

# CRAFTING BY CONCEPTS

Edited by  
SARAH-MARIE BELCASTRO  
CAROLYN YACKEL

*crafting by concepts*



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*fiber arts and mathematics*

*edited by*

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For my parents, Erna and Jim Yackel, who encouraged me in all of my creative pursuits, including mathematics and including fiber arts, and who instilled in me to view every endeavor intellectually and my life holistically so that my work in mathematical fiber arts was made possible.

—Carolyn Yackel

For the three friends, whose ideas inspired my knitting work during the editing of this book, and to whom I turn first when I need my mathematical knitting thoughts and questions to be instantly understood: Rachel Shorey, Amy Szczepański, and Carolyn Yackel.

—sarah-marie belcastro

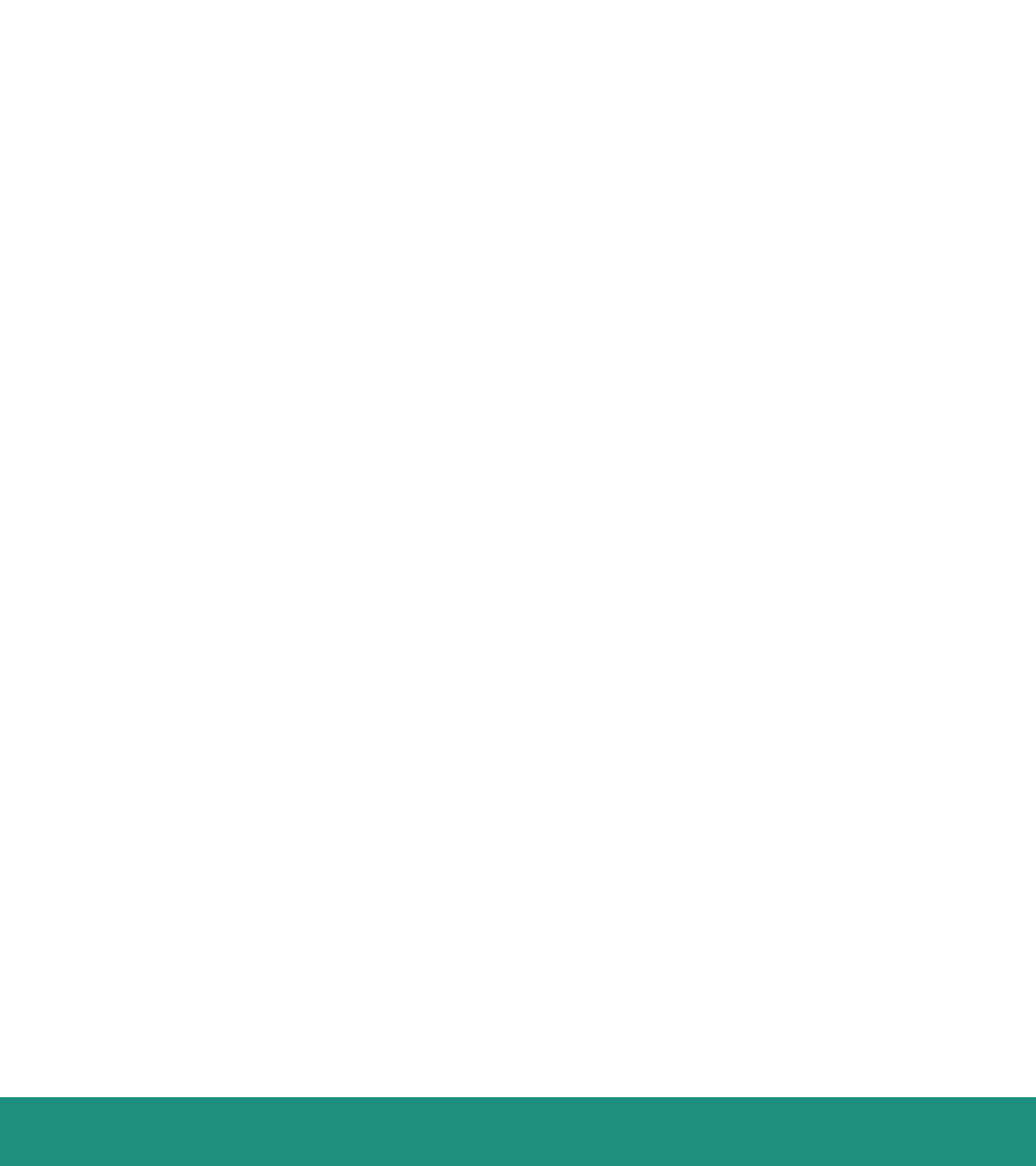
We would also like to remember our close pets Allie (Carolyn's dog) and Arzachel (sarah-marie's cat), who were our steadfast companions through *Making Mathematics with Needlework* but neither of whom made it to the completion of this volume.



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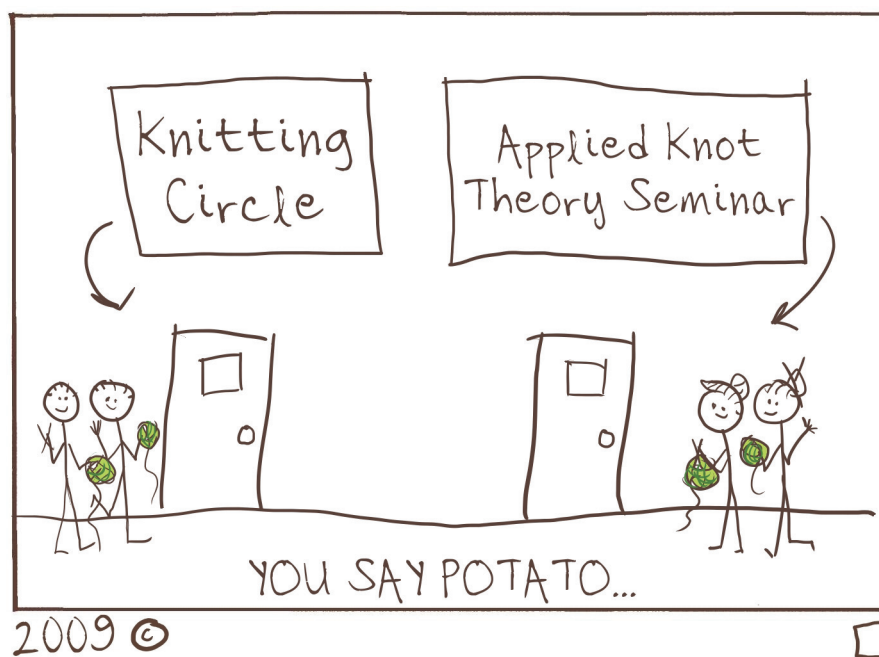
Since the publication of *Making Mathematics with Needlework*, we have discovered more individuals who enrich the community of fiber artists who intersect with mathematicians. These include

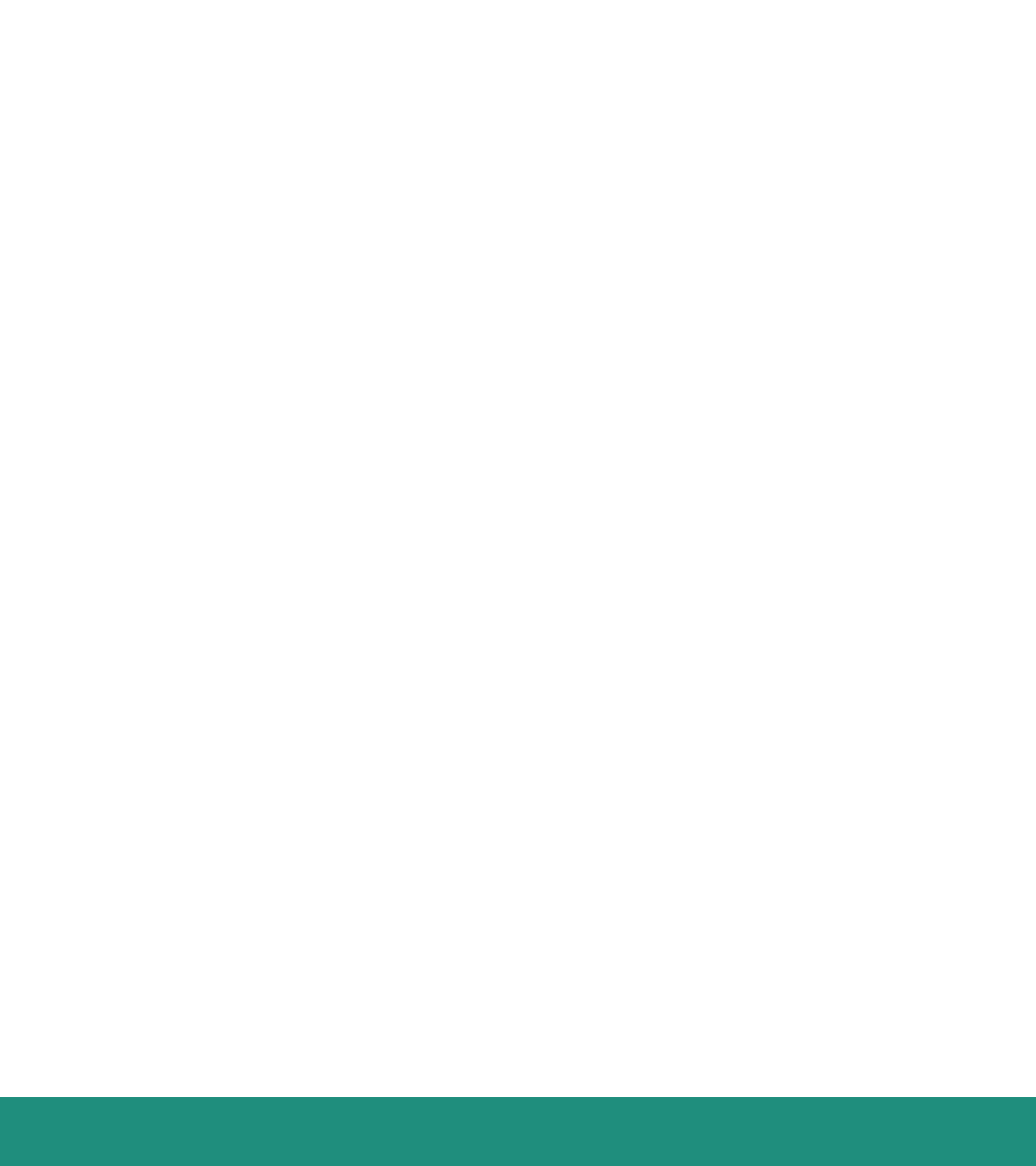
- ★ Miriam Quinn of the *Knit Science* podcast,
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- ★ Brent Annable, Ariel Barton, Hugh Griffiths, Jeny Carden Staiman, and Matthew Wright, for knitting and crocheting cool new mathy things, and

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INTRODUCTION

*welcomeb ack*

SARAH-MARIE BELCASTRO AND CAROLYN YACKEL

# 1 Prologue

When we began the project that became *Making Mathematics with Needlework* [1], we had no idea that material for a second book would be generated so quickly, or even that there would be enough interest to merit a second book. Yet, here we are introducing *Crafting by Concepts*, and as we send this volume to press, we are already aware of several works in progress for contribution to a third book.

The Introduction to [1] functioned as a survey of the field of mathematical work on the fiber arts. We will not cover that ground again here. Instead, we describe this book's structure in Section 2, mention recent research themes in Section 3, give comments to educators in Section 4, indicate how to use this book in Section 5, and conclude with Section 6.

## 2 Chapter Structure

Each of the Chapters and Minis in this book came into being because the authors, who are simultaneously mathematicians and crafters, noticed the potential for the expression of a mathematical idea within certain craft media.

Every Chapter begins with an Overview section in which the interplay between a mathematical notion and the craft expression of this notion is presented for a lay audience. This is followed by a Mathematics section, wherein the ideas are discussed via the formal language of mathematics and the interplay is made explicit. Next, a Teaching Ideas section contains suggestions for fiber-arts-related classroom activities that introduce or reinforce the mathematical concepts discussed in the Mathematics section. In addition, most chapters suggest extensions to the main material in the form of project ideas or open-ended problems. Each Crafting section contains projects that are specifically designed to illustrate

the mathematics in the chapter. Often mathematics is conveyed through the crafting technique, though for the initiated mathematician the result is usually apparent in the finished object.

The two Minis are shorter pieces (hence the name). Each summarizes the theory behind a mathematically inspired project and then moves directly to the Crafting section.

## 3 Research Update

In [1, pp. 4–7], we outlined four categories of mathematics research in the fiber arts. Since that time, the most work has been done in illustrating mathematical concepts using fiber arts. Anecdotal evidence [3] suggests that it is becoming more popular to answer problems arising in fiber arts by using mathematics. There has also been progress in determining which mathematical concepts can (and cannot) be constructed using a given fiber art. However, little work seems to have been done in describing the mathematics that is intrinsically present in a given fiber art.

We encourage readers to make their own contributions to the literature, formally or informally. We attempt to be cognizant of all publications that combine mathematics and fiber arts, and share the current state of our knowledge via the reference lists given on pages 243–245. These have been updated from [1, pp. 7–10].

## 4 A Note for Educators

In the Teaching Ideas sections of this book, the authors suggest instructional implementations of the concepts introduced in the Mathematics sections. The goal is to supply hands-on investigation of each mathematical construction. However, these are intentionally not given in the form of worksheets or lesson plans, though

the investigations can and should be turned into classroom materials by individual instructors. The reason that the authors have avoided specificity is that classrooms are too varied for such proscription to be useful. The grade level (e.g., secondary, graduate), pedagogy used (e.g., active learning, lecture), and even the context (e.g., homeschool, public institution, math club) all contribute to make each classroom highly individual.

Moreover, we see teaching and learning as highly personal endeavors, taking place in the context of relationships formed between teachers and students in individual classrooms. As such, we have provided the teaching ideas sections as resources from which a teacher can make use of what is most beneficial for her students, elaborating upon it and reworking it so as to fit seamlessly into her course. We hope the following text will assist educators in customizing the material supplied in this book and framing it for their classrooms.

Teaching students mathematics through the approach of crafting is both indirect and subtle. The mathematics that a mathematician sees may not be readily apparent to a student not yet familiar with that mathematical content. In general, we see what we already understand, not the other way around. Therefore, in order for students to learn mathematics through engaging in the fiber arts, it is incumbent upon the teacher to help students come to understand the relevant mathematical aspects of the kinesthetic activity in which they are engaged. In other words, students will not learn mathematics simply by knitting an object. Rather, it is through the accompanying discussion that the relationship between the mathematics and the fiber art activity begins to take on significance for the student and thereby has the potential to deepen the student's understanding of the relevant mathematical content.

Mathematicians are trained to notice numeric, geometric, and abstract relationships. Similarly, crafters are attuned to how stitches are made and how they lie together to create a piece. Both mathematicians and

crafters look for patterns, albeit in different forms. Yet students may not have experience attending to details of these kinds. Continuing in the vein of [2], because most students are novices, they are at the stage of only being able to notice or recognize the mathematical connections clearly pointed out to them by someone else, and are not yet at the stage of *marking*, where they can originate remarks about such observations. Recognizing that a shift in the level of noticing amongst the students will need to be a goal together with the mathematical and crafting goals is important as a teacher embarks upon the use of craft as a discovery-based tool in the classroom.

Although the additional layer of teaching involved in initiating students into the realm of coming-to-see mathematical ideas in the visual and kinesthetic world of craft may seem challenging, we argue that it significantly broadens students' utility with mathematics by giving them experience with transfer of knowledge from one domain to another.

To that end, here is a practical note. We recommend that students help each other, while the instructor surreptitiously listens in to make sure that their advice is correct. This builds confidence on the part of the student-as-instructor and reinforces the student's knowledge. In our experience, the student-as-receiver often finds the student-as-instructor easier to understand than the instructor, even if the student-as-instructor is using exactly the same words. Of course, if the advice is incorrect, this too yields valuable insight into the student's misunderstanding. This can allow for a gentle, yet powerful teaching moment.

## 5 How to Use This Book

While anyone can appreciate the visual aspects of this volume, it is more than a coffee-table book. (Still, if you like the look of any photo, the materials used and

people involved are listed in the Credits section.) There are some special audiences for whom we have guidance on the best use of this book.

- ★ Crafters should note that the Crafting sections are written as ordinary patterns, so their formatting and language will be familiar. Every project was tested by at least one non-mathematician (who was not given access to the rest of the chapter), so we are confident that you can complete any project you attempt.
- ★ Students and parents seeking ideas for science-fair or research projects are directed to the Project Ideas subsections of most of the Teaching Ideas sections. There are also some ideas for independent work embedded in other parts of the Teaching Ideas sections.
- ★ Faculty who teach mathematics for liberal arts courses will find that fiber arts connections to mathematics provide powerful and salient examples for students to explore. (However, not all students observe these connections readily; see Section 4.) Open-ended investigations as suggested in the Teaching Ideas and Project Ideas sections can allow even beginning students to formulate research questions—a feat that they find impressive.
- ★ The Mathematics sections have technical levels that range from that of college mathematics majors to professional mathematicians. Readers hoping to enhance their mathematical expertise will find that persistence is their best approach, as is true for all mathematical learning.
- ★ We believe the careful treatment of topics in this volume (and [1]) will be of particular use to beginning graduate students in mathematics. However, professional mathematicians may also find this book to be a useful source for quickly acquiring background and details on subfields or concepts that are new to them.

In the previous volume we highlight nonorientability through homology [1, Chapter 1], frieze pattern nomenclature [1, Chapter 5], and surface curvature [1, Chapter 10]. Here we note wallpaper pattern identification and notation (Chapter 5),  $m$ -color planar patterns (Chapter 6), group actions (Chapter 6), and discrete spherical symmetry types (Chapter 8).

- ★ Of course, the readership closest to our hearts is mathematicians who craft. We have no advice for you in how to approach this material; you already know.

We also have a few observations to share about particular chapters.

Chapter 1 combines knitting with calculus to create circles and surfaces of revolution. Some instructional ideas have been tested in Calculus II classes: sarahmarie’s students read an abbreviated version of the Mathematics section and she gave them a worksheet of the Designing a Cone questions. They had no trouble working through the problems until they encountered a question that specified a knitting gauge and head circumference and asked them to determine parameters  $a, b$  for the cone so that a wearable hat would result.

In Chapter 4, the educational activities are truly independent of educational level; the same activity is doable by sixth graders and graduate students alike (with different mental rewards). In the spirit of using many different crafts to create the same object, we note that the Chaos Game could be played using fabric paint dots instead of beads. Carolyn challenges readers to create a mola version of Sierpiński’s triangle, and also to tat Pascal’s triangle in multiple colors (avoiding cellular automata in the construction method, as Chapter 4 reveals is not useful).

Chapter 6 has two threads, one elementary and one advanced, throughout the Mathematics section. While the language is frequently technical, the reader can skip those parts, forge ahead, and see the concepts via example.

The Crafting section of Chapter 9 is characterized by unusual quilting techniques. Quilters are forewarned that several of the patterns are not for the algebraically faint of heart.

## 6 Conclusion

Gentle reader, we and the authors have endeavored to maintain a level of accessibility that will allow any reader to enjoy portions of the book. We do encourage you to stretch yourself by trying a new craft and learning some mathematics that is new to you (or new to the world). As you read and experiment more, you will appreciate the connections between the doing and learning of mathematics as and through craft.

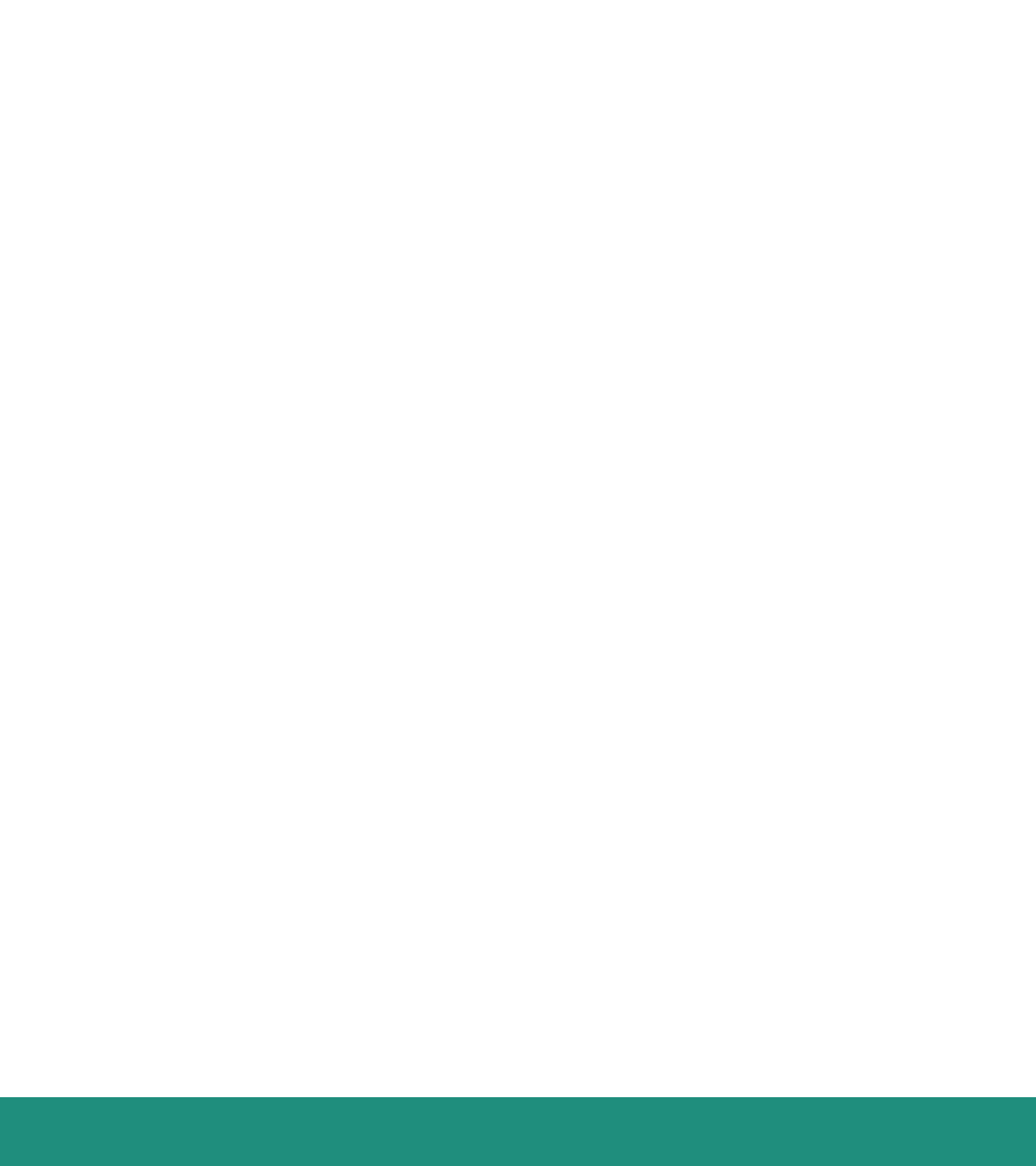
*Crafting by Concepts* has been edited, indeed curated, to address many different audiences. We hope

you find some part of this book that seems like it was written just for you.

*sarah-marie belcastro and Carolyn Yackel, June 2010*

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## CHAPTER 1

# *knit knit revolution*

AMY F. SZCZEPAŃSKI

# 1 Overview

For many knitters, the idea of knitting in the round brings to mind projects like socks, sweaters, hats, mittens, and gloves—anything that can be realized as modified cylindrical tubes. In addition to tubular projects, knitting in the round can also be used to knit shawls and other flat shapes. This chapter explores some of the mathematical techniques that can be used to design knitting patterns for flat circles and for large classes of shapes with circular cross sections. The mathematical and design techniques are not specific to knitting—all of these ideas work for crocheting shapes with circular cross sections as well. (A crocheter could simply replace every occurrence of the word “knit” with the word “crochet.”) By relying on the simple relationship between the radius of a circle and its circumference,  $C = 2\pi r$ , patterns can be designed for many such shapes by following a fairly straightforward algorithm. Patterns are included for three hats whose designs showcase some of the mathematical techniques.

## 1.1 Circles that Lie Flat

One of the more famous applications of this mathematical relationship is Elizabeth Zimmermann’s Pi Shawl [6]. In this pattern, Zimmerman periodically directs the knitter to double the number of stitches. The distance between increase rounds also continues to double. In the pattern in [6], the knitter is instructed to cast on nine stitches, knit one round even, double the number of stitches in rounds 2, 6, 13, 26, 51, 100, .... The number of rounds worked even between increase rounds will continue to double. Following the pattern, there will be 576 stitches in round 100. This process can be expressed in the language of algebra, as described in Section 2.1.

Zimmerman herself recognizes that there is a mathematical aspect of this pattern. In [6] she writes:

Have you begun to see the well-known geometric theory behind what you have been doing? ... It’s Pi; the geometry of the circle hinging on the mysterious relationship of the circumference of a circle to its radius. ... [I]n knitters’ terms, the distance between the increase-rounds, in which you double the number of stitches, goes 3, 6, 12, 24, 48, 96 rounds, and so on to 192, 394, 788, 1576 rounds for all I know. Theory is theory, and I have no intention of putting it into practice, as I do not plan to make a lace carpet for a football field.

As Zimmerman notes, this pattern has a reasonable mathematical grounding. Her algebra is a little bit off; she does not count the increase round anywhere in her calculations. It is ironic that she observes that  $\pi$  is at work here, yet her pattern makes no use of the actual value of  $\pi$ .

An Internet search will reveal many knitters’ takes on this pattern. It is also cited in [5], where it is described as the *rounds* method of knitting a circle. The plan for knitting a Pi Shawl is fairly simple to describe and does not require any fancy calculations on the part of the knitter. Additionally, if one is knitting in wool with a fairly open stitch pattern—as is common with lace—the finished shawl can be blocked to lie relatively flat.

If, however, one is working a fairly tight stitch pattern, especially over a great number of rounds, the Pi Shawl will start to show its limitations. As the knitter continues to work, there will be longer and longer stretches that are worked without increasing. These sections will yearn to be cylindrical, their stitches smushed close near the center and pulled taut near the edge as the knitter tries to coax the circle to lie flat. Doubling the number of stitches results in a significant increase in the amount of fabric, and it will tend to ripple and curl in the early rounds after the increase.

The reason that Zimmerman’s pattern works as well as it does is because it recognizes a fundamental fact about the relationship between the radius of a circle and its circumference: if the radius is multiplied by a specific constant (in this case, 2), then the circumference

must be multiplied by the same constant. (While Zimmerman’s pattern doesn’t follow this rule exactly, it’s pretty close.) Mathematically, this is the idea of “doing the same thing to both sides of the equation.” Her pattern neglects an important issue, though: the radius increases every round, so the circumference should also increase with every round.

In [5] this is addressed, to some extent, by offering the alternative *rays* method of knitting a circle. The pattern for the rays method instructs the knitter (working in garter stitch) to increase by 16 stitches every four rows. As suggested in the name, in the rays method, the increases are aligned in rays that extend from the center of the circle. If all of the increases are in the same place every round, this shape is more polygonal than circular. Depending on the number of rays, the location of the increases, and the aggressiveness of the blocking, the end result can be fairly round. One method of determining where to place the increases is to divide the number of stitches in the current round by the number of increases and then round down. This will tell you the number of stitches,  $s$ , to knit between evenly spaced increases. Choose a random number between 0 and  $s$ , and knit that many stitches before you do the first increase.

Both the rounds method pioneered by [6] and the rays method described by [5] can be improved by working more closely with the equation  $C = 2\pi r$ . When knitting in the round to make a circle that lies flat, the system of increase depends on the gauge. Measure the number of rows in 4" (10 cm) and the number of stitches in 4" (10 cm). Calculate  $2\pi \cdot \frac{\text{number of stitches}}{\text{number of rows}}$ . This is how many stitches we should increase every round. (This formula is explained in Section 2.1.) This won’t be a whole number, so we’ll need to round it to a whole number of stitches. If we always round up, we’ll eventually have too many stitches; if we always round down, we’ll eventually have too few stitches. This won’t matter when making a modest-size project out of a reasonably stretchy yarn. If, however, the goal is to make a gigantic circle out of an

unyielding medium: be warned. We call this method of adding  $2\pi \cdot \frac{\text{number of stitches}}{\text{number of rows}}$  stitches every round the *modified rays method*.

Both for the flat circle and for all of the other shapes described in this chapter there is a question of where to place the increases. Everywhere an increase is placed, the knitting will want to form a corner. For example, if there are four increase stitches each round, and they are evenly spaced about the round (thus lining the increases in each round up with the increases from the previous round), the shape will try very hard to have four corners. To make shapes that are as smooth as possible—especially when working a solid-colored pattern—increases should be scattered throughout the round, so that they do not line up with the increases from the round just below. When working a more complicated stitch pattern or color pattern, there may be fewer options for where to place the increases.

## 1.2 Surfaces of Revolution

Nothing stops us from taking this simple idea and extending it to a broader class of surfaces that are described by circles. In particular, surfaces of revolution have circular cross-sections. The obvious shapes to start with are spheres and tori (doughnut shapes), but nothing stops us from designing the patterns for any shape in this broader category—anything that is made up of circular slices. These sorts of shapes may interest the knitter making Christmas ornaments (especially if one wants to make knitted covers for the styrofoam balls from the craft store) or knitting models of food (such as doughnuts or ice cream cones). Additionally, these techniques can be applied to hats.

The only limit on the style of a knitted hat is the imagination of the knitter. Knitting a cylindrical tube (standard knitting in the round) and topping it with a flat circle makes a pillbox hat. Watch caps and other standard hats are cylinders topped with hemispheres. (Take

a good look at someone's head; it's roughly shaped more like a silo than like a ball.) The stocking cap (or dunce cap) is a cone. In Section 4 there are three patterns for hats. One is a stocking cap knit in the standard way. The other two are hats with hemispheric caps knit back and forth on straight needles.

Mathematically, the problem of knitting any shape with circular cross-sections is exactly the same as knitting the flat circle: for each circular round of the shape, figure out the radius of the circle, calculate its circumference, and knit enough stitches in the round so that the circle is the right size. This is explained in mathematical detail in Section 2.2.

When knitting these shapes, many of the same questions come up as did with the flat circle. Again there is the issue of where to place increases and decreases. As with the flat circle, if increases (or decreases) line up, a ridge will be formed. Unless one is knitting a very large shape (or using very small stitches), it's best to try to scatter the increases and decreases about the round.

## 2 Mathematics

The construction of both flat circles and surfaces of revolution relies on a single geometric formula, that of the circumference of a circle. This is the familiar  $C = 2\pi r$ . Throughout this section  $h$  will stand for the height of a knit stitch, and  $w$  will stand for the width of a knit stitch.

### 2.1 The Flat Circle

The first application of  $C = 2\pi r$  to knitting is the flat circle. The idea behind knitting a flat circle is to cast on a few stitches at the center of the circle and then to spiral out, increasing the number of stitches in each round. When knitting something circular, adding too few stitches per round will result in positive curvature (like a hat), and adding too many stitches per round will

result in negative curvature (like kale). The goal is to determine how many stitches should be added each round to achieve the proper growth rate. Therefore, when knitting in the round to make a circle that lies flat, we must calculate the number of stitches to increase each round so that the following round has the proper circumference.

The radius of round  $n$  will be  $nh$  ( $n$  times the height of a knit stitch), so the circumference of round  $n$  will be  $2\pi nh$ . To determine how many stitches to add when going from round  $n$  to round  $n + 1$ , we compare the circumferences. The desired circumference of round  $n + 1$  is given by  $2\pi(n + 1)h = 2\pi nh + 2\pi h$ . Therefore, the difference between the circumferences of round  $n + 1$  and round  $n$  is  $2\pi h$ . (This is essentially finite difference calculus: note that the derivative  $\frac{dC}{dn} = 2\pi h$ .) Dividing this by the width of a knit stitch, this means that for each round, one should increase by  $\frac{2\pi h}{w}$  stitches. This is the modified rays method described on page 8.

The number  $\frac{2\pi h}{w}$  will rarely be an integer, so the question of rounding arises. If  $\frac{2\pi h}{w}$  is very close to an integer, then round and everything will work out. However, if its decimal part is near  $\frac{1}{2}$ , it's best to sometimes round up and sometimes round down—especially for a large project.

In contrast, the rays method is a simplification of the modified rays method. The rays method will always add 16 stitches every four rounds—an average of four stitches per round. If there are  $c$  stitches cast on, then in round  $r$  there will be  $c + 4(r - 1)$  stitches. This method is independent of gauge. If the gauge is such that  $\frac{2\pi h}{w}$  is really close to 4, then the rays method will produce results nearly identical to those of the modified rays method. Each method has the knitter add stitches at a constant rate per round. In the improved (modified rays) method, if there are  $c$  stitches cast on, then in round  $r$  there will be  $c + (r - 1)\frac{2\pi h}{w}$  stitches.

With both of these methods, obtaining a result that is more round than polygonal relies on the placement

of the increases. Whenever possible, try to place the increases randomly throughout the round.

Finally, let's look at Elizabeth Zimmermann's formula. As given in [6], her Pi Shawl pattern does not follow the algebraic equation to which she refers,  $C = 2\pi r$ . If we let  $r_i$  be the number of the first increase row, the correct placement of the increase rounds that double the stitch count *should* be at rows  $r_i, 2r_i, 2^2r_i, \dots, 2^m r_i$ . The reason for this is the old maxim from algebra class: whatever is done to one side of the equation must be done to the other side of the equation. To double the circumference, one must also double the radius. The circumference is defined by the number of stitches, and the radius is defined by the number of rounds. Therefore, each increase round should be numbered as twice the previous increase round.

This is not what the actual pattern directs, although it is close. Zimmerman's pattern has increases in rows 2, 6, 13, 26, 51, 100, ... The difference between Zimmerman's pattern and the correct algebra arises because she neglects the fact that the increase round itself adds to the radius of the circle. It appears that additionally, Zimmerman doubled the number of rows between increase rounds instead of doubling the round numbers in which increases occur:

$$\begin{aligned} 6 &= 2 + 3 + 1 = 2 + 3 + 1, \\ 13 &= 6 + 6 + 1 = 6 + 2 \cdot 3 + 1, \\ 26 &= 13 + 12 + 1 = 13 + 2^2 \cdot 3 + 1, \\ 51 &= 26 + 24 + 1 = 26 + 2^3 \cdot 3 + 1, \\ 100 &= 51 + 48 + 1 = 51 + 2^4 \cdot 3 + 1. \end{aligned}$$

You will notice that twice the number of each increase round is usually close to, but not exactly equal to, the number of the next increase round:

$$\begin{aligned} 6 &= 2 \cdot 2 + 2, \\ 13 &= 2 \cdot 6 + 1, \\ 26 &= 2 \cdot 13 + 0, \end{aligned}$$

$$51 = 2 \cdot 26 - 1,$$

$$100 = 2 \cdot 51 - 2.$$

The slight differences accumulate. With each increase round, the relationship between Zimmerman's radius and her circumference gets a little more out of whack. Of course, after working 100 rounds and having 576 stitches, the error is small enough so as not to be noticeable.

The location of the increase rounds in Zimmerman's original pattern is governed by the difference equation  $A_{n+1} = 2 \cdot A_n + 3 - n$ , where  $A_i$  is the round number of an increase round, and  $A_1$  is defined by the pattern to be round 2. We solve the difference equation by attempting a solution of the form  $A_n = a \cdot 2^n + bn + c$ . We write  $A_{n+1}$  two ways, as  $a \cdot 2^{n+1} + b(n+1) + c = 2(a \cdot 2^n + bn + c) + 3 - n$ , and solve for  $b$  and  $c$ . We then use  $A_1 = 2$  to solve for  $a$ . The closed form of this difference equation is  $A_n = 3 \cdot 2^{n-1} + n - 2$ .

We will now fix Zimmerman's formula to give a mathematically accurate description of the circle and call this the *Modified Pi Shawl*. To do this, we will refocus on rows when increases should happen. The stitch count should double only when the number of rounds has doubled. Therefore, if the first increase round is round 2, then there should also be increases in rounds 4, 8, 16, 32, 64, etc. Similarly, if the first increase is in round 3, then there should also be increases in rounds 6, 12, 24, 48, 96, etc. Instead of doubling the space between the increase rounds, one should simply double the round number. As before, in each increase round the number of stitches should double.

In this case, the location of the increase rounds and the number of stitches in a round are fairly straightforward to calculate. If we begin the doubling process in round  $r_i$ , we will double the stitch count in rounds  $r_i, 2r_i, 2^2r_i, 2^3r_i, \dots$ . If we cast on  $c$  stitches, in each of the doubling rounds we will end up with  $2c, 2^2c, 2^3c, 2^4c, \dots$  stitches. From here it is fairly straightforward to

calculate how many stitches are in any given round  $r$ . The strategy is to determine how many times there has been an increase round and then multiply  $c$  by the suitable power of two.

As the round number of the increase rounds continues to double in this modified version of the Pi Shawl, finding their location will be done by taking logarithms base 2 to determine how many doublings there have been. To calculate the number of stitches in round  $r$ , we start by determining how many doublings there have been. The quantity  $\frac{r}{r_i}$  will be exactly equal to a power of two in each of the increase rounds, by construction. Calculating  $\log_2\left(\frac{r}{r_i}\right)$  will be a whole number in the doubling rounds and a decimal in other rounds; it counts the number of times the stitch count has been doubled before starting the row. Thus, we need  $\left\lfloor \log_2\left(\frac{r}{r_i}\right) \right\rfloor + 1$  to tell us the number of times we have doubled by the end of round  $r$ . (Recall that  $\lfloor x \rfloor$  is the greatest integer that does not exceed  $x$ .) So if we cast on  $c$  stitches and begin the doubling process in row  $r_i$ , then the number of stitches in row  $r$  (where  $r \geq r_i$ ) will be  $c \cdot 2^{\left\lfloor \log_2\left(\frac{r}{r_i}\right) \right\rfloor + 1}$ .

We have described four methods for knitting a circle that lies flat. The modified rays method is the only one that relies on the number  $\pi$  and that takes into account the gauge of the knitting. It directs the knitter to add roughly  $\frac{2\pi h}{w}$  per round, distributed as randomly as possible around the round. The other three methods do not involve the number  $\pi$ , nor do they take into account the gauge. They are the rays method from [5] that adds 16 stitches every four rounds (or four stitches each round), the Pi Shawl as described in [6], and the Modified Pi Shawl where the round number of the increases doubles rather than the space between the increases. We note, however, that for a lot of common yarns the number of stitches to add each round using the modified rays method with stockinette stitch comes out pretty close to the four prescribed by the rays method.

We prefer the Modified Pi Shawl to the Pi Shawl because it is more accurate from an algebraic point of view, and the arithmetic is nicer. Hence, we compare the remaining two of the three gauge-independent methods here (Modified Pi vs. rays). The number of stitches in the project will depend on which method one chooses as well as the number of stitches cast on. If one starts by casting on nine stitches, as is done in the original Pi Shawl, the number of stitches in each round of the Modified Pi Shawl far outnumbers the number of stitches in the rays method, as is shown in Table 1. We assume that we are adding four stitches each round in the rays method. From the beginning, the Modified Pi Shawl has greater than or equal to the number of stitches in each round as the rays method, and the difference continues to grow at a rate of  $6(2^n) - 5$  (as derived from the On-Line Encyclopedia of Integer Sequences [4]).

round	Modified Pi Shawl	rays
1	9	9
3	18	17
6	36	29
12	72	53
24	144	101
48	288	197

**Table 1.** Number of stitches in selected rows when casting on nine stitches.

However, if one starts out by casting on a smaller number of stitches, then the rays method may outpace the Modified Pi Shawl. There is an interesting pattern here. As seen in Table 2, the differences between the numbers of stitches in the first five increase rounds in the rays method and in the Modified Pi Shawl are 3, 5, 9, 17, and 33, or more generally  $2^n + 1$  (sequence A000051 in the OEIS [4]). Differences between adjacent terms in this sequence are consecutive powers of two.

This situation brings to mind the three bears from the fairy tale. With one initial number of stitches the Modified Pi Shawl dominates the rays. With another

round	Modified Pi Shawl	rays
1	5	5
3	10	13
6	20	25
12	40	49
24	80	97
48	160	193

**Table 2.** Number of stitches in selected rows when casting on five stitches.

initial number of stitches, the rays dominate the Modified Pi Shawl. Is there a number of stitches that is “just right”? The answer turns out to be yes, by casting on six stitches. Table 3 suggests that for the increase rounds in the Modified Pi Shawl (rounds 3, 6, 12, 24, ...), the number of stitches in the Modified Pi Shawl is two less than the number of stitches in the rays method.

round	Modified Pi Shawl	rays
1	6	6
3	12	14
6	24	26
12	48	50
24	96	98
48	192	194

**Table 3.** Number of stitches in selected rows when casting on six stitches.

Could it be a coincidence that the numbers work out this close for small-valued rounds? We can check this by looking at the expressions for the number of stitches in each round. Casting on six stitches and adding four stitches each round, we will have  $6 + 4(r - 1)$  stitches in round  $r$ . In the Modified Pi Shawl, we will have  $6 \cdot 2^{\lfloor \log_2(\frac{r}{3}) \rfloor + 1}$  stitches. If we restrict ourselves only to the increase rounds, we can omit the  $\lfloor$  and  $\rfloor$ . We wish to see whether these two expressions always differ by two. That is, we need to verify the equation  $6 \cdot 2^{\log_2(\frac{r}{3}) + 1} = 6 + 4(r - 1) - 2$ . Sure enough,  $4r = 6 \cdot \frac{r}{3} \cdot 2 = 6 \cdot 2^{\log_2(\frac{r}{3}) + 1}$  and  $4r = 6 + 4(r - 1) - 2$ . Therefore, in the case where we cast on six stitches, there is a balance between the

two methods. Let  $c$  be the number of stitches cast on in either method, let  $r_i$  be the location of the first doubling round in the Modified Pi Shawl, and let  $m$  be the number of stitches added each round in a generalized rays method. Then there will always be this balance between the number of stitches when  $c = \frac{r_i}{2}m$ . If  $r$  is an increase round in the Modified Pi Shawl and  $c = \frac{r}{2}m$ , then a round in the Modified Pi Shawl will have exactly  $m - c$  more stitches than the same-numbered round in the rays method.

## 2.2 Surfaces of Revolution

The fundamental equation  $C = 2\pi r$  is also used when designing patterns for surfaces of revolution. These shapes frequently come up in calculus classes; often the student is given an equation describing a graph in the plane and is asked to rotate this about an axis of revolution, generating a solid whose volume or surface area must be calculated.

The general method for creating a knitting pattern for such shapes is to imagine the original graph in the plane as made up of line segments each the length of one knit stitch. (This is somewhat like the approximation made when deriving the formula for arc length.) See Figure 1. Each of these knit stitches laid along the graph will be “rotated” about the axis of revolution by knitting a row with  $\frac{2\pi x_i}{w}$  stitches, where  $x_i$  is the distance from the top of the stitch to the axis of rotation.

The design of revolutionary patterns can be reduced to the question of how best to fit these polygonal segments to the original curve. For one, there is no guarantee that the height of a knit stitch will fit a whole number of times along the curve being described. Obviously, approximations to the curve by smaller-height knit stitches will be more accurate.

Additionally, it is not always easy to figure out where the endpoint of each segment will be. Given a point  $(x_i, y_i)$  on the curve, the point  $(x_{i+1}, y_{i+1})$  will also be on