"No one today writes about mathematics and mathematicians with more grace, knowledge, skill, and clarity than Constance Reid." —MARTIN GARDNER, author of *Mathematical Games*

the history, characteristics, and lore of the numbers we use every day

FROM zero TO infinity

Ø

WHAT MAKES NUMBERS INTERESTING

∞

Constance Reid

50TH ANNIVERSARY EDITION

What mathematicians and teachers write about *FROM ZERO TO INFINITY*

"I read *From Zero to Infinity* when I was a schoolboy in Oxford, England, and my only regret is that I was well into my teens (17) before it happened. Just last week I gave away my most recent copy to the 12-year-old daughter of a friend. I will be getting another copy for myself as soon as I can."

—John B. Cosgrave, St. Patrick's College, Dublin, Ireland

"After reading *From Zero to Infinity*, I was hooked. This book discussed many beautiful ideas and facts about the integers and posed several interesting problems. I tried to solve them. I failed. I tried to construct counter examples. I made my own conjectures and proved some related results. By the time I graduated from high school I had filled two 2-inch notebooks with my own ideas, results, and calculations."

—Nathaniel Dean, Mathematics Department,
Bell Communications Research

"Constance Reid's book *From Zero to Infinity* was translated into Japanese, and I found it when I was a junior high school boy. I was really impressed by Reid's book, and I read it repeatedly. Inspired by it, I even tried to solve Fermat's Last Theorem. Now I am working in analytic number theory, and I think one of the reasons for my choice is a sentence in Reid's book: 'Analytic number theory is said to be technically the most difficult in the whole of mathematics.' "

—Kohji Matsumoto, student at Nagoya University

"Yesterday, aboard a flight from Denver, I had a nice conversation with a gentleman, John Moulter. When I learned that Mr. Moulter is a retired Los Angeles mathematics teacher, I mentioned that I had the good fortune to know Connie Reid. Mr. Moulter abruptly demanded, 'Do you mean Constance R-E-I-D?' When I answered affirmatively, he told me that you had changed his life. About fifty years ago, then a high-school history teacher, he picked up a copy of *Esquire* magazine in a barbershop containing a review of *From Zero to Infinity*. The review prompted him to buy your book. And your book inspired him to switch from teaching history to teaching mathematics!"

—Letter from a friend with appended note
from John Moulter: "Thank you. Thank you."

"I was the sort of child who always carried a book wherever he went. In fifth and sixth grade that book was most frequently *From Zero to Infinity*. I was indeed born to be a mathematician, but *From Zero to Infinity* helped me to realize that I was part of a community of number-people. There can be few pleasures more satisfying than having the chance, as an adult, to help bring one's favorite childhood book back into print."

—Bruce Reznick, University of Illinois Urbana

"I want to thank you for having written such a wonderful book. It was pitched on just the right level for a young teenager but, more to the point, it expressed the right mix of beauty and wonder. I just had to learn more. I very much believe that this small book, which still occupies an important place in my personal library, enriched my life immeasurably. It is very rare that we find what we really want to do in life, and I am very grateful that your book led me in the right direction."

—Hugh Williams, University of Calgary, Canada,
author of *Edouard Lucas and Primality Testing*

· FROM ZERO TO INFINITY ·

· FIFTIETH ANNIVERSARY EDITION ·

From Zero to Infinity

What Makes Numbers Interesting

Constance Reid

A K Peters, Ltd.
Wellesley, Massachusetts

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite
300
Boca Raton, FL 33487-2742

© 2006 by Taylor & Francis Group, LLC
CRC Press is an imprint of Taylor & Francis Group, an Informa business

No claim to original U.S. Government works
Version Date: 20150311

International Standard Book Number-13: 978-1-4398-8123-1 (eBook - PDF)

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· IN MEMORY OF ·

Julia Bowman Robinson
1919–1985

and

Raphael Mitchel Robinson
1911–1995

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0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...

The natural numbers, which are the primary subject of this book, do not end with the digits with which we represent them. They continue indefinitely—as the three dots indicate—to infinity. And they are all interesting: for if there were any uninteresting numbers, there would of necessity be a smallest uninteresting number and it, for that reason alone, would be very interesting.

· ACKNOWLEDGMENTS ·

Throughout the half century during which *From Zero to Infinity* has been in print, it has had a number of different publishers. As its author I would like to express my special gratitude to four of them.

First, to Dennis Flanagan, the longtime editor of *Scientific American*, who accepted an article on "Perfect Numbers" from a freelancing housewife who was not even a mathematician.

Second, to Robert L. Crowell, who read her article in *Scientific American*, saw its possibilities for a book, and shepherded it and its author through three editions.

Third, to Donald J. Albers, publications director of the Mathematical Association of America, who, after I had retrieved the copyright from the last in the series of publishing companies that had come into possession of Crowell books, published a fourth edition of *From Zero to Infinity* under the imprint of the MAA.

Fourth, to Klaus Peters, the president of A K Peters, Ltd. I am particularly happy that Klaus will be republishing

my first "mathematical" book, because he was also the publisher, as mathematics editor of Springer-Verlag, who in 1969 accepted for publication my life of David Hilbert and thus opened up to me a new field of mathematical writing—the writing of mathematical lives.

—Constance Reid

· AUTHOR'S NOTE ·

There is a story behind the publication of this fiftieth anniversary edition of *From Zero to Infinity*.

It begins with a phone call from my sister, Julia Robinson, on the morning of January 31, 1952. She has called to tell me of an exciting event that occurred the night before at the Institute for Numerical Analysis on the UCLA campus, where the National Bureau of Standards has located its Western Automatic Computer—the SWAC.

Julia tells me that a program by her husband, Raphael Robinson, had turned up the first new "perfect numbers" in seventy-five years—not one but two of them. (I learn only later from others that Raphael had at this point never seen the SWAC and had programmed entirely from a copy of the manual.) Julia explains the problem simply: *perfect numbers*—the name itself is intriguing—are numbers like 6 that are the sum of all their divisors except themselves: $6 = 1 + 2 + 3$. Then she tells me there is a particular form of prime necessary for the formation of such numbers, the amount of calculation involved in determining their primality, the enormousness of such primes. For me the whole

thing is fascinating. I decide to write an article about the discovery of new perfect numbers.

I am lucky to be able to interview Dick Lehmer, the Director of the SWAC, while he and his wife Emma are visiting in Berkeley. It is Emma who suggests that I send my article to *Scientific American*. If you look up the March 1953 issue you will see a photo of the SWAC and be able to read a fairly detailed description of Raphael's program and how it was fed into the computer.

Of course a subscriber later wrote to Dennis Flanagan, the editor, to complain that when he read an article in *Scientific American* he expected the author to be a Ph.D. But my not being a Ph.D. did not seem to have concerned Dennis Flanagan anymore than it concerned the publisher Robert L. Crowell. After reading my article, Mr. Crowell immediately wrote to ask if I would be interested in writing a small book on numbers that he could pair with a book on the alphabet. Even I found the combination a bit incongruous, but it gave me an idea. The title of Mr. Crowell's book, already in print, was *Twenty-six Letters*. I would write a book about the ten digits; and because I had found what Julia had told me to be so interesting, I would call it *What Makes Numbers Interesting*.

I consulted Julia and Raphael. Robert Crowell's proposal was something of a joke to them—Constance writing a book about "mathematics"—but they thought I could do what I proposed. They would, they promised, read the manuscript before it went to the publisher. Otherwise I was on my own.

I promptly sent off a proposal and a sample chapter to Robert Crowell. He replied that although he had found my sample chapter on zero "pretty tough sledding," he was enclosing a contract.

Today I really don't know how I managed to write the book that I wrote. But I learned a great amount in the course of doing so, and I found what I was learning so very interesting that I didn't see how it could fail to interest others.

The book was finished in a little over a year,

Then came a problem. The sales department flatly vetoed my proposed title—*What Makes Numbers Interesting*. The word *interesting* bothered them. Nobody would buy a book about things that were described as "interesting".

Mr. Crowell agreed. Could Mrs. Reid come up with another title?

I submitted a dozen or so, none of which I liked. The sales department simply loved the one that I disliked the most—*From Zero to Infinity*. My reasons for disliking it were the following. First, in ascending order of importance, it was similar to the title of a then very popular novel, *From Here To Eternity*. (A later reviewer noted that the title sounded as if the book was a novel.) Second, it was too similar to George Gamow's *One, Two, Three . . . Infinity* (although Gamow had begun with the number 1 while I had begun with the number 0). My real objection to the proposed title, however, was that I had not written anything in my book about the theory of the infinite.

So I added three dots after the chapter on nine to indicate that the natural numbers continued "to infinity," and held out for my original title as a subtitle. It is still there fifty years later—*What Makes Numbers Interesting*—along with a neat little proof that there is no such thing as an *uninteresting* number.

The book did quite well. It was recommended for teachers and libraries and selected by science book clubs, even described as doing for number what George R. Stewart's *Storm* had done for weather: "breathing life into a seem-

ingly lifeless body." (Stewart had been the first to give girls' names to storms.)

The Russians put Sputnik into space in 1957. Americans became almost hysterical about the possibility of falling behind in mathematics and science. Mr. Crowell announced that it was time for a second edition. At this point the change in title, to which I had so strenuously objected, paid off. Since there must be a new chapter for a new edition, it would be a chapter on the theory of infinite sets.

Four years later Crowell wanted still another edition—and of course another chapter. What could follow "Infinity"?

Here Raphael came to the rescue, proposing a chapter on the base of the natural logarithm. As he reasonably pointed out, it was only with e that mathematicians had finally been able to establish—by mathematical proof—the distribution in the large of the prime numbers—in short, to prove the Prime Number Theorem. For this reason, in his opinion, e was not at all out of place in a book on the natural numbers.

Some time later, after the third edition of my book, Robert Crowell had to face the fact that none of his four sons wanted to carry on the publishing firm that had been founded by their great grandfather. The firm was sold. It became Lippincott, then Harper and Row, then, as I recall, HarperCollins. Eventually royalties were so meager that the book seemed essentially out of print. I asked for the copyright to be returned and HarperCollins agreed, retaining only the rights to a Japanese translation.

The fourth edition of *From Zero to Infinity* was published in 1992 by The Mathematical Association of America with an autobiographical author's note instead of still another chapter.

You are now reading the Fiftieth Anniversary Edition of *From Zero to Infinity*, which is being published by A K Peters, Ltd. It is the hope of both the publisher and the author that the story that began with the discovery of the first new perfect numbers in 75 years will continue and that, through this new edition, the book will continue to reach out to new generations of young people, some of whom may be inspired, as others have been, to become mathematicians—and all will gain a glimpse of what has made the natural numbers so eternally interesting.