SECOND EDITION

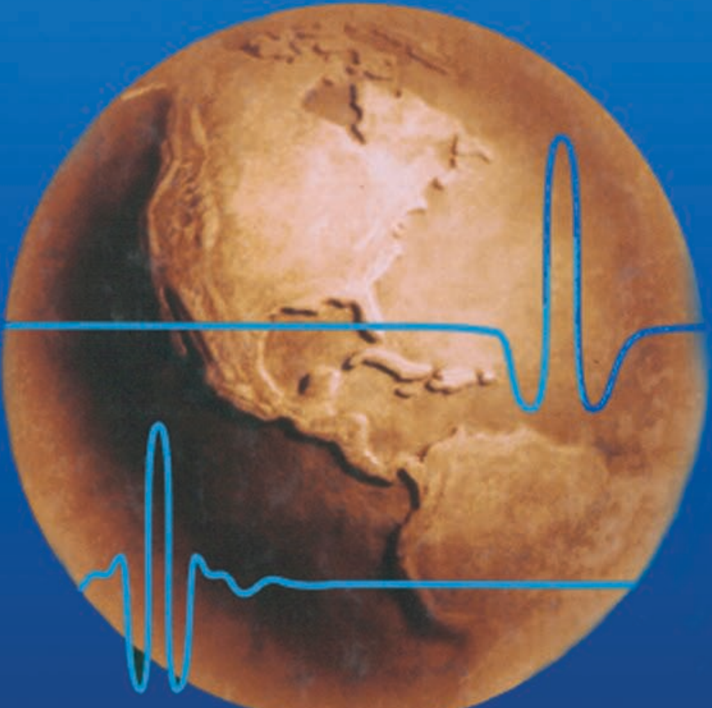
Barbara Burke Hubbard

THE WORLD

ACCORDING TO

WAVELETS

The Story of a Mathematical Technique in the Making



The World
According to Wavelets

And what is the use of a book, thought Alice,
without pictures or conversations?

—Lewis Carroll, *Alice's Adventures in Wonderland*

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an informa business

Reprinted 2010 by CRC Press
CRC Press
6000 Broken Sound Parkway, NW
Suite 300, Boca Raton, FL 33487
270 Madison Avenue
New York, NY 10016
2 Park Square, Milton Park
Abingdon, Oxon OX14 4RN, UK

Editorial, Sales, and Customer Service Office

A K Peters, Ltd.
63 South Avenue
Natick, MA 01760

Library of Congress Cataloging-in-Publication Data

Hubbard, Barbara Burke,
The world according to wavelets : the story of a mathematical technique in the making / Barbara Burke Hubbard. -- 2nd ed.
p. cm.
Includes bibliographical references (p. —) and index.
ISBN: 1-56881-072-5
1. Wavelets (Mathematics) I. Title.
QA403.3.H83 1998
515'.2433--dc21

97-48738
CIP

In memory of my father,
Vincent J. Burke,
who taught me that one must love one's work

And for my mother,
Velma Whitgrove Burke,
who sees wonders in the commonplace

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

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PART I The World According to Wavelets 1

To the Reader

When at the age of four or five I asked my mother just how babies are made, her answer was so obviously absurd that I didn't believe her, although I knew she never lied to me. I had at times the same feeling while writing this book: statements that the experts seemed to find natural, even commonplace, seemed barely credible. What I have tried to do in this book is to make them simultaneously surprising and believable.

The project started when the executive editor of the National Academy Press asked me to write an article about wavelets for a book, *A Positron Named Priscilla*, on the "frontiers of science." At the time I knew nothing about Fourier analysis and had never heard of wavelets; my only mathematical qualification—non-negligible—was my husband, a mathematician at Cornell University. (I never took calculus in high school because I spent my senior year in Moscow, where my father was sent as a newspaper correspondent; in college I carefully avoided all math courses.)

Rashly, I agreed to the assignment, and embarked on a period of work both passionate and disconcerting, which has led me to think about the possibilities of communicating mathematical ideas. Mathematicians claim that math is not a spectator sport: you cannot understand math, or enjoy it, without doing it. A mathematician who tries to communicate his subject to the layman soon finds himself in trouble. "This is getting vaguer and vaguer," lamented Robert Strichartz of Cornell as he tried to explain function spaces to me. "When I stick to the truth I don't communicate, and when I communicate I stray from the truth."

Talking without being understood is pointless, lying is painful, so most often mathematicians abandon the attempt, if they have the courage to try in the first place. It would be a shame to leave it at that. There exists, certainly, a limit to what someone without the right background can understand of mathematics, but I am convinced that we are far from having reached that limit. Mathematics contains ideas that can be, and deserve to be, communicated to a wider public—even if what is communicated is at the level of appreciation rather than practical knowledge.

No one claims that only geneticists doing recombinant DNA research should know what DNA is, or that only physicists and chemists should know that matter is made of atoms. In mathematics the reverse is often the case; too often we insist that children and students learn practical techniques without showing them that these techniques are based on interesting ideas, or that they make it possible to tackle interesting questions. No wonder some parents and educators are appalled when a child uses a calculator to solve problems; to them, mathematics is synonymous with computations, and if this work is given to a machine, nothing is left.

Fourier analysis and wavelets are good subjects with which to try another approach. The idea that information can be represented in different forms—that a mathematical function or a physical signal can become a *Fourier transform* or a *wavelet transform*, and that this transformation is useful—has had an enormous impact, both intellectual and practical, on our society.

This book is then both modest in scope and daring (not to say foolhardy) in design. It is only a brief introduction to a vast subject that itself is only a small part of mathematics; but I wanted to make this subject accessible to those who have no mathematical background to speak of, while giving enough details so that those who are more sophisticated will find it useful.

I have written the book on two levels. The main article contains no formulas. When mathematicians, physicists, or electrical engineers talk about Fourier analysis or wavelets, sooner or later they scribble down formulas. To them, formulas are the most precise and economical way of expressing their thoughts; naturally they think they are communicating something. But showing a formula to someone who can't remember whether $\sum$ means sum or integral is like asking someone who doesn't know which note on the musical staff is B-flat and which is D to analyze the score for a four-voice fugue by Bach.

This reaction isn't limited to English majors like me. "Is there any book that deals with fractal geometry in plain English instead of integrals, sigma notation, and a whole bunch of other funny symbols that I haven't learned yet?" a computer science major who had studied calculus asked my husband.

And in 1857 Michael Faraday, best known for his pioneering work connecting magnetism and electricity, wrote 26-year-old James Clerk Maxwell, 40 years his junior,

> When a mathematician engaged in investigating physical actions and results has arrived at his conclusions, may they not be expressed in common language as fully, clearly, and definitely as in mathematical formulae? If so, would it not be a great boon to such as I to express them so?—translating them out of their hieroglyphics, that we also might work upon them by experiment. I think it must be so, because I have always found that you could convey to me a perfectly clear idea of your conclusions, which, though they may give me no full understanding of the steps of your process, give me the results neither above nor below the truth, and so clear in character that I can think and work from them. If this be possible, would it not be a good thing if mathematicians, working on these subjects, were to give us the results in this popular, useful, working state, as well as in that which is their own and proper to them [Faraday, p. 206]?

In the beginning, formulas can seem more of a hindrance than a help. If they are to serve any purpose they must be translated into words, but the mere sight of a formula on the page can be paralyzing. So I have not used "funny symbols" or "hieroglyphics" in the main article. (The expression $f(x) = x^2$ slipped in, but you can ignore it.)

On the other hand, formulas were not invented simply as weapons of intimidation. Although it may seem paradoxical, it is easier to understand Fourier analysis and wavelets if you see how it actually works, rather than clinging to generalities and metaphors. (More interesting, too—as in the case of making babies...) But to explain these details, words are often clumsy and sometimes ambiguous.

"When I use a word," says Humpty Dumpty in *Through the Looking Glass*, "it means just what I choose it to mean—neither more nor less." Following this creed, wavelet researchers sometimes attach new meanings to established words. *Dilate* can mean contract, *large-scale* can mean small-scale, *decimate* can mean cut in half. Even with standard definitions, sentences can be interpreted in unintended ways. Along with the Fourier transform and the wavelet transform, there exists a transform that changes spoken or written information into information understood by the listener or reader; with this transform—as I, to my chagrin, had many occasions to note—perfect reconstruction of the original signal is all too rare.

I have, then, included formulas in the articles you will find at the end of the book in the section *Beyond Plain English*, although they introduce two dangers. The first is that formulas have the nasty trick of exposing errors that might remain decently veiled if one would only remain sufficiently vague. "If you force me to be precise, of course I'll make mistakes," protested the well-known French mathematician René Thom in a lecture when his ungrateful audience first insisted on more precision, and then found mistakes in his formulas.

The other danger is that some readers may be frightened away. "When I see a formula, I start to panic; I always wonder, am I going to understand?" a colleague told me. There's no reason to be afraid; at the risk of irritating those who don't need it, everything is explained in detail. (Some remnants of high-school algebra and trigonometry are assumed, but if those remnants are frayed, some of the basics are reviewed in the Appendix. Calculus, while useful, is not essential.)

The Appendix is schizophrenic. It began as a place to hide things that might seem too daunting or that require mathematical techniques beyond the scope of this book: proofs of the Heisenberg uncertainty principle and the sampling theorem, for example. But readers who are less sophisticated mathematically will also find, in addition to a brief review of trigonometry, a list of mathematical symbols and a discussion of integrals.

The bibliography includes technical books and articles that may be useful to some readers. I have also included a list of wavelet software. Neither list claims to be complete. I have given detailed references where possible, but for many quotations no references are given; they come either from conferences organized by the National Academy of Sciences in Irvine, California, in November 1992, or (especially) from subsequent interviews and personal communications. When possible I have used published translations of French texts, but generally the translations are my own.

. . .

Too often, I think, mathematics is taught as a collection of final results and techniques divorced from human thought or feeling. This is misleading. Mathematics is rewarding, but fickle and unforgiving; in an instant one can have an inspired idea that illuminates an entire field, but one can also watch helplessly as an edifice of results arduously constructed over months or years crumbles because of one small hole that refuses to be patched, or, that when patched, reopens elsewhere. The would-be mathematician who does not love mathematics will not withstand the moments of doubt and the empty weeks or months when ideas don't come, or come only to prove false. And despite the public image of the solitary mathematician immersed in abstractions, mathematicians work as intimately with other people as musicians in a string quartet. Collaboration is the rule, not the exception, and that collaboration often crosses national boundaries.

But the human side of mathematics is generally well hidden from outsiders; mathematical writing, as Philip J. Davis and Reuben Hersh [Davis, Hersh] write in *The Mathematical Experience*, generally "follows an unbreakable convention: to conceal any sign that the author or the intended reader is a human being."

In teaching, too, generally one starts at the end: everything is settled and there is no hint of the often tortuous path that led to those results. The newcomer may feel stupid not to understand immediately, although mathematicians often spent years arriving at results and feeling at ease with them. The Greeks were profoundly troubled by irrational numbers; in the sixteenth century, negative numbers were considered "impossible" solutions; in the eighteenth century, long division was taught in universities. Even those not easily intimidated may not realize that mathematics is an ongoing process, that there are questions that have not yet been answered and questions that have not yet been asked. And they may not see what is beautiful and surprising in facts presented, with reticence, like so many historical dates to memorize.

To me the process is as interesting as the result; I see Fourier analysis and wavelets not only as tools but also as a story to tell, of ideas and people. I am deeply grateful to the researchers, in France and in the United States, who shared with me not just the results of their work but also its human context.

Nevertheless this approach is inaccurate. Even when talking only about mathematics, attributing discoveries is difficult. How can one evaluate the influence of a conversation or an article, of an answer—or a question—at the right moment, without even considering all the existing mathematics that any new result builds upon? "The names of theorems are distributed like the names of streets," the French mathematician André Weil is reputed to have said. Doing justice to all those who have contributed to the development of wavelets is yet more difficult, since wavelets were developed independently in different fields. I could not, in this limited framework, mention all the researchers who are part of this story, and who deserve to find their names here. I apologize to them.

Acknowledgments

I could never have written this book without help from a great many people. Two mathematicians in particular made crucial contributions, one who is among the foremost wavelet researchers, another who works in an entirely different field. Yves Meyer spent countless hours, first in a series of interviews, or rather private lessons, and then reading the text, making many corrections and valuable suggestions. I will be forever indebted to him for his help, enthusiasm, and encouragement. The moral, material, and intellectual support of my husband, John Hubbard, was also essential. In addition to serving as translator and guide to mathematical concepts and formulas and providing technical assistance with illustrations and TeX, he kept me honest when I lapsed into fuzzy thinking and vague language. Although he claims not to believe me, his mathematical explanations gave me a real pleasure, for which I thank him too.

The wavelets researchers whom I asked for help responded with extraordinary generosity. Honoring Fourier's tradition, they displayed "an inexhaustible patience," whether explaining their subject in an interview or answering a question that appeared out of the blue on their computer screen. I have come to admire not just their professional competence, but also their ingenuity in finding simple words to explain technical matters, and the courtesy with which they spared my self-esteem. The pleasure I got in writing this book is largely due to them, and I cannot thank them enough.

I owe special thanks to Ingrid Daubechies for the care with which she read an earlier version of the text and for her corrections and suggestions, as well as for her lucid and prompt answers to many questions; Stéphane Mallat, who cleared up many troublesome points, helped me to avoid numerous pitfalls, and in particular insisted on a less magical portrayal of multiresolution theory (never complaining when, as a result, I deluged him with ill-formulated questions); Olivier Rioul, who showed me that I had misunderstood certain things and helped me to understand them with such good will that I gladly pardoned him the frustration his honesty originally caused me; and Victor Wickerhauser for his endless good will and good humor. While answering countless questions, they helped to keep up my spirits during long months when I was all too aware that I was out of my depth.

It's also a pleasure to thank David Donoho for his patient answers and useful suggestions; Marie Farge, who when I first arrived in her office, thoroughly confused, put her work aside and spent two hours explaining things from the beginning; David Field for a glimpse into how wavelets were developed and are used outside mathematics and signal processing; Michael Frazier for his valiant if doomed attempt at Irvine to teach me everything I ever wanted to know about harmonic analysis in two hours, as well as for verifying certain formulas; Leonard Gross for his suggestions and corrections concerning quantum mechanics; Alex Grossmann, whose gift for metaphor enlivened a very instructive interview; Jean Morlet for a very interesting historical perspective; and Robert Strichartz for his explanations of function spaces and his advice during final revisions.

Others provided assistance by mail or e-mail for which I would like to thank them: Edward Adelson, Christophe d'Alessandro, Michel Barlaud, Jonathan Berger, Gregory Beylkin, Ronald Coifman, Karen De Valois, Russell De Valois, Ronald DeVore, Uriel Frisch, Jeffrey Geronimo, Eric Goirand, Dennis Healy, Shubha Kadambe, Richard Kronland-Martinet, Bradley Lucier, Gilbert Strang, Gene Switkes, Bruno Torrésani, Michael Unser, and Martin Vetterli.

The list, as Virginia Woolf wrote in the preface to *Orlando*, "... threatens to grow too long and is already far too distinguished. For while it rouses in me memories of the pleasantest kind it will inevitably wake expectations on the reader which the book itself can only disappoint." So I will conclude by thanking Stephen Mautner of the National Academy

Press, who first asked me to write about wavelets; Philippe Boulanger, who suggested expanding the original English article into a book in French and, when I told him I couldn't, insisted; Ralph Oberste-Vorth, Ricardo Oliva, and Dierk Schleicher for technical assistance; Yuval Fisher for help with illustrations; my brother, Douglas Burke, for pointing out sections that were unclear; Fred Kochman for helpful comments; T.W. Körner for his book *Fourier Analysis*, which contains stories that are both instructive and entertaining; the research staff at Olin Library, Cornell University; and SIAM for permission to reprint a figure. I would also like to thank my children, Alex, Eleanor, Judith, and Diana Hubbard, for helping to avert complete disaster to house and garden during the past few months, and Judith and Diana for waiting so patiently for promised dance and horseback riding lessons.

Much of the work was done in France while my husband worked at the Institut des Hautes Etudes Scientifiques in Bures-sur-Yvette; I would like to thank the institute for its hospitality, and especially Françoise Schmit, who helped me find historical documents and research papers. Last but not least I wish to thank all those at A K Peters for what I consider one of the most remarkable transformations, turning a manuscript into a book: in particular, Alice and Klaus Peters, Alexandra Benis, Iris Kramer-Alcorn, and Erin Miles.

A number of those who helped with the first edition did so again with this second edition. In particular, I would like to thank Stéphane Mallat for his explanations and for giving me the opportunity to read his own wavelet book before it was published, Ted Adelson, Jonathan Berger, Gregory Beylkin, Michael Frazier, Leonard Gross, Michael Unser, and Victor Wickerhauser.

In addition, the following people contributed to the second edition through their suggestions, corrections, or responses to inquiries: Felix Abramovich, Jean-Pierre Antoine, Norman Bleistein, Christopher Brislawn, Sidney Burrus, Roberto Marcondes Cesar Jr., Richard Clark, Catherine Cooper, Luciano da F. Costa, Jean-Philippe Couderc, J. J. Duistermaat, Piotr Durka, Kevin Englehart, Gabriel Fernández, Fernando Gomez, Amara Graps, Peter Heller, Felix Herrmann, Xiaoming Huo, Harm Jonker, Gerald Kaiser, Frank Kurth, Tom Hopper, Irving Kaplansky, Tom Lane, Lars Lippert, François Meyer (who very kindly provided the brushlet illustration, as well as a clear explanation of his work), Colm Mulcahy, Ole M. Nielsen, Laurent Nottale, George Papanicolaou, Mary E. Patterson, Kannan Ramchandran, Vladimir Rokhlin, Mary Beth Ruskai, Matthias Schwab, Sergio Servetto, Jerry Shapiro, Eero Simoncelli, Wim Sweldens, Carl Taswell, Geert Uytterhoeven, and Hans van den Berg. I thank them for their help and apologize to anyone whose name was inadvertently omitted.

It should not be necessary to add that none of these people bears any responsibility for mistakes or omissions. Any virtues of this book have many authors, but its shortcomings are mine alone—especially as I did not always follow the good advice I was given. Since I wanted to write a book that would be accessible to readers with minimal mathematical background, I have chosen at times to follow my own intuition, reasoning that I am, regrettably, well placed to understand the needs of the mathematically innocent. The distinguished researchers who helped me may find it embarrassing to be associated with a text that is sometimes very elementary; I apologize to them for any discomfort and assure the reader that the unusual range of mathematical sophistication in this book is entirely my responsibility.

PART I

The World According to Wavelets

Mathematical Analysis

... mathematical analysis is as extensive as nature itself; it defines all perceptible relations, measures time, spaces, forces, temperatures; this difficult science is formed slowly, but it preserves every principle which it has once acquired....

Its chief attribute is clearness; it has no marks to express confused notions. It brings together phenomena the most diverse and discovers the hidden analogies which unite them. If matter escapes us, as that of air and light, by its extreme tenuity, if bodies are placed far from us in the immensity of space, if man wishes to know the aspect of the heavens at successive epochs separated by a great number of centuries, if the actions of gravity and of heat are exerted in the interior of the earth at depths which will be always inaccessible, mathematical analysis can yet lay hold of the laws of these phenomena. It makes them present and measurable, and seems to be a faculty of the human mind destined to supplement the shortness of life and the imperfection of the senses; and what is still more remarkable, it follows the same course in the study of all phenomena; it interprets them by the same language, as if to attest to the unity and simplicity of the plan of the universe....

—Joseph Fourier, *The Analytical Theory of Heat*, pp. 7–8

Prologue

Mathematician Michael Frazier of Michigan State University was taught that "real" mathematics by "real" mathematicians is and should be useless. "I never expected to do any applications—I was brought up to believe I should be proud of that," he says. "You did pure harmonic analysis for its own sake, and anything besides that was impure, by definition." But in the summers of 1990 and 1991, he found himself using a mathematical construction to pick out the pop of a submarine hull from surrounding ocean noise.

In St. Louis, Victor Wickerhauser was showing the FBI how the same mathematics might be used to store fingerprints more economically, while at Yale, Ronald Coifman used it to coax a battered recording of Brahms playing the piano into yielding its secrets. In France, Yves Meyer of the University of Paris-Dauphine found himself talking to astronomers about how these new techniques could be used to study the large-scale structure of the universe.

Over the past few years, a number of mathematicians used to the abstractions of pure research have been dirtying their hands—with enthusiasm—on a surprising range of practical projects. What these disparate tasks have in common is a new mathematical language, its alphabet consisting of undulations called wavelets, appropriately stretched, squeezed, or moved about.

A whole range of information—your voice, your fingerprints, a snapshot, x-rays ordered by your doctor, radio signals from outer space, seismic waves—can be translated into this new language, which emerged independently in a number of different fields; in fact, it was only recently understood to be a single language. In many cases this transformation into wavelets makes it easier to transmit, compress, and analyze information or to extract information from surrounding "noise"—even to do faster calculations.

Wavelets quickly won an impressive number of converts. When Olivier Rioul began his doctoral thesis at the Ecole Nationale Supérieure des Télécommunications in Paris in 1989, "wavelets were still a marginal subject in signal processing, as far as the scientific community at large was concerned...." Three years later one could no longer keep count of "the number of researchers publishing in the field,... of theses devoted to it, or of books that have appeared or are about to appear...." [Rioul, pp. 1–2].

In their initial excitement some researchers even thought that wavelets might virtually supplant the much older and very powerful mathematical language of Fourier analysis, used every time you talk on the telephone or turn on a television. But now they see the two as complementary and are exploring ways to combine them, or even to create languages beyond wavelets.

Different languages have different strengths and weaknesses, points out Meyer, one of the founders of the field. "French is effective for analyzing things, for precision, but bad for poetry and conveying emotion—perhaps that's why the French like mathematics so much. I'm told by friends who speak Hebrew that it is much more expressive of poetic images. So if we have information, we need to think, is it best expressed in French? Hebrew? English? The Lapps have 15 different words for snow, so if you wanted to talk about snow, that would be a good choice."

In data processing, some tasks are best tackled with Fourier transforms, others with wavelets; yet other tasks may require new languages. For the first time in a great many years—almost two centuries, if one goes back to the very birth of Fourier analysis—there is a choice.

CHAPTER I

Fourier Analysis: A Poem Transforms Our World

Although wavelets represent a departure from Fourier analysis, they are also a natural extension of it: the two languages clearly belong to the same family. The history of wavelets thus begins with the history of Fourier analysis. In turn, the roots of Fourier analysis predate Fourier himself (and much of what is now called Fourier analysis is due to his successors). But Fourier is a logical starting point; his influence on mathematics, science, and our daily lives has been incalculable, if to many people invisible. Yet he was not a professional mathematician or scientist; he fit these contributions into an otherwise very busy life.

His father's 12th child, and his mother's ninth, Jean Baptiste Joseph Fourier was born in 1768 in Auxerre, a town roughly halfway between Paris and Dijon. His mother died when he was nine, and his father died the following year. Although two younger siblings were abandoned to a foundling hospital after their mother's death, Fourier continued school and in 1780 entered the Royal Military Academy of Auxerre, where at age 13 he became fascinated by mathematics and took to creeping down at night to a classroom where he studied by candlelight.

"It is said that during the day he secretly collected candle stubs, and that at night, when everyone was asleep, he woke up and crept down to the classroom, lit the candles, and spent long hours working on mathematical problems," related V. Cousin in his *Notes biographiques pour faire suite à l'éloge de M. Fourier*, published in 1831. Fourier's academic success attracted the attention of the local bishop. But when at the end of his studies his application to join the artillery or the army engineers was rejected, he entered the abbey of St. Benoit-sur-Loire. (The popular story that he was rejected by the army because he was not of noble birth—and therefore ineligible "even if he were a second Newton"—is questioned by at least two of Fourier's contemporaries [Cousin, p.2]).

The French Revolution erupted before he took his vows. At first indifferent, he became increasingly committed to the cause of establishing "a free government exempt from kings and priests" [Fourier 2], and in 1793 joined the revolutionary committee of Auxerre. Twice he was arrested, once in the bloody days shortly before the fall of Robespierre and again in June 1795, on charges of terrorism, when he was roused from his bed and scarcely given time to get dressed. As he and his guards left his home, the concierge told Fourier she hoped he would soon be freed; according to Cousin, he never forgot the guard's reply: "You can come get him yourself—in two pieces."

Defending himself, Fourier pointed out that during the Terror no one in Auxerre was condemned to death; "no family in Auxerre," he wrote, "had to mourn a father or a relation" [Fourier 2]. Cousin even relates that once, to prevent a man he believed to be innocent from arrest and the guillotine, Fourier invited the agent charged with the arrest to lunch at an inn, and "having exhausted every means of retaining his guest voluntarily, left the room on a pretext, quietly locked the door, and ran to warn the man threatened by such imminent danger," returning later with excuses.

After the Revolution, Fourier taught in Paris, then accompanied Napoleon to Egypt and served as permanent secretary of the Institute of Egypt. Later he wrote a book on Egypt; even today some people know him as an Egyptologist and are unaware of his contributions to mathematics and physics.

On Fourier's return to France in 1802, Napoleon appointed him prefect of the department of Isère. He served as prefect for 14 years, in Grenoble, earning a reputation as an able administrator; one of his accomplishments was to persuade 37 communes to work together to drain some twenty thousand acres of swamps that had caused annual epidemics of fever, a task that, according to Cousin, required all his tact and "an inexhaustible patience." After Waterloo, denied a government pension because he had served under Napoleon, he found a safe haven in the Bureau of Statistics in Paris, and in 1817 (after an initial rebuff by King Louis XVIII) he was elected to the Academy of Sciences.

A Mathematical Poem

Despite administrative duties and his isolation from Paris for many years, Fourier managed to pursue his scientific and mathematical interests. Victor Hugo called him a man "whom posterity has forgotten" [Hugo], but Fourier's name is as familiar to countless scientists, mathematicians, and engineers as the names of their own children. This fame rests on ideas he set forth in a memoir in 1807 and published in 1822 in his book, *La Théorie Analytique de la Chaleur* (*The Analytic Theory of Heat*).

Physicist James Clark Maxwell called Fourier's book "a great mathematical poem" [Maxwell], but the description does not begin to give an idea of its influence. In the seventeenth century, Isaac Newton had a new insight: that forces are simpler than the motions they cause, and the way to understand the natural world is to use differential and partial differential equations to describe these forces ("perhaps," said Albert Einstein, "the greatest intellectual stride that it has ever been granted to any man to make" [Einstein, p. 68].)

Newton's differential equation showing how the gravitational pull between two objects is determined by their mass and the distance between them replaced countless observations; predictive science became possible, leading the French mathematician and astronomer Pierre Simon Laplace to imagine a single formula that would describe the motions of every object in the universe, for all time.

"An intelligence that knew, for a given instant, all the forces that animate nature and the respective states of everything that composes it, and that in addition was vast enough to submit these facts to analysis, would encompass in the same formula all the movements of the largest bodies of the universe and those of the lightest atom," he wrote, a century after Newton [Laplace, p. vi.]. "Nothing would be uncertain to it, and the future, like the past, would be present before its eyes."

This optimism foundered on the shoals of reality. Solving differential equations—actually predicting where we will be taken by forces that themselves depend at each moment on our changing position—is not easy. As Fourier wrote, although the equations that describe the propagation of heat have a very simple form,

> ...the known methods do not furnish any general mode of integrating them; we could not therefore deduce from them the values of the temperatures after a definite time. The numerical interpretation ... is however necessary.... So long as it is not obtained, the solutions may be said to remain incomplete and useless, and the truth which it is proposed to discover is no less hidden in the formulae of analysis than it was in the physical problem itself. We have applied ourselves with much care to this purpose, and we have been able to overcome the difficulty in all the problems of which we have treated [Fourier 1, p. 21].

Some 150 years after Newton, Fourier provided a practical way to extract the truth from a whole class of such equations: linear partial differential equations.

The reaction of his contemporaries was less enthusiastic than he might have hoped; his memoir was awarded the grand prize for mathematics in 1812, but with the comment that "his analysis ... leaves something to be desired in regards to its generality, and even from the point of view of rigor" [Carslaw, p. 7]. The judgment of posterity has been more generous. The English physicist Lord Kelvin found it "difficult to say whether [Fourier's results'] uniquely original quality, or their transcendently intense mathematical interest, or their perennially important instructiveness for physical science, is most to be praised" [Thomson, p. 578].

His ideas dominated mathematical analysis for a hundred years, having surprising ramifications even for number theory and probability. Outside mathematics their influence is difficult to exaggerate. Virtually every time scientists or engineers model systems or make predictions, they use Fourier analysis. Fourier's ideas are used in linear programming, crystallography, and in countless devices from telephones to radios and hospital x-ray machines. They are, in mathematician T.W. Körner's words [Körner, p. 221], "built into the commonsense of our society."

A Rabble of Functions

There are two parts to Fourier's contribution: first, a mathematical statement (actually proved later by Dirichlet); and second, an explanation of why this statement is useful. The mathematical statement is that any periodic function can be represented as a sum of sines and cosines—what is now called a *Fourier series*. Roughly, what this means is that any curve that periodically repeats itself, no matter how jagged or irregular, can be expressed as the sum of perfectly smooth oscillations (sines and cosines), as shown in Figure 1. The irregular curve and the sum of sines and cosines are two different representations of the same object in different "languages."

The trick is to multiply the sines and cosines by a coefficient to change their amplitude (the height of their waves) and to shift them so that they either add or cancel (changing the phase). Certain nonperiodic functions (those that decrease fast enough so that the area under their graphs is

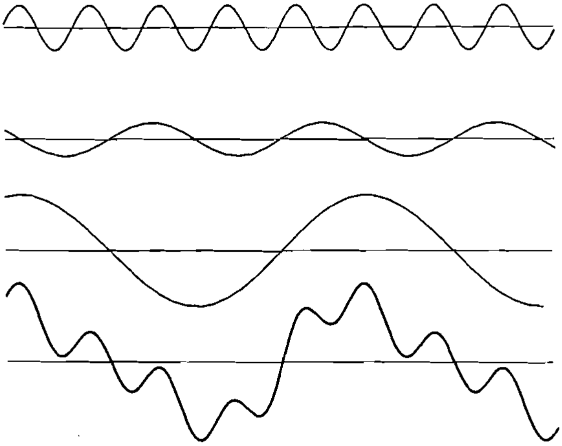

Figure 1. In 1807 Fourier showed that virtually any periodic function can be expressed as the sum of a series of sines and/or cosines. Here the bottom function is composed of the three functions above it. Fourier's realization contributed to a wrenching change in the way mathematicians thought of functions. It provided a straightforward way to solve certain differential equations, and paved the way for digital technology, including computers and compact disks. Fourier analysis is also a natural language for quantum mechanics.

finite) can also be treated this way, using the *Fourier transform.* From the Fourier series, or the Fourier transform, one can reconstruct the original function; no information is lost in translating from one language to the other.

Beyond Plain English 1

The Fourier Transform

The Fourier transform is a mathematical prism, breaking up a function into the frequencies that compose it, as a prism breaks up light into colors. It transforms a function f that depends on time (or on space) into a new function, $\hat{f}$, or "f hat," which depends on frequency. This new function is called the *Fourier transform* of the original function—or, when the original function is periodic, its *Fourier series.*

A function and its Fourier transform are two faces of the same information. The function displays the time (or space) information and hides the information about frequencies. The Fourier transform displays information about frequencies, but information about time or space is hidden in the *phases*: the displacement of the sines and cosines for each frequency, so that they add or subtract. The Fourier transform of music tells what notes (frequencies) are played, but it is virtually impossible to discern when the notes are played.

The Fourier series of a periodic function concerns only those sines and cosines that are integer multiples of the base frequency: for example, $\sin 2\pi k$, $\sin 2\pi 2k$... To make the Fourier transform of a (nonperiodic) function that decreases fast enough at infinity we must compute coefficients of all possible frequencies.

See p. 117.

Fourier himself found this statement "quite extraordinary," and it met with some hostility. Mathematicians were used to functions whose graphs are regular curves; the function $f(x) = x^2$, for example, produces a well-behaved, symmetrical parabola. (A *function* gives a rule for changing an arbitrary number into something else; the function $f(x) = x^2$ says to square any number x: if $x = 2$, then $f(x) = 4$.) The idea that an arbitrary periodic curve could be expressed as a series of sines and cosines and thus treated as a function came as a shock and contributed to a profound and sometimes disturbing change in mathematics. As we shall see in *Traveling from One Function Space to Another*, p. 223, mathematicians spent much of the nineteenth century coming to terms with these changes; eventually they were forced to admit as functions even some mathematical objects that are too wild to be graphed or imagined, but which translate into perfectly civilized Fourier series. "...we have seen a rabble of functions arise whose only job, it seems, is to look as little as possible like decent and useful functions," lamented the French mathematician Henri Poincaré in 1889 [Poincaré 1, pp. 130–131]. "No more continuity, or perhaps continuity but no derivatives.... Yesterday, if a new function was invented it was to serve some practical end; today they are specially invented only to show up the arguments of our fathers, and they will never have any other use."

Poincaré (who also wrote that Fourier's book "was of paramount importance in the history of mathematics and that pure analysis perhaps owed it even more than applied analysis" [Poincaré 2, p. 1]), worried about the effect these bizarre functions would have on a beginning student: "What will the poor student think? He will think that mathematical science is just an arbitrary heap of useless subtleties; either he will turn from it in aversion, or he will treat it like an amusing game...."

Ironically, Poincaré himself was ultimately responsible for showing that seemingly "pathological" functions are essential in describing nature (leading to such fields as chaos and fractals), and this new direction for mathematics proved enormously fruitful, giving new vigor to a discipline that some had found increasingly anemic, if not moribund.

In 1810 the French astronomer Jean-Baptiste Delambre (who had measured, with Pierre Méchain and under most difficult conditions, the arc of meridian between Dunkerque and Barcelona to establish the length of the new meter) had issued a report on mathematics expressing the fear that "the power of our methods is almost exhausted" [Delambre, p. 125]. Some 30 years earlier, Lagrange, then 45 years old, wrote to his friend d'Alembert that he wasn't sure he would be doing mathematics in another ten years. "It seems to me that the mine is already too deep, and that unless we find new veins, sooner or later we will have to abandon it.... It is not impossible that the mathematical positions in the academies will one day become what the university chairs in Arabic are now" [Lagrange].

"Looking back," writes Körner [Körner, p. 474], we can see Fourier's memoir "as heralding the surge of new mathematical methods and results which were to mark the new century."

Beyond Plain English 2

The Convergence of Fourier Series and the Stability of the Solar System

Virtually every periodic function can be represented as a series, or sum, of sines and cosines, but not every series of sines and cosines represents a function. If a series can be proved to converge, one can work with a finite number of terms, confident that adding more terms won't significantly change results. But if the coefficients of the series do not become small fast enough, the series diverges and does not represent a function.

Questions of convergence and divergence have provided a great deal of work for mathematicians. A particularly difficult problem is determining whether a series with an infinite number of small divisors is convergent. The great nineteenth-century German mathematician Karl Weierstrass struggled for years in an unsuccessful attempt to prove the stability of the solar system by proving the convergence of a series with small divisors. The problem was finally solved in the 1960s.

See p. 125.

The Explanation of Natural Phenomena

The German mathematician Carl Jacobi wrote that Fourier believed the chief goal of mathematics to be "the public good and the explanation of natural phenomena" [Jacobi, p. 454]. Fourier showed how his mathematical statement could be used to study natural phenomena such as heat diffusion, making it possible to numerically solve equations that had until then remained intractable. For a very important class of differential equations, the Fourier transform turns a difficult differential equation into a series of simple equations.

Suppose, for example, we want to predict the temperature at time t of each point along a metal bar that is cooling. As shown in Figure 2, we start by establishing the initial temperature—at time "zero"—which we consider as a function of distance along the bar. The differential equation that describes how the temperature varies doesn't look particularly forbidding, but it has two independent variables, time and distance; before Fourier, no one knew how to tackle it. But when that function is translated

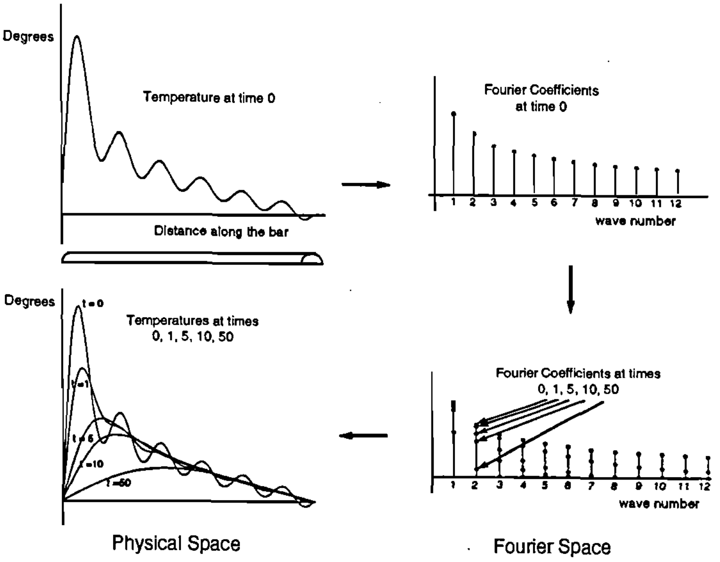

Figure 2. Different Equations in Fourier Space. To determine the temperature at time t of a metal bar that is cooling, one starts by measuring the bar's initial temperature (at time 0), representing it as a function f that depends on space (distance along the bar). Next one goes into Fourier space, calculating its Fourier transform $\hat{f}$, which tells us the coefficient for each wave number making up the function f at time 0. (For a function that depends on time, we would say frequency rather than wave number.) The Fourier coefficients at time 0, $c_n(0)$ (where n is the wave number, 1, 2, 3...), are given by the formula $c_n(0) = \frac{1}{\sqrt{n}}$. The coefficients at time t are computed with the formula

$$c_n(t) = c_n(0)\, e^{n^2 t/100} = \frac{e^{-n^2 t/100}}{\sqrt{n}}$$

We will consider here the coefficients for times $t = 1$, 5, 10, and 50. For each such time, the coefficients are the same for the entire bar: the information on space seems to have disappeared. But it reappears when we return to physical space: when we "invert" (undo) the Fourier transform $\hat{f}_t$, we obtain f_t, which gives the temperature for each point of the bar at time t.

into a Fourier series, a remarkable thing happens: this intractable differential equation *decouples*, becoming a series of independent differential equations, one for each sine or cosine making up the original function.

Each equation tells how the Fourier coefficient of one particular sine or cosine varies with time. (The coefficients, we will recall, are the numbers by which we multiply the sines and cosines of different frequencies to make them taller or shorter; they indicate how much of each frequency a function contains.) The equations are moreover very simple—the same as the equation that gives the value of a bank account earning compound interest (negative interest in this case).

One by one, we simply plug into the solutions of these equations the coefficients describing the temperature at time zero, and then we crank out the answers; these are the Fourier coefficients of the temperature at time t. Now we use these new coefficients to construct a new function giving the new temperature at each point on the bar. The procedure is no harder than the one your bank uses to compute the balance in your account each month.

Essentially we have made a little detour in Fourier space, where our calculations are immensely easier—as if, faced with the problem of multiplying the Roman numerals LXXXVI and XLI, we translated them into Arabic numerals to calculate 86 x 41 = 3526 and then translated the answer back into Roman numerals: LXXXVI × XLI = MMMDXXVI.

This example shows why Fourier needed a technique that could be applied to any function, even discontinuous; he couldn't expect the initial temperature to be so obliging as to take the form of a regular curve. "In order that these solutions might be general, and have an extent equal to that of the problem, it was requisite that they should accord with the initial state of the temperatures, which is arbitrary," he wrote [Fourier 1, p. 22].

The Public Good

The techniques Fourier invented have had an impact well beyond studies of heat, or even solutions to differential equations. Real data tend to be very irregular. Consider an electrocardiogram, or the readings of a seismograph. Such signals often look like "complicated arabesques," to use Yves Meyer's expression [Meyer 3, p. 2]—tantalizing curves that contain all the information of the signal but hide it from our comprehension. (We will speak of *signals* to be analyzed and of *functions*, such as wavelets, to analyze signals, but mathematically, signals are also functions.)

Fourier analysis translates these signals into a form that makes sense, transforming a signal that varies with time (or in some cases, with space) into a new function, the *Fourier transform* of the signal, which tells how much of each frequency the signal contains (more precisely, how much of the sine and cosine of each frequency it contains).

In many cases these frequencies are not simply a mathematical trick to make calculations easier; they correspond to the frequencies of the actual physical waves making up the signal. When we listen to music or conversation, we hear changes in air pressure caused by sound waves; high sounds have high frequency, with waves close together, and low sounds have lower frequency, the waves more spread out. (A piano can perform a kind of Fourier analysis: a loud sound near a piano with the damper off will cause certain strings to vibrate, corresponding to the different frequencies making up the sound.)

Similarly—although this was not known in Fourier's time—radio waves, microwaves, infrared, visible light, and x-rays are all electromagnetic waves differing only in frequency. Being able to break down sound waves and electromagnetic waves into frequencies has myriad uses, including tuning your radio to your favorite station, interpreting radiation from distant galaxies, using ultrasound to check the health of a developing fetus, and making cheap long-distance telephone calls.

With the discovery of quantum mechanics, it became clear that Fourier analysis is the language of nature itself. On the "position space" side of the Fourier transform, one can talk about an elementary particle's position; on the other side, in "Fourier space," one can talk about its momentum or think of it as a wave. The modern realization that matter at very small scales behaves differently from matter on a human scale—that an elementary particle does not simultaneously have a precise position and a precise momentum—is a natural consequence of Fourier analysis.