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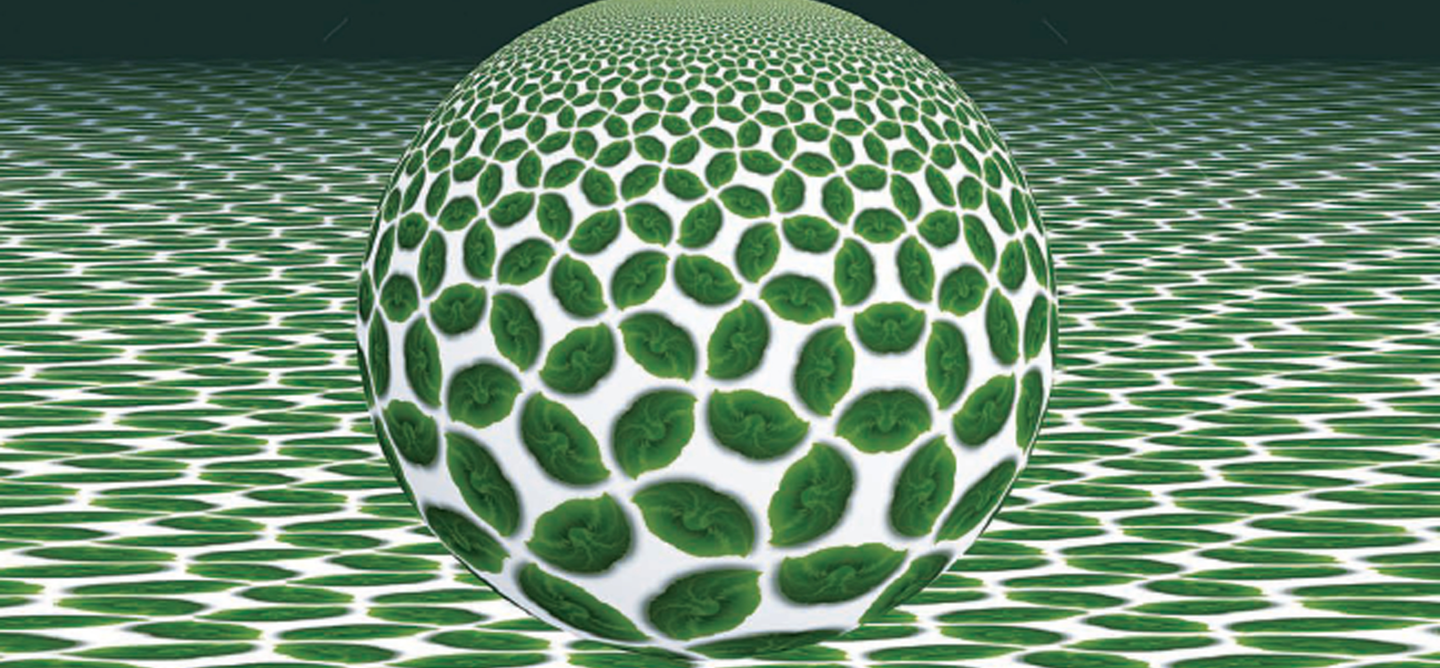
The Symmetries of Things



John H. Conway • Heidi Burgiel • Chaim Goodman-Strauss

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To

Gareth, Diana,
Eli, Zoe, Kendall, and
the memory of *Gay Lorraine Burgiel*

Contents

Preface

This book has been germinating for a long time. John Conway has always been interested in geometrical groups, for many of which he devised particular notations when he was teaching at Cambridge University. However, after he moved to Princeton University in 1985 and Bill Thurston told him of the orbifold idea, he dropped those notations forever and devised the signature notation used in this book. He then became Thurston's most avid prophet, lecturing on the theory to scores of audiences—ranging from the Princeton Rug Society to the International Congress of Mathematicians!

One of those audiences contained the young graduate student Heidi Burgiel, who was taking notes on the talk for distribution during the conference. Heidi went on to complete a graduate program in combinatorics and discrete geometry. Years later, when John spent some time at Northwestern University, Heidi offered to "write something else up" with John, but in the end they decided to write the same theory in more detail as a proposed book. That book has been growing ever since.

All they had intended to write was the content of what is now Part I—an elementary introduction to the orbifold signature notation. But then came the idea of writing a second part that would extend the signature to color symmetry. At this point it became clear that Chaim Goodman-Strauss would make an excellent addition to the team of authors. Chaim had been preaching the gospel of the orbifold signature on his own and was known for his gorgeous illustrations.

More topics burst into bloom at various seasons. When Conway, Delgado, Huson, and Thurston used the signature to re-enumerate the three-dimensional space groups, it seemed a good idea to incorporate this also in the second part. That "second part" is now Parts II and III.

Much of the book was written in hectic three-day sessions on the few occasions when all three of us could get together—this paragraph is being finalized on the way to the Tampa airport, days before the book is sent to press. We usually managed to write several chapters in each session, often including one that only arose just then. For example, at one session, Chaim said "we could perhaps do Heesch types," and an hour later Chapter 15 was complete. Just after completing the next section of this introduction (which describes what's new to this book), the three of us celebrated at a restaurant, discussed "Archimedean tilings," and Chaim and John discovered the "Archifold notation" that characterizes such things as they walked home after the meal. The next day this too was in the book. Of course, it often took Chaim years to catch up with the illustrations.

What's New in This Book?

Many of the results and proofs in this book are new, or nearly new, in the sense that their only previous appearances have been in the scholarly papers (often involving one of us) that are cited in the appropriate chapters. These new things are

- the orbifold signature,
- the statement of our Magic Theorem,
- its use to enumerate symmetry types
  (however, we should point out that a few decades ago, MacBeath introduced his own signature that is in fact equivalent to ours—but more complicated—and used it in the same way),
- Conway's "zip proof" of the classification of surfaces,
- uniform presentations for all the groups,
- their proof,
- our analysis and notation for color symmetry,
- the p-color types for all primes p,
- the simplified enumeration of Heesch types,

- the Besche-Eick-O'Brien table of group numbers,
- the extension of all of the above geometrical theory to hyperbolic groups,
- a new proof of the abstract distinctness of infinite groups with compact orbifolds,
- the explanation of isospectral "drums" via hyperbolic groups,
- the classification of Archimedean tilings in the hyperbolic plane,
- generalized Schläfli symbols,
- Architectonic 3-tessellations,
- the new space group notations and a panoply of objects with prime space symmetries,
- names and enumeration of platycosms,
- a list of Archimedean 4-polytopes.

Even when the results are old, our exposition is new.

We are also proud of our exposition and illustrations. Chaim Goodman-Strauss assures our readers that his software and illustrations are available for sale and licensing.

We are relieved that now the book is in print, bringing the orbifold signature to the world. This would not have happened without help from many people including Robert Strauss, Troy Gilbert, Marc Culler, Tom Moore, Charlotte Henderson, Alice and Klaus Peters, Bill Thurston, Silvio Levy, Peter Doyle, Natasha Jonaska, Daniel Huson, Olaf Delgado Friedrichs, Doris Schattschneider, Marjorie Senechal, Javier Bracho, our students, and our colleagues; and the patience and sympathy of our partners Diana, Kendall, and Rachel. We thank the institutions that supported our work, including Princeton University, the University of Arkansas, the Universidad Nacional Autómata de México, Northwestern University, the University of Illinois at Chicago, Bridgewater State College, and the National Science Foundation.

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All other illustrations by Chaim Goodman-Strauss.

Part I

Symmetries of Finite Objects and Plane Repeating Patterns

Introduction to Part I

Symmetries and symmetric patterns surround us throughout our lives. The aim of the first part of this book is to describe and enumerate all the symmetries found in repeating patterns on surfaces. To prove that our enumeration is accurate, we then explain the beautiful ideas from topology and algebra that form the basis for our conclusions.

We start with a problem—enumerating symmetric patterns. We then introduce tools for solving this problem and complete the enumeration. But then we are presented with a second problem—demonstrating that these tools work the way we claim, that there is a solid mathematical foundation beneath our results. Again, we solve this problem with some tools, then present the mathematics supporting the use of those tools. In this way, each chapter reduces the problems left by the preceding chapter to another problem whose solution is postponed to the following chapter.

This is a departure from the traditional practice of building a theory starting with basic principles and working toward the ultimate goal of proving some final theorem. We believe that our backward approach will be successful because it allows us to present one concept at a time, at the cost of always postponing the proof of just one thing to the next chapter. We hope also that the argument will be clearer when presented in a single logical thread, of the form $A \Leftarrow B \Leftarrow C \Leftarrow ... \Leftarrow Z$.

The first chapter is a gentle introduction to symmetry. Chapter 2 introduces the four fundamental features that we use to classify symmetry. In Chapter 3 we state our Magic Theorem and apply it to find the 17 possible types of repeating planar patterns, while Chapters 4 and 5 perform a similar service for spherical and frieze patterns, respectively. The Magic Theorem is deduced in Chapter 6 from Euler's Theorem, which is itself proved in Chapter 7. Finally, Chapter 8 gives our new proof of the classification of surfaces, and Chapter 9 illustrates the orbifolds that underlie our theory.

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Symmetries

Every day we are surrounded by symmetric objects and patterns. From furniture to flooring, symmetry is the rule. In art, symmetry is pleasing to the eye, and the intricacies of extremely symmetric patterns can entrance an audience. In architecture, symmetric designs are attractive for yet another reason—repetition of a design element means re-use, which ultimately requires less planning and testing. In manufacturing, it is simpler, cheaper and more efficient to repeat a pattern at regular intervals. Even Nature has reasons to use symmetry in her work.

Etymology

The word *symmetry* is a combination of the words *sym* (together) and *metron* (measuring). The meaning of *bilateral* is, literally, two-sided.

Recently, John H. Conway and William Thurston adapted Murray MacBeath's mathematical language for discussing symmetry. Now, the symmetries of a pattern can be defined by a single symbol that we call its signature: for example, 3*3, for the pattern on the left. With some practice, almost anyone with some knowledge of high-school geometry can read this signature and identify the symmetries it describes.

Kaleidoscopes

The simplest signature is just * (star). A * denotes a *mirror* or *kaleidoscopic* symmetry, and a * alone means that there are no other symmetries to the figure. The pair of gryphons (right) has a single line of mirror symmetry running between them.

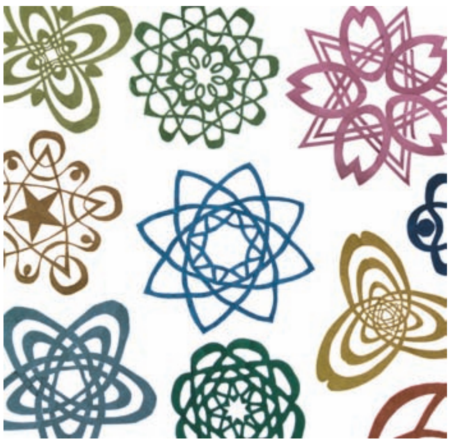
(opposite page) This pattern—which to a mathematician extends forever in every direction!—has reflections and gyrations.

"*Vesica piscis*" (fish bladder) is the traditional architectural name for patterns of this shape.

This *vesica piscis* (left) has signature *2•, pronounced "star two point symmetry" or, more formally, "period two kaleidoscopic symmetry fixing a point." We use stars for kaleidoscopes to suggest the star formed by the mirrors through a kaleidoscopic point. The *period* of a kaleidoscopic point is the number of mirror lines through it. In this case two lines of mirror symmetry—one vertical, the other horizontal—meet at the center of the flower. Finally, the point (•) indicates that all the symmetries fix a point.

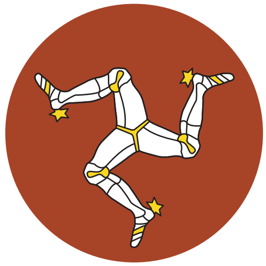
Many letters of the Roman alphabet have mirror symmetry (or approximately so)! Symmetry will vary from typeface to typeface.

You can probably guess that in a figure with signature *3•, three lines of mirror symmetry meet at its center, and similarly for signatures *4•, *5•, *6•, and so on. Mark the mirror lines and find the signatures of the tracery shown above.



For your first quiz, identify the mirror lines and signatures of these lovely cut-paper snowflakes.

Gyrations



This *triskelion* (right) appears on the coat of arms of the Isle of Man. This figure looks the same in three orientations; the rotation through 120 degrees is a congruence that takes the figure to itself. A triskelion has period 3 gyrational point symmetry and signature **3•**.

The snakes in the middle of the above figure entwine with a period 2 gyrational point symmetry and so have signature **2•**. The gothic tracery patterns to the left and right have signatures **4•** and **6•**, respectively.

NSZ ZSN

Three roman letters have gyrational symmetry.

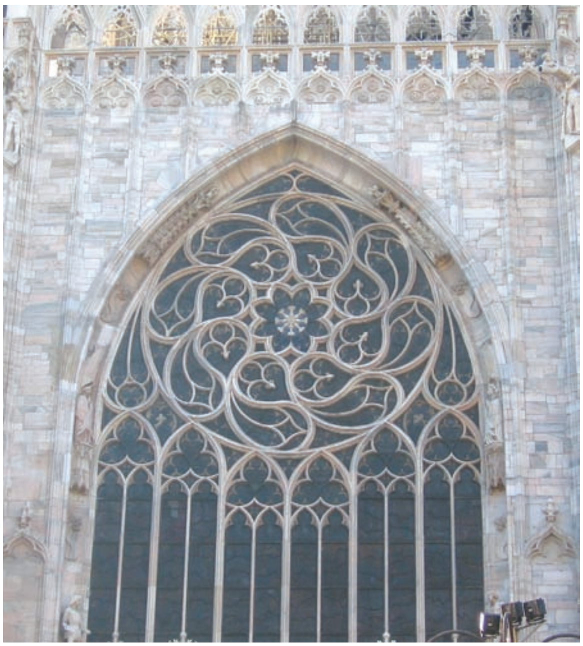
These hubcaps have gyrational symmetries, whose signatures you may identify for your second quiz.

Rosette Patterns

Obviously, we could keep going like this, generating pictures with period 37 kaleidoscopic point symmetry or period 42 gyrational point symmetry. But what else can we do?

For the finite *rosette patterns* like those on the last two pages, there are no other signatures. In a finite pattern, all symmetries of the pattern must fix (i.e., cannot move) the center of the pattern. Reflections across the center of the rosette and rotations about its center are the only symmetries that do this, so they're the only symmetries such a pattern can have.

By experimenting with different combinations of rotational and reflective symmetries, you can easily convince yourself that the types *•, *2•, *3•, *4•, . . . , *N• and **2•**, **3•**, **4•**, **5•**, . . . , **N•** are the only signatures possible for rosettes, to which we add **1•** = **•** for the case of no symmetry.



Milan Cathedral window.

Frieze Patterns

After isolated pictures on a page, the easiest patterns to understand are those made by repeating pictures in a row. We see patterns like this in friezes, ribbons, animal tracks and fences.

Frieze patterns photographed in downtown Chicago.

The difference between frieze patterns and isolated figures is that, in addition to any reflective and rotational symmetries of the figures that make up the pattern, a frieze pattern has a translational symmetry that takes the figure to a neighboring figure. The first half of the book concerns itself with patterns of this sort, called *repeating patterns*.

The "dart and egg" frieze pattern is truly ancient; like all frieze patterns with this type of symmetry, it is created by reflecting a motif across a line of kaleidoscopic symmetry, then repeating the pair of images forward and backward along the kaleidoscope.

Make Your Own Frieze Patterns

You can easily generate frieze patterns using symmetric letters! Here are some examples; can you make some others?

ppppppp

pdpdpdp

pqpqpqp

pbpbpbp

bbbbbbbbb
ppppppppp

bdbdbdbdb
pqpqpqpqp

bdbdbdbdb
qpqpqpqpq

Repeating Patterns on the Plane and Sphere

Frieze patterns have "forward and back" translational symmetry. Plane patterns add translational symmetry in another direction. These patterns can extend to cover an entire page, or beyond. We see them every day on the floors and walls around us.

In order to study the symmetries of common objects like hairbrushes and furniture, we will also need to learn about the symmetries of patterns on spheres. Basketballs have two planes of reflective symmetry, as do tennis balls. But these balls also have a 2-fold rotational symmetry. A cube has nine planes of mirror symmetry, while some soccer balls have fifteen! In order to classify such patterns we will study repeating patterns on spheres.

Where Are We?

At the beginning of this chapter we found all the possible types of symmetry for rosettes—namely • = 1•, *• = *1•, 2•, *2•, 3•, *3•, 4•, *4•, ... We've also introduced three categories of repeating pattern—repeating patterns in the Euclidean plane, frieze patterns, and patterns on the sphere. The focus of this book is to classify the different types of symmetry that objects in these categories can have. We've told you roughly what it means to say that two things have the same type of symmetry, but we'll have to postpone a precise definition of our problem until we've nearly solved it.

In fact, our book will have about as many postponements as chapters! For example, in the next chapter we'll introduce four features that in fact determine the notion of symmetry type, but will postpone the proof that they do so. These features determine the signatures that we use in Chapters 2–5 and 17 to list all possible types for each of our three categories. To do so, we employ a "Magic Theorem" whose proof is postponed to Chapter 6. In that chapter we also see that the signature really describes a topological surface called an orbifold that encapsulates all the symmetries of a pattern. The Magic Theorem is then revealed to describe a simple invariant, the Euler characteristic, of this orbifold; a detailed investigation of the Euler characteristic is in turn postponed to Chapter 7. An orbifold is a special kind of surface, and our last postponement is the fact that Euler's characteristic really does characterize the different possible topological types of surface. Our new "zip proof" of this wraps up the proof of all our results, and closes the first part of our book.

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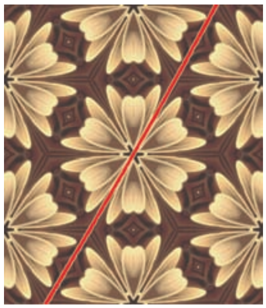
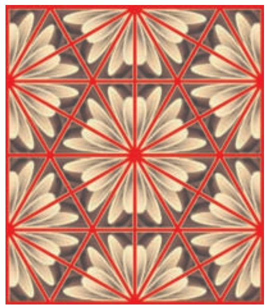
Planar Patterns

In this book we help you understand the symmetries of things. In this chapter we look at some repeating patterns and introduce you to the way we think about them. We describe the four fundamental features of a repeating pattern in the plane (or on any surface!) and introduce the signature we use to record these features of the pattern.

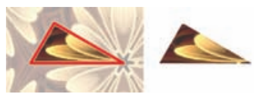
Mirror Lines

The floral pattern to the left has many symmetries. For example, the pattern is left-right symmetric: it has the vertical *mirror line* shown on the left below.





The figure in the middle shows another mirror line, which is a different kind because, unlike the first one, it runs *between*, rather than *along*, the petals. Drawing all the mirror lines we can, we get the figure on the right, which is at first sight rather confusing.



Fortunately, the small part we've highlighted in the margin contains enough information to reconstruct the whole pattern. This is because if we surround this small triangle by mirrors, as in a kaleidoscope, the reflections of the original triangle will fill in the neighboring triangular regions. The reflections of these reflections will fill in the neighbors of these neighbors, and so on, until the entire pattern is restored. With three small pieces of mirror (available at most hardware stores) and a little dexterity, you can try this yourself!

The patterns of Figures 2.1 and 2.2 are less ornate. The new patterns are somewhat simpler but have all the symmetries of the original; for our purposes all three patterns are identical.

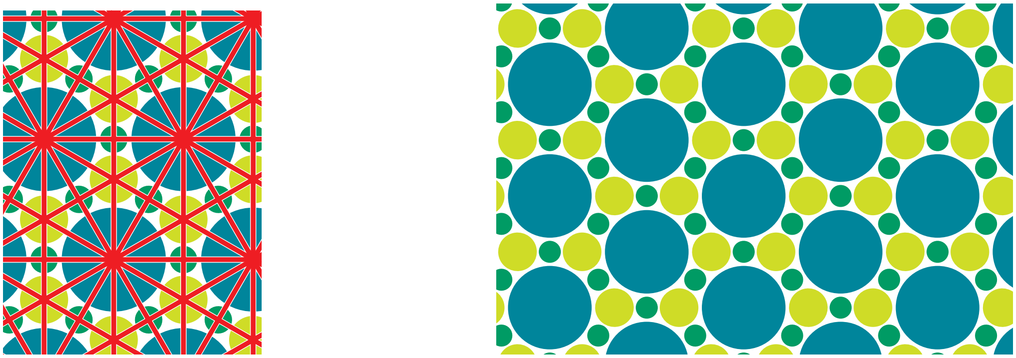


Figure 2.1. A simpler pattern.

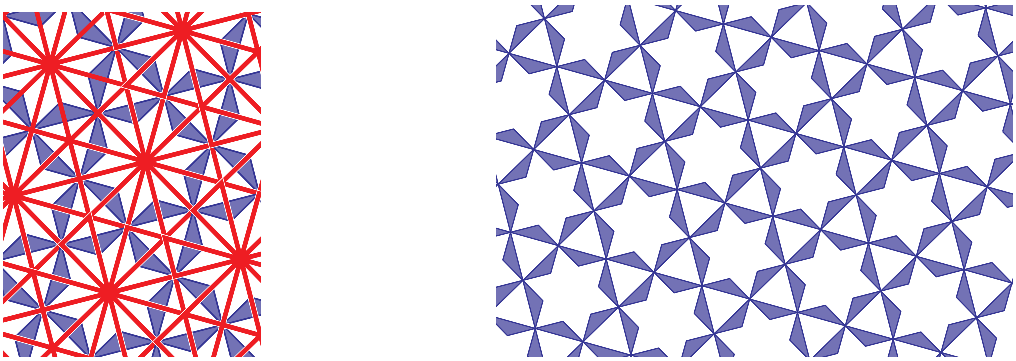


Figure 2.2. Another simple pattern.

Repeating patterns like the ones studied in this book are made up of many symmetric copies of a motif. What we are studying here are the symmetries relating each motif to each other motif in the pattern.

Describing Kaleidoscopes

Patterns whose symmetries are defined by reflections are called *kaleidoscopic* because of their similarity to the patterns seen in kaleidoscopes. They are classified by the way their lines of mirror symmetry intersect. So, for instance, in Figure 2.3 there are three particularly interesting kinds of point, one where six mirrors meet, one where three mirrors meet, and one where two mirrors meet. We call these 6-fold, 3-fold, and 2-fold kaleidoscopic points, respectively, because the local symmetries (right) are ∗6•, ∗3•, and ∗2•. The whole pattern has kaleidoscopic symmetry of signature ∗632, where there is no final point (•) because the symmetries don't all fix a point.

Figure 2.3. A kaleidoscope of type ∗632.

The numbers defining the type (or signature) of a kaleidoscope can be cyclically permuted, so that ∗632, ∗326, and ∗263 mean the same, or also reversed, equating these with ∗236, ∗362, and ∗623.

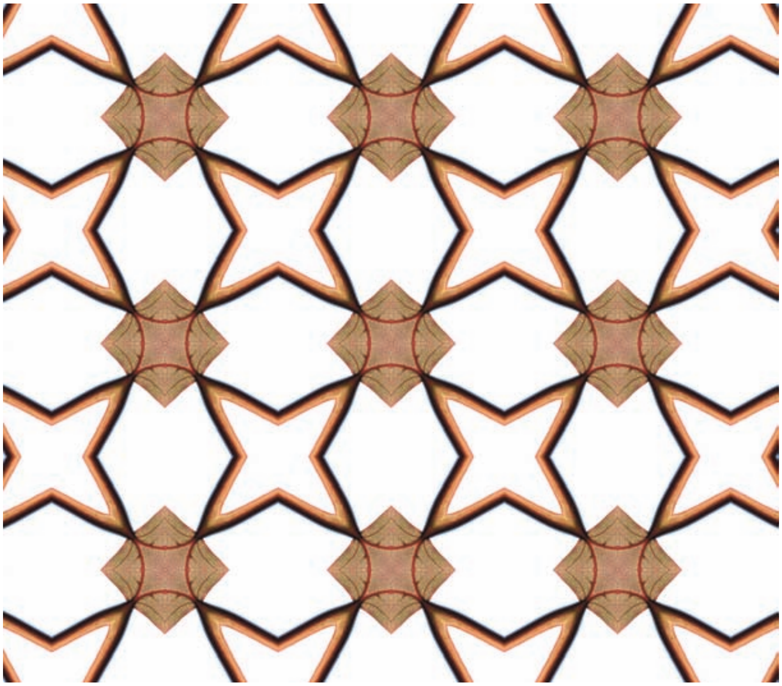


Figure 2.4. Decorated square tiling.

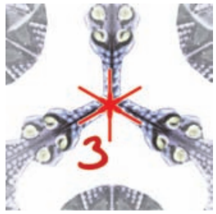
Patterns with a squarish sort of symmmetry, such as in Figure 2.4 are more common. The symmetry of this pattern is kaleidoscopic with signature *442. There are two 4's in the symbol because there are two different kinds of 4-fold kaleidoscopic points. The 2 in the symbol refers to the 2-fold kaleidoscopic point.

The fact that there can be several different kinds of kaleidoscopic points of the same order forces us to make it clear what *same kind* means for such points. We say, more generally, that any two features of a pattern are of the same kind only if they are related by a symmetry of the whole pattern. The points shown in the top two marginal figures are both 4-fold kaleidoscopic points but are obviously different. We will say that two points P and Q are the same if P can be moved to Q without changing the pattern's appearance in any way. (This "move" could include a reflection.)

Gyrations

The type of the kaleidoscope in Figure 2.5 is only *3 rather than *333, because all the kaleidoscopic points in that figure are of the same kind. However, the symmetries of this pattern are not purely kaleidoscopic. There is a new feature—a 3-fold rotational symmetry shown at right below.

Let's look at this more closely. The pattern would be undisturbed if the whole plane were to be rotated through 120 degrees around the point marked 3 in the middle of the figure. The same is also true of the point 3 in the top figure, but we've already accounted for this by calling it a 3-fold kaleidoscopic point—this rotation is "done by mirrors." Since the pattern has one kind of 3-fold gyration point and a kaleidoscope with one kind of 3-fold kaleidoscopic point, its signature is 3*3.



3*3

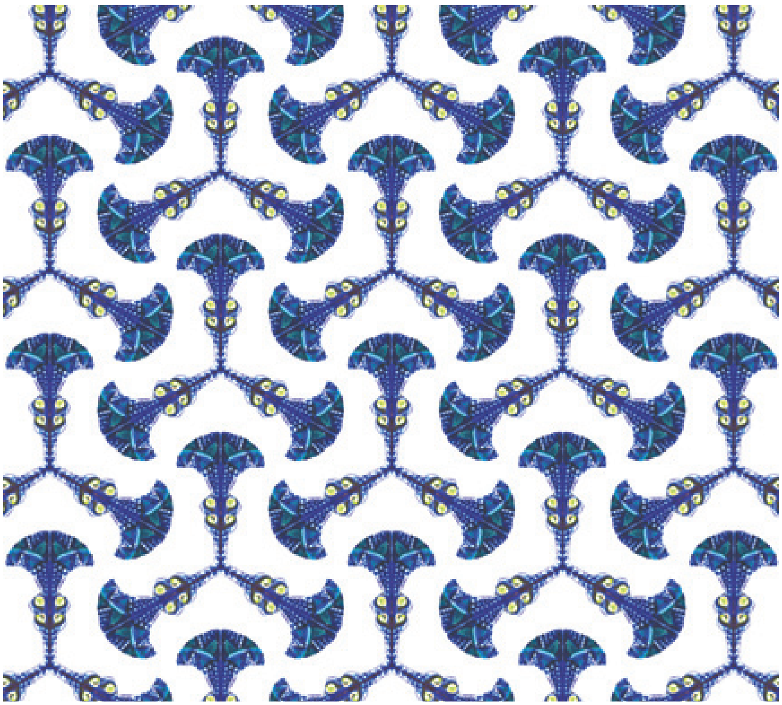


Figure 2.5. A pattern with signature 3*3.

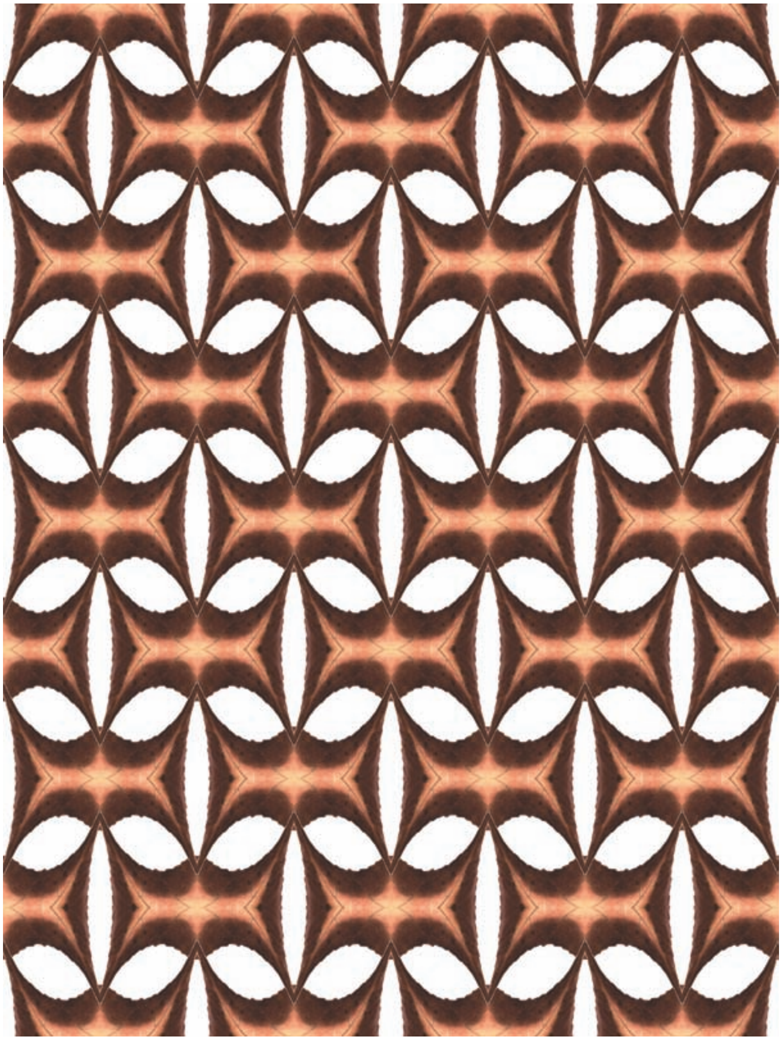
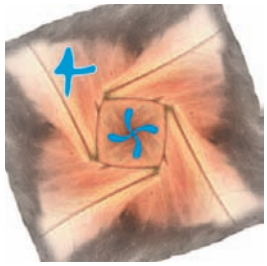


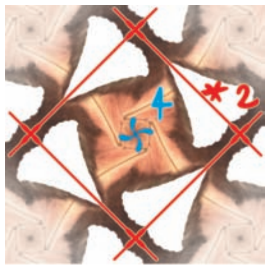
Figure 2.6. Pattern with signature 2*22.

The pattern in Figure 2.6 has two kinds of 2-fold kaleidoscopic points and one kind of 2-fold gyration point. The signature of this pattern is 2*22.

The * designating the presence of a kaleidoscope separates the digit representing the gyration point from those describing the kaleidoscopic points, which are read around the kaleidoscope.

Once you are familiar with this notation, you can tell immediately that the symbol 4*2 describes a pattern with one kind of 4-fold gyration point and one kind of 2-fold kaleidoscopic point. Figure 2.7 and the marginal figures show an example of such a pattern.





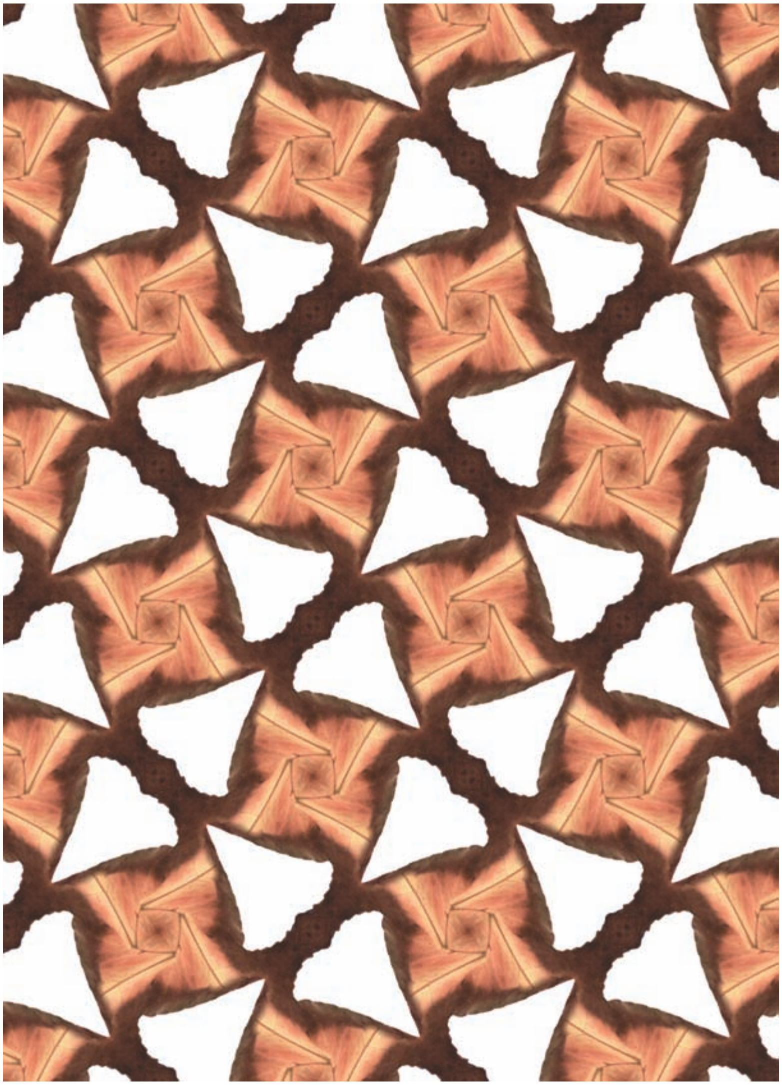
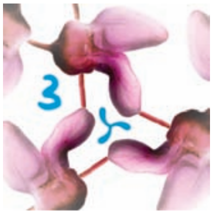
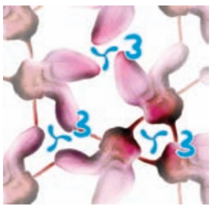


Figure 2.7. Pattern with signature 4*2.

In Figure 2.8 we see a pattern that has only gyration points and no kaleidoscopes. Since there are three kinds of 3-fold gyration point, the symmetry is of type **333**.





333

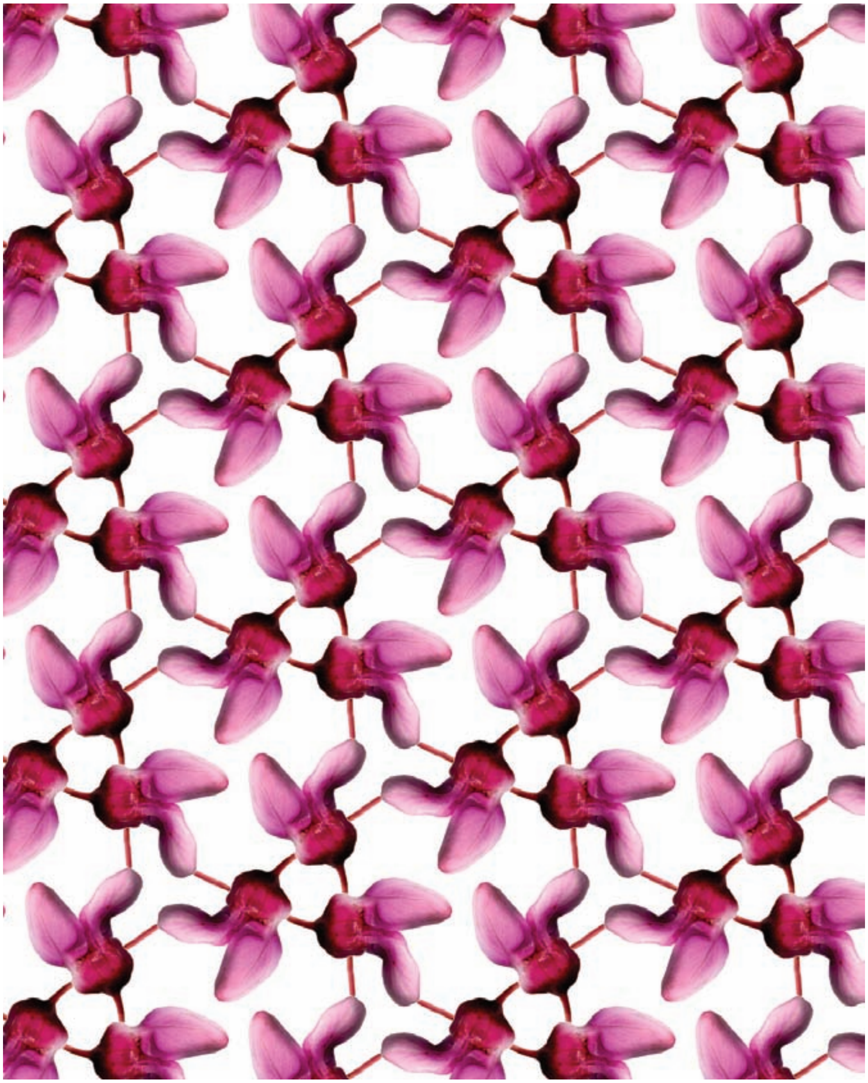
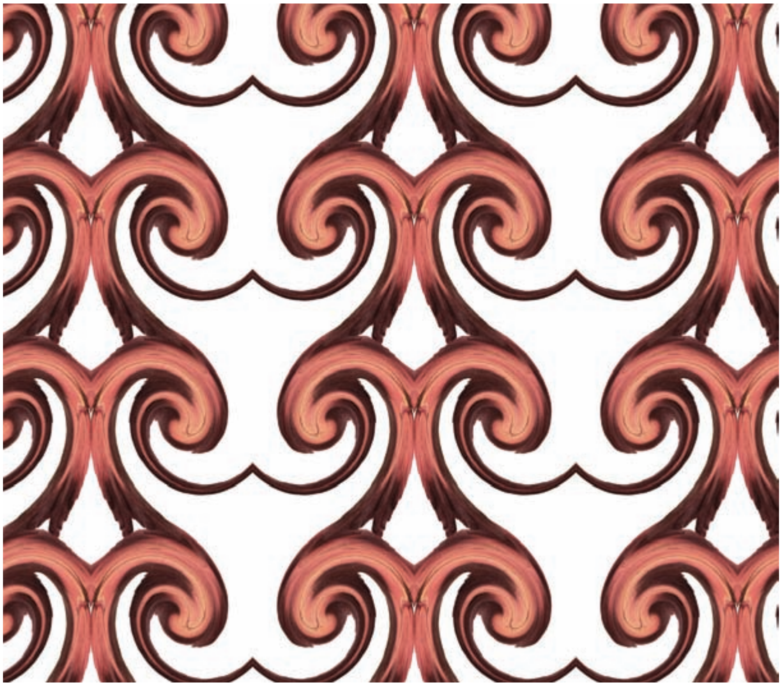


Figure 2.8. Pattern with signature **333**.

More Mirrors and Miracles

So far we have discussed two features of patterns in the plane: kaleidoscopes and gyration points. It is natural to ask in what ways these can occur in planar patterns. For instance, can a pattern have more than one kaleidoscope?



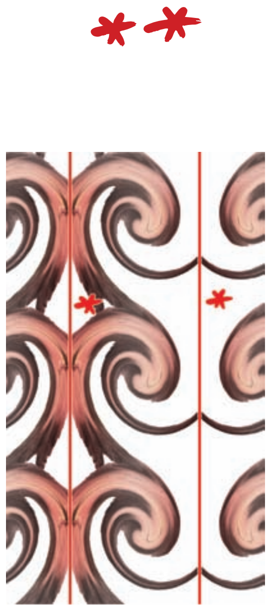


Figure 2.9. More than one kind of mirror signature **.

All the kaleidoscopes that we've seen so far have been defined by polygons enclosing part of our pattern, but that's not the only type there is. A single mirror line that has no other mirror lines crossing it is a kaleidoscope with signature *. Figure 2.9 shows a pattern with two of this kind of kaleidoscope in it, and its signature is **. (You should check that these two mirror lines really are different!)

We're also seeing something else for the first time here. The smallest subregion marked off by mirror lines in Figure 2.9 is infinite! There are several new features to be found in patterns like this, which will be presented in this section and the next.

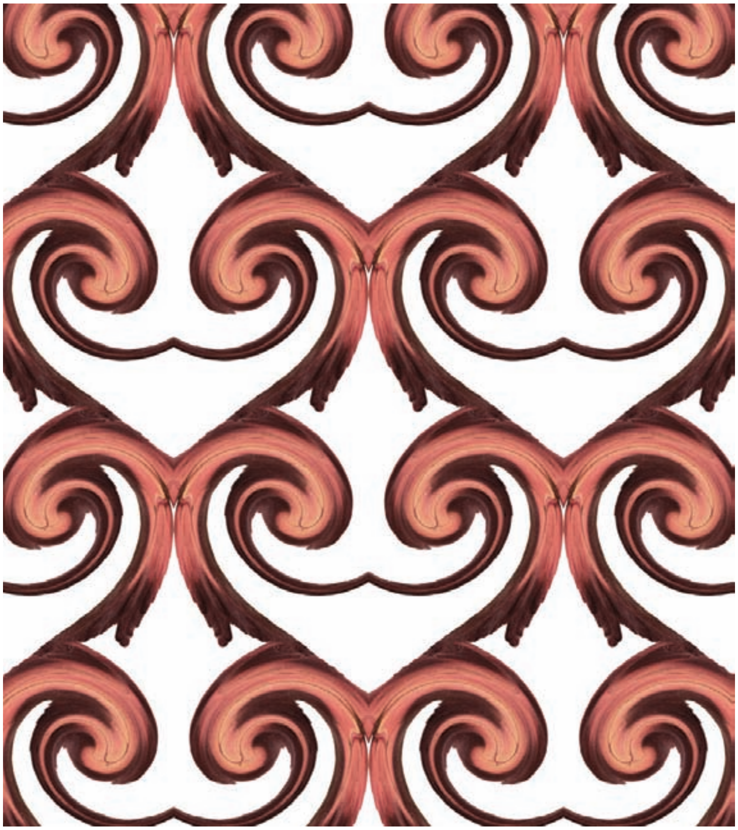


Figure 2.10. A pattern with a mirror and a miracle: signature ∗×.

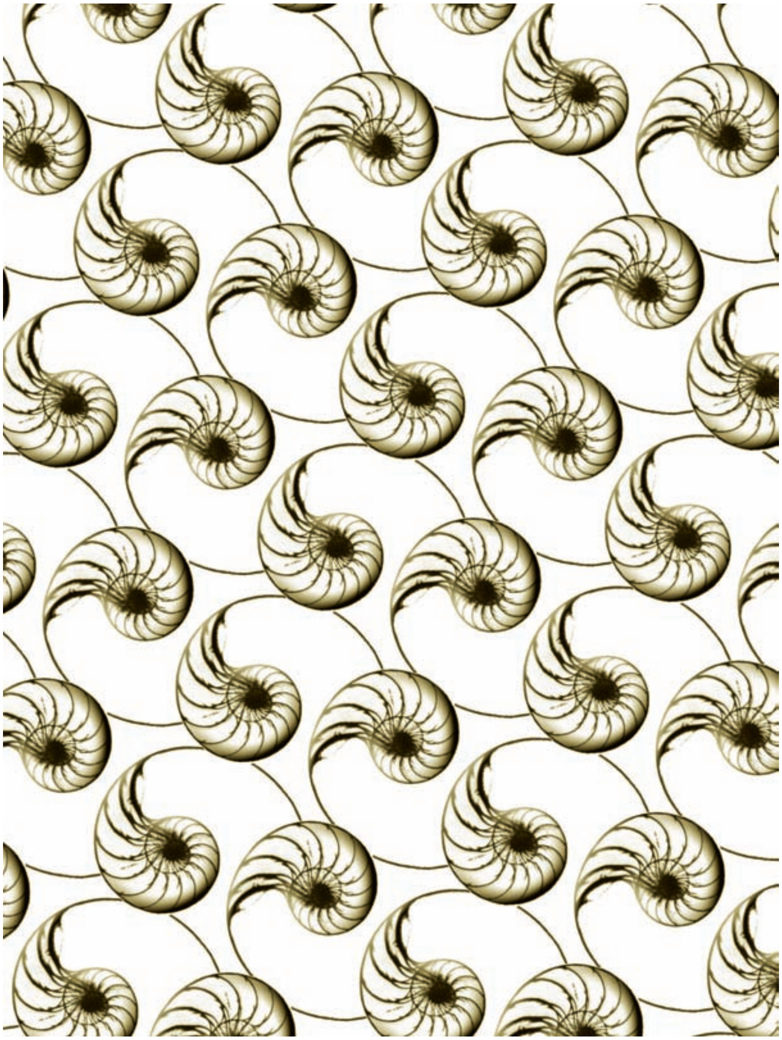
∗×

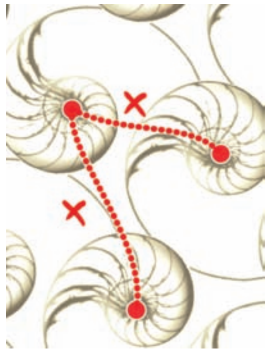
At first, Figure 2.10 looks very much like Figure 2.9. None of its mirror lines intersect, and the smallest subregion bounded by mirror lines is again infinite. But in this figure there is only one kind of mirror line!

And, there's a miracle here! There is a path from a left-handed spiral to a right-handed spiral that does not go through a mirror line. We will record the presence of such a path by a red cross (×) in the signature. We call this a "mirrorless crossing," or, for short, a *miracle*, and indicate it in figures by a red dotted line and cross.

Figure 2.10 has both mirrors and miracles, but only one kind of each, so its signature is ∗×.

We can have two miracles, just as we can have two different kinds of mirror. This happens in Figure 2.11, which has signature ××. (There are more than two paths from left-handed to right-handed spirals, but all of them can be made up of combinations of identical copies of the ones we've marked in the margin.)



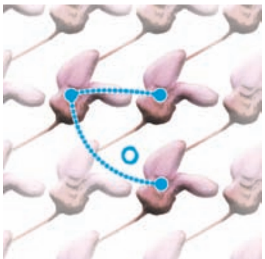


××

Figure 2.11. More than one kind of miracle: signature ××.

Wanderings and Wonder-Rings

Just as a miracle is a repetition-with-reflection of a fundamental region that's not "explained by" mirrors, it's possible to have a fundamental region repeated without reflection in a way that's not explained by gyrations, mirrors, or miracles. In fact, such repetitions always come in pairs. We call such a pair of paths a "wonderful wandering" and denote it by a blue "wonder-ring," ○. As in the figure in the margin, we draw such a pair of paths with blue dotted lines and with a blue ring nearby. The signature for Figure 2.12 is just ○.



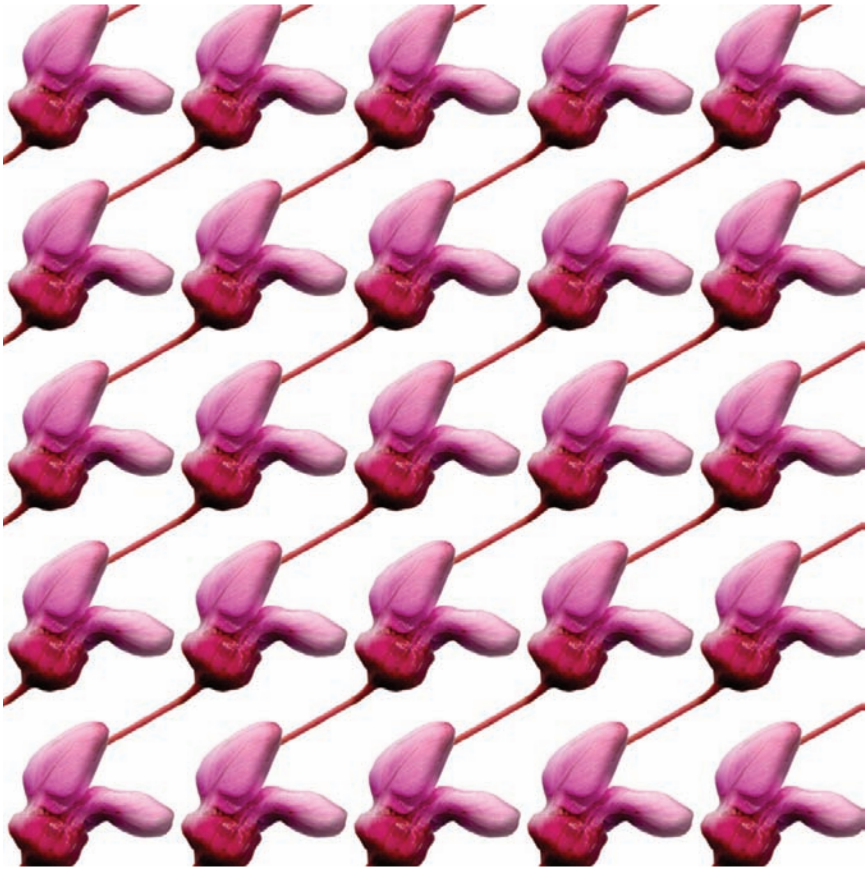


Figure 2.12. A wonderful wonder-ring: signature ○.