSS), *Systems and Control Letters*, *IEEE Transactions on Automatic Control*, *European Journal of Control*, and *Science China: Information Sciences.* Dr. Jiang is a recipient of the prestigious Queen Elizabeth II Fellowship Award from the Australian Research Council, the CAREER Award from the U.S. National Science Foundation, and the Young Investigator Award from the NSF of China. He received the Best Theory Paper Award (with Y. Wang) at the 2008 WCICA, and with T. Liu and D.J. Hill, the Guan Zhao Zhi Best Paper Award at the 2011 CCC. The paper with his PhD student Y. Jiang entitled "Robust Adaptive Dynamic Programming for Optimal Nonlinear Control Design" received the Shimemura Young Author Prize at the 2013

Asian Control Conference in Istanbul, Turkey.

Professor David J. Hill received BE (electrical engineering) and BSc (mathematics) degrees from the University of Queensland, Australia, in 1972 and 1974, respectively. He received a PhD degree in electrical engineering from the University of Newcastle, Australia, in 1977.

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His general research interests are in control systems, complex networks, power systems, and stability analysis. His work is now mainly on control and planning of future energy networks and basic stability questions for dynamic networks.

Professor Hill is a Fellow of the Institute of Electrical and Electronics Engineers, USA, the Society for Industrial and Applied Mathematics, USA, the Australian Academy of Science, and the Australian Academy of Technological Sciences and Engineering. He is also a foreign member of the Royal Swedish Academy of Engineering Sciences.

Notations

$\mathbb{C}$	The set of complex numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}_+$	The set of nonnegative real numbers
$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathbb{Z}$	The set of integers
$\mathbb{Z}_+$	The set of nonnegative integers
$\mathbb{N}$	The set of natural numbers
x^T	The transpose of vector x
$\|x\|$	Euclidean norm of vector x
$\|A\|$	Induced Euclidean norm of matrix A
$\mathrm{sgn}(x)$	The sign of $x \in \mathbb{R}$: $\mathrm{sgn}(x) = 1$ if $x > 0$; $\mathrm{sgn}(x) = 0$ if $x = 0$; $\mathrm{sgn}(x) = -1$ if $x < 0$
$a \bmod b$	Remainder of the Euclidean division of a by b for $a \in \mathbb{R}, b \in \mathbb{R}\backslash\{0\}$
$\\|u\\|_\Delta$	$\operatorname{ess\,sup}_{t\in\Delta} \|u(t)\|$ with $\Delta \subseteq \mathbb{R}_+$ for $u : \mathbb{R}_+ \to \mathbb{R}^n$
$\\|u\\|_\infty$	$\\|u\\|_\Delta$ with $\Delta = [0, \infty)$
$:=$ or $\stackrel{\mathrm{def}}{=}$	Equal by definition
$\equiv$	Identically equal
$f \circ g$	Composition of functions f and g
$\lambda_{\max}$ ($\lambda_{\min}$)	Largest (smallest) eigenvalue
t^+ (t^-)	Time right after (right before) t
∂	Partial derivative
$\nabla V(x)$	Gradient vector of function V at x
Id	The identity function
$\mathcal{B}^n$	The unit ball centered at the origin in $\mathbb{R}^n$
$\mathrm{cl}(\mathcal{S})$	The closure of set $\mathcal{S}$
$\mathrm{int}(\mathcal{S})$	The interior of set $\mathcal{S}$
$\mathrm{co}(\mathcal{S})$	The convex hull of set $\mathcal{S}$
$\overline{\mathrm{co}}(\mathcal{S})$	The closed convex hull of set $\mathcal{S}$
$\mathrm{dom}(F)$	The domain of map F
$\mathrm{graph}(F)$	The graph of map F
$\mathrm{range}(F)$	The range of map F

ABBREVIATIONS

AG	Asymptotic Gain
AS	Asymptotic Stability
GAS	Global Asymptotic Stability
GS	Global Stability
IOpS	Input-to-Output Practical Stability

IOS	Input-to-Output Stability
ISpS	Input-to-State Practical Stability
ISS	Input-to-State Stability
OAG	Output Asymptotic Gain
RS	Robust Stability
UBIBS	Uniform Bounded-Input Bounded-State Stability
UO	Unboundedness Observability
WRS	Weakly Robust Stability

1 Introduction

1.1 CONTROL PROBLEMS WITH DYNAMIC NETWORKS

The basic idea for control of dynamic networks is to consider complex systems as structural interconnections of subsystems with specific properties, and solve their control problems using the subsystem and structural features. Such ideas can be traced back to the original development of circuit theory. The rapid development of computing, communication, and sensing technology has enabled new potential applications of advanced control to complex systems. Significant problems are to integrally deal with the fundamental system characteristics, such as nonlinearity, dimensionality, uncertainty and information constraints, and diverse kinds of networked behaviors like quantization, data sampling, and impulsive events. With the development of new tools, this book studies the analysis and control problems of complex systems from the viewpoint of dynamic networks.

Even the single-loop control system may be considered as a dynamic network if detailed behaviors of the sensor and the actuator are taken into account. In a typical single-loop state-feedback control system, as shown in Figure 1.1, the state of the plant is measured by the sensor and sent to the controller, which computes the needed control actions. These are implemented by the actuator for a desired behavior of the plant. A key issue with control systems is stability. By designing an asymptotically stable control system, the error between the actual state signal and a desired signal is expected to converge to zero ultimately.

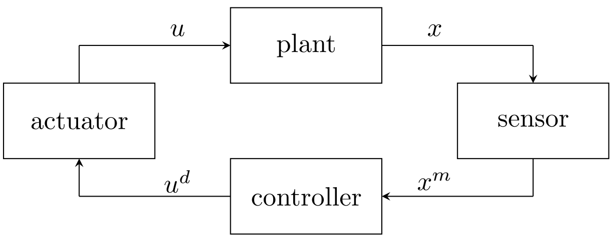

FIGURE 1.1 State-feedback control system: x is the state of the plant, u is the control input, x^m is the measurement of x, and u^d is the desired control input computed by the controller.

Practical control systems are inevitably subject to uncertainties, which may be caused by the sensing and actuation components, and the unmodeled dynamics of the plant. By considering a control system as an interconnection of the perturbation-free nominal system and the perturbation terms, the basic idea of robust control is to design the nominal system to be robust to the perturbations.

Based on this idea, a linear state-feedback control system

$$\dot{x} = Ax + Bu \tag{1.1}$$

$$u^d = -Kx^m \tag{1.2}$$

can be rewritten as the closed-loop nominal system with the perturbation terms:

$$\begin{aligned}\dot{x} &= Ax + B(-K(x + \tilde{x}) - \tilde{u}) \\ &= (A - BK)x - BK\tilde{x} - B\tilde{u},\end{aligned} \tag{1.3}$$

where $\tilde{x} = x^m - x$ and $\tilde{u} = u^d - u$. Suppose that the control objective is to make the system practically stable at the origin, i.e., to steer the state x to within a specific bounded neighborhood of the origin. If $\tilde{x}, \tilde{u}$ are bounded, then such an objective can be achieved if $(A - BK)$ is Hurwitz, i.e., all the eigenvalues of $(A - BK)$ are on the open left-half of the complex plane. For such a linear system, we can study the influence of $\tilde{u}$ and $\tilde{x}$ separately, due to the well-known Superposition Principle. If the eigenvalues of $(A - BK)$ can be arbitrarily assigned by an appropriate choice of K (with complex eigenvalues occurring in conjugate pairs), then the influence of $\tilde{u}$ can be attenuated to within an arbitrarily small level. But this may not be the case for the perturbation term $BK\tilde{x}$ (because it depends on K).

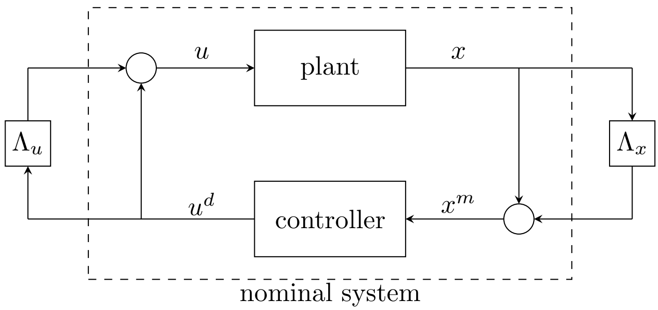

FIGURE 1.2 Robust control configuration, where Λ_u, Λ_x represent perturbation terms.

The problem can still be handled even if the perturbation terms may not be bounded. For example, they can satisfy the following properties:

$$|\tilde{x}| \leq \bar{\delta}_x |x| + \bar{c}_x \tag{1.4}$$

$$|\tilde{u}| \leq \bar{\delta}_u |u^d| + \bar{c}_u, \tag{1.5}$$

where $\bar{\delta}_x, \bar{c}_x, \bar{\delta}_u, \bar{c}_u$ are nonnegative constants. Such perturbations are said to have the sector bound property.

One may denote $\tilde{x} = \delta_x(t)x + c_x(t)$ and $\tilde{u} = \delta_u(t)K(1 + \delta_x(t))x + \delta_u(t)Kc_x(t) + c_u(t)$ with $|\delta_x(t)| \leq \bar{\delta}_x$, $|\delta_u(t)| \leq \bar{\delta}_u$, $|c_x(t)| \leq \bar{c}_x$, $|c_u(t)| \leq \bar{c}_u$ for $t \geq 0$. Then, system (1.3) can be represented by $\dot{x} = (A - BK)x + w$ with

$$\begin{aligned} w(t) &= -BK(\delta_x(t) + \delta_u(t)(1 + \delta_x(t)))x(t) \\ &\quad - B(Kc_x(t) + K\delta_u(t)c_x(t) + c_u(t)) \\ &:= \phi(x(t), c_x(t), c_u(t), t). \end{aligned} \tag{1.6}$$

It can be directly checked that $|\phi(x, c_x, c_u, t)| \leq a_1|x| + a_2|c_x| + a_3|c_u|$ for all $t \geq 0$ with constants $a_1, a_2, a_3 \geq 0$.

As shown in Figure 1.3, the system is transformed into the interconnection of the nominal system and the perturbation term. There have been standard methods to solve this kind of problem in robust linear control theory [288]. One of them is the classical small-gain theorem, due to Sandberg and Zames. Interested readers may consult [48, Chapter 5] and [54, Chapter 4] for the details. See also [207, Section V] for a small-gain result of large-scale systems.

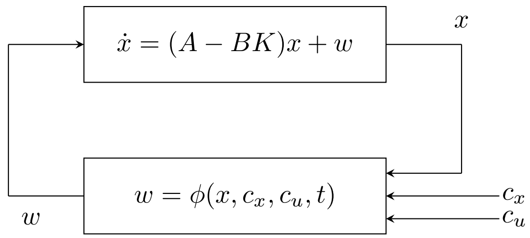

FIGURE 1.3 An interconnected system.

For genuinely nonlinear control systems, the problems discussed above will typically be more complicated. Consider the popular strict-feedback nonlinear system:

$$\dot{x}_i = \Delta_i(\bar{x}_i, w) + x_{i+1}, \quad i = 1, \ldots, n-1 \tag{1.7}$$

$$\dot{x}_n = \Delta_n(\bar{x}_n, w) + u, \tag{1.8}$$

where $[x_1, \ldots, x_n]^T := x \in \mathbb{R}^n$ is the state, $\bar{x}_i = [x_1, \ldots, x_i]^T$, $u \in \mathbb{R}$ is the control input, $w \in \mathbb{R}^{n_w}$ represents the external disturbances, and $\Delta_i : \mathbb{R}^i \to \mathbb{R}$ for $i = 1, \ldots, n$ are locally Lipschitz functions. For this system, we consider x_1 as the output. Recursive designs have proved to be useful for the control of such system; see e.g., [153, 235, 151]. By representing the system as a dynamic network composed of x_i-subsystems for $i = 1, \ldots, n$, the basic idea is to recursively design control laws for the $\bar{x}_i$-subsystems by considering x_{i+1} as the control inputs until the true control input u occurs. For such system, the influence of the disturbance w might be amplified through the numerous interconnections between the subsystems as shown in Figure 1.4. The problem would be more complicated if the system is subject to sensor noise. As shown in Example 4.1, even for a first-order nonlinear system, small sensor noise may drive the system state to infinity, although the state of the noise-free system asymptotically converges to the origin. Quantized control provides another interesting example for robust control of nonlinear systems with measurement errors satisfying the sector bound property; see Section 5.1 for details.

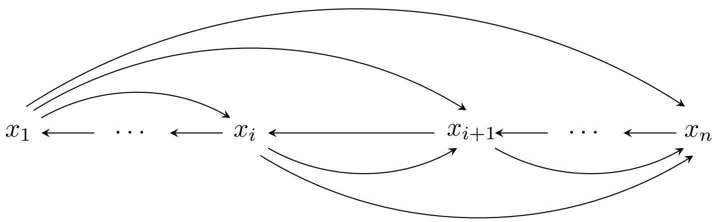

FIGURE 1.4 A high-order nonlinear system as a dynamic network.

Dynamic networks also occur in distributed control of interconnected systems, for which each i-th subsystem takes the following form:

$$\dot{x}_i = f_i(x_i, u_i) \tag{1.9}$$

$$y_i = h_i(x_i), \tag{1.10}$$

where $y_i \in \mathbb{R}^{p_i}$ is the output, $x_i \in \mathbb{R}^{n_i}$ is the state, $u_i \in \mathbb{R}^{m_i}$ is the control input, and $f_i : \mathbb{R}^{m_i+n_i} \to \mathbb{R}^{n_i}, h_i : \mathbb{R}^{n_i} \to \mathbb{R}^{p_i}$ are properly defined functions. In a distributed control structure, each subsystem may be equipped with a controller. Through information exchange, the controllers for the subsystems coordinate with each other, and the outputs of the subsystems achieve some desired group behavior, e.g., $\lim_{t\to\infty}(y_i(t) - y_j(t)) = 0$. In this case, the interconnections in the dynamic network are formed by the information exchange between the controllers, and in some cases, e.g., power systems and telephone networks, by direct physical interconnections.

This book develops new design tools for nonlinear control of dynamic networks, which are applicable to measurement feedback control, quantized control, and distributed control. With the new tools, the related control problems can be transformed into solvable stability problems of dynamic networks composed of subsystems admitting the input-to-state stability (ISS) property or the more general input-to-output stability (IOS) property. To introduce these basic notions, we begin with Lyapunov stability for systems without external inputs.

1.2 LYAPUNOV STABILITY

The stabilization problem is one of the most important problems in control theory. In general terms, a control system is stabilizable if one can find a control law which makes the closed-loop system stable at an equilibrium point. This section reviews some basic concepts of Lyapunov stability [195, 78, 144] for systems with no external inputs.

The comparison functions defined below are used to characterize Lyapunov stability and the notions of ISS and IOS.

Definition 1.1 *A function* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *is said to be positive definite if* $\alpha(0) = 0$ *and* $\alpha(s) > 0$ *for* $s > 0$.

Definition 1.2 *A continuous function* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *is said to be a class* $\mathcal{K}$ *function, denoted by* $\alpha \in \mathcal{K}$, *if it is strictly increasing and* $\alpha(0) = 0$*; it is said to be a class* $\mathcal{K}_\infty$ *function, denoted by* $\alpha \in \mathcal{K}_\infty$, *if it is a class* $\mathcal{K}$ *function and satisfies* $\alpha(s) \to \infty$ *as* $s \to \infty$.

Definition 1.3 *A continuous function* $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *is said to be a class* $\mathcal{KL}$ *function, denoted by* $\beta \in \mathcal{KL}$, *if, for each fixed* $t \in \mathbb{R}_+$, *function* $\beta(\cdot, t)$ *is a class* $\mathcal{K}$ *function and, for each fixed* $s \in \mathbb{R}_+$, *function* $\beta(s, \cdot)$ *is decreasing and satisfies* $\lim_{t\to\infty} \beta(s, t) = 0$.

For convenience of the further discussions, we also give the following definitions on Lipschitz continuity.

Definition 1.4 *A function* $h : \mathcal{X} \to \mathcal{Y}$ *with* $\mathcal{X} \subseteq \mathbb{R}^n$ *and* $\mathcal{Y} \subseteq \mathbb{R}^m$ *is said to be Lipschitz continuous, or simply Lipschitz, on* $\mathcal{X}$, *if there exists a constant* $L_h \geq 0$, *such that for any* $x_1, x_2 \in \mathcal{X}$,

$$|h(x_1) - h(x_2)| \leq L_h |x_1 - x_2|. \tag{1.11}$$

Definition 1.5 *A function* $h : \mathcal{X} \to \mathcal{Y}$ *with* $\mathcal{X} \subseteq \mathbb{R}^n$ *being open and connected, and* $\mathcal{Y} \subseteq \mathbb{R}^m$ *is said to be locally Lipschitz on* $\mathcal{X}$, *if each* $x \in \mathcal{X}$ *has a neighborhood* $\mathcal{X}_0 \subseteq \mathcal{X}$ *such that* h *is Lipschitz on* $\mathcal{X}_0$.

Definition 1.6 *A function* $h : \mathcal{X} \to \mathcal{Y}$ *with* $\mathcal{X} \subseteq \mathbb{R}^n$ *and* $\mathcal{Y} \subseteq \mathbb{R}^m$ *is said to be Lipschitz on compact sets, if* h *is Lipschitz on every compact set* $\mathcal{D} \subseteq \mathcal{X}$.

Consider the system

$$\dot{x} = f(x), \tag{1.12}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function. Assume that the origin is an equilibrium of the nonlinear system, i.e., $f(0) = 0$. Note that if an equilibrium other than the origin, say x^e, is of interest, one may use a coordinate transformation $x' = x - x^e$ to move the equilibrium to the origin. Therefore, the assumption of the equilibrium at the origin is with no loss of generality. Denote $x(t, x_0)$ or simply $x(t)$ as the solution of system (1.12) with initial condition $x(0) = x_0$, and let $[0, T_{\max})$ with $0 < T_{\max} \leq \infty$ be the right maximal interval for the definition of $x(t, x_0)$.

The standard definition of Lyapunov stability is usually given by using "ϵ-δ" terms, which can be found in the standard textbooks on nonlinear systems; see, e.g., [78] and [144, Chapter 4]. Definition 1.7 employs the comparison functions $\alpha \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ for convenience of the comparison between Lyapunov stability and ISS. A proof of the equivalence between the standard definition and Definition 1.7 can be found in [144, Appendix C.6]. See also the discussions in [78, Definitions 2.9 and 24.2].

Definition 1.7 *System* (1.12) *is*

- *stable at the origin if there exist an $\alpha \in \mathcal{K}$ and a constant $c > 0$ such that for any $|x_0| \leq c$,*

$$|x(t, x_0)| \leq \alpha(|x_0|) \tag{1.13}$$

  for all $t \geq 0$;
- *globally stable (GS) at the origin if property* (1.13) *holds for all initial states $x_0 \in \mathbb{R}^n$;*
- *asymptotically stable (AS) at the origin if there exist a $\beta \in \mathcal{KL}$ and a constant $c > 0$ such that for any $|x_0| \leq c$,*

$$|x(t, x_0)| \leq \beta(|x_0|, t) \tag{1.14}$$

  for all $t \geq 0$;
- *globally asymptotically stable (GAS) at the origin if condition* (1.14) *holds for any initial state $x_0 \in \mathbb{R}^n$.*

With the standard definition, GAS at the origin can be defined based on GS by adding the global convergence property at the origin: $\lim_{t\to\infty} x(t, x_0) = 0$ for all $x_0 \in \mathbb{R}^n$; see [144, Definition 4.1]. It can be observed that GAS is more than global convergence.

Theorem 1.1, which is known as Lyapunov's Second Theorem (or the Lyapunov Direct Method), gives sufficient conditions for stability and AS.

Theorem 1.1 *Let the origin be an equilibrium of system* (1.12) *and $\Omega \subset \mathbb{R}^n$ be a domain containing the origin. Let $V : \Omega \to \mathbb{R}_+$ be a continuously differentiable function such that*

$$V(0) = 0, \tag{1.15}$$

$$V(x) > 0 \text{ for } x \in \Omega\backslash\{0\}, \tag{1.16}$$

$$\nabla V(x) f(x) \leq 0 \text{ for } x \in \Omega. \tag{1.17}$$

Then, system (1.12) *is stable at the origin. Moreover, if*

$$\nabla V(x) f(x) < 0 \text{ for } x \in \Omega\backslash\{0\}, \tag{1.18}$$

then system (1.12) *is AS at the origin.*

A function V that satisfies (1.15)–(1.17) is called a Lyapunov function. If moreover, V satisfies (1.18), then it is called a strict Lyapunov function [16].

It is natural to ask whether the condition for AS in Theorem 1.1 can guarantee GAS by directly replacing the Ω with $\mathbb{R}^n$. Example 1.1, which was given in [78, p. 109], answers this question.

Example 1.1 *Consider system*

$$\dot{x}_1 = \frac{-6x_1}{(1+x_1^2)^2} + 2x_2, \tag{1.19}$$

$$\dot{x}_2 = \frac{-2(x_1+x_2)}{(1+x_1^2)^2}. \tag{1.20}$$

Let

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2. \tag{1.21}$$

It can be directly verified that V satisfies all the conditions for AS at the origin given by Theorem 1.1 with $n = 2$ and $\Omega = \mathbb{R}^2$. By testing the vector field on the boundary of hyperbola $x_2 = 2/(x_1 - \sqrt{2})$, the trajectories to the right of the branch in the first quadrant cannot cross that branch. This means that the system is not GAS at the origin.

Theorem 1.2 gives extra conditions on the Lyapunov function V for GAS.

Theorem 1.2 *Let the origin be an equilibrium of system* (1.12)*. Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable function such that*

$$V(0) = 0, \tag{1.22}$$

$$V(x) > 0 \text{ for } x \in \mathbb{R}^n \backslash \{0\}, \tag{1.23}$$

$$|x| \to \infty \Rightarrow V(x) \to \infty, \tag{1.24}$$

$$\nabla V(x) f(x) < 0 \text{ for } x \in \mathbb{R}^n \backslash \{0\}. \tag{1.25}$$

Then, system (1.12) *is globally asymptotically stable at the origin.*

According to Theorem 1.2, it is not sufficient to guarantee GAS by simply replacing the Ω in the condition for AS in Theorem 1.1 with $\mathbb{R}^n$. Condition (1.24) is also needed for GAS.

Condition (1.22)–(1.24) is equivalent to the statement that V is positive definite and radially unbounded, which can be represented with comparison functions $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ as

$$\underline{\alpha}(|x|) \le V(x) \le \overline{\alpha}(|x|) \tag{1.26}$$

for all $x \in \mathbb{R}^n$. Moreover, condition (1.25) is equivalent to the existence of a continuous and positive definite function α such that

$$\nabla V(x) f(x) \leq -\alpha(V(x)) \tag{1.27}$$

holds for all $x \in \mathbb{R}^n$. See [144, Lemma 4.3] for the details.

Theorems 1.1 and 1.2 give sufficient conditions for stability, AS and GAS. A proof of the converse Lyapunov theorem for the necessity of the conditions can be found in [144].

1.3 INPUT-TO-STATE STABILITY

For systems with external inputs, the notion of input-to-state stability (ISS), invented by Sontag, has proved to be powerful for evaluating the influence of the external inputs.

1.3.1 DEFINITION

Consider the system

$$\dot{x} = f(x, u), \tag{1.28}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ represents the input, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz function and satisfies $f(0,0) = 0$. By considering the input u as a function of time, assume that u is measurable and locally essentially bounded. Recall that u is locally essentially bounded if for any $t \geq 0$, $\|u\|_{[0,t]}$ exists. Denote $x(t, x_0, u)$, or simply $x(t)$, as the solution of system (1.12) with initial condition $x(0) = x_0$ and input u.

In [241], the original definition of ISS is given in the "plus" form; see (1.31). For convenience of discussions, we mainly use the definition in the equivalent "max" form. The equivalence is discussed later.

Definition 1.8 *System* (1.28) *is said to be input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(0) = x_0$ and any measurable and locally essentially bounded input u, the solution $x(t)$ satisfies*

$$|x(t)| \leq \max\{\beta(|x_0|, t), \gamma(\|u\|_\infty)\} \tag{1.29}$$

for all $t \geq 0$.

Here, γ is called the ISS gain of the system. Notice that, if $u \equiv 0$, then Definition 1.8 is reduced to Definition 1.7 for GAS at the origin. Due to causality, $x(t)$ depends on x_0 and the past inputs $\{u(\tau) : 0 \leq \tau \leq t\}$, and thus, the $\|u\|_\infty$ in (1.29) can be replaced with $\|u\|_{[0,t]}$.

Since

$$\max\{a, b\} \leq a + b \leq \max\{(1 + 1/\delta)a, (1 + \delta)b\} \tag{1.30}$$

for any $a, b \geq 0$ and any $\delta > 0$, property (1.29) in the "max" form is equivalent to

$$|x(t)| \leq \beta'(|x_0|, t) + \gamma'(\|u\|_\infty), \tag{1.31}$$

where $\beta' \in \mathcal{KL}$ and $\gamma' \in \mathcal{K}$. It should be noted that although the transformation from (1.29) to (1.31) can be done by directly replacing the "max" operation with the "+" operation without changing functions β and γ, the transformation from (1.31) to (1.29) may result in a pair of β and γ different from the pair of β' and γ'. To get a γ very close to γ', one may choose a very small δ for the transformation, but this could result in a very large β.

With property (1.31), $x(t)$ asymptotically converges to within the region defined by $|x| \leq \gamma'(\|u\|_\infty)$, i.e.,

$$\varlimsup_{t\to\infty} |x(t)| \leq \gamma'(\|u\|_\infty). \tag{1.32}$$

As shown in Figure 1.5, γ' describes the "steady-state" performance of the system, and is usually called the asymptotic gain (AG), while the "transient performance" is described by β'.

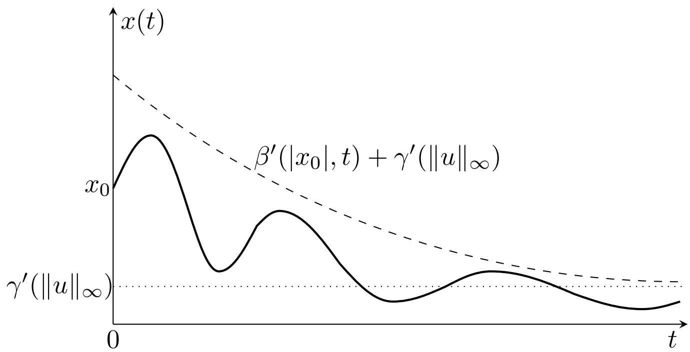

FIGURE 1.5 Asymptotic gain property.

Intuitively, since only large values of t determine the value $\varlimsup_{t\to\infty} |x(t)|$, one may replace the $\gamma'(\|u\|_\infty)$ in (1.32) with $\gamma'(\varlimsup_{t\to\infty} |u(t)|)$ or $\varlimsup_{t\to\infty} \gamma'(|u(t)|)$. See [250, 247] for more detailed discussions.

When system (1.28) is reduced to a linear system, a necessary and sufficient condition for the ISS property can be derived.

Theorem 1.3 *A linear time-invariant system*

$$\dot{x} = Ax + Bu \tag{1.33}$$

is ISS if and only if A is Hurwitz.

Proof. With initial condition $x(0) = x_0$ and input u, the solution of system (1.33) is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \tag{1.34}$$

which implies

$$|x(t)| \leq |e^{At}||x_0| + \left(\int_0^\infty |e^{A\tau}|d\tau\right)|B|\|u\|_\infty. \tag{1.35}$$

If A is Hurwitz, i.e., every eigenvalue of A has negative real part, then $\int_0^\infty |e^{As}|ds < \infty$. Define $\beta'(s,t) = |e^{At}|s$ and $\gamma'(s) = \left(\int_0^\infty |e^{A\tau}|d\tau\right)|B|s$ for $s, t \in \mathbb{R}_+$. Clearly, $\beta' \in \mathcal{KL}$ and $\gamma' \in \mathcal{K}_\infty$. Then, the linear system is ISS in the sense of (1.31). The sufficiency part is proved.

For the necessity, one may consider the case of $u \equiv 0$. In this case, the ISS of system (1.33) implies GAS of system

$$\dot{x} = Ax \tag{1.36}$$

at the origin. According to linear systems theory [28], system (1.36) is GAS at the origin if and only if A is Hurwitz. $\diamondsuit$

Based on the proof of Theorem 1.3, one may consider the ISS property (1.31) as a nonlinear modification of property (1.35) of linear systems. Lemma 1.1 shows that any $\mathcal{KL}$ function $\beta(s,t)$ can be considered as a nonlinear modification of function se^{-t}.

Lemma 1.1 *For any $\beta \in \mathcal{KL}$, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\beta(s,t) \leq \alpha_2(\alpha_1(s)e^{-t}) \tag{1.37}$$

for all $s, t \geq 0$.

See [243, Proposition 7] and its proof therein.

According to Lemma 1.1, if property (1.31) holds, then there exist $\alpha_1', \alpha_2' \in \mathcal{K}_\infty$ such that

$$|x(t)| \leq \alpha_2'(\alpha_1'(|x_0|)e^{-t}) + \gamma'(\|u\|_\infty), \tag{1.38}$$

which shows a close analogy of ISS to the solution property (1.35) of linear system (1.33) with A being Hurwitz.

Also, with Lemma 1.1, property (1.29) implies

$$|x(t)| \leq \max\{\alpha_2(\alpha_1(|x_0|)e^{-t}), \gamma(\|u\|_\infty)\}, \tag{1.39}$$

where α_1, α_2 are appropriate class $\mathcal{K}_\infty$ functions. This means, for any x_0 and $\|u\|_\infty$ satisfying $\alpha_2 \circ \alpha_1(|x_0|) > \gamma(\|u\|_\infty)$, there exists a finite time $t^* = \log(\alpha_1(|x_0|)) - \log(\alpha_2^{-1} \circ \gamma(\|u\|_\infty))$, after which solution $x(t)$ is within the range defined by $|x| \leq \gamma(\|u\|_\infty)$. This shows the difference between the ISS gain γ defined in (1.29) and the asymptotic gain γ' defined in (1.31).

Theorem 1.3 means that a linear system is ISS if the corresponding input-free system is GAS at the origin. But this may not be true for nonlinear systems. Consider Example 1.2 given by [243].

Example 1.2 *Consider the nonlinear system*

$$\dot{x} = -x + ux \tag{1.40}$$

with $x, u \in \mathbb{R}$. If $u \equiv 0$, then the resulting system $\dot{x} = -x$ is GAS at the origin. But system (1.40) *is not ISS. Just consider the class of constant inputs $u > 1$.*

However, it has been proved that AS at the origin of system (1.28) with $u \equiv 0$ is equivalent to a local ISS property of system (1.28) [250]. The definition of local ISS is given by Definition 1.9.

Definition 1.9 *System* (1.28) *is said to be locally input-to-state stable if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, and constants $\rho^x, \rho^u > 0$ such that for any initial state $x(0) = x_0$ satisfying $|x_0| \leq \rho^x$ and any measurable and locally essentially bounded input u satisfying $\|u\|_\infty \leq \rho^u$, the solution $x(t)$ satisfies*

$$|x(t)| \leq \max\{\beta(|x_0|, t), \gamma(\|u\|_\infty)\} \tag{1.41}$$

for all $t \geq 0$.

Theorem 1.4 presents the equivalence between AS and local ISS.

Theorem 1.4 *System* (1.28) *is locally ISS if and only if the zero-input system*

$$\dot{x} = f(x, 0) \tag{1.42}$$

is AS at the origin.

Proof. The proof of Theorem 1.4 is motivated by the proof of [78, Theorems 56.3 and 56.4] on the equivalence between total stability and AS at the origin, and the proof of [250, Lemma I.2] on the sufficiency of GAS for local ISS.

The necessity part is obvious. We prove the sufficiency part. By using the converse Lyapunov theorem (see e.g., [144]), the AS of system (1.42) at the origin implies the existence of a Lyapunov function $V : \Omega \to \mathbb{R}_+$ with $\Omega \subseteq \mathbb{R}^n$ being a domain containing the origin such that properties (1.15)–(1.18) hold. For such V, one can find an $\Omega' \subseteq \Omega$ still containing the origin such that for all $x \in \Omega'$,

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{1.43}$$

$$\nabla V(x) f(x, 0) \leq -\alpha(V(x)), \tag{1.44}$$

where $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ and α is a continuous and positive definite function.

By using the continuity of ∇V and f, for any $x \in \Omega' \backslash \{0\}$, one can find a $\delta > 0$ such that

$$|\nabla V(x) f(x, \epsilon) - \nabla V(x) f(x, 0)| \leq \frac{1}{2} \alpha(V(x)) \tag{1.45}$$

for all $|\epsilon| \leq \delta$. Thus, there is a positive definite function χ_0 such that for any $x \in \Omega'$, property (1.45) holds for all $|\epsilon| \leq \chi_0(|x|)$.

Then, we choose Ω_0 as a compact set containing the origin and belonging to Ω', and choose $\chi \in \mathcal{K}$ such that

$$\chi(s) \leq \chi_0(s) \tag{1.46}$$

for all $0 \leq s \leq \max\{|x| : x \in \Omega_0\}$. It can be directly proved that if $x \in \Omega_0$ and $\chi(|u|) \leq |x|$, then

$$\nabla V(x) f(x, u) \leq -\frac{1}{2} \alpha(V(x)). \tag{1.47}$$

Thus, with (1.43), property (1.47) holds if

$$V(x) \leq \max \left\{ \underline{\alpha}(|x|) : x \in \Omega_0 \right\}, \tag{1.48}$$

$$V(x) \geq \overline{\alpha} \circ \chi(\|u\|_\infty) := \gamma(\|u\|_\infty). \tag{1.49}$$

Then, the sufficiency part can be proved following the same line as (1.54)–(1.56) given later for ISS-Lyapunov functions. The interested reader may also consult the proof of [250, Lemma I.2]. $\diamondsuit$

From Definition 1.8, an ISS system is always forward complete, i.e., for any initial state $x(0) = x_0$ and any measurable and locally essentially bound input u, the solution $x(t)$ is defined for all $t \geq 0$. Moreover, it has the uniformly bounded-input bounded-state (UBIBS) property.

Definition 1.10 *System* (1.28) *is said to have the UBIBS property if there exists $\sigma_1, \sigma_2 \in \mathcal{K}$ such that for any initial state $x(0) = x_0$ and any measurable and locally essentially bounded input u,*

$$|x(t)| \leq \max\{\sigma_1(|x_0|), \sigma_2(\|u\|_\infty)\} \tag{1.50}$$

for all $t \geq 0$.

Recall Definition 1.3 for class $\mathcal{KL}$ functions. If system (1.28) is ISS satisfying (1.29), then it admits property (1.50) by defining $\sigma_1(s) = \beta(s, 0)$ and $\sigma_2(s) = \gamma(s)$ for $s \in \mathbb{R}_+$.

More importantly, ISS is equivalent to the conjunction of UBIBS and AG [250]. This result can be used for the proof of the ISS small-gain theorem for interconnected nonlinear systems; see detailed discussions in Chapter 2.

Theorem 1.5 *System* (1.28) *is ISS if and only if it has the properties of UBIBS and AG in the sense of* (1.50) *and* (1.32), *respectively.*