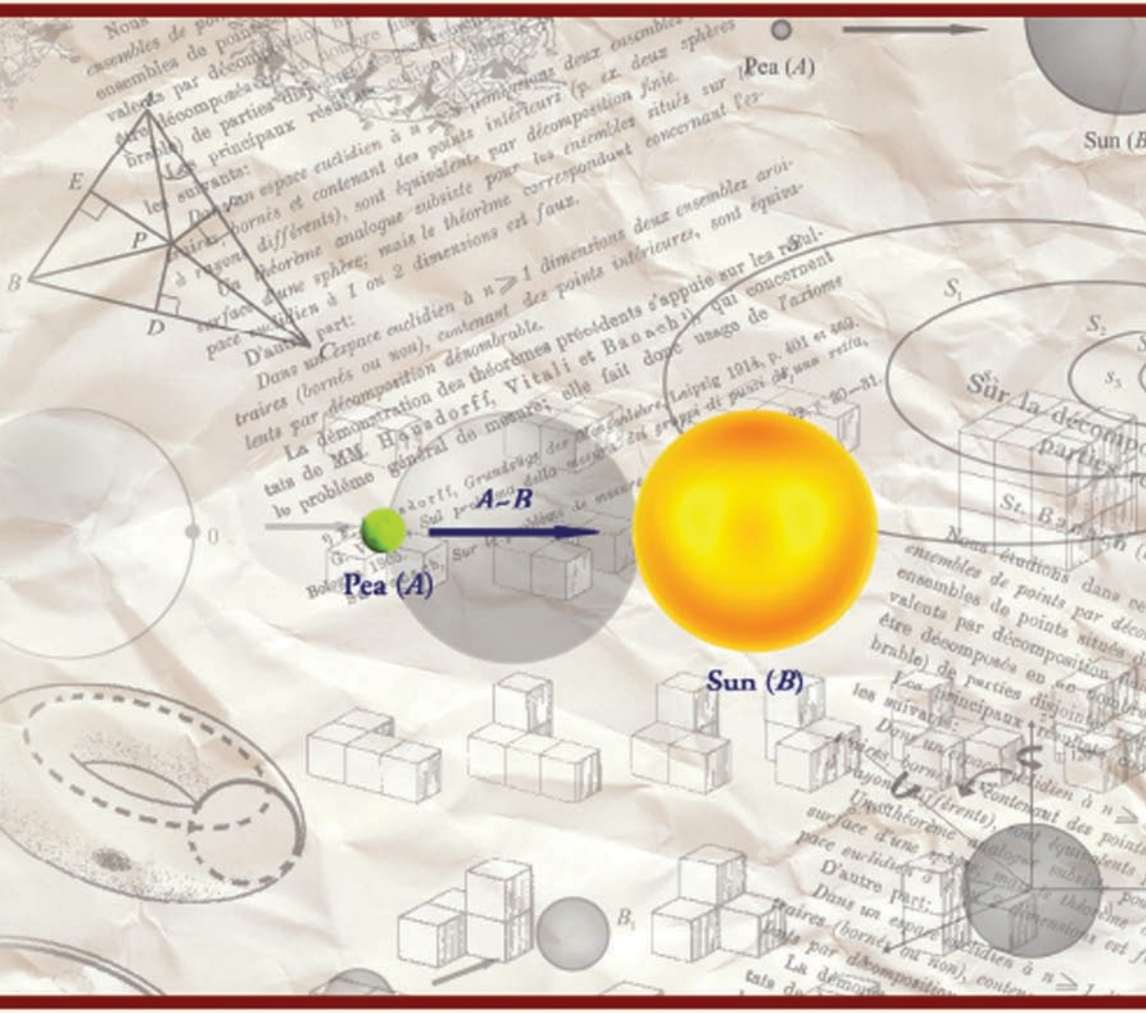
"A marvelous book"— Martin Gardner, *The New Criterion*

The Pea and the Sun

A Mathematical Paradox



Leonard M. Wapner

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To Kirsty
with love,
Dad

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Acknowledgments

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I am indebted to many and it is with great pleasure that I acknowledge those who have helped me write this book. I begin by thanking my teachers from elementary school through graduate study. I have modeled my teaching style after many of these teachers and I hope to inspire my students as my teachers have inspired me.

Since elementary school I've enjoyed reading popular (general audience) books about science and mathematics. These books and articles played no small part in my deciding to study and ultimately teach mathematics. Of all that I have read and continue to read, I single out Martin Gardner as having the most profound influence in my choice of study and profession. The reader may know that Mr. Gardner served as the Mathematical Games columnist for *Scientific American* from 1956 to 1981 and has written over 70 books on various topics. Though having no formal academic status in mathematics (his degree is in philosophy), he has received praise from mathematicians worldwide and I'm certain that I'm not alone in owing him a great debt of gratitude.

I thank my colleagues at El Camino College for patiently answering my questions and offering assistance. (They had no choice as I can be persistent.) I am particularly grateful to Bill Hemmer for encouraging me to continue writing at a time when I was considering giving up on this project. Bob Lewis deserves credit for convincing me the Cantor set is indeed remarkable. I thank my good friend and colleague Paul Wozniak for reviewing parts of the manuscript and offering constructive criticism. And I owe much to Michael Berg, professor of mathematics at Loyola Marymount University, undoubtedly one of the most talented mathematicians I know.

With the power of email I have apologetically imposed myself upon some world class authorities on this subject matter. Jan Mycielski, Stan Wagon, Matthew Foreman, Robert French, Karl Svozil, and Yiannis Moschovakis have graciously answered my questions and most have been cited in this book. I thank them all.

In this digital age where information is no more than a mouse click away via the Internet, there remains no more reliable a source than the *bricks and mortar* library. I have visited many while working on this book and I am especially indebted to the administration, reference librarians, and support staff of the Science Library at the University of California, Irvine. They are a talented and patient group whose assistance has been invaluable.

I give special thanks to my longtime friend, Howard Glassman, for answering an endless stream of computer questions and helping with the graphics.

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Introduction

Dear God—
If I have but one hour remaining to live,
please allow me to spend this time in a mathematics class
so that it will seem to last forever.

—A bored student's prayer

The clock had stopped and I was falling asleep.

In 1971, I was completing my graduate study in mathematics at UCLA, taking a class entitled *Measure and Integration*. I was bored. Despite the fact my major was mathematics, this particular course held no interest for me. I had then, and retain today, a strong love of mathematics and have spent most of my adult life teaching the subject at El Camino College. This course was a requirement for my major, however, and I just wanted it out of the way. I had no idea as I stared at the clock that I was about to be introduced to a truly remarkable theorem.

The professor was concluding his lecture and summed up by drawing a solid sphere (ball) on the board. He claimed he had just presented a proof of the fact that the ball could be partitioned into five pieces and then rearranged, much like a jigsaw puzzle, in such a way that two balls could be formed, each identical in shape and volume to the original.

I was immediately reminded of those red sponge balls a magician uses. He places one ball in his hand and closes it. When he opens his hand, there are two balls. Or doves? He puts one dove in the box,

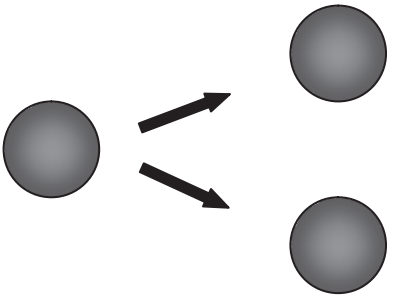

Figure I.1. Two solid spheres from one.

closes the lid, opens it and out fly two doves! So, I assumed there was some sort of joke or trick to this; I hadn't been paying attention and I had no idea as to the professor's intentions.

He continued. "So, we have the Banach-Tarski Theorem, or Banach-Tarski Paradox. An equivalent form of this theorem states that a solid of any size, say that of a small pea, can be partitioned into a finite number of pieces and then reassembled to form another solid of any specified shape and volume, say that of the sun. Consequently, this paradoxical theorem of Stefan Banach and Alfred Tarski is sometimes referred to as the *pea and the sun* paradox." (See Figure I.2.)

Is it conceivable *aus einer Mücke einen Elefanten machen*? Can an elephant be made out of a mosquito? It was not April Fools' Day! Was this a joke? And if so, what was the punch line? I was unwilling to hazard a potentially embarrassing question so I just looked around the room to see how others were taking all of this.

The student on my right raised his hand. "Clearly these results are nonsense, right? I mean, you're not suggesting that one can cut an apple into five pieces and reassemble the pieces to form two apples, are you? Are you suggesting that we can create something out of nothing?"

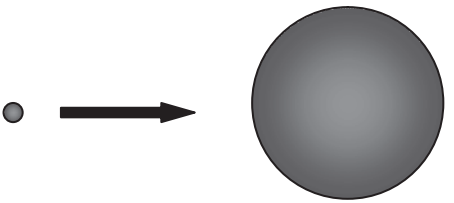

Figure I.2. The pea and the sun.

By now I had stopped watching the clock and was staring intently at the professor. I expected he would deliver a humorous punch line and dismiss the class or explain why, in fact, the theorem's proof was flawed. Instead, he just dismissed the student's questions with a blasé response.

"Well, you know, it's just one of those things. Results like this are common when we work with nonmeasurable sets, the Axiom of Choice, etc. The proof is valid and the theorem is accepted."

The class ended, and I left feeling a bit confused.

This was my introduction to the remarkable Banach-Tarski Theorem. In the foreword to *The Banach-Tarski Paradox* by Stan Wagon, Jan Mycielski refers to this theorem when he writes [Wagon 85, p. xi], "This, I believe, is the most surprising result of theoretical mathematics." For obvious reasons, the theorem's publication in 1924 was followed by a storm of controversy among mathematicians. How could such results, blatantly contradicting common sense, be accepted?

As the general population learned of the theorem, the controversy spread. An irate citizen once demanded of the Illinois legislature that they outlaw the teaching of this result in Illinois schools [Addison 83, p. 28]. So, two camps evolved—one accepting the beautifully counterintuitive results and the other rejecting it all as being meaningless.

Despite the proliferation of highly technical journal articles on the subject, little has been written for the general public. Indeed, browsing the library stacks and searching the Internet yield little of interest to anyone not having a graduate degree in mathematics. As a result, there is significant misunderstanding as to the nature of the theorem. I've had students ask, "What's this I hear about mathematicians being able to duplicate matter? They proved some sort of theorem suggesting we can build duplicating machines, right?"

The spirit of this book is to make this topic accessible by providing a journalistic, as opposed to mathematically intensive, look at the theorem. Chapter 1 addresses historical matters by presenting the *cast of characters*. The stars are Georg Cantor, the founder of modern set theory; Stefan Banach and Alfred Tarski, the leads; Kurt Gödel, the foremost logician of the twentieth century; and finally, Paul Cohen, professor of mathematics at Stanford University, who puts a sense of closure on the matter.

Chapter 2 presents a collection of mathematical recreations involving the geometrical dissection and reassembly of figures where something is magically gained, or lost, in the process. Though clever in construction, it must be stressed that these are included as recreations in comparison to the mathematically sound Banach-Tarski Theorem.

Chapter 3 presents the prerequisite mathematics necessary to fully appreciate the theorem. Being written for a general audience having a mathematics background including algebra and geometry, mathematical formality is excluded. Readers interested in pursuing the subject and having the appropriate mathematics background will find the bibliography helpful.

I whimsically entitle Chapter 4 "Baby BTs." These are mathematical curiosities, involving an apparent gain by decomposition and reassembly. Mathematically, they are somewhere between the jigsaw recreations given in Chapter 2 and the Banach-Tarski Theorem.

In Chapter 5, the statement and proof of the Banach-Tarski Theorem are given. Mathematical formality is omitted to reach a wider audience. This is not done at the expense of correctness.

The resolution of the paradox is given in Chapter 6. In some ways, a paradox loses its character, once resolved; but no discussion of this beautiful theorem would be complete without some explanation of its magic. Magicians and mathematicians treat similar mysteries in opposite ways. Magicians never reveal secrets; mathematicians strive to expose and clarify secrets.

Is there a physical reality to the consequences of the Banach-Tarski Theorem, or has mathematics just gone off the deep end? Chapter 7 will provide some answers.

Chapter 8 closes the presentation with a look at the past and future of mathematical discovery.

As a lifelong mathematics educator, I have more respect for questions than answers. Therefore, if the reader is to conclude this book with more questions than when he or she began, I will be gratified.

Len Wapner

1 History: A Cast of Characters

The good Christian should beware of mathematicians,
and all those who make empty prophecies.
The danger already exists that mathematicians have
made a covenant with the devil to darken the spirit
and to confine man in the bonds of Hell.

—St. Augustine

Significant mathematical achievement is best understood when viewed in correct historical and mathematical context. Mathematics is not created in a vacuum and, in the case of Banach and Tarski's remarkable work, there are at least three other mathematicians to acknowledge—Georg Cantor, Kurt Gödel, and Paul Cohen. This chapter gives a four part history of the Banach-Tarski Theorem:

1. Georg Cantor introduces the concepts of set theory and transfinite arithmetic.
2. Stefan Banach and Alfred Tarski publish the Banach-Tarski Theorem.
3. Kurt Gödel shows the Axiom of Choice is consistent with the other axioms set theory.
4. Paul Cohen shows the Axiom of Choice is independent of the other axioms of set theory.

With a nod to Jules Verne, this chapter could have been entitled "Around the World in Eighty *Years*," as it is truly an international story with Germany, Poland, Austria, and the United States well represented.

Before beginning, let's briefly consider the nature of mathematical achievement. Do mathematicians discover or create? Did Stefan Banach and Alfred Tarski discover their mysterious paradox lurking in the depths of mathematical truth, or did they create it? Similar questions could be asked of the scenic or portrait photographer. Environmentalist Ansel Adams is best known for his stunning black and white photographs of Yosemite Valley and the Sierra Nevada mountains in California. Did he *record* or *create*? As black and white images with extreme light and shadow effects, the photographs are not realistic impressions of what one actually sees when hiking these mountains. In this sense, the photographs are created (artistically) with a good eye and state of the art equipment. But these beautiful photographs may also represent nature at its best and one gets a sense of seeing *God's gifts* when looking at Adams' photographs. Did Adams create this beauty, or simply discover and record it with the right technique and equipment?

With respect to mathematics, there are two opposing viewpoints. The Platonic view of mathematics—*Platonism* (or *mathematical realism*)—holds that mathematical objects exist *out there*, independent of the human mind. "Pi" in the sky! Theorems, proofs, constructions, and solutions to unsolved problems are waiting to be discovered by mathematical researchers much the same way gemstones are waiting to be unearthed by geologists. According to the Platonist, mathematical shapes, quantities, and relationships have always existed, at least in a theoretical sense. They are no more creations of the human mind than a diamond is a creation of the geologist. A talented mathematician does not create mathematics. Mathematics is discovered. Thus, it may be that the popular view of the mathematician is that of a Platonist, as many think of the mathematician as a scientist, rather than a creator or artist. Georg Cantor and Kurt Gödel have generally been regarded as Platonists. It has been suggested that most of today's mathematicians are Platonists, but few are willing to admit it. Mathematician and human rights advocate Lipman Bers has stated [Albers, Alexanderson, and Reid 90, p. xiii], "A working mathematician is always a Platonist. It doesn't

matter what he says . . . I think that in mathematics he always has that feeling of discovery . . . Mathematics is, as Ron Graham has said, the ultimate reality . . . ."

The opposing viewpoint, known as *formalism*, holds that mathematics is a language consisting of symbols, and conventions for manipulating these symbols which, when the *rules of the game* are followed, generate theorems, proofs, constructions, etc. It is a construct of the human mind. These theorems, sets, etc. need not be applied to the physical world. Mathematics stands separated from physical reality as a human creation, much like a spoken language or work of art.

A third viewpoint is that of the *constructivist* (*intuitionist, finitist*), believing that only mathematical objects which can be constructed in a finite way have meaning. Constructivists tend to oppose infinite processes and existence theorems which do not construct the object being considered. Think of Platonists and formalists as having opposing viewpoints (see Figure 1.1), with constructivists having issues with both groups.

Are you, the reader, a Platonist? Read the following and decide for yourself.

The number π is the ratio of the circumference of a circle to its diameter. It has been proven that π is irrational. (A real number is rational if it is the quotient of two integers. If a number is rational, it will have a repeating or terminating decimal expansion. If a real number is not rational, it is called irrational.) In fact, it has been proven that π is transcendental, meaning that it is not the root of any algebraic equation. Consequently, it is impossible to compute π (the decimal expansion π) exactly by algebraic means (addition, subtraction, multiplication, division, square roots, etc.). Because there are ways to represent π as the sum of an infinite series (e.g. $\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \ldots$) it is theoretically possible to calculate π to any specified number of decimal places. Despite the fact computers have calculated π to trillions of decimal places, there remain many open questions regarding possible patterns in the decimal expansion of π. Do all digits occur infinitely often? Do they occur with equal frequency? Are there patterns of digits within the expansion of π? There are endless questions of this sort, some of which may never be answered.

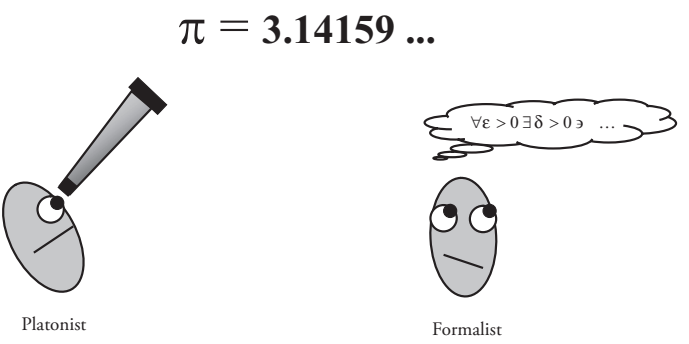

Figure 1.1. "Pi" in the sky.

Later in this chapter when considering the work of Kurt Gödel, we will see that there are mathematical questions and conjectures which are independent of our axiomatic system. That is, there are questions which can never be answered, and are called *undecidable*. For the sake of this test, let's assume that the following question has been proven undecidable: "In our axiomatic system, does the decimal expansion of π contain infinitely many zeros?"

Do you, the reader, believe the decimal expansion of π contains infinitely many zeros?

If your answer is either yes or no, then you are a Platonist. The Platonist sees the decimal expansion of π as being out there, somewhere, and acknowledges the fact that our axiomatic system will never yield an answer to our question. However, the answer must exist, and there may be additional mathematical evidence, perhaps in the form of an extended or alternative axiomatic system, which will decide the issue.

The formalist and constructivist dismiss the question as meaningless. If we do not have the means to obtain the answer, then there is no answer. (Philosophically, this is reminiscent of the old question, "If a tree falls in the forest and there is no one there to hear it fall, does it make a sound?")

So decide for yourself. Are you a Platonist?

Bear in mind that Platonists and formalists work on similar mathematical problems and tend to agree on most mathematical matters. Both prove new theorems, find new proofs to old theorems, and find solutions to unsolved problems. Each group generally accepts the mathematics of the other, but they disagree as to what it all ultimately represents.

Figure 1.2. Georg Cantor (1845–1918).
(From UA Halle Rep. 40 I C 11. Reprinted with the permission of Martin-Luther-Universität Halle-Wittenberg.)

The five major players of the Banach-Tarski story are (chronologically) Georg Cantor, Stefan Banach, Alfred Tarski, Kurt Gödel, and Paul Cohen. The story is international, beginning in Germany, moving on to Poland, then to Austria, and concluding in the United States. We begin with the story of Georg Cantor, a Platonist by his own admission, and generally recognized as the founder of modern set theory.

Georg Cantor—The Founder of Modern Set Theory

Georg Cantor (see Figure 1.2), born in St. Petersburg in 1845, revolutionized mathematics in establishing set theory (*Mengenlehre)* as a mathematical discipline. In doing so, he was able to actualize or consummate the notion of infinity creating an arithmetic of infinities—transfinite arithmetic. As a Platonist with theological interests, he often saw himself as a secretary or messenger for God.

The proof of the Banach-Tarski Theorem, as presented in Chapter 5, requires the manipulation of infinitely many points of the solid sphere and the manipulation of infinitely many rotations of these points. Without Cantor's revolutionary ideas of set theory and transfinite arithmetic, Stefan Banach and Alfred Tarski would surely not have published their theorem. In fact, twentieth century mathematics would not exist, as we know it.

Dreams of infinite space and time come naturally to anyone gazing at the heavens or contemplating the periodic cycles of day into night. Infinity is historically mystical, having been contemplated by scientists,

philosophers, and theologians, as well as mathematicians. By its very nature, it is difficult to define, and in some discussions it may be convenient to reject the concept as meaningless. The history of infinity may be infinite in itself. Cantor was certainly not the first to consider the concept; so, to fully comprehend what Cantor accomplished, we begin with the fourth century BC Greek philosopher, Zeno.

Known for his paradoxes of motion and continuity, Zeno was one of the first to pose serious questions about infinite processes. His Dichotomy Paradox asserts that a runner can never reach the end of his race; for in order to do so he must first reach the halfway mark, then the halfway mark of the remaining half, and so on. So the fraction of the course completed can be thought of as the infinite sum $1/2+1/4+1/8+\ldots$. Since infinitely many fractional parts of the course would have to be completed in a finite amount of time, the end of the course could never be reached. Surely it is impossible to complete an infinite amount of tasks in finite time. Right? Another well known paradox of Zeno involves Achilles racing a tortoise with the tortoise being given a head start. Zeno argues that Achilles can never overtake the tortoise. To do so, Achilles must first reach the point where the tortoise started, by which time the tortoise will have advanced to another point. When Achilles reaches that point, the tortoise will have advanced again to a third point. Since the process goes on ad infinitum, there is no hope that the tortoise can be overtaken in a finite amount of time.

Clearly Zeno knew that in actuality the runner of the Dichotomy Paradox completes the course and that Achilles would catch up to the tortoise. Yet, he made no attempt to resolve these paradoxes. It would be over two thousand years before mathematics would resolve such problems.

It was Aristotle in the third century BC who first made the distinction between the actual infinite and the potential infinite. Aristotle writes [Aristotle, 207b]:

> Hence this infinite is potential, . . . and not a permanent actuality but consists in a process of coming to be, like time . . . . With magnitudes the contrary holds . . . In point of fact they (mathematicians) do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish . . . Hence, for the purposes of proof, it will make no difference to them to have such an infinite instead, while its existence will be in the sphere of real magnitude.

Thirteenth century Christian theologian Saint Thomas Aquinas writes [Aquinas, Ia 7.4.]:

> The existence of an actual infinite multitude is impossible. For any set of things one considers must be a specific set. And sets of things are specified by the number of things in them. Now no number is infinite, for number results from counting through a set of units. So no set of things can actually be inherently unlimited, nor can it happen to be unlimited.

The great German mathematician Carl Friedrich Gauss (1777–1855), considered by some as the greatest mathematician the world has known, wrote in a letter to a friend [Maor 87, p. 55]:

> I must protest most vehemently against your use of the infinite as something consummated, as this is never permitted in mathematics. The infinite is but a *façon de parler*...

And to this day, mathematics education through geometry, algebra, and calculus generally treats the infinite not as a real number, but rather as the quality of being unlimited in size. Elementary mathematics almost never makes such statements as "$x = \infty$." In its place we see "$x \to \infty$," suggesting infinity as a potential, rather than achievable quantity.

(The popular symbol ∞ used for infinity was first introduced by the English mathematician John Wallis in the seventeenth century. He may have taken it from the Roman numeral for 100 million, which consists of the lazy eight placed within a rectangle. The symbol itself is symbolic of an endless process, perhaps that of a snake devouring itself.)

While mathematicians and philosophers before Georg Cantor could only look to infinity with mathematical telescopes, treating it as a potential, Cantor consummated, or actualized the infinite, dropping it in our laps to be manipulated and explored. His work, revolutionary in its time, brought with it controversy and sadness which ultimately may have contributed to his death.

Born to parents of Jewish descent, Georg moved with his family to Frankfurt, Germany in 1856. His father had converted to Protestantism and his mother was born Catholic. His father encouraged him to pursue a career in engineering but the young Cantor was more interested in the philosophy of medieval theologians, showing interest and talent in philosophy, physics, and mathematics. In the end, his father gave him

permission to pursue a career in mathematics. Cantor was grateful and would always feel a need to live up to his father's expectations. Some have suggested that this pressure may have contributed to Cantor's mental health problems later in life.

He earned his doctorate at the University of Berlin in 1867 with a thesis in number theory, then took an entry level position as *Privatdozent* at the University of Halle. It was a low level position at an institution lacking reputation. He was promoted to associate professor and then to professor of mathematics, yet never achieved his dream of a professorship at the University of Berlin. For this failure he blamed his lifelong archenemy Leopold Kronecker (1823–1891), who was highly critical of Cantor's concept of the infinite.

The concept of set, as developed by Cantor, became entwined with the infinite when he began to consider the size, or *cardinality*, of sets. The cardinality of a finite set simply is the number of elements in the set. So, the cardinality of $\{2, 4, 6\}$ is 3. But what could be said of the cardinality of the set of counting numbers {1, 2, 3, . . .} or of the set of all real numbers? Do they have infinite cardinalities? Are their cardinalities equal? Does it make sense to say that some infinities are greater than other infinities?

To answer such questions, Cantor looked first to finite sets and pointed out that two finite sets would have the same cardinality if their members could be put in one-to-one correspondence with each other. So, the sets {2, 4, 6} and {8, 9, 10} have the same cardinality (3) because of the correspondence $2 \leftrightarrow 8, 4 \leftrightarrow 9, 6 \leftrightarrow 10$. He then suggested the same could be said of infinite sets. For example, the set of even numbers {2, 4, 6, . . .} would have the same infinite cardinality as the set of counting numbers {1, 2, 3, . . .} because there is a clear one-to-one correspondence between the two sets (Figure 1.3).

So, despite the fact that the even numbers form a proper subset of the counting numbers, both sets contain the same (infinite) number of elements and are said to be of the same cardinality. (A proper subset of a given set is a subset of the given set not equal to the given set.) This would be the first of many set theoretic paradoxes as it seemingly contradicts Euclid's common notion of the whole being greater than the part. Paradoxical as this may be, Cantor used this as the very definition of an infinite set. He defined an infinite set as one which could be put in one-to-one correspondence with a proper subset of itself, removing all vagueness from previous notions of infinitely large sets.

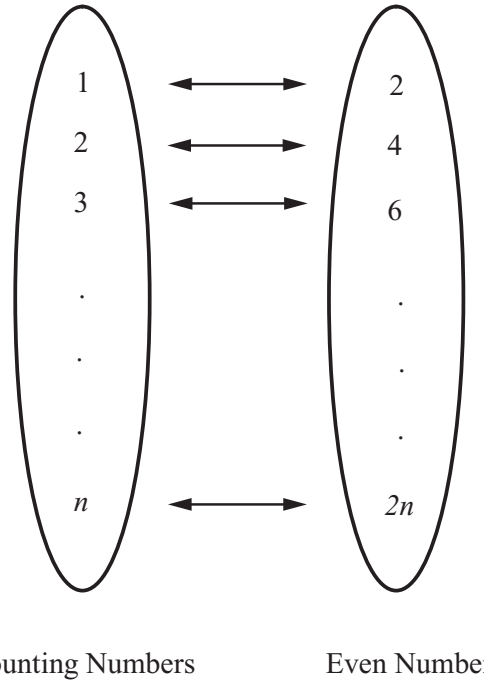

Figure 1.3. One-to-one correspondence between two infinite sets.

Other sets having the same cardinality as the counting numbers include the set of odd numbers, the set of all perfect squares, and the set of primes. In each case, the members of the set are *denumerable*, or *countable*, in that they can be *listed*, putting them in one-to-one correspondence with the counting numbers. The question naturally arises if all infinite sets have the cardinality of the counting numbers: is there a one-to-one correspondence between the rational numbers and the counting numbers? What about the set of all real numbers? Cantor hypothesized a ranking, or hierarchy of infinities, and thus was born the subject of *transfinite arithmetic*, the arithmetic of the infinities.

To investigate, Cantor considered the set of positive rational numbers—all positive numbers which can be written as a ratio of two integers. Cantor had reason to believe that this set might be more numerous than the set of counting numbers, because between any two rational numbers there exists another rational number. In fact, between