A Gardner's Workout

Training the Mind and Entertaining the Spirit

Martin Gardner

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To all the underpaid teachers of mathematics,
everywhere, who love their subject
and are able to communicate that love
to their students.

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Preface

For 25 years I had the honor and pleasure of writing the Mathematical Games*f* column in *Scientific American*. All those columns have now been reprinted, with updating, in fifteen volumes, starting with *The Scientific American Book of Mathematical Puzzles and Diversions* and ending with *Last Recreations*.

Since I stopped writing the column I have from time to time contributed articles and book reviews about mathematics to both academic journals and popular magazines. Forty-one of these pieces are gathered here. The most controversial is the final review in which I criticize a current teaching fad known as the new new Math.*f*

By the time this book is published I would guess and hope that new new math is being abandoned almost as rapidly as the old new math faded. I could be wrong. In any case, it may be decades before our public education is able to attract competent teachers who have learned how to teach math to pre-college students without putting them to sleep. There are, of course, many teachers who deserve nothing but praise. It is to them I have dedicated this book.

Martin Gardner
Hendersonville, NC

Part I

Chapter 1
The Opaque Cube

The Opaque Cube

I want to propose the following unsolved problem. As far as I know, I am the first to ask it.

⋆ ⋆ ⋆

> What is the minimal area of surfaces inside a transparent cube that will render it opaque?

⋆ ⋆ ⋆

By opaque I mean *if the surfaces are opaque, no ray of light, entering the cube from any direction, will pass through it.*

The answer may or may not be the *minimal surface* spanning the twelve edges of the cube. This question also is unanswered. See the discussion of it by Courant and Robbins ([1], Ch. 7).

That the minimal spanning surface may not be the answer to the opaque cube problem is suggested by the fact that the minimal Steiner tree spanning the four corners of a square is not the answer to the opaque square problem. The best known solution for the square (also not proved minimal) is shown in Figure 1. The square problem is discussed by Ross Honsberger ([2], p. 22).

My best solution for the cube is to join the center to all the corners. These lines outline 12 triangles with a combined area of $3\sqrt{2}$.

I believe the opaque cube problem to be extremely difficult. It is keeping me awake at night![1]

[1]The note appeared in the Dutch periodical *Cubism for Fun* (No. 23, March 1990, p. 15). I followed this with a second note, "The Opaque Cube Again," in the same periodical (No. 25, December 1990, Part 1, p. 14).

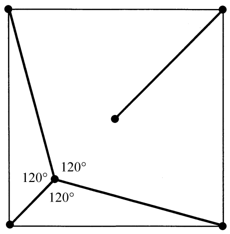

Figure 1. Total length = 2.639+.

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[1] R. Courant and H. Robbins. *What is Mathematics?*, 4th edition, Oxford University Press, 1947.

[2] R. Honsburger. Mathematical Morsels,ƒThe Mathematical Association of America, 1978.

The Opaque Cube Again

So far as I know, no one has yet proved that the solution for the opaque square is minimal, nor has anyone proved a minimal surface spanning the edges of a cube.

I refer to the problem posed in CFF 23 (March 1990), p. 15: Find the least-area surface that will block any light ray trying to pass through a unit cube.ƒ

From Stephen Harvey, Dunedin, New Zealand, I received a calculation of the area of a surface such as pictured in Figure 240 of Courant and Robbins *What Is Mathematics?*ƒ. This area ($A = 4.2425$) is slightly less than $3\sqrt{2}$.

From H.S.M. Coxeter, Toronto, Canada, I received a letter in which he shows that the spanning surface of Courant and Robbins cannot have straight edges, which makes this area very difficult to compute.

Even this surface can be improved slightly, however, as is shown by Kenneth Brakke, Mathematics Department, Susquehanna University, Selingsgrove, Pennsylvania. He has a marvelous computer program that searches for minimal surfaces. He found a minimal surface spanning the cube s edges that has an area of $A = 4.2324$. This is the best solution yet for the Opaque Cube.

Opaque Cubes by Kenneth Brakke

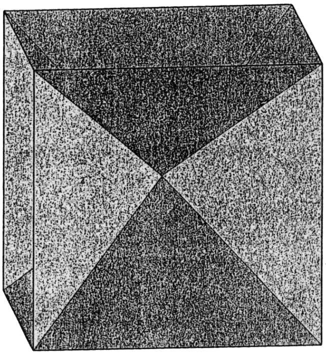

Figure 2. Twelve triangles from the edges to the centre. Area = $3\sqrt{2} \approx 4.2426$.

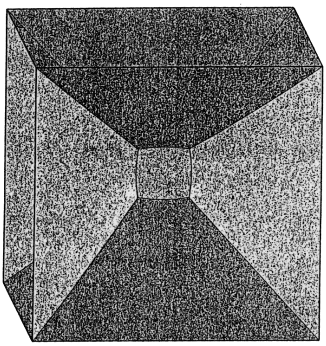

Figure 3. Soap film formed by dipping a cubical frame in a soap solution. Area ≈ 4.2398.

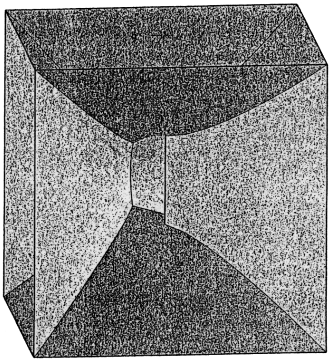

Figure 4. Generalization of opaque square solution to three dimensions. Topologically, this is the opaque square solution stretched vertically with the top and bottom faces of the cube added. Area ≈ 4.2343.

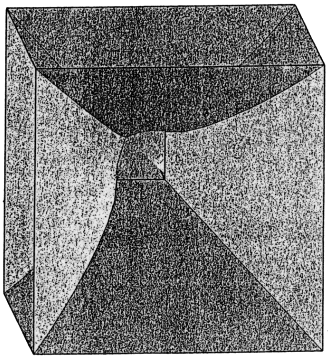

Figure 5. My best solution. This is like Figure 4, except it has three-fold symmetry in place of two-fold. Area ≈ 4.2324.

Kenneth Brakke will give a fuller treatment of his results elsewhere. Meanwhile he kindly allowed the editors of CFF to publish some of his computer printouts (see Figures 2¡ 5). If they inspire any reader to find a better solution, he would like to hear about it.

At the moment it seems unlikely that the minimum solution for the opaque cube will consist of (disconnected) surfaces that do not span the cube s edges with a single surface. However, this is far from proved.

Postscript

The picture of Brakke s best solution to the opaque cube, reproduced here as Figure 5, made the cover of *The American Mathematical Monthly* Vol. 99, (November 1992). The cover illustrated Brakke s paper, The Opaque Cube Problem,ƒ that ran in Richard Guy s Unsolved Problemsƒ column. Brakke also discusses the opaque sphere problem.

The opaque square, with its single Steiner point, obviously generalizes to opaque regular polygons. The limiting case, as the number of polygon sides increases, is, of course, the circle. The opaque circle is probably solved by a fenceƒ of length 2π that is the limit of an infinite series. The solver is Bernd Kawohl, a professor of mathematics at the University of Cologne s Mathematical Institute. His solution is given in The Opaque Square and the Opaque Circle,ƒ in the proceedings of a conference in Oberwolfach, reprinted in the *International Series of Numerical Mathematics*, Vol. 123 (1997), pp. 339¡ 346.

Figure 6 and 7 show the best known solutions for the opaque pentagon and hexagon. Note that each is totally lacking in symmetry. Kawohl s conjectured solution for the opaque circle is based on extrapolating from the polygonal cases. See also Kawohl s article Symmetry or Not?ƒ, in *The Mathematical Intelligencer*, Vol. 20 (1998), pp. 16¡ 22, in which both the square and circle cases are discussed.

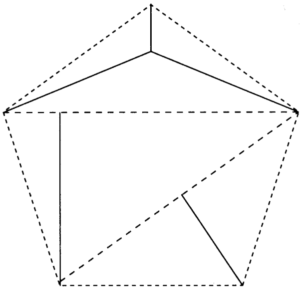

Figure 6. The opaque pentagon. Length ≈ 3.528.

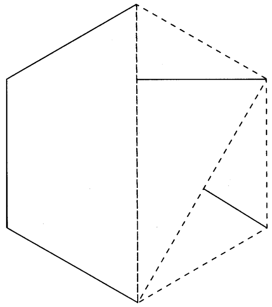

Figure 7. The opaque hexagon. Length = $(7+\sqrt{3})/2 \approx 4.366$.

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Chapter 2

The Square Root of 2 = 1.414 213 562 373 095 . . .

Roses are red,
Violets are blue.
One point 414 . . .
Is the square root of two.

I confess that I wrote the above jingle only to have some light verse top this article. The dots at the end of the third line indicate that the decimal fraction is endless and nonrepeating. In other words, $\sqrt{2}$ is irrational. Although its decimal digits, like those of other famous irrationals such as π and e, *look* like a sequence of random digits, they are far from random because if you know what the number is you can always calculate the next digit after any break in the sequence. Such irrationals also should not be called patternlessƒ because they have a pattern provided by any formula that calculates them. The square root of two, for example, is the limit of the following continued (endless) fraction:

$$\sqrt{2} = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + 1}}}$$

From this continued fraction one can derive rational fractions (fractions with integers above and below the line) that give $\sqrt{2}$ to any desired accuracy. The sequence 1/1, 3/2, 7/5, 17/12, 41/29, 99/70, 239/169, 577/408, 1303/985 . . . is sometimes called Eudoxus ladderƒ after an ancient Greek astronomer and geometrician. The fractions are alternately higher and lower than their limit, which is $\sqrt{2}$. Each fraction is closer to $\sqrt{2}$ than its predecessor. The best approximation with numerator and denominator not exceeding three digits is 577/408. It gives

This article first appeared in *Math Horizons* (April 1997).

$\sqrt{2}$ to five decimal places. If a fraction in this sequence is represented by a/b, the next fraction will be $(a+2b)/(a+b)$. Note than on each rungf of the ladder the numerator is the sum of its denominator and the denominator of the preceding fraction.

David Wells, in his *Penguin Dictionary of Curious and Interesting Numbers* (pages 34¡ 35) gives some strange properties of the multiples of $\sqrt{2}$. For example, write in a line the multiples, omitting the fractional part. For example, 1 times $\sqrt{2}$, ignoring the decimal digits is 1. Twice $\sqrt{2}$, ignoring the decimals, is 2. In this way you obtain the following sequence: 1, 2, 4, 5, 7, 8, ...

Beneath this sequence put down the numbers *missing* from the first sequence:

1	2	4	5	7	8	9	11	12	...
3	6	10	13	17	20	23	27	30	...

The difference between the top and bottom numbers at each n^{th} position is always twice n.

Normal Numbers

Any n^{th} root of a positive integer (in all that follows integerf will mean a positive integer) not itself an n^{th} power is irrational. Although all such irrational roots have decimal digits that are neither random nor patternless, they are all, so far as anyone knows, normal.f This means that if you specify any pattern of digits, such as a single digit, pairs of adjacent digits, triplets of adjacent digits, and so on, in the long run the pattern will appear with just the frequency you would expect on the assumption that the probability of finding any given digit at any given place is always 1/10.

The pattern need not involve adjacent digits. They can be spaced any way you like. For example, you might consider the pattern abc, where a and b are separated by, say, seven digits, and b and c are separated by, say, 100 digits. All tests so far to determine the frequency of such patterns have shown that all irrational roots, in any base notation, are normal.

The most extensive tests for the normalcy of certain irrationals have been made for π because π has now been calculated to hundreds of millions of digits, but similar tests of other famous irrationals such as e and the golden ratio have shown no deviations from normalcy. I do not know how far $\sqrt{2}$ has been calculated, though I have a reference

(*TIME*, October 25, 1971) to it having been carried to more than a million digits in 1971 by Jacques Dutka, then a Columbia University mathematician.

One might imagine that all irrationals are normal, but it is easy to see that this is not the case. A popular example is the binary fraction .10100100010000 . . . . The number clearly is not rational and just as clearly is far from normal.

$\sqrt{2}$ and Drowning at Sea

The discovery of irrational roots was first made by the Pythagoreans, a secret brotherhood that flourished in ancient Greece. Their discovery of the first irrational number, the square root of 2, was a milestone in the history of mathematics. In geometrical form this says that the diagonal of a unit square is incommensurable with the square s side. No ruler, no matter how finely graduated, can accurately measure the two line segments. If the side of a square is rational, the diagonal will be irrational, and vice versa.

There are two legends about the explosive effect of this discovery. One is that a Pythagorean named Hippasus was sworn not to reveal the discovery because it shattered the Pythagorean belief that integers accurately measure all things. Hippasus broke the vow. As a result he was drowned at sea either by suicide, murder, or the wrath of the gods the legend has many variations. The other legend has the Pythagoreans celebrating their great discovery by sacrificing many oxen to the gods. The discovery of incommensurable line segments had a profound influence on Plato, who wrote about it in his *Laws*.

Infinite Descent

The Greeks proved the incommensurability of a square s side and diagonal by a clever infinite descentƒ proof using the diagram shown in Figure 1.

Assume that the side of the largest square is commensurable with its diagonal. If so, each of the two line segments will be multiples of a unit that we call k. Draw a smaller square of side b, choosing point x so that $a = c$. Side b of this square will be commensurable with its

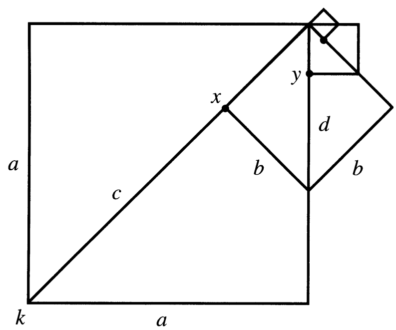

Figure 1. An infinite descent proof that $\sqrt{2}$ is irrational.

diagonal because each is a multiple of k. Next we select point y so that $d = b$. Again, the side and diagonal of this smaller square will be commensurable with respect to k.

This process can be continued to infinity as suggested by the fourth tiny square. The sides of all these squares cannot be zero, but at some point in the endless construction we reach a square with a side less than k. A length less than k cannot be a multiple of k, so we have encountered a contradiction proving that our assumption, that the side and diagonal of a square are commensurable, is false. If the square s side is 1, the diagonal is $\sqrt{2}$. We have shown that $\sqrt{2}$ is irrational.

We can express the proof another way. We seem to get an infinite series of integers (multiples of k), each smaller than the previous one, but such a series obviously must be finite.

Hugo Steinhaus, in the first chapter of *Mathematical Snapshots*, gives a different geometrical proof by infinite descent. It is based on the rectangle shown in Figure 2. Its sides are in a ratio such that if the rectangle is sliced in half as shown, each half will be a rectangle similar to the original one. If the sides are labeled as indicated, a and b will be in the same ratio as $a/2$ and b. The equation reduces to $a^2 = 2b^2$, so if $b = 1$, a will be $\sqrt{2}$.

Assume that a and b are commensurable, each side a multiple of unit k. Of course, k can be any unit, inches, centimeters, or whatever.

In Figure 3 we have attached to the long side of rectangle ab a congruent rectangle that has been given a quarter turn clockwise. This produces a larger rectangle of sides b and $(a + b)$. By cutting two squares of side b from this large rectangle we produce the smaller

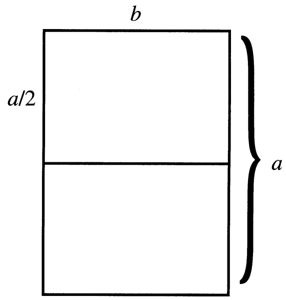

Figure 2. Another infinite descent proof.

shaded rectangle. Its sides are b and $(a - b)$. Because a and b are integers, $(a - b)$ must also be an integer. Therefore the shaded rectangle must have sides that are multiples of k.

We can repeat the procedure by cutting two squares from the shaded rectangle to create a still smaller rectangle, similar to the shaded one, with sides that also must be multiples of k. As in the previous proof, if this process is continued we soon produce a rectangle with sides smaller than k. We have reached a contradiction. The procedure can be carried to infinity, but one cannot have an infinite sequence of integers that keep getting smaller and smaller. Therefore a and b are incommensurable, and $\sqrt{2}$ is irrational. Infinite descent proofs can be given algebraic forms, many of which generalize to proving that any n^{th} root not an n^{th} power is irrational.

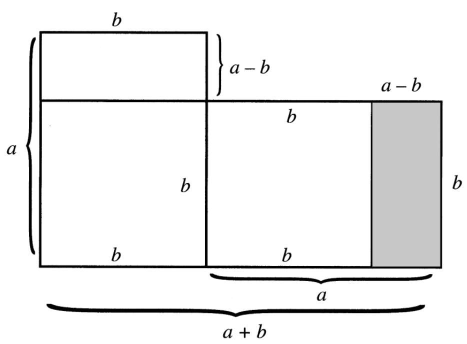

Figure 3. Steinhaus infinite descent proof.

For an application of the $1 \times \sqrt{2}$ rectangle to a magic trick involving the repeated folding of a playing card, see the chapter on rep-tiles in my *Unexpected Hanging and Other Mathematical Diversions*. The rectangle is called an order-2 rep-tile because it can be cut into two parts each similar to itself. British and European sheets of paper usually have sides in a 1 to $\sqrt{2}$ ratio so that when halved, quartered, and so on, the sheets remain similar.

Odds and Evens

The ancient Greeks also had an elegant way of using the laws of odd and even numbers to prove $\sqrt{2}$ is irrational. It can be expressed in numerous ways, but the following seems the simplest.

Let a stand for the hypotenuse of a right isosceles triangle and b for its side. We know from the Pythagorean theorem that $a^2 = 2b^2$, or $a^2/b^2 = 2$. The fraction a/b obviously is between 1 and 2. Assume it is reduced to lowest terms that is, its top and bottom numbers have no common divisor other than 1. We know b is greater that 1, otherwise a/b would be an integer.

The right side of $a^2 = 2b^2$ is even, therefore the left side a^2 is also even, and a is even because the square root of any even number is even. For a we can substitute $2x$ where x is any integer. Squaring $2x$ gives $4x^2$, so we can write $4x^2 = 2b^2$. This reduces to $2x^2 = b^2$. The left side is even, therefore b^2 is even and b is even. Because both a and b are even, each can be divided by 2. This contradicts the assumption that a/b has been reduced to lowest terms. We have proved that a/b cannot be a rational fraction between 1 and 2, therefore $\sqrt{2}$ is irrational.

Euclid gave this proof in Book 10, and Aristotle alludes to it in many places. According to Plato in his dialogue *Theaetetus* (section 147), Theodorus of Cyrene, a brilliant philosopher and geometrician, also proved the irrationality of the square roots of all nonsquares of 3 through 17. Alas, none of his writings survive, so we don t know how he did it, or why he stopped at 17. Incidentally, Theodorus was banished from Cyrene because he doubted the existence of the Greek gods. With suitable modifications, parity (odd¡ even) proofs of $\sqrt{2}$ can be generalized to all n^{th} roots of integers that are not n^{th} powers.

Each of the foregoing proofs is a *reductio ad absurdum* or indirect*f* proof in which an assumption is made then later proved false by a contradiction. A whimsical indirect proof of the irrationality of $\sqrt{2}$ is based on the final digit of square numbers. It is easy to see that this digit must be 0, 1, 4, 5, 6, or 9. Consider again the equation $a^2 = 2b^2$, where a/b is reduced to lowest terms, b greater than 1.

The terminal digit of both a^2 and b^2 must be one of the six listed above. On the right side of $a^2 = 2b^2$, b^2 is multiplied by 2, therefore the final digit of $2b^2$ must be 0, 2, or 8. It cannot be 2 or 8 because there is no 2 or 8 as the last digit of a^2. The only match is 0. So a^2 and $2b^2$ must each end in zero. It follows that a must end in 0, and b must end in 0 or 5. In either case both a and b are divisible by 5, contradicting the assumption that a/b is reduced to lowest terms. Hence a/b is irrational and $\sqrt{2}$ is irrational.

Similar terminal digit proofs of the irrationality of $\sqrt{2}$ can be formulated in other base notations. In binary notation, for example, the proof is unusually simple. The left side of $a^2 = 2b^2$ terminates in an even number of zeros and the right side terminates in an odd number of zeros.

Many elegant proofs of the irrationality of $\sqrt{2}$ are based on the fundamental theorem of arithmetic, which states that every integer is the product of a unique set of primes. Here is one of the easiest to follow.

As before, we use the equation $a^2 = 2b^2$ where a/b is a rational fraction reduced to lowest terms, b greater than 1. The term a^2 must have an even number of prime factors. Why? Because if a is the product of either an odd or an even number of primes, its square will have twice as many prime factors.

Consider now the right side of $a^2 = 2b^2$. It will have an odd number of prime factors because to the even number of prime factors of b^2 we add the prime factor 2. We have produced a contradiction because the number of prime factors for the two sides of the equation cannot be even on one side and odd on the other. It is not difficult to see that the proof applies to the square root of any prime, or to an integer with an odd number of prime factors.

Prime divisors provide a simple proof that any square root not an integer is irrational. We apply it first to $\sqrt{2}$. From $a^2 = 2b^2$ we can derive the equation $b^2 = a^2/2$ which is the same as a times $a/2$. If a prime divides the product of two integers x and y, it obviously must divide either x or y. Let a^2 and a be the two integers whose product is $a^2/2$. There must be a prime that divides b^2 because b is greater than 1. This same prime must divide the right side of the equation, therefore it must divide $a/2$ or a. In either case it divides a because if it divides half of a, it will also divide a. Contradiction! We have shown that a prime divides both a and b, therefore a/b cannot be a rational fraction reduced to lowest terms.

Substitute for 2 any integer whose square root is not an integer and the foregoing proof holds. With further generalizations the proof will apply to all n^{th} roots of integers that are not n^{th} powers.

Another simple proof of the irrationality of $\sqrt{2}$ is based on inequalities. If a/b is $\sqrt{2}$ reduced to lowest terms, then b is less than a, and a is less than $2b$, therefore $(a-b)$ is less than b. Start with $a^2 = 2b^2$, and make the following changes:

$$\begin{aligned} a^2 - ab &= 2b^2 - ab \\ a(a-b) &= b(2b-a) \\ a/b &= (2b-a)/(a-b) \end{aligned}$$

As we have seen, $(a-b)$ is smaller than b. We have contradicted the assumption that a/b is reduced to lowest terms. This proof also generalizes to any n^{th} root of any number not an n^{th} power.

There are dozens of other ways to prove the irrationality of the square roots of integers that are not squares, many of which extend easily to n^{th} roots. They all come down to the following theorem: If a/b is a rational fraction in lowest terms, b greater than 1, then any power of a/b will also be a rational fraction that cannot be reduced to lower terms.

This can be proved by the following argument involving prime factors. Assume that a/b, with b greater than 1, is reduced to lowest terms. The prime factors of a will have no factors in common with b, otherwise the common factors cancel out and a/b is reduced. Consider now the square of a/b. The factors above the line will be the same as before, each repeated twice, and the same for the prime factors below the line. There are still no common factors to cancel. This means that the square of a rational fraction reduced to lowest terms is another fraction reduced to lowest terms, so it cannot be an integer. In brief, no integer not a square can have a square root that is rational.

The argument obviously applies to cubes and all higher roots. For example, a^3/b^3 is $(a \times a \times a)/(b \times b \times b)$. This too is a nonreducible fraction because there are no common prime factors above and below the line to be canceled. Is there any simpler, easier to comprehend, way to show that n^{th} roots of integers not n^{th} powers are irrational?

When I was in high school and first learned that $\sqrt{2}$ could not be expressed as a rational fraction, I couldn t believe it. I squandered many hours in study periods trying to find such a fraction. Eventually I convinced myself it couldn t be done, but today I have no memory of how I proved it, if indeed I did. I like to think it was one of the proofs given in this article. It would be interesting to know how many mathematicians, far greater than I, had a similar experience when they were very young.

Note that all the proofs in this article are *reductio ad absurdem* proofs. They illustrate how powerful this type of proof is. As G. H. Hardy put it in his famous *Mathematician's Apology*:

> It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game*.

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Hundreds of books contain proofs of the irrationality of $\sqrt{2}$ and more general proofs of the irrationality of any n^{th} root of an integer not an n^{th} power. What follows are references in easily accessible periodicals.

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Postscript

Here are two short indirect proofs, not in my article, that the square root of 2 is irrational.

1. If $\sqrt{2}$ is a rational fraction there must be a smallest positive integer k that would make $k\sqrt{2}$ an integer. But $k\sqrt{2} - k$ is a *smaller* such integer. Contradiction.

2. As made clear in this chapter, if the square root of 2 is an integral fraction between 1 and 2, the equation $a^2 = 2b^2$ would have a solution in integers. In base-3 notation, the last non-zero digit of a square must be 1. This applies to the left side of the equation. But the last non-zero digit of $2b^2$ is 2, a contradiction proving that $a^2 = 2b^2$ has no solution in integers.

Monte Zerger called my attention to an amazing way to generate all the convergents of the square root of 2. He found this explained in a paper by D. V. Anderson in *The Mathematical Gazette* (November 1996, pp. 574¡ 575). Start with the sequence of powers of 2, each power appearing twice, then add adjacent digits in the manner of Pascal s triangle. The first two entries of each row are the successive convergents that I listed earlier as the rungs of Eudoxus ladder!

1 1 2 2 4 4 8 8

2 3 4 6 8 12 16

5 7 10 14 20 28

12 17 24 34 48

29 41 58 82

70 99 140

169 239

I was surprised and honored when the foregoing article won the 1998 Trevor Evans Award of $250, given each year to articles published in *Math Horizons*. Trevor Evans was a distinguished mathematician at Emory University.

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Chapter 3
Flip, the Psychic Robot

Can You Outwit a Mindless Automaton?

Here s a chance to take on an opponent who has no control over the moves he makes. Still, winning may be tougher than you think.

The game is matching pennies. You flip a coin and then Flip, the robot, will do the same. If Flip s flip matches yours (heads after you have thrown heads, or tails after your tails), Flip wins. If not (tails after your heads, or heads after your tails), you win. Flip has challenged you to a 25-game match.

Here s how it works. For each game (1¡ 25), throw a coin and note whether you threw heads or tails. Being a mere sheet of paper, Flip can t toss for himself, so you now get to tell him what to throw. Thus, if you threw heads, you may want to tell Flip to throw tails. But there s a catch: Sometimes Flip will obey your commands, sometimes he won t. You have no way of knowing in advance when he ll obey and when he ll disobey.

Once you ve chosen heads or tails for Flip (and not before), follow a line that leads from that response all the way to its end to discover Flip s true flip. In following the lines, you may not change directions at intersections. Check the result of each game before proceeding to the next.

Keep a running total of the number of games won by you and the number won by Flip.

You may be surprised to find that every time you play this game Flip will win. Here s why:

The game is designed to counter the way people make choices when trying to beat a machine. Research has shown that most people in such a situation fall into a predictable psychological pattern; the game uses this pattern in designing the robot s responses (whether he ll obey or

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