NON-EUCLIDEAN GEOMETRY

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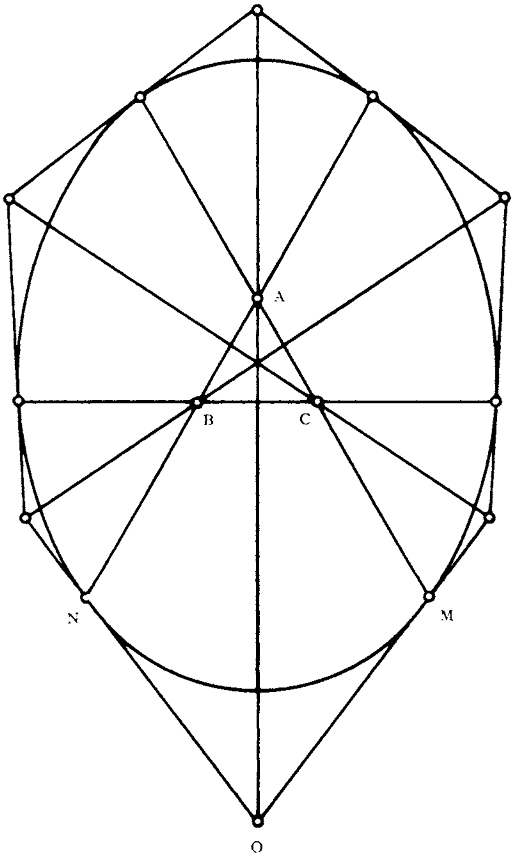
"Mighty is the charm
Of those abstractions to a mind beset
With images, and haunted by herself."

WORDSWORTH
(*The Prelude*, BOOK VI)



Brianchon's Theorem
and the concurrence of angle-bisectors,
(See pages 59 and 200)

MATHEMATICAL EXPOSITIONS. No. 2

NON-EUCLIDEAN GEOMETRY

by

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PREFACE

THE name *non-Euclidean* was used by Gauss to describe a system of geometry which differs from Euclid's in its properties of parallelism. Such a system was developed independently by Bolyai in Hungary and Lobatschewsky in Russia, about 120 years ago. Another system, differing more radically from Euclid's, was suggested later by Riemann in Germany and Schläfli in Switzerland. The subject was unified in 1871 by Klein, who gave the names *parabolic*, *hyperbolic*, and *elliptic* to the respective systems of Euclid, Bolyai-Lobatschewsky, and Riemann-Schläfli. Since then, a vast literature has accumulated, and it is with some diffidence that I venture to add a fresh exposition.

After an historical introductory chapter (which can be omitted without impairing the main development), I devote three chapters to a survey of real projective geometry. Although many text-books on that subject have appeared, most of those in English stress the connection with Euclidean geometry. Moreover, it is customary to define a conic and then derive the relation of pole and polar, whereas the application to non-Euclidean geometry makes it more desirable to define the polarity first and then look for a conic (which may or may not exist)! This treatment of projective geometry, due to von Staudt, has been found satisfactory in a course of lectures for undergraduates (Coxeter [6]).

In Chapters VIII and IX, the Euclidean and hyperbolic geometries are built up axiomatically as special cases of a more general "descriptive geometry." Following Veblen, I develop the properties of parallel lines (§8.9) *before* introducing congruence. For the introduction of *ideal* elements, such as points at infinity, I employ the method of Pasch and F. Schur. In this manner, hyperbolic geometry is eventually identified with the geometry of Klein's projective metric as applied to a real conic or quadric (Cayley's Absolute, §§8.1, 9.7). This elaborate process of identification is unnecessary in the case of *elliptic* geometry. For, the axioms of real projective geometry (§2.1) can be taken over as they stand. Any axioms of congruence that might be proposed would quickly lead to the absolute polarity, and so are conveniently replaced by the simple statement that one uniform polarity is singled out as a means for *defining* congruence.

Von Staudt's extension of real space to complex space is logically similar to Pasch's extension of descriptive space to projective space, but is far harder for students to grasp; so I prefer to deal with real space alone, expressing distance and angle in terms of real cross ratios. I hope this restriction to real space will remove some of the mystery that is apt to surround such concepts as Clifford parallels (§§7.2, 7.5). But Klein's complex treatment is given as an alternative (at the end of Chapters IV-VII).

In order to emphasize purely geometrical ideas, I introduce the various geometries synthetically. But coordinates are used for the derivation of trigonometrical formulae in Chapter XII.

Roughly speaking, the chapters increase in difficulty to the middle of the book. (Chapter VII may well be omitted on first reading, although it is my own favourite.) Then they become progressively easier. For a rapid survey of the subject, just read the first and the last two

"Mathematical Expositions, No. 2" is to some extent a sequel to No. 1. (See especially pages 46, 50, 57, 156 of No. 1.) But No. 1 naturally has a far wider scope, and No. 2 can be read independently. A certain overlapping of subject-matter was inevitable.

For reading various parts of the manuscript in preparation, and making valuable suggestions for its improvement, I offer cordial thanks to my colleagues on the Editorial Board, especially Richard Brauer and G. de B. Robinson; also to N. S. Mendelsohn of the Department of Mathematics, to S. H. Gould of the Department of Classics, and to A. W. Tucker of Princeton University.

H. S. M. COXETER

The University of Toronto,
May, 1942.

PREFACE TO THE THIRD EDITION

Apart from a simplification of p. 206, most of the changes are additions. Accordingly, it has seemed best to put the extra material together as a new chapter (XV). This includes a description of the two families of "mid-lines" between two given lines, an elementary derivation of the basic formulae of spherical trigonometry and hyperbolic trigonometry, a computation of the Gaussian curvature of the elliptic and hyperbolic planes, and a proof of Schläfli's remarkable formula for the differential of the volume of a tetrahedron.

I gratefully acknowledge the help of L. J. Mordell and Frans Handest in the preparation of §15.6 (on quadratic forms) and §15.8 (on problems of construction), respectively.

H. S. M. C.

December, 1956.

PREFACE TO THE FIFTH EDITION

The principal change for this edition is on pp. 265–266, where Poincaré's conformal model for hyperbolic space is extended in the manner advocated by Liebmann ([1], on p. 54 of his first edition, 1905), Sommerville ([2], pp. 230, 238), Veblen and Young ([2,] II, p. 370), and Klein ([3], pp. 309–311). This amounts to a representation of the planes in hyperbolic space by circles in the real inversive plane, enabling us to interpret the known results on homographies and antihomographies as properties of isometries (or congruent transformations) in hyperbolic space. Conversely, some rather tricky problems on circles become almost trivial when expressed in hyperbolic terms. For instance, an *elliptic bundle of circles* is transformed into the set of planes through a point! Again, the *loxodromic homography*

$$z' = e^{d+i\delta} z$$

becomes a screw displacement involving a translation through distance d and a rotation through angle δ. For further details, see a forthcoming paper on "The inversive plane and hyperbolic space" in *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg.*[*]

H.S.M.C.

April, 1965.

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[*] Vol. 29 (1966), pp. 217–241.

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CHAPTER I

THE HISTORICAL DEVELOPMENT OF NON-EUCLIDEAN GEOMETRY

1.1. Euclid. Geometry, as we see from its name, began as a practical science of measurement. As such, it was used in Egypt about 2000 B.C. Thence it was brought to Greece by Thales (640-546 B.C.), who began the process of abstraction by which positions and straight edges are idealized into points and lines. Much progress was made by Pythagoras and his disciples. Among others, Hippocrates attempted a logical presentation in the form of a chain of propositions based on a few definitions and assumptions. This was greatly improved by Euclid (about 300 B.C.), whose Elements became one of the most widely read books in the world. The geometry taught in high school today is essentially a part of the Elements, with a few unimportant changes.

According to the best editions, Euclid's basic assumptions consist of five "common notions" concerning magnitudes, and the following five Postulates:

I. *A straight line may be drawn from any one point to any other point.*

II. *A finite straight line may be produced to any length in a straight line.*

III. *A circle may be described with any centre at any distance from that centre.*

IV. *All right angles are equal.*

V. *If a straight line meet two other straight lines, so as to make the two interior angles on one side of it together less than two right angles, the other straight lines will meet if produced on that side on which the angles are less than two right angles.*

According to the modern view, these postulates are incomplete and somewhat misleading. (For the rigorous axioms that replace them, see §§ 8.3, 9.1, 9.5.) Still, they give some idea of the kind of assumptions that have to be made, and are of interest historically.

Postulate I is generally regarded as implying that any two points determine a unique line, Postulate II that a line is of infinite length. Euclid showed the great strength of his genius by introducing Postulate V, which is not self-evident like the others. (Moreover, his reluctance to introduce it provides a case for calling him the first non-Euclidean geometer!) Between his time and our own, hundreds of people, finding it complicated and artificial, have tried to deduce it as a proposition. But they only succeeded in replacing it by various equivalent assumptions, such as the following five:

1.11. *Two parallel lines are equidistant.* (Posidonius, first century B.C.)

1.12. *If a line intersects one of two parallels, it also intersects the other.* (Proclus, 410-485 A.D.)

1.13. *Given a triangle, we can construct a similar triangle of any size whatever.* (Wallis, 1616-1703.)

1.14. *The sum of the angles of a triangle is equal to two right angles.* (Legendre, 1752-1833.)

1.15. *Three non-collinear points always lie on a circle.* (Bolyai Farkas,* 1775-1856.)

According to Euclid's definition, two lines are parallel if they are coplanar without intersecting. (Following Gauss and Lobatschewsky, we shall modify this definition later.) The

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*In Hungarian, the surname is put first. The "l" in "Bolyai" is mute.

existence of such pairs of lines follows from Euclid I, 27, which depends on Postulate II but not on Postulate V.

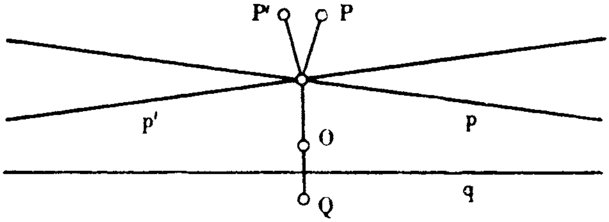

FIG. 1.1A

It is an interesting exercise to establish the equivalence of all the above statements, assuming Postulates I-IV (and the consequent propositions Euclid I, 1-28).* For instance, to deduce 1.12 from 1.15 we may proceed as follows.

Let **p** and **p′** be two intersecting lines, let **O** be any point on the perpendicular from (**p**, **p′**) to another line **q**, and let **P**, **P′**, **Q** be the reflected images of **O** in **p**, **p′**, **q**. (See Fig. 1.1A.) By Postulate IV, **p** and **p′** are not both perpendicular to **OQ**: thus **P** and **P′** cannot both lie on **OQ**. Suppose **P** does not lie on **OQ**; then, by 1.15, **p** meets **q** at the centre of the circle **POQ**. Hence **p** and **p′** cannot both be parallel to **q**. Thus 1.15 implies that two intersecting lines cannot both be parallel to the same line. This statement (commonly known as Playfair's Axiom, though Playfair copied it from Ludlam) is clearly equivalent to 1.12.

The necessity of making some such assumption has been finally established only during the last hundred years. Nowadays, anyone who tries to prove Postulate V is classed with circle-squarers and angle-trisectors. For we know that there is a perfectly logical geometry in which the lines in question

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*See Bonola [2], pp. 61, 119; Sommerville [1], pp. 288-293. All such references are to the bibliography on pp. 293-300.

may fail to meet, even when the interior angles are quite small. This remark is so easy to make today that we are apt to forget what a heresy it seemed to a generation brought up in the belief that Euclid's was the only true geometry. We now learn of many different geometries, but for historical reasons we reserve the name ***non-Euclidean*** for two special kinds: ***hyperbolic*** geometry, in which all the "self-evident" postulates I-IV are satisfied though Postulate V is denied, and ***elliptic*** geometry, in which the traditional interpretation of Postulate II is modified so as to allow the total length of a line to be finite.

As a first glimpse of hyperbolic geometry, here are the statements that replace 1.11-1.15: Two lines cannot be equidistant; a line may intersect one of two parallels without intersecting the other; similar triangles are necessarily congruent; the sum of the angles of a triangle is less than two right angles; three points may be neither collinear nor concyclic. In elliptic geometry, on the other hand, any two coplanar lines intersect, so there are no parallels in Euclid's sense, and no equidistant lines in a plane. (We shall see, however, that equidistant lines are possible in space.)

Each of these geometries, Euclidean and non-Euclidean, is *consistent*, in the sense that the assumptions imply no contradiction. But which geometry is valid in physical space? It is important to realize that this question is meaningless until we have assigned physical equivalents for the geometrical concepts Even the notion of a point, "position without magnitude. can only be realized by a process of approximation. Then, what is the physical counterpart for a straight line? The two most obvious answers are: a taut string, and a ray of light. According to recent developments in physics, these are not precisely the same! But the discrepancy is due to the presence of matter, and so a theoretical geometry of empty space remains significant. Consider, then, two rays of light, perpendicular to one plane. Certainly they remain equidistant according to all terrestrial experiments; but it is quite conceivable that they might ultimately diverge (as in hyperbolic geometry) or converge (as in elliptic).

1.2. Saccheri and Lambert. The most elaborate attempt to prove the "parallel postulate" was that of the Jesuit Saccheri (1667-1733), who based his work on an *isosceles birectangle*, i.e. a quadrangle **ABED** with **AD** = **BE** and right angles at **D** and **E**. It is obvious that the angles at **A** and **B** are equal. He considered the three hypotheses that they are obtuse, right, or acute, and showed that the assumption of any one of these hypotheses for a single isosceles birectangle implies the same for every isosceles birectangle. It was his intention to establish the hypothesis of the right angle by showing that either of the other hypotheses leads to a contradiction. He found that the hypothesis of the obtuse angle implies Postulate V, which in turn implies the hypothesis of the right angle. From the hypothesis of the acute angle he made many interesting deductions, always hoping for an eventual contradiction. We know now that his hope could never have been realized (without his making a mistake); but in the attempt he was unwittingly discovering many of the theorems of what was later to be known as hyperbolic geometry.

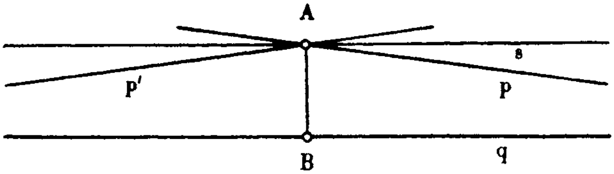

FIG. 1.2A

In particular, he considered a point **A** and a line **q** (not through **A**), and showed that, on the hypothesis of the acute angle, the flat pencil of lines through **A** contains two special lines **p** and **p**′, which divide the pencil into two parts, the first part consisting of the lines which intersect **q**, and the second of those which have a common perpendicular with **q**. (See Fig. 1.2A.) The line **p** (and likewise **p**′) is *asymptotic* to **q**, in the sense that the distance to **q** from a point proceeding along **p** continually diminishes, and eventually becomes smaller than any segment, taken as small as we please. The consequence which Saccheri imagined to be "contrary to the nature of a straight line" is that *the lines* **p** *and* **q** *have a common perpendicular at their common point at infinity.* (We shall see in §10.1 that this statement can in fact be justified.)

Fifty years later, Lambert (1728-1777) followed the same general program, using a *trirectangle* which can be regarded as one half of Saccheri's isosceles birectangle (divided along the join of the midpoints of **AB** and **DE**). He likewise rejected the hypothesis of the obtuse angle (for the fourth angle of his trirectangle), but he carried the consequences of the hypothesis of the acute angle still farther.* He defined the *defect* of a polygon as the difference between its angle-sum and that of a polygon of the same number of sides in the Euclidean plane; and, observing that the defect is additive for juxtaposed polygons, he concluded that, on the hypothesis of the acute angle, *the defect of a polygon is proportional to its area.* Comparing this with the well-known result concerning the angular excess of a spherical polygon, he suggested that the hypothesis of the acute angle would hold in the case of a sphere of imaginary radius.

In Euclidean geometry, on account of 1.13, *lengths* are measured in terms of an entirely arbitrary unit which has no structural significance. In measuring *angles*, on the other hand, we can make use of a natural unit, such as a right angle or a radian, which has particular geometrical properties. In this

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*Stäckel and Engel [1], pp. 152-207.

sense we may say that Euclidean lengths are relative, whereas angles are absolute. Lambert made the notable discovery that, when Postulate V is denied, angles are still absolute, but lengths are absolute too. In fact, for every segment there is a corresponding angle, e.g. the angle of an equilateral triangle based on the given segment.

1.3. Gauss, Wachter, Schweikart, Taurinus. Gauss (1777-1855) was the first to take the modern point of view, that a geometry denying Postulate V should be developed for its intrinsic interest, without expecting any contradiction to arise. But, fearing ridicule, he kept these revolutionary ideas to himself until others had published them independently. From 1792 to 1813, he too tried to prove the parallel postulate; but after 1813 his letters show that he had overcome the customary prejudice, and developed an "anti-Euclidean" or "non-Euclidean" geometry, which is in fact the geometry of Saccheri's hypothesis of the acute angle. He discussed this with his pupil Wachter (1792-1817), who remarked in 1816 that the limiting form assumed by a sphere as its radius becomes infinite is a surface on which all the propositions of Euclid (including Postulate V) are valid; or, as we should say nowadays, that *the intrinsic geometry of a horosphere is Euclidean.*

Independently of Gauss, Schweikart (1780-1859) developed what he called "astral" geometry, in which the angle-sum of a triangle is less than two right angles and (consequently) diminishes as the area increases. In a Memorandum dated 1818 he observed that "the altitude of an isosceles right-angled triangle continually grows, as the sides increase, but it can never become greater than a certain length, which I call the Constant." Gauss complimented Schweikart on his results, and remarked that, if the Constant is called $k \log(1+\sqrt{2})$, the area of a triangle has the upper bound πk^2.

Thus encouraged, Schweikart persuaded his nephew

Taurinus (1794-1874) to devote himself to the subject; and it was in a letter written to this young man in 1824 that Gauss gave the fullest account of his own discoveries. Taurinus developed a "logarithmic-spherical" geometry by writing ik for the radius k in the formulae of spherical trigonometry. (Compare Lambert's suggestion about an imaginary sphere.) Thus, for a triangle with angles A, B, C and sides a, b, c, he found

1.31. $$\cos C = \sin A \sin B \cosh (c/k) - \cos A \cos B,$$

whence $$\cos C > -\cos (A+B), \text{ and}$$

1.32. $$A+B+C<\pi.$$

For the circumference and area of a circle of radius r, he obtained

$$2\pi k \sinh \frac{r}{k} \text{ and } 2\pi k^2 \left(\cosh \frac{r}{k} - 1\right),$$

and for the surface and volume of a sphere,

$$4\pi k^2 \sinh^2 \frac{r}{k} \text{ and } 2\pi k^3 \left(\sinh \frac{r}{k} \cosh \frac{r}{k} - \frac{r}{k}\right)$$

(Notice that, if k tends to ∞, these tend to the usual expressions in Euclidean geometry.)

1.4. Lobatschewsky. Formulae equivalent to these were derived rigorously (and quite independently) by the Russian mathematician Lobatschewsky (1793-1856), who shares with Gauss and the younger Bolyai the honour of having made the first really systematic study of what we now call hyperbolic geometry. His earliest paper was read in 1826, published in 1830, and a number of others followed.* Like Gauss, he defined parallelism in such a way that there are just two lines through a given point **A** parallel to a given line **q** (Fig. 1.2A), these being

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*Lobatschewsky [1].

asymptotic to **q**, as Saccheri had already shown. Drawing **AB** perpendicular to **q**, he defined the *angle of parallelism* $\Pi(\mathbf{AB})$ as the (acute) angle between **AB** and either of the parallels. In terms of a convenient form of the absolute unit of length (Taurinus's $k=1$), he found that the angle of parallelism for the distance $\mathbf{AB}=c$ is

$$\Pi(c)=2 \text{ arc tan } e^{-c}= \text{arc cot}(\sinh c)= \text{arc cos}(\tanh c),$$

which decreases from $\frac{1}{2}\pi$ to 0 as c increases from 0 to ∞. The same result, in the form $\sin \Pi(c) \cosh c=1$, could have been obtained by Taurinus also, if he had put $A=\Pi(c)$, $B=\frac{1}{2}\pi$, $C=0$, $k=1$ in 1.31.

Lobatschewsky derived his trigonometrical formulae from a study of the horocycle (circle of infinite radius) and horosphere (sphere of infinite radius), in the course of which he rediscovered Wachter's theorem that the geometry of horocycles on a horosphere is identical with the geometry of straight lines in the Euclidean plane. Having also rediscovered Lambert's formula $\pi-A-B-C$ for the area of a triangle **ABC**, he proceeded to calculate the volume of a tetrahedron,* expressing the result in terms of his famous transcendental function

$$L(x)=\int_0^x \log \sec y \, dy.$$

Observing that the trigonometrical formulae of Euclidean geometry are valid in the infinitesimal neighbourhood of a point in the new geometry, Lobatschewsky considered the possibility that his geometry might replace Euclid's in the exploration of astronomical space. The crucial experiment would consist in finding a positive lower bound for the parallax of stars. For, if c is a diameter of the Earth's orbit, measured in terms of the (unknown) absolute unit, the parallax of any star should exceed $\frac{1}{2}\pi-\Pi(c)$. But this remains an open question, since such a lower bound, if it exists, is smaller than the

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*Cf. Schläfli [1], p. 97; Richmond [1]; Coxeter [1].

allowance for experimental error. The failure of the experiment merely tells us that, if space is in fact hyperbolic, the absolute unit must be many millions of times as long as the diameter of the Earth's orbit.

In this connection we must bear in mind that, although a geometry may seem more interesting if we can compare it with the real world, its validity as a logical structure is not affected, but depends only on its internal consistency. In order to show that his "imaginary" geometry or "pangeometry" is as consistent as Euclidean geometry, Lobatschewsky pointed out that it is all based on his formulae for a triangle, which lead to the familiar formulae for a spherical triangle when the sides a, b, c are replaced by ia, ib, ic. Any inconsistency in the new geometry could be "translated" into an inconsistency in spherical geometry (which is part of Euclidean geometry). Thus, after two thousand years of doubt, the independence of Euclid's Postulate V was finally established.

1.5. **Bolyai.** Many of the same results were discovered about the same time by Bolyai János (1802-1860), who wrote to his father, Bolyai Farkas, in 1823: "I have resolved to publish a work on the theory of parallels, as soon as I shall have put the material in order. . . . The goal is not yet reached, but I have made such wonderful discoveries that I have been almost overwhelmed by them. . . . *I have created a new universe from nothing.*"

Bolyai Farkas expressed the wish to insert his son's discoveries in his own book, as an Appendix.* In making this offer, he remarked, more appropriately than he realized, that "many things have an epoch, in which they are found at the same time in several places, just as the violets appear on every side in spring."

The younger Bolyai's speciality was the "absolute science

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*Bolyai [1], [2].

of space" (or ***absolute geometry***), consisting of those propositions which are independent of Postulate V, so that they hold in both Euclidean and hyperbolic geometry. For instance, he expressed the "sine rule" for a triangle **ABC** in the form

$$\bigcirc a : \bigcirc b : \bigcirc c :: \sin A : \sin B : \sin C,$$

where $\bigcirc a$ denotes the circumference of a circle of radius a. He observed that such formulae hold also in ***spherical*** geometry.

1.6. Riemann. The full recognition that spherical geometry is itself a kind of non-Euclidean geometry, without parallels, is due to Riemann (1826-1866). He realized that Saccheri's hypothesis of the obtuse angle becomes valid as soon as Postulates I, II and V are modified to read:

I. *Any two points determine at least one line.*
II. *A line is unbounded.*
V. *Any two lines in a plane will meet.*

For a line to be unbounded and yet of finite length, it merely has to be re-entrant, like a circle. The great circles on a sphere provide a model for the finite lines on a finite plane, and, when so interpreted, satisfy the modified postulates. But if a line and a plane can each be finite and yet unbounded, why not also an *n*-dimensional manifold, and in particular the three-dimensional space of the real world? In Riemann's words of 1854: "The unboundedness of space possesses a greater empirical certainty than any other external experience. But its infinite extent by no means follows from this; on the other hand, if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite provided this curvature has ever so small a positive value."*

According to the General Theory of Relativity, astronomical space has positive curvature locally (wherever there is matter), but we cannot tell whether the curvature of

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*Riemann [1], p. 36.

"empty" space is exactly zero or has a very small positive or negative value. In other words, we still cannot decide whether the real world is approximately Euclidean or approximately non-Euclidean.

Riemann employed the "infinitesimal approach" to geometry, wherein the differential of distance is expressed as the square root of a quadratic form in the differentials of the coordinates. In the special case of constant curvature, his formula is

1.61. $$ds = \frac{\sqrt{(\Sigma dx^2)}}{1+\frac{1}{4}K\Sigma x^2}$$

A year or two before Riemann read his epoch-making *Habilitationsschrift*, quoted above, Schläfli (1814-1895) developed the analytical geometry of n-dimensional Euclidean space,* and considered in particular the *hyper-sphere* $\Sigma x^2 = k^2$, which provides a model for Riemann's $(n-1)$-dimensional spherical space.

In the differential geometry of a surface in ordinary space, the product of the maximum and minimum "normal curvatures" is usually denoted by K. (Thus $K = k^{-2}$ for a sphere of radius k.) Gauss made the notable discovery that this *specific curvature* can be expressed in terms of quantities measured on the surface itself, without using properties of the underlying Euclidean space (e.g. normals). Thus it could still be defined if the underlying space did not exist. Riemann's "constant curvature" is the n-dimensional analogue of this K. Although the geometry of astronomical space, according to his hypothesis, may be identical with that of a hyper-sphere in four-dimensional Euclidean space, it does not follow that there is in any physical sense a Euclidean four-space in which the spherical three-space is imbedded.† Thus spherical space is like

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*Schläfli [1].

†Sommerville [2], p. 199 (§7).

the substance of a balloon with an extra dimension; but the simile breaks down if we seek a meaning for the air inside or outside the balloon.

1.7. Klein. Riemann developed the differential geometry of spherical space. On the other hand, Cayley (1821-1895) considered space "in the large," defining distance in terms of homogeneous coordinates. But it was Klein (1849-1925) who first saw clearly how to rid spherical geometry of its one blemish: the fact that two coplanar lines (being two great circles of a sphere) have not just one but *two common points.* Since every point determines a unique antipodal point, and every figure is thus duplicated at the antipodes, he realized that nothing would be lost, but much gained, by abstractly identifying each pair of antipodal points, i.e. by changing the meaning of the word "point" so as to call such a pair *one point.*

The word "line" will then be used for a great circle with every pair of diametrically opposite points identified (or a great semicircle with its two ends identified). So also, the word "plane" will be used for a great sphere with every pair of antipodal points identified, and analogous definitions can be made in any number of dimensions. With this meaning for the words, *any* two points determine a unique line; for, antipodal points are no longer two but one. Thus the traditional form of Postulate I is restored. As for Postulate II, a line is still unbounded, though finite, its length being half that of the great circle. Right angles retain their ordinary meaning, but a circle appears as a pair of antipodal circles.

It was to this modification of spherical geometry that Klein gave the name *elliptic* geometry. It is in many ways simpler than either spherical or Euclidean geometry, and can be developed quite independently. The geometry of pairs of antipodal points is merely a *model* for it, a convenient representation in terms of more familiar concepts. Another model is obtained by considering the diameters which join such pairs of points. In this manner the points and lines in elliptic space of n dimensions are represented by the lines and planes through a fixed point O in Euclidean space of $n+1$ dimensions. In particular, elliptic geometry of two dimensions is represented as the geometry of a *bundle* in ordinary space. To interpret the elliptic concepts in terms of Euclidean concepts, we translate them according to the following "dictionary":

The elliptic plane	Euclidean space, in the neighbourhood of a fixed point O
Point	Line through O
Line	Plane through O
Segment	Angle
Angle	Dihedral angle
Perpendicular lines	Perpendicular planes
Triangle	Trihedron
Circle	Right circular cone
Rotation about a point	Rotation about a line through O
Reflection in a line	Reflection in a plane through O
etc.	etc.

A third model (for elliptic plane geometry) can be derived from this second model by considering the section of the bundle by an arbitrary plane, not passing through O. This has the advantage of representing points by points, and lines by lines.* But distances and angles are inevitably distorted, since the distance between two points has to be re-defined as the angle subtended at O. Moreover, certain points of the elliptic geometry are left out, since certain lines of the bundle are parallel to the chosen plane. In order to accommodate these extra points, it is natural to augment the Euclidean plane by postulating *points at infinity*, one for every direction, in the manner advocated by Kepler (1571-1630) and Desargues (1593-1662). When this is done, we have the *projective* plane,

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*Klein [1], p. 604; [3], p. 148.

in which every two lines intersect (either at an ordinary point or at infinity). Thus, if metrical ideas are left out of consideration, elliptic geometry is the same as real projective geometry.

Conversely, real projective geometry (which we shall develop in Chapters II, III, IV) contains certain correspondences which enable us to define the elliptic metric in the whole space (see Chapters V, VI, VII), and to define either the Euclidean or the hyperbolic metric in a suitable part of space (Chapter IX).

The study of elliptic geometry is almost forced upon us as soon as we have added points and lines at infinity to Euclidean space. For, such points and lines form a plane—the *plane at infinity*—whose intrinsic geometry is elliptic. (See §9.5.)

To sum up, the metrical geometries with which we are concerned are Euclidean, hyperbolic, spherical, and elliptic. Our preoccupation with these four, as against all other continuous geometries, is justified by the fact that only in these cases is space *completely isotropic*, in the sense that all the lines through each point are alike. It is an interesting result in differential geometry that, if space is continuous and isotropic, it is also homogeneous or, as Riemann would say, of constant curvature.

CHAPTER II

REAL PROJECTIVE GEOMETRY: FOUNDATIONS

2.1. Definitions and axioms. In any geometry, logically developed, each definition of an entity or relation involves other entities and relations; therefore certain particular entities and relations must remain undefined. Similarly, the proof of each proposition uses other propositions; therefore certain particular propositions must remain unproved; these are the *axioms*. We take for granted the machinery of logical deduction, and the primitive concept of a *class* (or "set of all").

Unless the contrary is stated, the word *correspondence* will be used in the sense of *one-to-one* correspondence. Thus a set of entities is said to correspond to another set if every entity in each set is associated with a unique entity in the other set. In geometry the entities are usually points or lines, and the set of entities is called a *figure*. Thus we speak of a correspondence between two figures. It is often convenient to regard the correspondence as an operation which changes the first figure into the second. (Familiar instances are rotation, reflection, inversion, and reciprocation.) The general technique for discussing correspondences belongs properly to the theory of groups; but the following outline will suffice for our purposes.

We shall find it convenient to denote a correspondence by a capital Greek letter, such as Θ, writing $F\Theta = F'$ to mean that Θ relates the figure F to F' (or that the figure corresponding to F is F'). If a second correspondence Φ relates the figure F' to F_1', we write $F'\Phi = F_1'$, or $F\Theta\Phi = F_1'$, and say that the *product* $\Theta\Phi$ relates F to F_1'. The trivial correspondence that relates every entity to itself is called the *identity*, and is denoted by 1 (since its product with Θ is Θ itself). If $\Theta\Phi = 1$, we call Φ the

inverse of Θ, writing $\Phi = \Theta^{-1}$. (Thus the relation $F\Theta = F'$ is equivalent to $F = F'\Theta^{-1}$.) Some authors write $\Theta(F)$ instead of $F\Theta$, so as to exhibit it as a ***function*** of F. (Note that, in one of the accepted notations, we write $x = \sin^{-1} x'$ when $\sin x = x'$.)

If a correspondence Θ relates F to F′, while another correspondence Φ relates the pair of figures (F, F') to (F_1, F_1'), we say that Φ *transforms* Θ into the correspondence between F_1 and F_1'. Since

$$F_1' = F'\Phi = F\Theta\Phi = F_1\Phi^{-1}\Theta\Phi,$$

this transformed correspondence is $\Phi^{-1}\Theta\Phi$. It may happen that Θ itself relates F_1 to F_1', so that Φ transforms Θ into itself. We then say that Θ is ***invariant*** under transformation by Φ. Since the relation $\Phi^{-1}\Theta\Phi = \Theta$ may be written $\Theta\Phi = \Phi\Theta$, an equivalent statement is that Θ and Φ are ***permutable***. (As a familiar example of correspondences which are ***not*** permutable, consider the reflections in two planes not perpendicular to one another.)

According to Klein, the character of any geometry is determined by the type of correspondence under which its relations are invariant; e.g. Euclidean geometry is invariant under "similarity transformations."* The title of this book refers strictly to just two geometries, elliptic and hyperbolic; but certain others are so closely interwoven with these as to compel our attention. The concept of ***similarity***, which plays such a vital role in Euclidean geometry, has no analogue in either of the non-Euclidean geometries. On the other hand, the concept of ***parallelism*** (for lines in one plane) belongs to both Euclidean and hyperbolic geometry, but is lacking in elliptic. Bolyai János (§1.5) gave the name ***absolute*** geometry to the large body of propositions common to Euclidean and hyperbolic geometry. Some of these propositions will be used in Chapter IX, before

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*Veblen and Young [2], I, pp. 64-68; II, pp. 78, 119.