

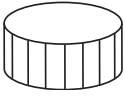
JOHN H. CONWAY  
DEREK A. SMITH

# ON QUATERNIONS AND OCTONIONS

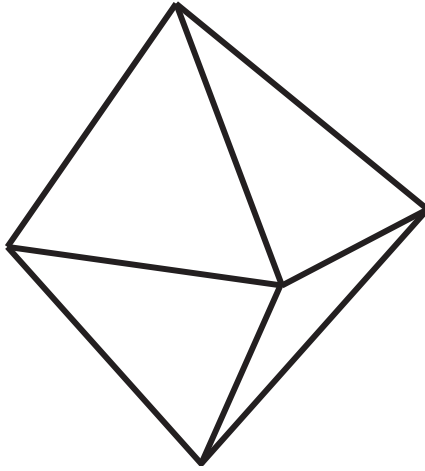


THEIR  
GEOMETRY,  
ARITHMETIC,  
AND SYMMETRY

 **CRC Press**  
Taylor & Francis Group  
AN A K PETERS BOOK



## On Quaternions and Octonions



# On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry

John H. Conway  
Derek A. Smith



**CRC Press**

Taylor & Francis Group

Boca Raton London New York

---

CRC Press is an imprint of the  
Taylor & Francis Group, an **informa** business

AN A K PETERS BOOK

CRC Press  
Taylor & Francis Group  
6000 Broken Sound Parkway NW, Suite 300  
Boca Raton, FL 33487-2742

© 2003 by Taylor and Francis Group, LLC  
CRC Press is an imprint of Taylor & Francis Group, an Informa business

No claim to original U.S. Government works  
Printed on acid-free paper  
Version Date: 20131202

International Standard Book Number-13: 978-1-56881-134-5 (Hardcover)

This book contains information obtained from authentic and highly regarded sources. Reprinted material is quoted with permission, and sources are indicated. A wide variety of references are listed. Reasonable efforts have been made to publish reliable data and information, but the author and the publisher cannot assume responsibility for the validity of all materials or for the consequences of their use.

Except as permitted by U.S. Copyright law, no part of this book may be reprinted, reproduced, transmitted, or utilized in any form by any electronic, mechanical, or other means, now known or hereafter invented, including photocopying, microfilming, and recording, or in any information storage or retrieval system, without written permission from the publishers.

For permission to photocopy or use material electronically from this work, please access [www.copyright.com](http://www.copyright.com) (<http://www.copyright.com/>) or contact the Copyright Clearance Center, Inc. (CCC) 222 Rosewood Drive, Danvers, MA 01923, 978-750-8400. CCC is a not-for-profit organization that provides licenses and registration for a variety of users. For organizations that have been granted a photocopy license by the CCC, a separate system of payment has been arranged.

**Trademark Notice:** Product or corporate names may be trademarks or registered trademarks, and are used only for identification and explanation without intent to infringe.

---

#### Library of Congress Cataloging-in-Publication Data

---

Conway, John Horton.

On quaternions and octonions / John H. Conway, Derek A. Smith.

p. cm.

ISBN 978-1-56881-134-5

I. Quaternions. 2. Cayley numbers. I. Smith, Derek Alan, 1970-. II. Title.

QA196.C66 2002

512'.5-dc21

2002035555

---

Visit the Taylor & Francis Web site at  
<http://www.taylorandfrancis.com>

and the CRC Press Web site at  
<http://www.crcpress.com>

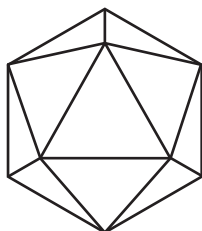
This book is dedicated to Lilian Smith and Gareth Conway,  
without whom we would have finished this book much sooner.



**Taylor & Francis**

Taylor & Francis Group

<http://taylorandfrancis.com>



# Contents

Preface	xii
<b>I The Complex Numbers</b>	<b>1</b>
1 Introduction	3
1.1 The Algebra $\mathbb{R}$ of Real Numbers	3
1.2 Higher Dimensions	5
1.3 The Orthogonal Groups	6
1.4 The History of Quaternions and Octonions	6
<b>2 Complex Numbers and 2-Dimensional Geometry</b>	<b>11</b>
2.1 Rotations and Reflections	11
2.2 Finite Subgroups of $GO_2$ and $SO_2$	14
2.3 The Gaussian Integers	15
2.4 The Kleinian Integers	18
2.5 The 2-Dimensional Space Groups	18
<b>II The Quaternions</b>	<b>21</b>
<b>3 Quaternions and 3-Dimensional Groups</b>	<b>23</b>
3.1 The Quaternions and 3-Dimensional Rotations	23
3.2 Some Spherical Geometry	26

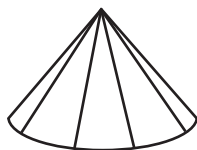


3.3	The Enumeration of Rotation Groups . . . . .	29
3.4	Discussion of the Groups . . . . .	30
3.5	The Finite Groups of Quaternions . . . . .	33
3.6	Chiral and Achiral, Diploid and Haploid . . . . .	33
3.7	The Projective or Elliptic Groups . . . . .	34
3.8	The Projective Groups Tell Us All . . . . .	35
3.9	Geometric Description of the Groups . . . . .	36
	Appendix: $v \rightarrow \bar{v}qv$ Is a Simple Rotation . . . . .	40
<b>4</b>	<b>Quaternions and 4-Dimensional Groups</b>	<b>41</b>
4.1	Introduction . . . . .	41
4.2	Two 2-to-1 Maps . . . . .	42
4.3	Naming the Groups . . . . .	43
4.4	Coxeter's Notations for the Polyhedral Groups . . . . .	45
4.5	Previous Enumerations . . . . .	48
4.6	A Note on Chirality . . . . .	49
	Appendix: Completeness of the Tables . . . . .	50
<b>5</b>	<b>The Hurwitz Integral Quaternions</b>	<b>55</b>
5.1	The Hurwitz Integral Quaternions . . . . .	55
5.2	Primes and Units . . . . .	56
5.3	Quaternionic Factorization of Ordinary Primes . . . . .	58
5.4	The Metacommutation Problem . . . . .	61
5.5	Factoring the Lipschitz Integers . . . . .	61
<b>III</b>	<b>The Octonions</b>	<b>65</b>
<b>6</b>	<b>The Composition Algebras</b>	<b>67</b>
6.1	The Multiplication Laws . . . . .	68
6.2	The Conjugation Laws . . . . .	68
6.3	The Doubling Laws . . . . .	69
6.4	Completing Hurwitz's Theorem . . . . .	70
6.5	Other Properties of the Algebras . . . . .	72
6.6	The Maps $L_x$ , $R_x$ , and $B_x$ . . . . .	73
6.7	Coordinates for the Quaternions and Octonions . . . . .	75
6.8	Symmetries of the Octonions: Diassociativity . . . . .	76
6.9	The Algebras over Other Fields . . . . .	76
6.10	The 1-, 2-, 4-, and 8-Square Identities . . . . .	77
6.11	Higher Square Identities: Pfister Theory . . . . .	78
	Appendix: What Fixes a Quaternion Subalgebra? . . . . .	80

7	Moufang Loops	83
7.1	Inverse Loops . . . . .	83
7.2	Isotopies . . . . .	84
7.3	Monotopies and Their Companions . . . . .	86
7.4	Different Forms of the Moufang Laws . . . . .	88
8	Octonions and 8-Dimensional Geometry	89
8.1	Isotopies and $SO_8$ . . . . .	89
8.2	Orthogonal Isotopies and the Spin Group . . . . .	91
8.3	Triality . . . . .	92
8.4	Seven Rights Can Make a Left . . . . .	92
8.5	Other Multiplication Theorems . . . . .	94
8.6	Three 7-Dimensional Groups in an 8-Dimensional One . . . . .	95
8.7	On Companions . . . . .	97
9	The Octavian Integers $\mathbb{O}$	99
9.1	Defining Integrality . . . . .	99
9.2	Toward the Octavian Integers . . . . .	100
9.3	The $E_8$ Lattice of Korkine, Zolotarev, and Gosset . . . . .	105
9.4	Division with Remainder, and Ideals . . . . .	109
9.5	Factorization in $\mathbb{O}^8$ . . . . .	111
9.6	The Number of Prime Factorizations . . . . .	114
9.7	“Meta-Problems” for Octavian Factorization . . . . .	116
10	Automorphisms and Subrings of $\mathbb{O}$	119
10.1	The 240 Octavian Units . . . . .	119
10.2	Two Kinds of Orthogonality . . . . .	120
10.3	The Automorphism Group of $\mathbb{O}$ . . . . .	121
10.4	The Octavian Unit Rings . . . . .	125
10.5	Stabilizing the Unit Subrings . . . . .	128
	Appendix: Proof of Theorem 5 . . . . .	131
11	Reading $\mathbb{O}$ Mod 2	133
11.1	Why Read Mod 2? . . . . .	133
11.2	The $E_8$ Lattice, Mod 2 . . . . .	135
11.3	What Fixes $\langle \lambda \rangle$ ? . . . . .	138
11.4	The Remaining Subrings Modulo 2 . . . . .	140



12 The Octonion Projective Plane $\mathbb{O}P^2$	143
12.1 The Exceptional Lie Groups and Freudenthal's "Magic Square" . . . . .	143
12.2 The Octonion Projective Plane . . . . .	144
12.3 Coordinates for $\mathbb{O}P^2$ . . . . .	145
Bibliography	149
Index	153



# Preface

This is a book on the geometry and arithmetic of the quaternion and octonion algebras. These algebras are intimately connected with special features of the geometry of the appropriate Euclidean spaces, which makes them a useful tool for understanding symmetry groups in low dimensions. For example, there is a special relationship between 3- and 4-dimensional groups that is clearly revealed by the quaternions because a 3-dimensional rotation can be specified by a single quaternion, and a 4-dimensional one by a pair of quaternions. The details are subtle, because certain maps are 2-to-1 instead of 1-to-1.

Many people are familiar with quaternions, so in the [first part](#) of our book we take their properties for granted and use them to enumerate the finite groups in 3 and 4 dimensions (following a similar treatment of 2-dimensional groups using complex numbers). We close the [first part](#) with a discussion of what geometry says about the arithmetic of Hurwitz's integral quaternions, and in particular establish the unique factorization theorem.

A major theme of the [second part](#) of our book is the remarkable “triviality symmetry” that arises in connection with the octonions. However, the properties of the octonions are not so familiar, so we start by proving the celebrated theorem of Hurwitz that  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$ , and  $\mathbb{O}$  are the only algebras of their kind, since this also yields the best way to construct them. Using the methods of the proof, we show that these algebras form Moufang loops, and we study Moufang loops in a way that exhibits the triality, applying this to the particular case of the 8-dimensional orthogonal group.

The study of the arithmetic of the integral octonions defined by Dickson, Bruck, and Coxeter has not progressed very far. In the final chapters of our book, we improve this situation in several ways. We use a method due to Rehm to create a new factorization theory for the integral octonions.

We also describe the action of their automorphism group on important subrings, finding maximal subgroups to be stabilizers of certain of these. In one case, we are led to read the integral octonions mod 2.

We close with a very brief chapter on the celebrated octonion projective plane.

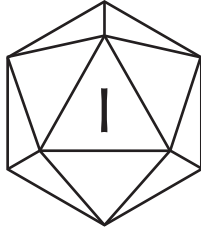
Very complicated arguments have been needed to deduce properties of the algebras over arbitrary fields from minimal hypotheses such as the alternative law. We have preferred to keep our arguments simple by starting from the composition property and restricting to the classical algebras over the real field.

The programmers of video games make heavy use of quaternions, as do the controllers of spacecraft, since in both these disciplines it is necessary to compose rotations with minimal computation. We have eschewed writing of these and other practical applications, which are amply treated in the recent book of Kuipers [29].

We thank John Baez, Warren Smith, Daniel Allcock, Warren Johnson, and Mohammed Abouzaid for their useful comments on various things in the book. Derek thanks the Academic Research Committee at Lafayette College for its financial support through a summer research fellowship. Alice and Klaus Peters have been the most helpful publishers one could imagine. We thank them, Jonathan Peters, Heather Holcombe, Darren Wotherspoon, Ariel Jaffe, and Susannah Peters for their work on this book. Finally, we thank our wives Barbara and Diana for their patience.

John Conway and Derek Smith

November 2002



# The Complex Numbers and Their Applications to 1- and 2-Dimensional Geometry

We suppose you understand the real numbers! The complex numbers are formal expressions  $x_0 + x_1i$  (with  $x_0, x_1$  real), combined by

$$(x_0 + x_1i) + (y_0 + y_1i) = (x_0 + y_0) + (x_1 + y_1)i$$

$$(x_0 + x_1i)(y_0 + y_1i) = (x_0y_0 - x_1y_1) + (x_0y_1 + x_1y_0)i;$$

that is to say, they constitute the algebra over the reals generated by a basic unit  $i$  that satisfies

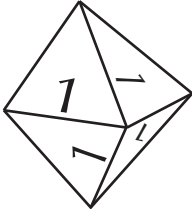
$$i^2 = -1.$$



**Taylor & Francis**

Taylor & Francis Group

<http://taylorandfrancis.com>



# Introduction

## 1.1 The Algebra $\mathbb{R}$ of Real Numbers

After the work of the ancient Greek and later geometers, it is customary to parametrize the Euclidean line by the algebra  $\mathbb{R}$  of real numbers. In this parametrization, the distance between  $a$  and  $b$  is the absolute value of their difference,  $|a - b|$ , which can also be written as  $\sqrt{(a - b)^2}$ . It is an important property of  $\mathbb{R}^1$  that  $|xy| = |x||y|$ .

The isometries (i.e., the maps that preserve distance) of  $\mathbb{R}^1$  are the

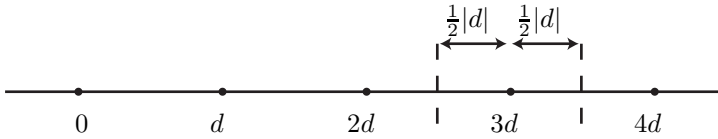
**translations**  $x \rightarrow k + x$  and **reflections**  $x \rightarrow k - x$ .

So the 1-dimensional general orthogonal group  $GO_1$  consists of the two isometries  $x \rightarrow \pm x$  that fix the origin.

The real numbers  $\mathbb{R}$  contain the rational numbers  $\mathbb{Q}$ , and in particular, the (rational) integers  $\mathbb{Z}$  that are the subject of number theory. In particular, they satisfy the unique factorization theorem (the essentials of which were also discovered by Euclid) whose traditional statement is that each positive integer is a product of (positive) prime numbers in a way that is unique up to order.

For the purposes of this book, it is better to delete the positivity requirement, when the statement becomes that each positive or negative integer is a product of positive or negative primes in a way that is unique up to order and sign change.

We briefly sketch the traditional proof, which makes essential use of the fact that any number  $n$  of  $\mathbb{Z}$  can be divided by any nonzero number  $d$  to leave a remainder  $r$  strictly smaller than the divisor. (In fact, we can make  $|r| \leq \frac{1}{2}|d|$ ; see [Figure 1.1.](#))



**Figure 1.1.** Maximal remainder for  $\mathbb{Z}$ .

An **ideal** of  $\mathbb{Z}$  is a subset  $\mathcal{I}$  with the following properties:

- $0 \in \mathcal{I}$
- The sum of any two elements in  $\mathcal{I}$  is in  $\mathcal{I}$
- All of the multiples of any member of  $\mathcal{I}$  by arbitrary integers are again in  $\mathcal{I}$

It is a **principal ideal** only if it consists of all multiples of some integer  $g$ , called the generator.

Then we have

**Lemma 1.** *Any ideal of  $\mathbb{Z}$  is principal.*

**Proof.**  $\{0\}$  is principal. If  $\mathcal{I} \neq \{0\}$ , let  $d$  be a non-zero element of  $\mathcal{I}$  with smallest absolute value. We show that  $\mathcal{I}$  must consist of just the multiples of  $d$ . For if  $\mathcal{I}$  contains any other integer, say  $n$ , we can write

$$n = qd + r, \quad \text{where } 0 \leq r < |d|.$$

But since  $n$  and  $d$  are in the ideal, so is  $r$  (of smaller absolute value than  $d$ ), a contradiction.

We use this to prove

**Lemma 2.** *If  $p$  is a prime number, then  $p$  divides a product if and only if it divides one of the factors.*

**Proof.** It will suffice to suppose that  $p$  divides  $ab$ . Then the ideal of numbers of the form  $mp + na$  must be a principal ideal whose generator  $g$  must be a divisor of  $p$ , and so can be chosen to be  $p$  or 1. But if  $g = p$ , then  $p$  divides  $a$ , while  $g = 1$  implies  $1 = mp + na$  for some  $m, n$ , and so  $b = mpb + nab$ , which is a multiple of  $p$ .

These lemmas combine to prove the theorem, for if

$$p_1 p_2 \dots \quad \text{and} \quad q_1 q_2 \dots$$

are two prime factorizations of the same number, we can deduce that  $q_1$  must divide one of the  $p_i$ , and so be that  $p_i$ —say  $p_1$ —up to sign. After cancelling  $p_1$ , we find that  $q_2$  must equal  $p_2$  (say) up to sign, etc.

## 1.2 Higher Dimensions

The results with which we have just opened [Chapter 1](#) concern 1-dimensional space  $\mathbb{R}^1$  and its sublattice  $\mathbb{Z}$ . It is the purpose of this book to generalize these results to certain higher dimensions.

[Chapter 2](#) handles the 2-dimensional case, where we discuss plane geometry in terms of the algebra  $\mathbb{C}$  of complex numbers, and say something about its two most famous arithmetics  $\mathbb{Z}[i]$  of Gaussian integers and  $\mathbb{Z}[\omega]$  of Eisenstein integers.

It is a remarkable fact that the analogous discussion of 3-dimensional geometry requires a 4-dimensional algebra, that of *quaternions*. This made them hard to discover, as is seen in the famous story Baez quotes later in this chapter.

In [Chapter 3](#) we define the quaternions  $\mathbb{H}$  and use them to enumerate the finite groups of 3-dimensional isometries. In [Chapter 4](#), we provide a similar service for 4-dimensional space, which is of course the natural setting for the quaternions.

The correspondence between chapters and dimensions is broken in the next few chapters. [Chapter 5](#) discuss what it means for a quaternion to be “integral.” We find that Hurwitz’s system of integers has a form of unique factorization, and so must be preferred to the more naive system of Lipschitz.

What other algebras have properties like those of the quaternions? It turns out that the most important property  $[xy] = [x][y]$  is the one that defines “composition algebras.” In [Chapter 6](#) we prove Hurwitz’s famous result that the only composition algebras are the famous ones in dimensions 1, 2, 4, and 8. It turns out that multiplication in the 8-dimensional composition algebra  $\mathbb{O}$  of *octonions* is not associative, but it satisfies the Moufang laws, an intriguing substitute for associativity. In [Chapter 7](#), we explain how the Moufang laws may be regarded as a symmetry condition.

[Chapter 8](#), analogous to [Chapter 4](#), discusses octonions and 8-dimensional geometry. We discuss Cartan’s wonderful “triviality” of  $PSO_8$ , and use it to prove our “7-multiplication theorem.”

[Chapter 9](#), analogous to [Chapter 5](#), explores the correct definition of integrality for octonions and then erects the correct factorization theory for it. The next [two chapters](#) study the automorphisms of the octavian integers in great detail. [Chapter 10](#) explores the units, and [Chapter 11](#) shows how more information can be obtained by working modulo 2.

Finally, [Chapter 12](#) uses the octonions to construct a most intriguing projective plane.