

# Group Theory for High Energy Physicists

Mohammad Saleem  
Muhammad Rafique

 CRC Press  
Taylor & Francis Group

A TAYLOR & FRANCIS BOOK

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for  
High Energy  
Physicists



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**CRC Press**

Taylor & Francis Group  
Boca Raton London New York

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CRC Press  
Taylor & Francis Group  
6000 Broken Sound Parkway NW, Suite 300  
Boca Raton, FL 33487-2742

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Version Date: 20120725

International Standard Book Number-13: 978-1-4665-1064-7 (eBook - PDF)

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# *Preface*

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Group theory has played an exciting, fascinating, and significant role in the development of various disciplines of physics. However, it is amazing that during the last three decades no book has been written that starts *ab initio* and then builds on and considers applications for group theory from the point of view of high energy physicists. This book is an attempt to achieve this objective.

Chapter 1 introduces the concept of a *group* and then presents the characteristics that are imperative for developing group theory as it applies to high energy physics. Chapter 2 describes group representations, as physicists have always found it more convenient to deal with matrix representations of a group than the abstract group itself. Group representations are important because various irreducible invariant subspaces of a group can be used to accommodate elementary particles with specific characteristics. Chapter 3 discusses continuous groups, which have given rise to spectacular progress in our understanding of particles and their interactions. The subject, however, has not been developed in a mathematically rigorous manner. The aim of this book is to introduce the concept of continuous groups, especially Lie groups, and their characteristics in a way that is easily comprehensible to physicists. The root structure of some important groups is analyzed, and the weights of various representations of these groups are obtained. All three chapters are interspersed with examples and problems. These chapters can therefore form the content of a group theory course for undergraduate students interested in specializing in high energy physics.

In the first three chapters, group theory is developed to a level that it can be applied to solve high energy physics problems. Chapter 4 shows how symmetry principles associated with group theoretical techniques can be used to interpret some experimental results and make predictions. The appendices at the end of the book prove some important relations and theorems given in the text.

It is hoped that the book will be useful to undergraduate as well as graduate students in physics and mathematics and researchers in high energy physics.

Dr. Muhammad Rafique, coauthor of the book, breathed his last while the book was still in preparation. May his soul rest in peace.



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## *About the Author*

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Dr. Mohammad Saleem obtained his BA (Hons), MA (math), MSc (physics), and PhD from University of the Punjab and his BSc (Special Hons) from the London University. He has been professor and chair of the Department of Physics, founder and director of the Centre for High Energy Physics, professor in the School of Physical Sciences and dean of the Faculty of Science at University of the Punjab. At present, he is professor emeritus at the University of the Punjab. He is also professor at the Institute for Basic Research in Palm Harbor, Florida. He has written more than 150 research papers on high energy physics, most of which have been published in standard foreign journals or were accepted for presentation at international conferences. His books on special relativity and high energy physics, written in collaboration with his colleagues, have been published in the United Kingdom and the United States. He is also an editor of the *Hadronic Journal*.

Dr. Muhammad Rafique graduated from University of the Punjab in 1962. He obtained his MSc and PhD from University of North Wales. He spent a year at the International Centre for Theoretical Physics on a postdoctoral fellowship and then worked for five years as associate professor of applied mathematics at Alfateh University in Tripoli, Libya. He was appointed professor of applied mathematics at University of the Punjab in 1983. He has written more than 50 research papers on high energy physics.



# 1

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## *Elements of Group Theory*

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This chapter introduces the concept of a group and presents the characteristics that are imperative for developing group theory as it applies to high energy physics.

---

### 1.1 Definition of a Group

What is a group? This can best be answered by reference to a few examples. Consider the set  $S = \{1, -1, i, -i\}$ . The numbers in curly brackets are called its *elements* or *members*. With the law for the combination of elements, called the *law of composition* or the *binary operation*, as a multiplication of complex numbers, the elements of  $S$  possess the following four properties:

1. If an element of  $S$  is multiplied, according to the prescribed law of composition, by an element of the same set, the resulting number is again an element of  $S$ . This is called the *closure property*. For example,

$$1 \times i = i, i \times (-i) = 1, (-1) \times (-1) = 1$$

2. The multiplication is *associative*; that is, the product does not depend on the order in which the elements are multiplied. Thus, if  $a, b,$  and  $c$  are arbitrary elements of  $S$ , then  $a \times (b \times c) = (a \times b) \times c$ . For example:

$$1 \times \{(-1) \times i\} = 1 \times (-i) = -i$$

and

$$\{1 \times (-1)\} \times i = (-1) \times i = -i$$

so that

$$1 \times \{(-1) \times i\} = \{1 \times (-1)\} \times i$$

3. The set  $S$  contains an element that, when multiplied by any one of its elements, either from the left or from the right, *reproduces* that element. The element 1 in  $S$  possesses this characteristic:

$$1 \times 1 = 1, 1 \times (-1) = -1, i \times 1 = i, 1 \times (-i) = -i$$

Such an element is said to be the **identity element**.

4. To every element of the set  $S$ , there corresponds an element of the same set such that the product of two elements, irrespective of their order, is the identity element. Then any one of these elements is called the *inverse* of the other. For instance, multiplying the elements  $i$  and  $-i$  of  $S$ , we get 1 (i.e., the identity element). By virtue of the previous definition,  $-i$  is the inverse of  $i$  and  $i$  is the inverse of  $-i$ .

The set of the cube roots of unity,  $\{1, \omega, \omega^2\}$ , with  $\omega^3 = 1$ , also possesses these four characteristics with respect to ordinary multiplication with 1 as the identity and the inverse elements forming the pairs  $(1, 1)$ ,  $(\omega, \omega^2)$ ,  $(\omega^2, \omega)$ .

The set  $S$  of two nonsingular matrices  $I$  and  $A$  given by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

exhibits the same four characteristics with the law of composition as matrix multiplication. (A nonsingular matrix is that whose determinant is different from zero.) We notice that:

1. The product of an arbitrary element of the set  $S$  with any one of its elements produces an element of this very set. For instance:

$$AA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

2. The multiplication of matrices is always associative.  
 3.  $I$  is the identity element because, for instance,  $IA = A = AI$ .  
 4. The inverse of every element exists and is an element of the set. For instance, as  $AA = I$ ,  $A$  is its own inverse.

The fact that different sets of various elements possess the same four characteristics with respect to some given binary operation (law of composition) suggests that they may be given a common name. Such sets are called *groups*.

Thus, a finite or an infinite set  $G = \{a, b, c, \dots\}$  forms a group with respect to a binary operation, usually called multiplication, if

1. The product of any two elements of  $G$  is also an element of  $G$ ; this includes the product of an element with itself. This is called the *closure property*.
2. The multiplication is associative.
3. The identity element, which occurs when multiplication with any element of the set, either from the right or from the left, reproduces that element, exists, and is an element of  $G$ .
4. The inverse of an element, which occurs when multiplication with the element, irrespective of the order, yields the identity element, exists, and is an element of  $G$ .

Terms like *product* and *multiplication*, when applied to groups, do not always have their conventional meaning. In the example  $(1, \omega, \omega^2)$ , the law of combination of elements is multiplication of complex numbers, while in (4) the law of composition is matrix multiplication. The law of composition may be multiplication of numbers, addition of numbers, matrix multiplication, or something else; it is customary to use *product* or *multiplication* for every kind of binary operation. Since in the sequel we shall often have to refer to the elements of the various sets by phrases *belongs to* or *is a member of* or *an element of* or *is contained in*, it is convenient to introduce the symbol  $\in$ , which stands for these phrases. Thus,  $a \in G$  means that  $a$  belongs to  $G$  or is a member of  $G$  or is contained in  $G$ .

It may be noticed that if we combine the elements of the set  $S = \{1, -1, i, -i\}$  with the addition of complex numbers as the binary operation, the set does not contain the identity element zero and therefore does not form a group.

#### PROBLEM 1.1

Show that *under addition of complex numbers*, the set  $\{1, \omega, \omega^2\}$  does not form a group.

To crystallize our ideas, we recast the definition of a group as follows. A finite or infinite nonempty set  $G = \{a, b, c, \dots\}$  is said to form a group *under a specified law of composition* if

- (1) For every  $a, b \in G$ , the product  $ab \in G$ .
- (2) For every  $a, b, c \in G$ ,  $a(bc) = (ab)c$ .
- (3) There exists an element  $e \in G$  such that  $ae = a = ea$  for every  $a \in G$ . The element  $e$  is called the identity element.
- (4) For every  $a \in G$ , there exists an element  $a^{-1} \in G$ , known as the inverse of  $a$ , such that  $aa^{-1} = e = a^{-1}a$ .

Sometimes postulates (3) and (4) are replaced by weaker postulates that the left identity and the left inverse exist. From these weaker postulates, (3) and (4) can be derived.

A group is said to be *finite* if it has a finite number of elements. Otherwise, it is said to be *infinite*. The number of elements of a finite group is called the *order of the group*. Thus, the finite group  $G = \{1, -1, i, -i\}$  is of order 4. The group of all integers negative, zero, and positive, that is,  $G = \{\dots, -2, -1, 0, 1, 2, \dots\}$  is an infinite group under addition. Such a group is said to be an *additive group*.

The group elements do not necessarily commute with respect to the underlying law of composition. If all the elements of a group  $G$  commute with one another, that is,  $ab = ba$  for all  $a, b \in G$ , the group is said to be *commutative* or *Abelian*. Otherwise, it is said to be noncommutative or non-Abelian. In all the examples given above, the groups are Abelian.

**PROBLEM 1.2**

Give an example of a noncommutative finite group.

**PROBLEM 1.3**

Is an additive group always Abelian?

## 1.2 Some Characteristics of Group Elements

We will now show that a group contains only one identity element and there is only one inverse of each element. In other words, in every group, *identity element is unique*, and to each element of the group there corresponds a *unique inverse*. These results can be proved as follows:

1. Suppose that a group contains two identity elements  $e$  and  $e'$ . Then, by the very definition of an identity element, we must have  $ee' = e$  and  $ee' = e'$  so that  $e = e'$ . This shows that a group can contain only one identity element: identity element of a group is unique.
2. We will next show that inverse of any element of a group is also unique. Consider an arbitrary element  $a$  of a group  $G$ . Suppose that the element  $a$  has two inverses,  $b$  and  $c$ . By definition, both the inverses belong to  $G$  and must be such that  $ba = ab = e$  and  $ac = ca = e$ . Multiplying  $ba = e$  from the right by  $c$ , we get  $bac = ec = c$ . Similarly, multiplying  $ac = e$  from the left by  $b$ , we get  $bac = be = b$ . Comparing the equations  $bac = c$  and  $bac = b$ , we obtain  $c = b$ . That is, the two inverses of  $a$  must be equal. This shows that inverse of an element is unique.
3. It is clear from the definition of the inverse of an element  $a$  of a group that if  $a^{-1}$  is the inverse of  $a$ , then  $a$  is the inverse of  $a^{-1}$ . Therefore, we may write  $(a^{-1})^{-1} = a$ .
4. We next prove that the inverse of the product of any two elements of a group is equal to the product of the inverses of those elements taken in the reverse order, that is,  $(ab)^{-1} = b^{-1}a^{-1}$  for all  $a, b \in G$ .

To prove this, we note that since the multiplication is associative, we must have

$$(ab)(b^{-1}a^{-1}) = abb^{-1}a^{-1} = aea^{-1} = aa^{-1} = e$$

This equation shows that  $b^{-1}a^{-1}$  is the inverse of  $ab$ :  $b^{-1}a^{-1} = (ab)^{-1}$ . The result can be generalized to any finite number of factors.

**PROBLEM 1.4**

Show that if  $a, b, c$  belong to a group  $G$  such that  $ab = ac$ , then  $b = c$ .

**PROBLEM 1.5**

Prove that if  $a$  is an arbitrary element of a group  $G$ , then  $aG = G$ , where  $aG$  stands for the set of all the elements obtained by multiplying each element of  $G$  from the left by  $a$ .

**PROBLEM 1.6**

A rational number can be expressed as a ratio  $m/n$  of two integers,  $m$  and  $n$ , where  $n \neq 0$ . Show that the set of all positive rational numbers is a group under ordinary multiplication. Can the set of all negative rational numbers form a group under ordinary multiplication?

**PROBLEM 1.7**

Show that the set of all rational numbers is a group under ordinary addition.

**PROBLEM 1.8**

Prove that the set of all nonzero rational numbers is a group under ordinary multiplication.

**Example 1.1**

**PROBLEM**

Show that the set of all nonsingular  $2 \times 2$  matrices forms a group under matrix multiplication.

**SOLUTION**

Consider the set of all nonsingular  $2 \times 2$  matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \dots$$

Then

- (i) The product  $AB$  of two arbitrary members  $A$  and  $B$  of the set is also a square matrix of order 2. Denoting it by  $C$ , we have

$$AB = C$$

Taking the determinant of both sides, we get

$$\det(AB) = \det A \cdot \det B = \det C$$

Since A and B are nonsingular,  $\det A \neq 0$ ,  $\det B \neq 0$ . Therefore, the previous equation shows that  $\det C \neq 0$ . That is, the  $2 \times 2$  matrix C is also nonsingular and hence a member of the set.

- (ii) The multiplication, being matrix multiplication, is associative.
- (iii) The nonsingular  $2 \times 2$  unit matrix I serves as the identity element.
- (iv) Since the matrices are nonsingular, the inverse of each matrix exists and is nonsingular. Moreover, as the inverse of every matrix is of the same order as the matrix itself, its order is 2. Hence, the inverse of every matrix, being a nonsingular and  $2 \times 2$  matrix, is a member of the set.

Since the set of all nonsingular  $2 \times 2$  matrices satisfies all the group postulates, it forms a group under matrix multiplication.

### 1.3 Permutation Groups

Consider the set of all permutations of digits 1, 2, 3, that is, 123, 132, 213, 231, 312, 321.

We may write these as

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, T_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, T_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, T_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

The permutation  $T_1$  takes 1 into 1, 2 into 3, and 3 into 2. We can therefore write  $T_1$  also as

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

or

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

since these permutations also take 1 into 1, 2 into 3, and 3 into 2. That is, the permutation is unchanged by shifting the elements in the first row if the corresponding elements in the second row are also shifted in the same manner. We shall show that the set of permutations  $I, T_1, T_2, T_3, T_4, T_5$  forms a group under successive operation of permutations from right to left. For instance, the product  $T_1 T_2$  of two permutations  $T_1$  and  $T_2$  is defined as the permutation obtained by carrying out first the permutation  $T_2$  and then the permutation  $T_1$ .

(i) Now

$$T_1 T_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

This may be written as

$$T_1 T_2 = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

This shows that  $T_2$  changes 1 2 3 to 2 1 3 and  $T_1$  changes 2 1 3 to 3 1 2. The net effect is that 1 2 3 is changed to 3 1 2. Hence,

$$T_1 T_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = T_4$$

which is a member of the set. In fact, we can show that the product of any two members of the set is also a member of the set.

(ii) The multiplication is associative. We will show it for one case only:

$$T_1 (T_2 T_3) = (T_1 T_2) T_3$$

We have

$$\begin{aligned} T_1 (T_2 T_3) &= T_1 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\ &= T_1 \begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\ &= T_1 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}
 (T_1 T_2) T_3 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} T_3 \\
 &= \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} T_3 \\
 &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} T_3 \\
 &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}
 \end{aligned}$$

This shows that the multiplication is associative.

(iii) The permutation

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

is the identity element. This can be easily verified. For instance,

$$\begin{aligned}
 IT_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = T_1
 \end{aligned}$$

and

$$T_1 I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = T_1$$

(iv) Since the correspondence can always be reversed, the inverse of each permutation exists. Let us calculate the inverse of  $T_4$ . Let  $T_i$  be the

inverse of  $T_4$ . Then the product of  $T_4$  and  $T_i$  must be equal to the identity element  $I$ :

$$T_i T_4 = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

or

$$\begin{pmatrix} 3 & 1 & 2 \\ c & a & b \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 2 & 3 \\ c & a & b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

This yields  $a = 2, b = 3, c = 1$  so that

$$T_i = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = T_3$$

Thus, the inverse of  $T_4$  is  $T_3$  which is a member of the set. Similarly, we can show that  $I, T_1, T_2, T_4, T_5$  are, respectively, the inverses of  $I, T_1, T_2, T_3, T_5$ . Hence, all the inverses are members of the set of permutations.

The previous analysis shows that the set of permutations  $I, T_1, T_2, T_3, T_4, T_5$  forms a group and that the law of composition is the application of permutations from right to left.

In general, the set of all permutations

$$T = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ i_1 & i_2 & i_3 & \dots & i_n \end{pmatrix}$$

of  $n$  objects forms a group. This group is called the *permutation group* or the *symmetric group* and is denoted by  $S_n$ . Since the number of permutations of  $n$  objects is  $n!$ , there are  $n!$  transformations and hence  $n!$  elements in the group.