Finite Element Analysis of Composite Materials Using Abaqus™

Ever J. Barbero
Finite Element Analysis of Composite Materials Using Abaqus™
Composite Materials: Analysis and Design

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Finite Element Analysis of Composite Materials Using Abaqus™

Ever J. Barbero
Dedicated to my graduate students, who taught me as much as I taught them.
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Series Preface

Half a century after their commercial introduction, composite materials are of widespread use in many industries. Applications such as aerospace, windmill blades, highway bridge retrofit, and many more require designs that assure safe and reliable operation for twenty years or more. Using composite materials, virtually any property, such as stiffness, strength, thermal conductivity, and fire resistance, can be tailored to the users needs by selecting the constituent material, their proportion and geometrical arrangement, and so on. In other words, the engineer is able to design the material concurrently with the structure. Also, modes of failure are much more complex in composites than in classical materials. Such demands for performance, safety, and reliability require that engineers consider a variety of phenomena during the design. Therefore, the aim of the Composite Materials: Analysis and Design book series is to bring to the design engineer a collection of works written by experts on every aspect of composite materials that is relevant to their design.

Variety and sophistication of material systems and processing techniques has grown exponentially in response to an ever-increasing number and type of applications. Given the variety of composite materials available as well as their continuous change and improvement, understanding of composite materials is by no means complete. Therefore, this book series serves not only the practicing engineer but also the researcher and student who are looking to advance the state-of-the-art in understanding material and structural response and developing new engineering tools for modeling and predicting such responses.

Thus, the series is focused on bringing to the public existing and developing knowledge about the material-property relationships, processing-property relationships, and structural response of composite materials and structures. The series scope includes analytical, experimental, and numerical methods that have a clear impact on the design of composite structures.

Ever Barbero, book series editor
West Virginia University, Morgantown, WV
Finite Element Analysis of Composite Materials deals with the analysis of structures made of composite materials, also called composites. The analysis of composites treated in this textbook includes the analysis of the material itself, at the micro-level, and the analysis of structures made of composite materials. This textbook evolved from the class notes of MAE 646 Advanced Mechanics of Composite Materials that I teach as a graduate course at West Virginia University. Although this is also a textbook on advanced mechanics of composite materials, the use of the finite element method is essential for the solution of the complex boundary value problems encountered in the advanced analysis of composites, and thus the title of the book.

There are a number of good textbooks on advanced mechanics of composite materials, but none carries the theory to a practical level by actually solving problems, as it is done in this textbook. Some books devoted exclusively to finite element analysis include some examples about modeling composites but fall quite short of dealing with the actual analysis and design issues of composite materials and composite structures. This textbook includes an explanation of the concepts involved in the detailed analysis of composites, a sound explanation of the mechanics needed to translate those concepts into a mathematical representation of the physical reality, and a detailed explanation of the solution of the resulting boundary value problems by using commercial Finite Element Analysis software such as Abaqus™. Furthermore, this textbook includes more than fifty fully developed examples interspersed with the theory, as well as more than seventy-five exercises at the end of chapters, and more than fifty separate pieces of Abaqus pseudocode used to explain in detail the solution of example problems. The reader will be able to reproduce the examples and complete the exercises. When a finite element analysis is called for, the reader will be able to do it with commercially or otherwise available software. A Web site is set up with links to download the necessary software unless it is easily available from Finite Element Analysis software vendors. Use of Abaqus and MATLAB™ is explained with numerous examples, and the relevant code can be downloaded from the Web site. Furthermore, the reader will be able to extend the capabilities of Abaqus by use of user material subroutines and Python scripting, as demonstrated in the examples included in this textbook.

Chapters 1 through 7 can be covered in a one-semester graduate course. Chapter 2 (Introduction to Finite Element Analysis) contains a brief introduction intended for those readers who have not had a formal course or prior knowledge about the finite element method. Chapter 4 (Buckling) is not referenced in the remainder of
the textbook and thus it could be omitted in favor of more exhaustive coverage of content in later chapters. Chapters 7 (Viscoelasticity), 8 (Continuum Damage Mechanics), and 9 (Discrete Damage Mechanics) are placed consecutively to emphasize hereditary phenomena. However, Chapter 7 can be skipped if more emphasis on damage and/or delaminations is desired in a one-semester course.

The inductive method is applied as much as possible in this textbook. That is, topics are introduced with examples of increasing complexity, until sufficient physical understanding is reached to introduce the general theory without difficulty. This method will sometimes require that, at earlier stages of the presentation, certain facts, models, and relationships be accepted as fact, until they are completely proven later on. For example, in Chapter 7, viscoelastic models are introduced early to aid the reader in gaining an appreciation for the response of viscoelastic materials. This is done simultaneously with a cursory introduction to the superposition principle and the Laplace transform, which are formally introduced only later in the chapter. For those readers accustomed to the deductive method, this may seem odd, but many years of teaching have convinced me that students acquire and retain knowledge more efficiently in this way.

It is assumed that the reader is familiar with basic mechanics of composites as covered in introductory level textbooks such as my previous textbook, Introduction to Composite Material Design–Second Edition. Furthermore, it is assumed that the reader masters a body of knowledge that is commonly acquired as part of a bachelor of science degree in any of the following disciplines: Aerospace, Mechanical, Civil, or similar. References to books and to other sections in this textbook, as well as footnotes, are used to assist the reader in refreshing those concepts and to clarify the notation used. Prior knowledge of continuum mechanics, tensor analysis, and the finite element method would enhance the learning experience but are not necessary for studying with this textbook. The finite element method is used as a tool to solve practical problems. For the most part, Abaqus is used throughout the book. Computing programming using Fortran, Python, and MATLAB is limited to programming material models and post-processing algorithms. Basic knowledge of these programming languages is useful but not essential.

Only three software packages are used throughout the book. Abaqus is needed for finite element solution of numerous examples and suggested problems. MATLAB is needed for both symbolic and numerical solution of examples and suggested problems. Additionally, BMI3, which is available free of charge on the book’s Web site, is used in Chapter 4. Several other programs such as ANSYS Mechanical®, LS-DYNA®, MSC-MARC®, SolidWorks™ are cited, but not used in the examples. Relevant code used in the examples is available in the book’s Web site http://barbero.cadec-online.com/feacm-abaqus/.

Composite materials are now ubiquitous in the marketplace, including extensive applications in aerospace, automotive, civil infrastructure, sporting goods, and so on. Their design is especially challenging because, unlike conventional materials such as metals, the composite material itself is designed concurrently with the composite structure. Preliminary design of composites is based on the assumption of a state of plane stress in the laminate. Furthermore, rough approximations are made
about the geometry of the part, as well as the loading and support conditions. In this way, relatively simple analysis methods exist and computations can be carried out simply using algebra. However, preliminary analysis methods have a number of shortcomings that are remedied with advanced mechanics and finite element analysis, as explained in this textbook. Recent advances in commercial finite element analysis packages, with user-friendly pre- and post-processing, as well as powerful user-programmable features, have made detailed analysis of composites quite accessible to the designer. This textbook bridges the gap between powerful finite element tools and practical problems in structural analysis of composites. I expect that many graduate students, practicing engineers, and instructors will find this to be a useful and practical textbook on finite element analysis of composite materials based on sound understanding of advanced mechanics of composite materials.

Ever J. Barbero, 2013

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List of Symbols

Symbols Related to Mechanics of Orthotropic Materials

\( \epsilon \) Strain tensor
\( \varepsilon_{ij} \) Strain components in tensor notation
\( \epsilon_\alpha \) Strain components in contracted notation
\( \varepsilon_\alpha^e \) Elastic strain
\( \varepsilon_\alpha^p \) Plastic strain
\( \lambda \) Lame constant
\( \nu \) Poisson’s ratio
\( \nu_{12} \) In-plane Poisson’s ratio
\( \nu_{23}, \nu_{13} \) Interlaminar Poisson’s ratios
\( \nu_{xy} \) Apparent laminate Poisson’s ratio x-y
\( \sigma \) Stress tensor
\( \sigma_{ij} \) Stress components in tensor notation
\( \sigma_\alpha \) Stress components in contracted notation
\( [a] \) Transformation matrix for vectors
\( e_i \) Unit vector components in global coordinates
\( e'_i \) Unit vector components in materials coordinates
\( f_i, f_{ij} \) Tsai-Wu coefficients
\( K \) Bulk modulus
\( l, m, n \) Direction cosines
\( \tilde{u}(\varepsilon_{ij}) \) Strain energy per unit volume
\( u_i \) Displacement vector components
\( x_i \) Global directions or axes
\( x'_i \) Materials directions or axes
\( C \) Stiffness tensor
\( C_{ijkl} \) Stiffness in index notation
\( C_{\alpha,\beta} \) Stiffness in contracted notation
\( E \) Young’s modulus
\( E_1 \) Longitudinal modulus
\( E_2 \) Transverse modulus
\( E_2 \) Transverse-thickness modulus
\( E_x \) Apparent laminate modulus in the global x-direction
\( G = \mu \) Shear modulus
\( G_{12} \) In-plane shear modulus
\( G_{23}, G_{13} \) Interlaminar shear moduli
\( G_{xy} \)  
Apparent laminate shear modulus x-y

\( I_{ij} \)  
Second-order identity tensor

\( I_{ijkl} \)  
Fourth-order identity tensor

\( Q'_{ij} \)  
Lamina stiffness components in lamina coordinates

\([R]\)  
Reuter matrix

\( S \)  
Compliance tensor

\( S_{ijkl} \)  
Compliance in index notation

\( S_{\alpha,\beta} \)  
Compliance in contracted notation

\([T]\)  
Coordinate transformation matrix for stress

\([\bar{T}]\)  
Coordinate transformation matrix for strain

Symbols Related to Finite Element Analysis

\( \partial \)  
Strain-displacement equations in matrix form

\( \xi \)  
Six-element array of strain components

\( \theta_x, \theta_y, \theta_z \)  
Rotation angles following the right-hand rule \( \text{(Figure 2.19)} \)

\( \sigma \)  
Six-element array of stress components

\( \phi_x, \phi_y \)  
Rotation angles used in plate and shell theory

\( a \)  
Nodal displacement array

\( u^e \)  
Unknown parameters in the discretization

\( B \)  
Strain-displacement matrix

\( C \)  
Stiffness matrix

\( K \)  
Assembled global stiffness matrix

\( K^e \)  
Element stiffness matrix

\( N \)  
Interpolation function array

\( N_j^e \)  
Interpolation functions in the discretization

\( P^e \)  
Element force array

\( P \)  
Assembled global force array

Symbols Related to Elasticity and Strength of Laminates

\( \gamma_{xy}^0 \)  
In-plane shear strain

\( \gamma_{4u} \)  
Ultimate interlaminar shear strain in the 2-3 plane

\( \gamma_{5u} \)  
Ultimate interlaminar shear strain in the 1-3 plane

\( \gamma_{6u} \)  
Ultimate in-plane shear strain

\( \epsilon^0, \epsilon_y^0 \)  
In-plane strains

\( \epsilon_{1t} \)  
Ultimate longitudinal tensile strain

\( \epsilon_{2t} \)  
Ultimate transverse tensile strain

\( \epsilon_{3t} \)  
Ultimate transverse-thickness tensile strain

\( \epsilon_{1c} \)  
Ultimate longitudinal compressive strain

\( \epsilon_{2c} \)  
Ultimate transverse compressive strain

\( \epsilon_{3c} \)  
Ultimate transverse-thickness compressive strain

\( \kappa_x, \kappa_y \)  
Bending curvatures

\( \kappa_{xy} \)  
Twisting curvature
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**Symbols Related to Buckling**

<table>
<thead>
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<td>Stress stiffness matrix</td>
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<td>$P_{CR}$</td>
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</tr>
</tbody>
</table>
Symbols Related to Free Edge Stresses

\( \eta_{xy,x} \), \( \eta_{xy,y} \)  
Coefficients of mutual influence

\( \eta_{x,x}, \eta_{y,y} \)  
Alternate coefficients of mutual influence

\( F_{yz} \)  
Interlaminar shear force y-z

\( F_{xz} \)  
Interlaminar shear force x-z

\( M_z \)  
Interlaminar moment

Symbols Related to Micromechanics

\( \bar{\varepsilon}_\alpha \)  
Average engineering strain components

\( \bar{\varepsilon}_{ij} \)  
Average tensor strain components

\( \varepsilon^0_{\alpha}, \varepsilon^0_{ij} \)  
Far-field applied strain components

\( \bar{\sigma}_\alpha \)  
Average stress components

\( A^i \)  
Strain concentration tensor, i-th phase, contracted notation

\( 2a_1, 2a_2, 2a_3 \)  
Dimensions of the RVE

\( A_{ijkl} \)  
Components of the strain concentration tensor

\( B^i \)  
Stress concentration tensor, i-th phase, contracted notation

\( B_{ijkl} \)  
Components of the stress concentration tensor

\( I \)  
6 \times 6 identity matrix

\( P_{ijkl} \)  
Eshelby tensor

\( V_f \)  
Fiber volume fraction

\( V_m \)  
Matrix volume fraction

Symbols Related to Viscoelasticity

\( \dot{\varepsilon} \)  
Stress rate

\( \eta \)  
Viscosity

\( \theta \)  
Age or aging time

\( \dot{\sigma} \)  
Stress rate

\( \tau \)  
Time constant of the material or system

\( \Gamma \)  
Gamma function

\( s \)  
Laplace variable

\( t \)  
Time

\( C_{\alpha,\beta}(t) \)  
Stiffness tensor in the time domain

\( C_{\alpha,\beta}(s) \)  
Stiffness tensor in the Laplace domain

\( \tilde{C}_{\alpha,\beta}(s) \)  
Stiffness tensor in the Carson domain

\( D(t) \)  
Compliance

\( D_0, (D_i)_0 \)  
Initial compliance values

\( D_c(t) \)  
Creep component of the total compliance \( D(t) \)

\( D', D'' \)  
Storage and loss compliances

\( E_0, (E_i)_0 \)  
Initial moduli

\( E_\infty \)  
Equilibrium modulus

\( E, E_0, E_1, E_2 \)  
Parameters in the viscoelastic models (Figure 7.1)

\( E(t) \)  
Relaxation
List of Symbols

\( E', E'' \)  Storage and loss moduli
\( F[] \)  Fourier transform
\( (G_{ij})_0 \)  Initial shear moduli
\( H(t - t_0) \)  Heaviside step function
\( H(\theta) \)  Relaxation spectrum
\( L[] \)  Laplace transform
\( L[]^{-1} \)  Inverse Laplace transform

Symbols Related to Damage

\( \alpha \)  Laminate CTE
\( \alpha^{(k)} \)  CTE of lamina \( k \)
\( \alpha_{cr} \)  Critical misalignment angle at longitudinal compression failure
\( \alpha_{\sigma} \)  Standard deviation of fiber misalignment
\( \gamma(\delta) \)  Damage hardening function
\( \gamma_0 \)  Damage threshold
\( \delta_{ij} \)  Kronecker delta
\( \delta \)  Damage hardening variable
\( \varepsilon \)  Effective strain
\( \varepsilon^p \)  Plastic strain
\( \dot{\gamma} \)  Heat dissipation rate per unit volume
\( \dot{\gamma}_s \)  Internal entropy production rate
\( \lambda \)  Crack density
\( \lambda_{lim} \)  Saturation crack density
\( \dot{\lambda}, \dot{\lambda}^d \)  Damage multiplier
\( \dot{\lambda}^p \)  Yield multiplier
\( \rho \)  Density
\( \sigma \)  Effective stress
\( \varepsilon \)  Undamaged strain
\( \tau_{13}, \tau_{23} \)  Intralaminar shear stress components
\( \varphi, \varphi^* \)  Strain energy density, and complementary SED
\( \chi \)  Gibbs energy density
\( \psi \)  Helmholtz free energy density
\( \Delta T \)  Change in temperature
\( \Omega = \Omega_{ij} \)  Integrity tensor
\( 2a_0 \)  Representative crack size
\( d_i \)  Eigenvalues of the damage tensor
\( f^d \)  Damage flow surface
\( f^p \)  Yield flow surface
\( f(x), F(x) \)  Probability density, and its cumulative probability
\( g \)  Damage activation function
\( g^d \)  Damage surface
\( g^p \)  Yield surface
\( h \)  Laminate thickness
Finite Element Analysis of Composite Materials

$h_k$  Thickness of lamina $k$
$m$  Weibull modulus
$p$  Yield hardening variable
$\hat{p}$  Thickness average of quantity $p$
$\bar{p}$  Virgin value of quantity $p$
$\overline{p}$  Volume average of quantity $p$
$q$  Heat flow vector per unit area
$r$  Radiation heat per unit mass
$s$  Specific entropy
$u(\varepsilon_{ij})$  Internal energy density
$A$  Crack area
$[A]$  Laminate in-plane stiffness matrix
$A_{ijkl}$  Tension-compression damage constitutive tensor
$B_{ijkl}$  Shear damage constitutive tensor
$B_a$  Dimensionless number (8.57)
$C_{\alpha,\beta}$  Stiffness matrix in the undamaged configuration
$C^{ed}$  Tangent stiffness tensor
$D_{ij}$  Damage tensor
$D_{1t}^{cr}$  Critical damage at longitudinal tensile failure
$D_{1c}^{cr}$  Critical damage at longitudinal compression failure
$D_{2t}^{cr}$  Critical damage at transverse tensile failure
$D_2, D_6$  Damage variables
$E(D)$  Effective modulus
$\overline{E}$  Undamaged (virgin) modulus
$G_c = 2\gamma_c$  Surface energy
$G_{Ic}, G_{IIc}$  Critical energy release rate in modes I and II
$J_{ijkl}$  Normal damage constitutive tensor
$M_{ijkl}$  Damage effect tensor
$N$  Number of laminas in the laminate
$\{N\}$  Membrane stress resultant array
$Q$  Degraded 3x3 stiffness matrix of the laminate
$R(p)$  Yield hardening function
$R_0$  Yield threshold
$S$  Entropy or Laminate complinace matrix, depending on context
$T$  Temperature
$U$  Strain energy
$V$  Volume of the RVE
$Y_{ij}$  Thermodynamic RVE tensor

Symbols Related to Delaminations

$\alpha$  Mixed mode crack propagation exponent
$\beta_\delta, \beta_G$  Mixed mode ratios
$\delta$  CZM separation of the interface
$\delta_m$  Mixed mode separation
List of Symbols

\( \delta^0_m \) Mixed mode separation at damage onset
\( \delta^0 \) Mixed mode separation at fracture
\( \sigma^0 \) CZM critical separation at damage onset
\( \ell \) Delamination length for 2D delaminations
\( \sigma^0 \) CZM strength of the interface
\( \psi_{xi}, \psi_{yi} \) Rotation of normals to the middle surface of the plate
\( \Omega \) Volume of the body
\( \Omega_D \) Delaminated region
\( \Pi_e \) Potential energy, elastic
\( \Pi^r \) Potential energy, total
\( \dot{\Gamma} \) Dissipation rate
\( \Lambda \) Interface strain energy density per unit area
\( \partial \Omega \) Boundary of the body
\( d \) One-dimensional damage state variable
\( k_{xy}, k_z \) Displacement continuity parameters
\( [A_i], [B_i], [D_i] \) Laminate stiffness sub-matrices
\( D_I, D_{II}, D_{III} \) Damage variables for modes I, II, and III of CZM
\( G(\ell) \) Energy release rate (ERR), total, in 2D
\( G \) Energy release rate (ERR), total, in 3D
\( G_I, G_{II}, G_{III} \) Energy release rate (ERR) of modes I, II, and III
\( G_c \) Critical energy release rate (ERR), total, in 3D
\( G_I^c \) Critical energy release rate mode I
\( [H_i] \) Laminate interlaminar shear stiffness matrix
\( K \) Penalty stiffness
\( \tilde{K} \) Virgin penalty stiffness
\( K_I, K_{II}, K_{III} \) Stress intensity factors (SIF) of modes I, II, and III
\( N_i, M_i, T_i \) Stress resultants
\( U \) Internal energy
\( W \) Work done by the body on its surroundings
\( W_{closure} \) Crack closure work
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Chapter 1

Mechanics of Orthotropic Materials

This chapter provides the foundation for the rest of the book. Basic concepts of mechanics, tailored for composite materials, are presented, including coordinate transformations, constitutive equations, and so on. Continuum mechanics is used to describe deformation and stress in an orthotropic material. The basic equations are reviewed in Sections 1.2 to 1.9. Tensor operations are reviewed in Section 1.10 because they are used in the rest of the chapter. Coordinate transformations are required to express quantities such as stress, strain, and stiffness in lamina coordinates, in laminate coordinates, and so on. They are reviewed in Sections 1.10 to 1.11. This chapter is heavily referenced in the rest of the book, and thus readers who are already versed in continuum mechanics may choose to come back to review this material as needed.

1.1 Lamina Coordinate System

A single lamina of fiber reinforced composite behaves as an orthotropic material. That is, the material has three mutually perpendicular planes of symmetry. The intersection of these three planes defines three axes that coincide with the fiber direction \( x'_1 \), the thickness coordinate \( x'_3 \), and a third direction \( x'_2 = x'_3 \times x'_1 \) perpendicular to the other two.\(^1\) [1].

1.2 Displacements

Under the action of forces, every point in a body may translate and rotate as a rigid body as well as deform to occupy a new region. The displacements \( u_i \) of any point \( P \) in the body (Figure 1.1) are defined in terms of the three components of the vector \( u_i \) (in a rectangular Cartesian coordinate system) as \( u_i = (u_1, u_2, u_3) \). An

\(^1\times\) denotes vector cross product.
alternate notation for displacements is \( u_i = (u, v, w) \). Displacement is a vector or first-order tensor quantity

\[
\mathbf{u} = u_i = (u_1, u_2, u_3) \ ; \ i = 1...3
\]

(1.1)

where boldface (e.g., \( \mathbf{u} \)) indicates a tensor written in tensor notation, in this case a vector (or first-order tensor). In this book, all tensors are boldfaced (e.g., \( \mathbf{\sigma} \)), but their components are not (e.g., \( \sigma_{ij} \)). The order of the tensor (i.e., first, second, fourth, etc.) must be inferred from context, or as in (1.1), by looking at the number of subscripts of the same entity written in index notation (e.g., \( u_i \)).

1.3 Strain

For geometric nonlinear analysis, the components of the Lagrangian strain tensor are [2]

\[
L_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{r,i}u_{r,j})
\]

(1.2)

where

\[
u_{i,j} = \frac{\partial u_i}{\partial x_j}
\]

(1.3)

If the gradients of the displacements are so small that products of partial derivatives of \( u_i \) are negligible compared with linear (first-order) derivative terms, then the (infinitesimal) strain tensor \( \varepsilon_{ij} \) is given by [2]

\[
\varepsilon = \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})
\]

(1.4)
Again, boldface indicates a tensor, the order of which is implied from the context. For example $\varepsilon$ is a one-dimensional strain and $\boldsymbol{\varepsilon}$ is the second-order tensor of strain. Index notation (e.g., $= \varepsilon_{ij}$) is used most of the time and the tensor character of variables (scalar, vector, second order, and so on) is easily understood from context.

From the definition (1.4), strain is a second-order, symmetric tensor (i.e., $\varepsilon_{ij} = \varepsilon_{ji}$). In expanded form the strains are defined by

\[ \begin{align*}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} = \epsilon_1 ; \\
2\varepsilon_{12} &= 2\varepsilon_{21} = \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \gamma_6 = \epsilon_6 \\
\varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} = \epsilon_2 ; \\
2\varepsilon_{23} &= 2\varepsilon_{31} = \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) = \gamma_5 = \epsilon_5 \\
\varepsilon_{33} &= \frac{\partial u_3}{\partial x_3} = \epsilon_3 ; \\
2\varepsilon_{32} &= 2\varepsilon_{23} = \left( \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right) = \gamma_4 = \epsilon_4
\end{align*} \] (1.5)

where $\epsilon_\alpha$ with $\alpha = 1..6$ are defined in Section 1.5. The normal components of strain ($i = j$) represent the change in length per unit length (Figure 1.2). The shear components of strain ($i \neq j$) represent one-half the change in an original right angle (Figure 1.3). The engineering shear strain $\gamma_\alpha = 2\varepsilon_{ij}$, for $i \neq j$, is often used instead of the tensor shear strain because the shear modulus $G$ is defined by $\tau = G\gamma$ in mechanics of materials [3]. The strain tensor, being of second order, can be displayed as a matrix

\[ [\varepsilon] = \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}
\end{bmatrix} = \begin{bmatrix}
\epsilon_1 & \epsilon_6/2 & \epsilon_5/2 \\
\epsilon_6/2 & \epsilon_2 & \epsilon_4/2 \\
\epsilon_5/2 & \epsilon_4/2 & \epsilon_3
\end{bmatrix} \] (1.6)

where $[\ ]$ is used to denote matrices.

1.4 Stress

The stress vector associated to a plane passing through a point is the force per unit area acting on the plane passing through the point. A second-order tensor, called stress tensor, completely describes the state of stress at a point. The stress tensor
can be expressed in terms of the components acting on three mutually perpendicular planes aligned with the orthogonal coordinate directions as indicated in Figure 1.4. The tensor notation for stress is $\sigma_{ij}$ with $(i, j = 1, 2, 3)$, where the first subscript corresponds to the direction of the normal to the plane of interest and the second subscript corresponds to the direction of the stress. Tensile normal stresses ($i = j$) are defined to be positive when the normal to the plane and the stress component directions are either both positive or both negative. All components of stress depicted in Figure 1.4 have a positive sense. Force and moment equilibrium of the element in Figure 1.4 requires that the stress tensor be symmetric (i.e., $\sigma_{ij} = \sigma_{ji}$) [3]. The stress tensor, being of second order, can be displayed as a matrix

$$\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33}
\end{bmatrix} =
\begin{bmatrix}
\sigma_1 & \sigma_6 & \sigma_5 \\
\sigma_6 & \sigma_2 & \sigma_4 \\
\sigma_5 & \sigma_4 & \sigma_3
\end{bmatrix}$$

(1.7)

### 1.5 Contracted Notation

Since the stress is symmetric, it can be written in Voigt contracted notation as

$$\sigma_\alpha = \sigma_{ij} = \sigma_{ji}$$

(1.8)

with the contraction rule defined as follows

$$\alpha = i \quad \text{if} \quad i = j$$

$$\alpha = 9 - i - j \quad \text{if} \quad i \neq j$$

(1.9)
resulting in the contracted version of stress components shown in (1.7). The same applies to the strain tensor, resulting in the contracted version of strain shown in (1.6). Note that the six components of stress $\sigma_\alpha$ with $\alpha = 1 \ldots 6$ can be arranged into a column array, denoted by curly brackets $\{\}$ as in (1.10), but $\{\sigma\}$ is not a vector, but just a convenient way to arrange the six unique components of a symmetric second-order tensor.

### 1.5.1 Alternate Contracted Notation

Some FEA software packages use different contracted notations, as shown in Table 1.1. For example, to transform stresses or strains from standard notation to Abaqus notation, a transformation matrix can be used as follows.

---

**Table 1.1: Contracted notation convention used by various FEA software packages.**

<table>
<thead>
<tr>
<th>Standard Convention</th>
<th>Abaqus/Standard</th>
<th>Abaqus/Explicit</th>
<th>ANSYS/Mechanical</th>
</tr>
</thead>
<tbody>
<tr>
<td>11 $\rightarrow$ 1</td>
<td>11 $\rightarrow$ 1</td>
<td>11 $\rightarrow$ 1</td>
<td>11 $\rightarrow$ 1</td>
</tr>
<tr>
<td>22 $\rightarrow$ 2</td>
<td>22 $\rightarrow$ 2</td>
<td>22 $\rightarrow$ 2</td>
<td>22 $\rightarrow$ 2</td>
</tr>
<tr>
<td>33 $\rightarrow$ 3</td>
<td>33 $\rightarrow$ 3</td>
<td>33 $\rightarrow$ 3</td>
<td>33 $\rightarrow$ 3</td>
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<tr>
<td>23 $\rightarrow$ 4</td>
<td>12 $\rightarrow$ 4</td>
<td>12 $\rightarrow$ 4</td>
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<tr>
<td>13 $\rightarrow$ 5</td>
<td>13 $\rightarrow$ 5</td>
<td>23 $\rightarrow$ 5</td>
<td>23 $\rightarrow$ 5</td>
</tr>
<tr>
<td>12 $\rightarrow$ 6</td>
<td>23 $\rightarrow$ 6</td>
<td>13 $\rightarrow$ 6</td>
<td>13 $\rightarrow$ 6</td>
</tr>
</tbody>
</table>
\( \{\sigma_A\} = [T]\{\sigma\} \) \hspace{1cm} (1.10)

where the subscript \( ()_A \) denotes a quantity in ABAQUS notation. Also note that \( \{ \} \) denotes a column array, in this case of six elements, and \([ \] \) denotes a matrix, in this case the \( 6 \times 6 \) rotation matrix given by

\[
[T] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\] \hspace{1cm} (1.11)

The stiffness matrix transforms as follows

\[
[C_A] = [T]^T[C][T]
\] \hspace{1cm} (1.12)

For LS-DYNA and ANSYS, the transformation matrix is

\[
[T] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\] \hspace{1cm} (1.13)

1.6 Equilibrium and Virtual Work

The three equations of equilibrium at every point in a body are written in tensor notation as

\[
\sigma_{ij,j} + f_i = 0
\] \hspace{1cm} (1.14)

where \( f_i \) is the body force per unit volume and \( ( )_j = \frac{\partial}{\partial x_j} \). When body forces are negligible, the expanded form of the equilibrium equations, written in the laminate coordinate system \( x-y-z \), is

\[
\begin{align*}
\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} &= 0 \\
\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} &= 0 \\
\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} &= 0
\end{align*}
\] \hspace{1cm} (1.15)
The principle of virtual work (PVW) provides an alternative to the equations of equilibrium [4]. Since the PVW is an integral expression, it is more convenient than (1.14) for finite element formulation. The PVW reads

\[ \int_V \sigma_{ij} \delta \epsilon_{ij} dV - \int_S t_i \delta u_i dS - \int_V f_i \delta u_i dV = 0 \]  

(1.16)

where \( t_i \) are the surface tractions per unit area acting on the surface \( S \). The negative sign means that work is done by external forces \( (t_i, f_i) \) on the body. The forces and the displacements follow the same sign convention; that is, a component is positive when it points in the positive direction of the respective axis. The first term in (1.16) is the virtual work performed by the internal stresses and it is positive following the same sign convention.

**Example 1.1** Find the displacement function \( u(x) \) for a slender rod of cross-sectional area \( A \), length \( L \), modulus \( E \), and density \( \rho \), hanging from the top end and subjected to its own weight. Use a coordinate \( x \) pointing downward with origin at the top end.

**Solution to Example 1.1** We assume a quadratic displacement function

\[ u(x) = C_0 + C_1 x + C_2 x^2 \]

Using the boundary condition (B.C.) at the top yields \( C_0 = 0 \). The PVW (1.16) simplifies because the only non-zero strain is \( \epsilon_x \) and there is no surface tractions. Using Hooke’s law

\[ \int_0^L E \epsilon_x \delta \epsilon_x Adx - \int_0^L \rho g \delta u Adx = 0 \]

From the assumed displacement

\[ \delta u = x \delta C_1 + x^2 \delta C_2 \]
\[ \epsilon_x = \frac{du}{dx} = C_1 + 2xC_2 \]
\[ \delta \epsilon_x = \delta C_1 + 2x \delta C_2 \]

Substituting

\[ EA \int_0^L (C_1 + 2xC_2)(\delta C_1 + 2x \delta C_2)dx - \rho g A \int_0^L (x \delta C_1 + x^2 \delta C_2)dx = 0 \]

Integrating and collecting terms in \( \delta C_1 \) and \( \delta C_2 \) separately

\[ (EC_2 L^2 + EC_1 L - \frac{\rho g L^2}{2}) \delta C_1 + (\frac{A}{3} EC_2 L^3 + EC_1 L^2 - \frac{\rho g L^3}{3}) \delta C_2 = 0 \]

Since \( \delta C_1 \) and \( \delta C_2 \) have arbitrary (virtual) values, two equations in two unknowns are obtained, one inside each parenthesis. Solving them we get

\[ C_1 = \frac{L \rho g}{E} \; ; \; C_2 = -\frac{\rho g}{2E} \]

Substituting back into \( u(x) \)

\[ u(x) = \frac{\rho g}{2E} (2L - x) \]

which coincides with the exact solution from mechanics of materials.
1.7 Boundary Conditions

1.7.1 Traction Boundary Conditions

The solution of problems in solid mechanics requires that boundary conditions be specified. The boundary conditions may be specified in terms of components of displacement, stress, or a combination of both. For any point on an arbitrary surface, the traction \( T_i \) is defined as the vector consisting of the three components of stress acting on the surface at the point of interest. As indicated in Figure 1.4 the traction vector consists of one component of normal stress, \( \sigma_{nn} \), and two components of shear stress, \( \sigma_{nt} \) and \( \sigma_{ns} \). The traction vector can be written using Cauchy’s law

\[
T_i = \sigma_{ji} n_j = \sum_{j} \sigma_{ji} n_j \tag{1.17}
\]

where \( n_j \) is the unit normal to the surface at the point under consideration.\(^2\) For a plane perpendicular to the \( x_1 \) axis \( n_i = (1, 0, 0) \) and the components of the traction are \( T_1 = \sigma_{11}, T_2 = \sigma_{12}, \) and \( T_3 = \sigma_{13} \).

1.7.2 Free Surface Boundary Conditions

The condition that a surface be free of stress is equivalent to all components of traction being zero, i.e., \( T_n = \sigma_{nn} = 0, T_t = \sigma_{nt} = 0, \) and \( T_s = \sigma_{ns} = 0 \). It is possible that only selected components of the traction be zero while others are non-zero. For example, pure pressure loading corresponds to non-zero normal stress and zero shear stresses.

1.8 Continuity Conditions

1.8.1 Traction Continuity

Equilibrium (action and reaction) requires that the traction components \( T_i \) must be continuous across any surface. Mathematically this is stated as \( T_i^+ - T_i^- = 0 \). Using (1.17), \( T_i^+ = \sigma_{ji}^+ n_j \). Since \( n_j^+ = -n_j^- \), we have \( \sigma_{ji}^+ = \sigma_{ji}^- \). In terms of individual stress components, \( \sigma_{nn}^+ = \sigma_{nn}^-, \sigma_{nt}^+ = \sigma_{nt}^-, \) and \( \sigma_{ns}^+ = \sigma_{ns}^- \) (Figure 1.5). Thus, the normal and shear components of stress acting on a surface must be continuous across that surface. There are no continuity requirements on the other three components of stress. That is, it is possible that \( \sigma_{tt}^+ \neq \sigma_{tt}^-, \sigma_{ss}^+ \neq \sigma_{ss}^-, \) and \( \sigma_{ts}^+ \neq \sigma_{ts}^- \). Lack of continuity of the two normal and one shear components of stress is very common because the material properties are discontinuous across layer boundaries.

\(^2\)Einstein’s summation convention can be introduced with (1.17) as an example. Any pair of repeated indices implies a summation over all the values of the index in question. Furthermore, each pair of repeated indices represents a contraction. That is, the order of resulting tensor, in this case order one for \( T_i \), is two less than the sum of the orders of the tensors involved in the operation. The resulting tensor keeps only the free indices that are not involved in the contraction—in this case only \( i \) remains.
1.8.2 Displacement Continuity

Certain conditions on displacements must be satisfied along any surface in a perfectly bonded continuum. Consider for example buckling of a cylinder under external pressure (Figure 1.6). The displacements associated with the material from either side of the line A-A must be identical \( u^+_i = u^-_i \). The continuity conditions must be satisfied at every point in a perfectly bonded continuum. However, continuity is not required in the presence of de-bonding or sliding between regions or phases of a material. For the example shown, continuity of slope \( \frac{\partial w^+}{\partial \theta} = \frac{\partial w^-}{\partial \theta} \), must be satisfied, where \( w \) is the radial displacement.

1.9 Compatibility

The strain displacement equations (1.5) provide six equations for only three unknown displacements \( u_i \). Thus, integration of equations (1.5) to determine the unknown displacements will not have a single-valued solution unless the strains \( \varepsilon_{ij} \) satisfy certain conditions. Arbitrary specification of the \( \varepsilon_{ij} \) could result in discontinuities in the material, including gaps and/or overlapping regions.

The necessary conditions for single-valued displacements are the compatibility conditions. Although these six equations are available [2], they are not used here because the displacement method, which is used throughout this book, does not require them. That is, in solving problems, the form of displacements \( u_i \) is always assumed a priori. Then, the strains are computed with (1.5), and the stress with (1.46). Finally, equilibrium is enforced by using the PVW (1.16).
1.10 Coordinate Transformations

The coordinates of point \( P \) in the prime coordinate system can be found from its coordinates in the unprimed system. From Figure 1.7, the coordinates of point \( P \) are

\[
\begin{align*}
    x'_1 &= x_1 \cos \theta + x_2 \sin \theta \\
    x'_2 &= -x_1 \sin \theta + x_2 \cos \theta \\
    x'_3 &= x_3
\end{align*}
\]

or

\[
x'_i = a_{ij} x_j
\]

or in matrix notation

\[
\{ x' \} = [a] \{ x \}
\]

where \( a_{ij} \) are the components of the unit vectors of the primed system \( e'_i \) on the unprimed system \( e_j \), by rows [2]

\[
a_{ij} = \cos(e'_i, e_j) = \begin{vmatrix} e'_1 & e'_2 & e'_3 \\ e_1 & e_2 & e_3 \\ e'_2 & e_2 & e_3 \\ e'_3 & e_3 & a_{31} & a_{32} & a_{33} \end{vmatrix}
\]
If primed coordinates denote the lamina coordinates and unprimed denote the laminate coordinates, then (1.19) transforms vectors from laminate to lamina coordinates. The inverse transformation simply uses the transpose matrix

\[ \{x\} = [a]^T \{x'\} \]  \hspace{1cm} (1.22)

**Example 1.2** A composite layer has fiber orientation \( \theta = 30^\circ \). Construct the \([a]\) matrix by calculating the direction cosines of the lamina system, i.e., the components of the unit vectors of the lamina system \((x'_1)\) on the laminate system \((x_j)\).

**Solution to Example 1.2** From Figure 1.7 and (1.19) we have

\[
\begin{align*}
a_{11} &= \cos \theta = \frac{\sqrt{3}}{2} \\
a_{12} &= \sin \theta = \frac{1}{2} \\
a_{13} &= 0 \\
a_{21} &= -\sin \theta = -\frac{1}{2} \\
a_{22} &= \cos \theta = \frac{\sqrt{3}}{2} \\
a_{23} &= 0 \\
a_{31} &= 0 \\
a_{32} &= 0 \\
a_{33} &= 1
\end{align*}
\]

**Example 1.3** A fiber reinforced composite tube is wound in the hoop direction (1-direction). Formulas for the stiffness values \((E_1, E_2, etc.)\) are given in that system. However, when analyzing the cross-section of this material with generalized plane strain elements (CAX4 in
Abaqus), the model is typically constructed in the structural X,Y,Z system. It is therefore necessary to provide the stiffness values in the structural system as $E_x$, $E_y$, etc. Construct the transformation matrix $[a]^T$ to go from lamina coordinates (1-2-3) to structural coordinates in Figure 1.8.

**Solution to Example 1.3** First, construct $[a]$ using the definition (1.21). Taking each unit vector (1-2-3) at a time we construct the matrix $[a]$ by rows. The $i$-th row contains the components of $(i=1,2,3)$ along (X-Y-Z).

\[
[a] = \begin{bmatrix}
1 & 0 & 0 & 1 \\
2 & 0 & -1 & 0 \\
3 & 1 & 0 & 0 \\
\end{bmatrix}
\]

The required transformation is just the transpose of the matrix above.

### 1.10.1 Stress Transformation

A second-order tensor $\sigma_{pq}$ can be thought as the (un-contracted) outer product\(^3\) of two vectors $V_p$ and $V_q$

\[
\sigma_{pq} = V_p \otimes V_q \tag{1.23}
\]

each of which transforms as (1.19)

\[
\sigma'_{ij} = a_{ip}V_p \otimes a_{jq}V_q \tag{1.24}
\]

Therefore,

\(^3\)The outer product preserves all indices of the entities involved, thus creating a tensor of order equal to the sum of the order of the entities involved.
\[ \sigma'_{ij} = a_{ip}a_{jq}\sigma_{pq} \]  

or, in matrix notation

\[ \{\sigma'\} = [a]\{\sigma\}[a]^T \]  

For example, expand \( \sigma'_{11} \) in contracted notation

\[ \sigma'_{11} = a_{21}^2 + 2a_{11}a_{12}\sigma_6 + 2a_{11}a_{13}\sigma_5 + 2a_{12}a_{13}\sigma_4 \]  

Expanding \( \sigma'_{12} \) in contracted notation yields

\[ \sigma'_{12} = a_{11}a_{21}\sigma_1 + (a_{11}a_{22} + a_{12}a_{21})\sigma_6 + (a_{11}a_{23} + a_{13}a_{21})\sigma_5 + (a_{12}a_{23} + a_{13}a_{22})\sigma_4 \]

The following algorithm is used to obtain a \( 6 \times 6 \) coordinate transformation matrix \([T]\) such that (1.25) is rewritten in contracted notation as

\[ \sigma'_{\alpha} = T_{\alpha\beta}\sigma_{\beta} \]  

If \( \alpha \leq 3 \) and \( \beta \leq 3 \) then \( i = j \) and \( p = q \), so

\[ T_{\alpha\beta} = a_{ip}a_{ip} = a_{ip}^2 \quad \text{no sum on } i, p \]

If \( \alpha \leq 3 \) and \( \beta > 3 \) then \( i = j \) but \( p \neq q \), and taking into account that switching \( p \) by \( q \) yields the same value of \( \beta = 9 - p - q \) as per (1.9) we have

\[ T_{\alpha\beta} = a_{ip}a_{iq} + a_{ip}a_{ip} = 2a_{ip}a_{iq} \quad \text{no sum on } i, p \]

If \( \alpha > 3 \), then \( i \neq j \), but we want only one stress, say \( \sigma_{ij} \), not \( \sigma_{ji} \) because they are numerically equal. In fact \( \sigma_{\alpha} = \sigma_{ij} = \sigma_{ji} \) with \( \alpha = 9 - i - j \). If in addition \( \beta \leq 3 \) then \( p = q \) and we get

\[ T_{\alpha\beta} = a_{ip}a_{jp} \quad \text{no sum on } i, p \]

When \( \alpha > 3 \) and \( \beta > 3 \), \( i \neq j \) and \( p \neq q \) so we get

\[ T_{\alpha\beta} = a_{ip}a_{jq} + a_{iq}a_{jp} \]  

which completes the derivation of \( T_{\alpha\beta} \). Expanding (1.30–1.33) and using (1.21) we get

\[
[T] = 
\begin{bmatrix}
  a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{12}a_{13} & 2a_{11}a_{13} & 2a_{11}a_{12} \\
  a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{22}a_{23} & 2a_{21}a_{23} & 2a_{21}a_{22} \\
  a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{32}a_{33} & 2a_{31}a_{33} & 2a_{31}a_{32} \\
  a_{21}a_{31} & a_{22}a_{32} & a_{23}a_{33} & a_{22}a_{33} + a_{23}a_{32} & a_{21}a_{33} + a_{23}a_{31} & a_{21}a_{32} + a_{22}a_{31} \\
  a_{11}a_{31} & a_{12}a_{32} & a_{13}a_{33} & a_{12}a_{33} + a_{13}a_{32} & a_{11}a_{33} + a_{13}a_{31} & a_{11}a_{32} + a_{12}a_{31} \\
  a_{11}a_{21} & a_{12}a_{22} & a_{13}a_{23} & a_{12}a_{23} + a_{13}a_{22} & a_{11}a_{23} + a_{13}a_{21} & a_{11}a_{22} + a_{12}a_{21} \\
\end{bmatrix}
\]  

(1.34)
A *MATLAB* program that can be used to generate (1.34) is shown next (also available in [5]).

```matlab
% Derivation of the transformation matrix [T]
clear all;
syms T alpha R
syms a a11 a12 a13 a21 a22 a23 a31 a32 a33
a = [a11,a12,a13; a21,a22,a23; a31,a32,a33];
T(1:6,1:6) = 0;
for i=1:1:3
  for j=1:1:3
    if i==j; alpha = j; else alpha = 9-i-j; end
    for p=1:1:3
      for q=1:1:3
        if p==q beta = p; else beta = 9-p-q; end
        T(alpha,beta) = 0;
        if alpha<=3 & beta<= 3; T(alpha,beta)=a(i,p)*a(i,p); end
        if alpha> 3 & beta<= 3; T(alpha,beta)=a(i,p)*a(j,p); end
        if alpha<=3 & beta>3; T(alpha,beta)=a(i,q)*a(i,p)+a(i,p)*a(i,q);end
        if alpha>3 & beta>3; T(alpha,beta)=a(i,p)*a(j,q)+a(i,q)*a(j,p);end
      end
    end
  end
end
T
R = eye(6,6); R(4,4)=2; R(5,5)=2; R(6,6)=2; % Reuter matrix
Tbar = R*T*R^(-1)
```

1.10.2 Strain Transformation

The tensor components of strain $\varepsilon_{ij}$ transform in the same way as the stress components

$$
\varepsilon'_{ij} = a_{ip}a_{jq}\varepsilon_{pq}
$$

or

$$
\varepsilon'_{\alpha} = T_{\alpha\beta}\varepsilon_{\beta}
$$

with $T_{\alpha\beta}$ given by (1.34). However, the three engineering shear strains $\gamma_{xz}, \gamma_{yz}, \gamma_{xy}$ are normally used instead of tensor shear strains $\varepsilon_{xz}, \varepsilon_{yz}, \varepsilon_{xy}$. The engineering strains ($\epsilon$ instead of $\varepsilon$) are defined in (1.5). They can be obtained from the tensor components by the following relationship

$$
\epsilon_{\delta} = R_{\delta\gamma}\varepsilon_{\gamma}
$$

with the Reuter matrix given by