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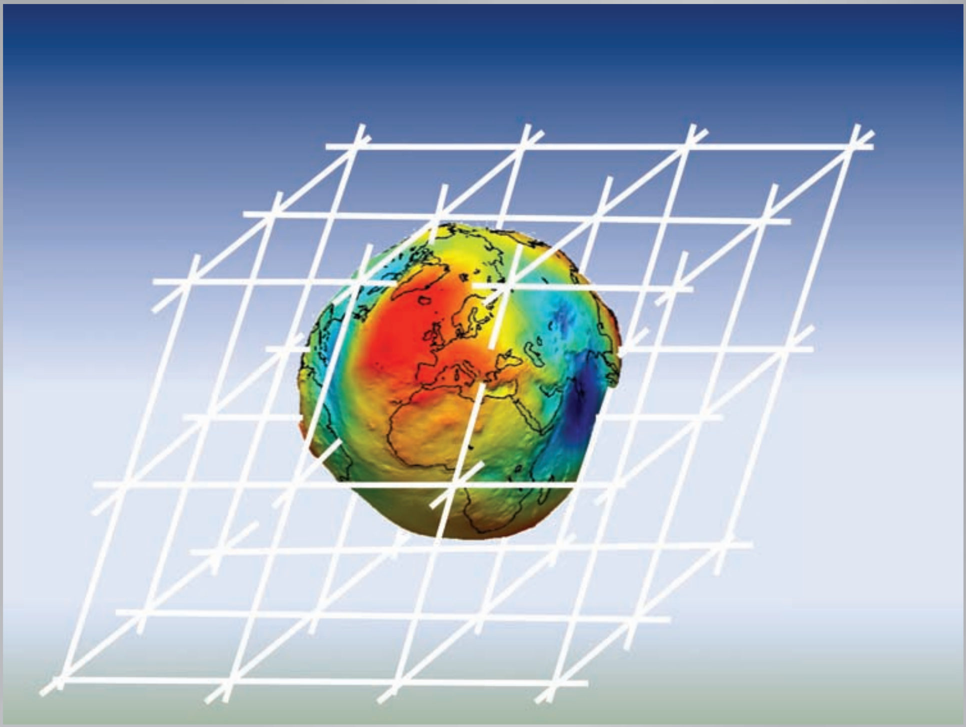

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Metaharmonic Lattice Point Theory

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Metaharmonic Lattice Point Theory

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Preface

These lecture notes are the result of an interrelated "transfer" of methods, settings, and tools of (spherically oriented) geomathematics and of (periodically reflected) analytic theory of numbers. The essential ingredients of mathematical (geo-)physics in this work are special function systems of the Laplace equation and the Helmholtz equation, i.e., harmonic and metaharmonic functions, problem-adapted constructions of Green's functions, and eigenvalue-based solution theory in terms of "Green type" integral formulas. Surprisingly, these fundamental techniques relevant for geomathematical research in gravitation, magnetics, geothermal research, etc. enable us to recover significant topics of lattice point theory in Euclidean spaces (such as Hardy–Landau identities determining the total number of lattice points inside spheres, weighted (radial and angular) lattice point summation, non-uniform distribution of lattice points, etc.). Even more, multi-dimensional alternating series become attackable by convergence criteria relating the specific oscillation properties of a summand to an appropriate choice of a Helmholtz operator. In addition, new classes of lattice point identities can be developed by adapted procedures of periodization within "Green type" integral formulas, i.e., Euler and Poisson summation.

More specifically, the main objectives of this work are multi-dimensional generalizations of the Euler summation formula by suitably interpreting the classical "Bernoulli polynomials" as Green's functions and by appropriately establishing the link to Zeta and Theta functions. The multi-dimensional Euler summation formulas are generated on arbitrary lattices by the conversion of the Helmholtz wave equation into an associated integral equation based on the concept of Green's functions as a bridging tool. In doing so, we are able to compare weighted sums of functional values for a prescribed system of lattice points with the corresponding integral over the function, plus a remainder term that is adaptable to the (oscillating) function under consideration. The remainder term is particularly useful for two aspects of multi-variate lattice point theory, viz. to guarantee the convergence of multi-dimensional alternating series and to formulate appropriate criteria for the validity of the Poisson summation formula. Since the infinite lattice point sums occurring in our approach usually offer the pointwise, but refuse the absolute convergence, the specification of the multi-dimensional summation process is a decisive feature. Throughout this book, with respect to the rotational invariance of the Laplace operator, (pointwise) convergence is understood in the spherical sense. In other words, multi-dimensional summation is consistently extended over balls, if the series expansion under consideration turns out to refuse absolute convergence.

The title of our work can be reformulated in more detail as the Helmholtz equation induced verification of the Hardy–Landau type lattice point identities with particular interest in characterizing radial and angular distributions of (planar) lattice points. Altogether, the book can be characterized briefly as a lecture note in the analytic theory of numbers in Euclidean spaces based on methods and procedures of mathematical physics. Its essential purpose is to establish multi-dimensional Euler and Poisson summation formulas corresponding to (iterated) Helmholtz operators for the adaptive determination and calculation of formulas and identities involving weighted lattice point numbers.

The roots of the book are threefold: (i) the basic results due to L.J. Mordell on one-dimensional Euler and Poisson summation formulas as well as the one-dimensional Zeta and Theta function (ii) the work by C. Müller on two-dimensional periodical Euler (Green) functions and their representation in the framework of complex analysis, and (iii) my own work on multi-dimensional generalizations of the Euler summation formula to elliptic operators and some attempts to extend the multi-dimensional Poisson summation formula to regular ("potato"-like) regions. In consequence, the number theoretical understanding of the book requires that the reader has mastered some material usually covered in courses on elliptic partial differential equations and special functions of mathematical physics, especially related to the theory of iterated Laplace as well as Helmholtz equations. The book can be used as a graduate text or as a reference for researchers.

The idea of writing this book first occurred to me while teaching graduate courses given during the last years at the University of Kaiserslautern, when I presented various topics on Green's functions in different fields of geomathematical application. Indeed, the lecture notes represent the link between my former PhD activities at the RWTH Aachen in analytic theory of numbers and my present work in geomathematics at the University of Kaiserslautern.

The preparation of the final version of this work was supported by important remarks and suggestions of many colleagues. I am deeply obliged to Z. Nashed, Orlando, USA, and T. Sonar, Braunschweig, Germany, for friendly collaboration and continuous support over the last years. It is a great pleasure to express my particular appreciation to my colleague G. Malle, University of Kaiserslautern, Germany, who helped me to clarify some concepts. I am indebted to M. Schreiner, NTB Buchs, Switzerland and M. A. Slawinski, Memorial University of Newfoundland, St. John's NL, Canada, for helpful comments and remarks.

Thanks also go to my co-workers, especially to M. Augustin, C. Gerhards, M. Gutting, S. Möhringer, and I. Ostermann, for eliminating inconsistencies in an earlier version. I am obliged to L. Hämmerling, Aachen, for providing me with the phase-dependent numerical computation and graphical illustration (Figure 14.5) of the radial distribution of lattice points in the plane.

The cover illustration shows the geoid of the Earth (i.e., the equipotential surface at sea level as it will be seen by the satellite GOCE) imbedded in a three-dimensional lattice. The “geoidal potato” constitutes a typical (geophysically relevant) regular region as discussed in this work. I am obliged to R. Haagmans, Head, Earth Surfaces and Interior Section, Mission Science Division, ESA–European Space Agency, ESTEC, Noordwijk, the Netherlands, for providing me with the image (ESA ID number SEMLXEOA90E).

I wish to express my particular gratitude to Claudia Korb, Geomathematics Group, TU Kaiserslautern, for her support in handling the typing job.

Finally, it is a pleasure to acknowledge the courtesy and the ready cooperation of Taylor & Francis and all staff members there who were involved in the publication of the manuscript. My particular thanks go to Bob Stern, Amber Donley, and Karen Simon.

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Kaiserslautern

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About the Book

This book is dedicated to the work of Claus Müller, Rheinisch–Westfälische Technische Hochschule Aachen (RWTH Aachen). His fascinating lecture on the two-dimensional Euler Summation Formula and its applications to the analytic theory of numbers held in the winter semester of 1969/1970 (and derived from C. Müller [1954a]) and his excellent guidance leading to the (unpublished) "Diplom" and "Staatsexamen" theses, at the RWTH Aachen, motivated the author more than four decades later to publish this book on metaharmonic lattice point theory. It presents a mathematical collection of promising fruits for the cross-fertilization of two disciplines, namely classical analytic and geometric number theory and future oriented geomathematics involving geophysically relevant regions (such as ball, "geoid(al potato)", (real) Earth's body).

Claus Müller (born February, 1920 in Solingen, Germany, died February, 2008 in Aachen, Germany).

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Willi Freeden (born March, 1948 in Nettetal–Kaldenkirchen, Germany).

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List of Symbols

$\mathbb{N}_0$ .... set of non-negative integers
$\mathbb{N}$ .... set of positive integers
$\mathbb{Z}$ .... set of integers
$\mathbb{R}$ .... set of real numbers
$\mathbb{C}$ .... set of complex numbers
$\Re(s)$ .... real part of $s \in \mathbb{C}$
$\Im(s)$ .... imaginary part of $s \in \mathbb{C}$
$\mathbb{R}^q$ .... q-dimensional Euclidean space
x, y, z .... elements of $\mathbb{R}^q$
$x \cdot y$ .... scalar product of vectors
$|x|$ .... Euclidean norm of $x \in \mathbb{R}^q$
$\epsilon^i,\ i = 1, \ldots, q$ .... canonical orthonormal basis in $\mathbb{R}^q$
δ_{ij} .... Kronecker symbol
$\mathrm{C}^{(k)}, \mathrm{L}^p$ .... classes of functions
F, G .... scalar-valued functions
f, g .... vector-valued functions
$\mathbf{i}$ .... identity tensor
$\mathbf{t}$ .... orthogonal matrix
$\mathbf{t}^T$ .... transpose of the matrix $\mathbf{t}$
$\det \mathbf{t}$ .... determinant of the matrix $\mathbf{t}$
Γ .... point set in $\mathbb{R}^q$
$\partial\Gamma$ .... boundary of Γ
$\overline{\Gamma}$ .... closure of Γ
$F|M$ .... restriction of F to M
∇ .... gradient
$\nabla\,\cdot$ .... divergence
Δ .... Laplace operator
$\Delta + \lambda$ .... Helmholtz operator
$\{\ldots\}$ .... set of the elements
$\emptyset$ .... empty set
$\mathbb{B}^q$ .... unit ball in $\mathbb{R}^q$ around 0
$\mathbb{B}^q_N(y)$ .... ball in $\mathbb{R}^q$ with radius N around y
$\mathbb{B}^q_{\rho,N}(y)$ .... ball ring with radii ρ and N around y
$\mathbb{S}^{q-1}$ .... unit sphere in $\mathbb{R}^q$ around 0
$\mathbb{S}^{q-1}_N(y)$ .... sphere in $\mathbb{R}^q$ with radius N around y
t, φ .... polar coordinates in $\mathbb{R}^2$
ξ, η, ζ .... elements of $\mathbb{S}^{q-1}$

$dV, dV_{(q)}$ .... volume element in $\mathbb{R}^q$
$dS, dS_{(q-1)}$ .... surface element in $\mathbb{R}^q$
∇^* .... surface gradient in $\mathbb{R}^q$
$\nabla^* \cdot$ .... surface divergence in $\mathbb{R}^q$
Δ^* .... Laplace–Beltrami operator in $\mathbb{R}^q$
$\Delta^* + \lambda$ .... Helmholtz–Beltrami operator in $\mathbb{R}^q$
$(\Delta^*)^\wedge(n)$ .... $\mathbb{S}^{q-1}$-symbol of the Laplace–Beltrami operator in $\mathbb{R}^q$
$(\Delta^* + \lambda)^\wedge(n)$ .... $\mathbb{S}^{q-1}$-symbol of the Helmholtz–Beltrami operator in $\mathbb{R}^q$
$G(\Delta^* ; \cdot)$ .... $\mathbb{S}^{q-1}$-sphere function for Δ^*
$G(\Delta^* + \lambda ; \cdot)$ .... $\mathbb{S}^{q-1}$-sphere function for $\Delta^* + \lambda$
$\mathbb{Z}^q$ .... unit lattice in $\mathbb{R}^q$
$\tau\mathbb{Z}^q$ .... dilated unit lattice in $\mathbb{R}^q$
$\mathbb{Z}^q + \{x\}$ .... translated unit lattice based at x in $\mathbb{R}^q$
Λ .... (general) lattice in $\mathbb{R}^q$
Λ^{-1} .... inverse lattice of Λ in $\mathbb{R}^q$
$\mathcal{F}$ .... fundamental cell of Λ in $\mathbb{R}^q$
$\mathcal{F}^{-1}$ .... fundamental cell of Λ^{-1} in $\mathbb{R}^q$
$||\mathcal{F}||$ .... volume of the fundamental cell Λ
g .... lattice points of Λ
h .... lattice points of Λ^{-1}
$\mathrm{Spect}_\Delta(\Lambda)$ .... eigenspectrum of the Laplace operator in $\mathbb{R}^q$
$\Phi_h,\ h \in \Lambda^{-1}$ .... (orthonormal) Λ-periodical polynomials in $\mathbb{R}^q$
$\mathrm{C}^{(k)}_\Lambda, \mathrm{L}^p_\Lambda$ .... classes of Λ-periodical functions in $\mathbb{R}^q$
$F^\wedge_\Lambda(h)$ .... Fourier transform of F at $h \in \Lambda^{-1}$ with respect to Λ
$F^\wedge_{\mathbb{R}^q}(x)$ .... Fourier transform of F at $x \in \mathbb{R}^q$ with respect to $\mathbb{R}^q$
$\mathcal{X}_\mathcal{M}$ .... characteristic function for a set $\mathcal{M}$
$\tau\Lambda$ .... dilated point lattice in $\mathbb{R}^q$
$\Lambda - \{x\}$ .... shifted point lattice in $\mathbb{R}^q$
$(\Delta)^\wedge(h)$ .... Λ-symbol of the Laplace operator in $\mathbb{R}^q$
$(\Delta + \lambda)^\wedge(h)$ .... Λ-symbol of the Helmholtz operator in $\mathbb{R}^q$
$G(\Delta ; \cdot)$ .... Λ-lattice function for Δ
$G(\Delta + \lambda ; \cdot)$ .... Λ-lattice function for $\Delta + \lambda$
$\mathcal{M} + \Lambda$ .... figure lattice associated to $\mathcal{M}$ by Λ
$\mathbb{B}^q_\tau + \Lambda$ .... ball lattice associated to $\mathbb{B}^q_\tau$ by Λ
$\mathcal{G}$ .... regular region in $\mathbb{R}^q$
$\overline{\mathcal{G}} = \mathcal{G} \cup \partial\mathcal{G}$ .... closure of a regular region in $\mathbb{R}^q$
$\partial\mathcal{G}$ .... boundary of the regular region $\mathcal{G}$ in $\mathbb{R}^q$
$||\mathcal{G}||$ .... volume of the regular region $\mathcal{G}$ in $\mathbb{R}^q$
$\alpha(x)$ .... solid angle at x subtended by $\partial\mathcal{G}$

o, O, Ω .... Landau symbols
$\sim$ .... asymptotically equal
$\simeq$ .... (B^2-)Besicovitch almost periodical
$\lfloor \delta \rfloor$ .... the largest integer $\leq \delta$ (floor-function)
$\lceil \delta \rceil$ .... the smallest integer $\geq \delta$ (ceil-function)
L .... L function
ζ .... Zeta function
$\vartheta, \Theta,\ \theta$ .... Theta functions
p, p_i .... primes
p^m, p_i^l .... prime powers
$d|n$ .... d divides n
$d \nmid n$ .... d does not divide n
$u \equiv (v \bmod m)$ .... m is divisor of $u - v$, i.e., $m|u - v$
$u \not\equiv (v \bmod m)$ .... m is not divisor of $u - v$, i.e., $m \nmid u - v$
(n, m) .... the greatest common divisor (gcd) of n and m
$\{n, m\}$ .... the least common multiple (lcm) of n and m
$n = \prod_p p^{l(p)}$ .... the prime factorization of an integer $n > 1$
.... with p running through all primes and exponents $l(p)$
$r(n)$ $(= r_2(n))$ .... the solution number of the equation
.... $n_1^2 + n_2^2 = n$ with $n_1, n_2 \in \mathbb{Z}, n \in \mathbb{N}_0$,
$r_q(n)$ .... the solution number of the equation
.... $n_1^2 + \ldots + n_q^2 = n$ with $n_1, \ldots, n_q \in \mathbb{Z}, n \in \mathbb{N}_0$,
$\#_\Lambda\left(\overline{\mathcal{G}}\right) = \sum_{\substack{g \in \overline{\mathcal{G}} \\ g \in \Lambda}} 1$ .... total number of lattice points of Λ inside $\overline{\mathcal{G}}$
$P^\lambda\left(F; \overline{\mathcal{G}}\right)$ .... Λ-lattice point discrepancy of F in $\overline{\mathcal{G}}$ w.r.t. $\Delta + \lambda$
$P\left(\overline{\mathcal{G}}\right) = P^0\left(1; \overline{\mathcal{G}}\right)$ .... Λ-lattice point discrepancy in $\overline{\mathcal{G}}$ w.r.t. Δ
$P_\tau^\lambda\left(F; \overline{\mathcal{G}}\right)$ .... Λ-lattice τ-ball discrepancy of F in $\overline{\mathcal{G}}$ w.r.t. $\Delta + \lambda$
$P_\tau\left(\overline{\mathcal{G}}\right) = P_\tau^0\left(1; \overline{\mathcal{G}}\right)$ .... Λ-lattice τ-ball discrepancy in $\overline{\mathcal{G}}$ w.r.t. Δ

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Introduction

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1.1 Historical Aspects

Leonhard Euler (1707–1783) discovered his powerful "summation formula" in the early 1730s. He used it in 1736 to compute the first 20 decimal places for the alternating sum

$$\sum_{g=0}^{\infty} \frac{(-1)^g}{2g+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \ldots = \frac{\pi}{4}. \tag{1.1}$$

Since, aside from the geometric series, very few infinite series then had a known sum, Euler's remarkable sum enticed mathematicians like G. Leibniz (1646–1716) and the Bernoulli brothers Jakob (1654–1705) and Johann (1667–1748) to seek sums of other series, particularly the sum of the reciprocal squares. But it was L. Euler, within the next two decades up to 1750, who did a "broadening of the context" to formulate his "summation formula" for the general sum

$$\sum_{g=0}^{n} F(g) \ = \ \sum_{\substack{g\in[0,n] \\ g\in\mathbb{Z}}} F(g) \ = \ \sum_{\substack{0\le g\le n \\ g\in\mathbb{Z}}} F(g) \tag{1.2}$$

(with n possibly infinite). More concretely, under the assumption of second order continuous derivatives of F on the interval $[0, n]$, $n \in \mathbb{N}$, Euler succeeded in finding the summation formula

$$\begin{aligned} &\sum_{\substack{0\le g\le n \\ g\in\mathbb{Z}}} F(g) - \frac{1}{2}\left(F(0) + F(n)\right) &(1.3)\\ &= \int_0^n F(x)\ dx + \frac{1}{12}\left(F'(n) - F'(0)\right) + \int_0^n \underbrace{\left(-\frac{1}{2}B_2(x)\right)}_{=G(\Delta;x)} F''(x)\ dx, \end{aligned}$$

where B_2 given by

$$B_2(x) = (x - \lfloor x \rfloor)^2 - (x - \lfloor x \rfloor) + \frac{1}{6} \tag{1.4}$$

is the "Bernoulli function" of degree 2. In particular, Euler's new setting also encompassed the quest for closed formulas for sums of powers

$$\sum_{k=0}^{n} k^l \simeq \int_0^n x^l \, dx, \tag{1.5}$$

which had been sought since antiquity for area and volume investigations. In addition, this setting provided a canonical basis for the introduction of the Zeta function.

From the mathematical point of view, Euler's summation formula is a fine illustration of how a generalization can lead to the solution of seemingly independent problems. The particular structure of his summation formula also captures the delicate details of the connection between integration, i.e., "continuous summation", and its various discretizations, viz. summation. Obviously, it subsumes and resolves the appropriate bridge between continuous and discrete summation within a single exposition. But it should be pointed out that Leonhard Euler himself used this interrelation between continuous and discrete sums only for estimating sums and series by virtue of integrals. It was actually Colin Maclaurin (1698–1746), who discovered the summation formula (1.3) independently in 1742, to use it for the evaluation of integrals in terms of sums.

Altogether, the classical Euler summation formula provides a powerful tool of connecting integrals and sums. It can be used in diverse areas to approximate integrals by finite sums, or conversely to evaluate finite sums and infinite series based on the integral calculus.

More specifically, the Euler summation formula offers two important perspectives:

- to compute (slowly) converging infinite series as well as to specify convergence criteria for (alternating) infinite series and to verify limits and asymptotic relations of infinite lattice point sums,

- to evaluate integrals (numerically) as well as to estimate and to optimize the error and to provide multi-dimensional settings of constructive approximation.

1.2 Preparatory Ideas and Concepts

In this book we follow Euler's interest, i.e., the first of the aforementioned perspectives including its applications to relevant lattice point sums of analytic theory of numbers. The essential idea is based on the interpretation of the Bernoulli function (1.4) occurring in the classical (one-dimensional) Euler summation formula (1.3) by means of mathematical physics as the Green function $G(\Delta;\cdot)$ for the (one-dimensional) Laplace operator Δ corresponding to the "boundary condition" of $\mathbb{Z}$-periodicity (note that $\Delta = (\frac{d}{dx})^2$ is the operator of the second order derivative). More concretely, the periodical Green function $G(\Delta;\cdot)$ for the Laplace operator Δ is constructed so as to have the bilinear expansion

$$G(\Delta; x-y) \;=\; \sum_{\substack{\Delta^\wedge(h)\neq 0 \\ h\in\mathbb{Z}}} \frac{e^{2\pi i h x} e^{-2\pi i h y}}{-\Delta^\wedge(h)}, \quad x,y\in\mathbb{R}, \tag{1.6}$$

where the sequence $\{\Delta^\wedge(h)\}_{h\in\mathbb{Z}}$ forms the spectrum $\mathrm{Spect}_\Delta(\mathbb{Z})$ of the Laplace operator Δ, i.e.,

$$(\Delta + \Delta^\wedge(h))\; e^{2\pi i h x} \;=\; 0, \quad x\in\mathbb{R}, \tag{1.7}$$

where

$$\Delta^\wedge(h) = 4\pi^2 h^2, \qquad h\in\mathbb{Z}. \tag{1.8}$$

In doing so, the Bernoulli function - in the jargon of mathematical physics, the Green function - acts as a connecting tool to convert a differential equation involving the Laplace operator corresponding to periodical boundary conditions into an associated integral equation, i.e., the Euler summation formula (1.3). Observing the special values

$$G(\Delta;0) \;=\; G(\Delta;n) \;=\; \sum_{\substack{\Delta^\wedge(h)\neq 0 \\ h\in\mathbb{Z}}} \frac{1}{-\Delta^\wedge(h)} \;=\; -\frac{1}{12} \tag{1.9}$$

and the explicit representation of the Fourier series expansion (1.6) we are able to reformulate the Euler summation formula (1.3). Partial integration yields (by letting $F'(x) = \nabla F(x),\; F''(x) = \Delta F(x)$)

$$\begin{aligned} &\sum_{\substack{0\leq g\leq n \\ g\in\mathbb{Z}}} F(g) - \frac{1}{2}\left(F(0)+F(n)\right) \qquad (1.10)\\ &= \int_0^n F(x)\, dx + \sum_{\substack{h\neq 0 \\ h\in\mathbb{Z}}} \left(\frac{\nabla F(n) - \nabla F(0)}{4\pi^2 h^2} - \int_0^n \frac{e^{2\pi i h x}}{4\pi^2 h^2}\, \Delta F(x)\, dx \right) \\ &= \int_0^n F(x)\, dx - \lim_{N\to\infty} \sum_{\substack{|h|\leq N \\ h\neq 0}} \frac{1}{2\pi i h} \int_0^n \nabla F(x)\, e^{2\pi i h x}\, dx, \end{aligned}$$

such that the Poisson summation formula comes into play

$$\sum_{\substack{0\le g\le n\\ g\in\mathbb{Z}}} F(g) - \frac{1}{2}\left(F(0)+F(n)\right) = \lim_{N\to\infty} \sum_{\substack{|h|\le N\\ h\in\mathbb{Z}}} \int_0^n F(x)\ e^{2\pi i h x}\ dx. \tag{1.11}$$

Surprisingly, in spite of their apparent dissimilarity, the Euler summation formula (1.3) and the Poisson summation formula (1.11) are equivalent for twice continuously differentiable functions on the interval $[0, n]$. Moreover, the Green function for the Laplace operator and the "boundary condition" of $\mathbb{Z}$-periodicity acts as the canonical bridge between both identities.

The "building blocks" of the bridge between the two equivalent formulas (1.3) and (1.11) are the *defining constituents of the Green function* $G(\Delta;\cdot)$, which can be uniquely characterized in the following way:

(*Periodicity*) $G(\Delta;\cdot)$ is continuous in $\mathbb{R}$ and $\mathbb{Z}$-periodical

$$G(\Delta;x)\ =\ G(\Delta;x+g),\ x\in\mathbb{R},\ g\in\mathbb{Z}, \tag{1.12}$$

(*Differential equation*) $\Delta G(\Delta;\cdot)$ "coincides" apart from an additive constant with the Dirac function(al)

$$\Delta_x G(\Delta;x)\ =\ -1,\ \ x\in\mathbb{R}\backslash\mathbb{Z}, \tag{1.13}$$

(*Characteristic singularity*) $G(\Delta;\cdot)$ possesses the singularity of the fundamental solution of the (one-dimensional) Laplace operator

$$G(\Delta;x)\ -\ \frac{1}{2}\ |x|\ =\ O(1),\ \ x\to 0, \tag{1.14}$$

(*Normalization*) $G(\Delta;\cdot)$ integrated over a whole period interval of length 1 is assumed to be zero

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} G(\Delta;x)\ dx\ =\ 0. \tag{1.15}$$

Even more generally, for *arbitrary* intervals $[a,b]\subset\mathbb{R}$, $a<b$, and *arbitrary* twice continuously differentiable (weight) functions F on $[a,b]$, the constituents (1.12)–(1.15) of the Green function $G(\Delta;\cdot)$ enable us to guarantee the equivalence of the Euler summation formula

$$\begin{aligned}\sum_{\substack{a\le g\le b\\ g\in\mathbb{Z}}}{}' F(g) \quad &= \quad \int_b^a F(x)\ dx + \left\{F(x)\ \left(\nabla G(\Delta;x)\right) - G(\Delta;x)\ \left(\nabla F(x)\right)\right\}\Big|_a^b \\ &\quad + \int_a^b G(\Delta;x)\ \Delta F(x)\ dx \end{aligned} \tag{1.16}$$

and the Poisson summation formula

$$\sum_{\substack{a\le g\le b\\ g\in\mathbb{Z}}}{}' F(g) \quad = \quad \lim_{N\to\infty} \sum_{\substack{|h|\le N\\ h\in\mathbb{Z}}} \int_a^b F(x)\ e^{2\pi i h x}\ dx, \tag{1.17}$$

where we have used the abbreviation

$$\sum_{\substack{a \le g \le b \\ g \in \mathbb{Z}}}{}' \, F(g) = \sum_{\substack{a < g < b \\ g \in \mathbb{Z}}} F(g) \; + \; \frac{1}{2} \sum_{\substack{g = a, b \\ g \in \mathbb{Z}}} F(g), \tag{1.18}$$

and the second sum on the right side of (1.18) occurs only if a and/or b is an integer.

Of particular interest in lattice point theory is the special case of a constant weight function (i.e., $F = 1$), i.e., the *one-dimensional "Hardy–Landau identity"*

$$\sum_{\substack{a \le g \le b \\ g \in \mathbb{Z}}}{}' \, 1 \; = \; b - a \; + \; \sum_{\substack{|h| \neq 0 \\ h \in \mathbb{Z}}} \int_a^b e^{2\pi i h x} \, dx. \tag{1.19}$$

The formula (1.19) compares the number of $\mathbb{Z}$-lattice points inside an interval $[a, b]$ with the length $b - a$ of the interval $[a, b]$ under the explicit knowledge of the remainder term (usually called, the *$\mathbb{Z}$-lattice point discrepancy*) as a one-dimensional alternating series.

In addition, the close relation between the Hardy–Landau summation and the metaharmonicity of the summands becomes obvious in the identity (1.19) since the function $e(h\cdot) = e^{2\pi i h \cdot}, h \in \mathbb{Z}$, satisfies *the one-dimensional Helmholtz equation* $\left(\Delta + 4\pi^2 h^2\right) e(hx) = 0, \; x \in \mathbb{R}$, corresponding to the "wave number" $\Delta^\wedge(h) = 4\pi^2 h^2, \; h \in \mathbb{Z}$; i.e., the $\mathbb{Z}$-periodical polynomial $e(h\cdot), \; h \in \mathbb{Z}$, is *metaharmonic* in $\mathbb{R}$.

1.3 Tasks and Perspectives

This book is devoted to the generalization of the univariate features, settings, and methods involving Euler and Poisson summation to higher dimensions. The key points are the defining properties (1.12)–(1.15) of the $\mathbb{Z}$-periodical Green function, which can be easily transferred to the multi-dimensional case (in contrast to the bilinear expansion (1.6)). Our tasks actually show plenty of essential aspects, namely the supply of multi-variate tools for the Laplace operator and Helmholtz operators, the feasible construction of the multi-dimensional Green function with respect to (iterated) Laplace and Helmholtz operators and the "boundary condition" of periodicity, the realization of associated Euler summation formulas, the introduction and explanation of special function systems of harmonic as well as metaharmonic nature such as spherical harmonics as well as Bessel and Kelvin functions, the formulation of adequate convergence criteria for multi-dimensional alternating sums, some pointwise inversion procedures of Fourier and other integral transforms such as Gauß–Weierstraß and Abel–Poisson transforms, suitable concepts to establish the validity of the multi-dimensional Poisson summation formula, and finally their applications to problems of the analytic and geometric theory of numbers, for example, in the field of radial and angular lattice point distribution.

Altogether, keeping in mind the physically motivated character of the key ingredient, i.e., the Green function with respect to the "boundary condition" of $\mathbb{Z}$-periodicity for Euler summation, we are able to guarantee various directions of extension to

- replace the one-dimensional lattice $\mathbb{Z}$ (consisting of the integers) by *multi-dimensional "lattices"* Λ (such as $\mathbb{Z}^q, \tau\mathbb{Z}^q$, etc).
- consider instead of the finite one-dimensional interval $[a, b]$ *other "geometries"* for summation/integration such as the fundamental cell $\mathcal{F}$ of a multi-dimensional lattice Λ, a (regular) region ("potato") $\mathcal{G}$ in $\mathbb{R}^q$, or the whole Euclidean space $\mathbb{R}^q$ (of course, under additional asymptotic relations at infinity).
- substitute the operator of the second order derivative, i.e., the one-dimensional Laplace operator by *special elliptic differential operators* such as iterated Laplace or Helmholtz operators.
- transfer the classical "boundary conditions" of periodicity into *other boundary conditions* (e.g., of Dirichlet's/Neumann's type).
- "blow up" the multi-dimensional lattice points to *"lattice balls"* for establishing lattice ball analogues of Euler and Poisson summation formulas.

The critical ingredients of our approach to Euler summation are twofold:

On the one hand, the multi-dimensional generalization of the Bernoulli function, i.e., the Green function for a Helmholtz operator and "periodical boundary conditions", is not (yet) available as an elementary function. In addition, the multi-dimensional counterparts of the Fourier series expansion (1.6) are divergent. This is the reason why we first condense the original Bernoulli functions to their constituting properties (see (1.12) – (1.15) for more details) in order to find a setup of a uniquely determined definition in terms of specific features. In turn, the constituting properties of a Green function $G(\Delta + \lambda; \cdot)$ (namely, boundary condition, differential equation, characteristic singularity, normalization) can be used as the keystones to characterize the role of Green's function in generalized variants of the Euler summation formula.

On the other hand, from a structural point of view (every generalization of) the Euler summation formula rests on the basic idea to relate a sum of values of a function at finitely or infinitely many successive nodes to certain sums involving (Helmholtz) derivatives of this function. Clearly, this makes things complicated. But it opens the perspective to stop sums with an expectation that the particular value in which one is interested lies between any partial sum and another one, all of them being explicitly calculable. In doing so, the summation formula provides appropriate approximation, and the (infinite) series or integrals can also be attacked asymptotically even if they diverge. Moreover, the "wave number" $\lambda \in \mathbb{R}$ of an Helmholtz operator $\Delta + \lambda$ may be used to adapt the operator to the specific properties of a summand showing an alternating character, e.g., in order to force the convergence of the associated infinite series.

Following our approach of establishing extensions of the Euler summation formula by specifying particular classes of Green's function within a certain framework (once more, boundary condition, Helmholtz differential operator, singularity and normalization (if necessary)), we are able to make essential scientific progress in both formulating convergence criteria for a multi-dimensional series in adaptation to the (oscillating) properties of the summand and representing the series in terms of certain volume and surface integrals to come up with relevant lattice point identities of number theoretical significance.

Indeed, the list of significant topics and innovative results based on the Euler summation with respect to Helmholtz operators is long. It enables us to

- interrelate Green functions to Zeta and Theta functions,
- develop convergence criteria for (alternating) multi-dimensional series (always understood here in spherical summation),
- formulate adapted conditions for the validity of the multi-dimensional Poisson summation formula in Euclidean spaces,
- outline Euler summation formulas for regular ("potato"-like) regions,
- deduce Poisson summation formulas for regular ("potato"-like) regions in the sense of Gauß–Weierstraß or Abel–Poisson summability,
- verify extended Hardy–Landau identities for "lattice point" sums as well as "lattice ball" sums in spheres,
- derive asymptotic relations for weighted lattice point sums,
- explain non-uniform radial and angular distributions of lattice points.
- develop comparisons of asymptotic laws between lattice ball and lattice point sums.

As essential ingredients to establish our results we need a number of auxiliary means and tools such as

- fundamental solutions for the iterated Laplace equation, asymptotic laws for spherical integrals involving Green functions with respect to (iterated) Helmholtz operators, the appropriate ball averaging of the Green functions,

- an (alternative) approach to the theory of spherical harmonics,
- fundamental solutions for the iterated (spherical) Laplace–Beltrami equation, spherical integral formulas for the Beltrami operator, discrepancy representations by means of the Green function on the sphere for the Beltrami operator,
- the (metaharmonic) theory of cylinder functions (Bessel, Hankel, Kelvin, Neumann functions, etc.), asymptotic rules for entire solutions of the Helmholtz equation (i.e., the reduced wave equation),
- integral transforms, e.g., Abel–Poisson transform, Gauß–Weierstraß transform,
- the Fourier inversion formula for discontinuous functions, however, in the pointwise sense,
- Hankel transform involving discontinuous integrals in terms of Bessel functions,
- functional equations of Zeta and Theta functions,
- lattice point and lattice ball discrepancies as specific expressions in terms of periodical Green functions,
- the theory of almost periodicity in the (B^2)-Besicovitch sense,
- asymptotic expansions for weighted "lattice point" and "lattice ball" discrepancies,
- "width" and "phase" dependent quantification of planar radial lattice point distributions.

2

Basic Notation

CONTENTS

2.1 Cartesian Nomenclature

Throughout this book we base our considerations on the following notational background.

The letters $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote the set of positive, non-negative integers, integers, real numbers, and complex numbers, respectively.

As usual, we write $x, y, \ldots$ to represent the elements of the q-dimensional (real) Euclidean space $\mathbb{R}^q$ ($q \geq 1$). In Cartesian coordinates we have the component representation (q-tuples of real numbers)

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_q \end{pmatrix}. \tag{2.1}$$

If necessary we write $x_{(q)}$ instead of x to point out that x is an element of $\mathbb{R}^q$. The canonical orthonormal system in $\mathbb{R}^q$ is denoted by $\epsilon^1, \ldots, \epsilon^q$. More explicitly,

$$\epsilon^1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots, \ \epsilon^q = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{2.2}$$

Any $x \in \mathbb{R}^q$ may be represented in Cartesian coordinates $x_i, i = 1, \ldots, q$, by

$$x = \sum_{i=1}^{q} x_i \epsilon^i. \tag{2.3}$$

In Cartesian coordinates the *inner (scalar) product* of two elements $x, y \in \mathbb{R}^q$ is given by

$$x \cdot y = x^T y = \sum_{i=1}^{q} x_i y_i. \tag{2.4}$$

Clearly,

$$x^2 = |x|^2 = x \cdot x = x^T x, \quad x \in \mathbb{R}^q, \tag{2.5}$$

i.e., *the norm* in $\mathbb{R}^q$ is given

$$|x| = \sqrt{x \cdot x} = \sqrt{x^T x}, \quad x \in \mathbb{R}^q. \tag{2.6}$$

Given a vector $a \in \mathbb{R}^q$ and a set $\mathcal{M} \subset \mathbb{R}^q$. Let $\mathcal{M} + \{a\}$ denote the set of all points $y = x + a$, as x runs through the points of $\mathcal{M}$. $\mathcal{M} + \{a\}$ is the *translate* of the set $\mathcal{M}$ by a. More generally, if $\mathcal{N}$ denotes some set of vectors from $\mathbb{R}^q$ then by $\mathcal{M} + \mathcal{N}$ we understand the set of all points $y = x + a$ for an arbitrary x from $\mathcal{M}$ and an arbitrary a from $\mathcal{N}$.

If $\mathcal{G}$ is a set of points in $\mathbb{R}^q$, $\partial\mathcal{G}$ will denote its *boundary*. The set $\overline{\mathcal{G}} = \mathcal{G} \cup \partial\mathcal{G}$ is called the *closure* of $\mathcal{G}$. A set $\mathcal{G} \subset \mathbb{R}^q$ is called a *region* if and only if it is open and connected.

By a scalar or vector function (field) on a region $\mathcal{G} \subset \mathbb{R}^q$, we mean a function that assigns to each point of $\mathcal{G}$, a scalar or vectorial function value, respectively. Unless otherwise specified, all functions are assumed to be complex valued. It will be of advantage to use the following general scheme of notation:

capital letters F, G	:	scalar functions,
lower-case letters f, g	:	vector fields.

The restriction of a scalar-valued function F or a vector-valued function f to a subset M of its domain is denoted by $F|M$ or $f|M$, respectively. For a set S of functions, we set $S|M = \{F|M \;\big|F \in S\}$.

Differential Operators

Let $\mathcal{G} \subset \mathbb{R}^q$ be a region. Suppose that $F : \mathcal{G} \to \mathbb{C}$ is differentiable. $\nabla F : x \mapsto (\nabla F)(x)$, $x \in \mathcal{G}$, denotes the *gradient* of F on $\mathcal{G}$. The *partial derivatives* of F at $x \in \mathcal{G}$, sometimes briefly written as $F_{|i}$, $i \in \{1, \ldots, q\}$, are given by

$$F_{|i}(x) = \frac{\partial F}{\partial x_i}(x) = (\mathrm{grad}_x\ F)(x) \cdot \epsilon^i = (\nabla_x F)(x) \cdot \epsilon^i = ((\nabla_x F)(x))_i \ . \tag{2.7}$$

Let $u : \mathcal{G} \to \mathbb{C}^q$ be a vector field, and suppose, in addition, that u is differentiable at a point $x \in \mathcal{G}$. The partial derivatives of u at $x \in \mathcal{G}$ are given by

$$u_{i|j}(x) = \frac{\partial u_i}{\partial x_j}(x) = \epsilon^i \cdot (\nabla u)(x)\epsilon^j \ . \tag{2.8}$$

The *divergence* of u at $x \in \mathcal{G}$ is the scalar value

$$\nabla_x \cdot u(x) = \mathrm{div}_x u(x) = \mathrm{tr}\ (\nabla u)(x)\ . \tag{2.9}$$

Thus, we have the identity

$$\nabla_x \cdot u(x) = \mathrm{div}_x u(x) = \sum_{i=1}^{q} u_{i|i}(x)\ . \tag{2.10}$$

Let F be a differentiable scalar field on $\mathcal{G}$, and suppose, in addition, that ∇F is differentiable at $x \in \mathcal{G}$. Then we introduce the *Laplace operator (Laplacian)* of F at $x \in \mathcal{G}$ by

$$\Delta_x F(x) = \mathrm{div}_x\left((\mathrm{grad}_x\ F)(x)\right) = \nabla_x \cdot \left(\left(\nabla_x F\right)(x)\right). \tag{2.11}$$

Analogously, we define the *Laplacian* of a vector field $f : \mathcal{G} \to \mathbb{C}^q$ (with ∇f being differentiable at $x \in \mathcal{G}$) by

$$\Delta_x f(x) = \mathrm{div}_x\left((\mathrm{grad}_x\ f)(x)\right) = \nabla_x \cdot \left(\left(\nabla_x f\right)(x)\right). \tag{2.12}$$

Clearly, for sufficiently often differentiable fields F, f, we have

$$\Delta_x F(x) = \sum_{i=1}^{q} F_{|i|i}(x), \tag{2.13}$$

$$\Delta_x f(x) \cdot \epsilon^i = \sum_{j=1}^{q} f_{i|j|j}(x)\ . \tag{2.14}$$

Multi-Indices

Let $\alpha = (\alpha_1, \ldots, \alpha_q)^{\mathrm{T}}$ be a q-tuple of non-negative integers $\alpha_1, \ldots, \alpha_q$, i.e., $\alpha \in \mathbb{N}_0{}^q$. We set

$$\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_q!, \tag{2.15}$$

$$[\alpha] = \alpha_1 + \ldots + \alpha_q, \tag{2.16}$$

$$|\alpha| = \sqrt{\alpha_1^2 + \ldots + \alpha_q^2}. \tag{2.17}$$

We say $\alpha = (\alpha_1, \ldots, \alpha_q)^{\mathrm{T}}$ is a q-dimensional *multi-index of degree* n if $[\alpha] = n$. As usual, we set

$$x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_q^{\alpha_q},\ \ x \in \mathbb{R}^q,\ \alpha \in \mathbb{N}_0^q, \tag{2.18}$$

$$(\nabla_x)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_q}\right)^{\alpha_q} = \frac{\partial^{[\alpha]}}{(\partial x_1)^{\alpha_1} \ldots (\partial x_q)^{\alpha_q}}. \tag{2.19}$$

Clearly, for $[\alpha] = [\beta]$, we have

$$(\nabla_x)^\alpha \, x^\beta = \begin{cases} 0 & , \quad \alpha \neq \beta, \\ \alpha! & , \quad \alpha = \beta. \end{cases} \tag{2.20}$$

In this notation of multi-indices we have

$$\left(\sum_{i=1}^{q} x_i \right)^n = \sum_{[\alpha]=n} \frac{n!}{\alpha!} x^\alpha. \tag{2.21}$$

As is well known, for $x, y \in \mathbb{R}^q$, the *binomial theorem* reads

$$(x \cdot y)^n = \left(\sum_{i=1}^{q} x_i y_i \right)^n = \sum_{[\alpha]=n} \frac{n!}{\alpha!} x^\alpha y^\alpha. \tag{2.22}$$

2.2 Regular Regions

A bounded region $\mathcal{G} \subset \mathbb{R}^q$ is called *regular*, if its boundary $\partial\mathcal{G}$ is an orientable piecewise smooth Lipschitzian manifold of dimension $q-1$ (for more details about regular regions the reader is referred to textbooks of vector analysis). Examples are ball, cube, other polyhedra, geoid(al potato), (real) Earth's body, etc.

$F \in \mathrm{C}^{(k)}(\overline{\mathcal{G}}),\ 0 \leq k \leq \infty$, means that the function $F : \overline{\mathcal{G}} \to \mathbb{C}$ is k-times continuously differentiable in $\overline{\mathcal{G}} = \mathcal{G} \cup \partial\mathcal{G}$. By convention, $F \in \mathrm{C}^{(k-1)}(\overline{\mathcal{G}}) \cap \mathrm{C}^{(k)}(\mathcal{G})$ means that the function $F : \overline{\mathcal{G}} \to \mathbb{C}$ is $(k-1)$-times continuously differentiable in $\overline{\mathcal{G}}$ such that $F|\mathcal{G}$ is k-times continuously differentiable.

The *volume of a regular region* $\mathcal{G} \subset \mathbb{R}^q$ is given by

$$||\mathcal{G}|| \ = \ \int_{\mathcal{G}} dV_{(q)}(x), \tag{2.23}$$

where

$$dV_{(q)}(x) = dx_1 \ldots dx_q \tag{2.24}$$

is the volume element.

The *area of the boundary* $\partial\mathcal{G}$ *of a regular region* $\mathcal{G} \subset \mathbb{R}^q$ is given by

$$||\partial\mathcal{G}|| \ = \ \int_{\partial\mathcal{G}} dS_{(q-1)}(x), \tag{2.25}$$

where $dS_{(q-1)}(x)$ is the surface element

$$\begin{aligned} dS_{(q-1)}(x) \quad &= \quad x_1 \; dx_2 \ldots dx_q \\ &\quad\ -x_2 \; dx_1 dx_3 \ldots dx_q \\ &\quad\ + - \quad \ldots \\ &\quad\ +(-1)^{q-1} x_q \; dx_1 \ldots dx_{q-1}. \end{aligned} \tag{2.26}$$

Remark 2.1. *Throughout this work, for integration in the q-dimensional Euclidean space and on the boundary surface $\partial\mathcal{G}$ in $\mathbb{R}^q$, we use the traditional (non-oriented) notations dV and dS, respectively. If the dimension and the variable of integration must be specified, the notations $dV_{(q)}(x)$ and $dS_{(q-1)}(x)$ are used, respectively.*

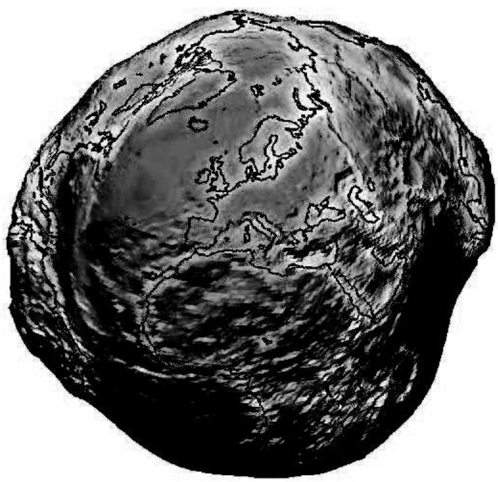

FIGURE 2.1
Typical (geomathematically relevant) regular region ("geoidal potato").

2.3 Spherical Nomenclature

As usual, the unit sphere in $\mathbb{R}^q$ is denoted by $\mathbb{S}^{q-1}$:

$$\mathbb{S}^{q-1} = \left\{ x \in \mathbb{R}^q \middle| \, |x| = 1 \right\}. \tag{2.27}$$

Each $x \in \mathbb{R}^q$, $x = (x_1, \ldots, x_q)^{\mathrm{T}}$, $|x| \neq 0$, admits a *representation in polar coordinates* of the form

$$x = r\xi, \; r = |x|, \; \xi = (\xi_1, \ldots, \xi_q)^{\mathrm{T}}, \tag{2.28}$$

where $\xi \in \mathbb{S}^{q-1}$ is the uniquely determined (unit) vector of x.

Using the canonical orthonormal basis $\epsilon^1, \ldots, \epsilon^q$ in $\mathbb{R}^q$ (more accurately, $\epsilon^1_{(q)}, \ldots, \epsilon^q_{(q)}$ in $\mathbb{R}^q$) we are able to write $\xi_{(q)} \in \mathbb{S}^{q-1}$, $q \geq 3$, in the form

$$\xi_{(q)} = t\epsilon^q_{(q)} + \sqrt{1-t^2}\,\xi_{(q-1)},\ t \in [-1,1],\ \xi_{(q-1)} \in \mathbb{S}^{q-2}, \tag{2.29}$$

$$\xi_{(2)} = (\cos\varphi, \sin\varphi)^T,\ \xi_{(2)} \in \mathbb{S}^1,\ \varphi \in [0, 2\pi). \tag{2.30}$$

Differential Operators

By means of polar coordinates $x_{(q)} = r\xi_{(q)}$, $r = |x_{(q)}|$, $\xi_{(q)} \in \mathbb{S}^{q-1}$, the *gradient* ∇ in $\mathbb{R}^q$ can be represented in the form

$$\nabla_{x_{(q)}} = \xi_{(q)}\frac{\partial}{\partial r} + \frac{1}{r}\nabla^*_{\xi_{(q)}}, \tag{2.31}$$

where ∇^* is the *surface gradient* on $\mathbb{S}^{q-1}$. Moreover, in terms of spherical coordinates the *Laplace operator (Laplacian)* $\Delta = \nabla \cdot \nabla$ in $\mathbb{R}^q$

$$\Delta_{x_{(q)}} = \left(\frac{\partial}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial}{\partial x_q}\right)^2 \tag{2.32}$$

has the representation

$$\Delta_{x_{(q)}} = r^{1-q}\frac{\partial}{\partial r}r^{q-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^*_{\xi_{(q)}}, \tag{2.33}$$

where Δ^* describes the *Laplace–Beltrami operator* of the unit sphere $\mathbb{S}^{q-1}$ recursively given by

$$\Delta^*_{\xi_{(q)}} = (1-t^2)\left(\frac{\partial}{\partial t}\right)^2 - (q-1)t\frac{\partial}{\partial t} + \frac{1}{1-t^2}\Delta^*_{\xi_{(q-1)}},\ q \geq 3, \tag{2.34}$$

$$\Delta^*_{\xi_{(2)}} = \left(\frac{\partial}{\partial\varphi}\right)^2 \tag{2.35}$$

(if no confusion is likely to arise the Laplace–Beltrami operator is simply called the Beltrami operator).

Clearly,

$$\Delta^* = \nabla^* \cdot \nabla^*, \tag{2.36}$$

where $\nabla^*\cdot$ is the *surface divergence* on $\mathbb{S}^{q-1}$ (for more details concerning the differential operators in the three-dimensional case see, e.g., W. Freeden, M. Schreiner [2009]).

Spheres and Balls

The *sphere in* $\mathbb{R}^q$ *with radius* R *around* $y \in \mathbb{R}^q$ is denoted by $\mathbb{S}^{q-1}_R(y)$

$$\mathbb{S}^{q-1}_R(y) = \left\{x \in \mathbb{R}^q \;\middle|\; |x-y| = R\right\}, \tag{2.37}$$

and $\mathbb{S}_R^{q-1}$ is the sphere with radius R around 0 (i.e., $\mathbb{S}_R^{q-1} = \mathbb{S}_R^{q-1}(0)$).

$\mathbb{B}_R^q(y)$ denotes *the (open) ball in the Euclidean space* $\mathbb{R}^q$ *with center* $y \in \mathbb{R}^q$ *and radius* R:

$$\mathbb{B}_R^q(y) = \{x \in \mathbb{R}^q \big| \; |x-y| < R\}. \tag{2.38}$$

The closure of the ball $\mathbb{B}_R^q(y) \subset \mathbb{R}^q$ is given by

$$\overline{\mathbb{B}_R^q(y)} = \{x \in \mathbb{R}^q \big| \; |x-y| \leq R\}. \tag{2.39}$$

We simply write $\mathbb{B}_R^q$ and $\overline{\mathbb{B}_R^q}$, respectively, for the open and closed ball with radius R around the origin 0.

By $\mathbb{B}_{\rho,R}^q(y)$, $0 \leq \rho < R$, we denote the *ball ring in the Euclidean space* $\mathbb{R}^q$ *with center* $y \in \mathbb{R}^q$ *and radii* ρ *and* R given by

$$\mathbb{B}_{\rho,R}^q(y) = \{x \in \mathbb{R}^q \big| \; \rho < |x-y| < R\}, \tag{2.40}$$

i.e.,

$$\mathbb{B}_{\rho,R}^q(y) \;=\; \mathbb{B}_R^q(y) \backslash \overline{\mathbb{B}_\rho^q(y)}. \tag{2.41}$$

2.4 Radial and Angular Functions

A function $G : \overline{\mathbb{B}_{\rho,N}^q} \to \mathbb{C}$ is called *radial* in $\overline{\mathbb{B}_{\rho,N}^q}$, $0 \leq \rho \leq N$, if for all $x \in \overline{\mathbb{B}_{\rho,N}^q}$

$$G(x) = G(r\xi) = G(r), \quad x = r\xi,\; r = |x|. \tag{2.42}$$

A function $H : \overline{\mathbb{B}_{\rho,N}^q} \to \mathbb{C}$ is called *angular* in $\overline{\mathbb{B}_{\rho,N}^q}$, $0 < \rho \leq N$, if for all $x \in \overline{\mathbb{B}_{\rho,N}^q}$

$$H(x) = H(r\xi) = H(\xi), \quad x = r\xi,\; r = |x|. \tag{2.43}$$

The Laplace derivative of a radial and angular function, respectively, is of particular significance for our later work

$$\Delta_x G(x) = r^{1-q} \frac{\partial}{\partial r} r^{q-1} \frac{\partial}{\partial r} G(r), \quad r \in [\rho, N], \tag{2.44}$$

$$\Delta_x H(x) = \frac{1}{r^2} \Delta_\xi^* H(\xi), \quad \xi \in \mathbb{S}^{q-1}. \tag{2.45}$$

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3

One-Dimensional Auxiliary Material

CONTENTS

In this chapter we provide well known one-dimensional tools and methods of basic importance for this work. The point of departure is the Gamma function. A central topic is the Stirling formula. Particular attention is paid to generalizations of the Riemann–Lebesgue theorem known from the Fourier theory. We continue with some procedures of the stationary phase that turn out to be extremely helpful to secure the convergence of weighted lattice point sums including Fourier integrals (as discussed, for example, in Subsection 13.2). Finally, our considerations are dedicated to Abel–Poisson and Gauß–Weierstraß limit relations as canonical preparations for the Abel–Poisson and Gauß–Weierstraß transforms in multi-dimensional Euclidean spaces $\mathbb{R}^q$ (as studied in Section 7.4 and applied to the lattice point theory in Section 12.1).

3.1 Gamma Function and Its Properties

First our purpose is to introduce the classical Gamma function. Its essential properties are explained (for a more detailed discussion the reader is referred, e.g., to N. Nielsen [1906], E.T. Whittaker, G.N. Watson [1948], N.N. Lebedev [1973], C. Müller [1998], and the references therein). In particular, the Stirling formula is verified. The extension of the Gamma function to complex values is studied.

Definition and Functional Equation

For real values $x > 0$ we consider the integrals

$$(\alpha) \qquad \int_0^1 e^{-t} t^{x-1} dt \tag{3.1}$$

and

$$(\beta) \qquad \int_1^\infty e^{-t} t^{x-1} dt. \tag{3.2}$$

In order to show the convergence of (α), we observe that $0 < e^{-t} t^{x-1} \leq t^{x-1}$ holds true for all $t \in (0,1]$. Therefore, for $\varepsilon > 0$ sufficiently small, we have

$$\int_\varepsilon^1 e^{-t} t^{x-1} dt \leq \int_\varepsilon^1 t^{x-1} dt = \left. \frac{t^x}{x} \right|_\varepsilon^1 = \frac{1}{x} - \frac{\varepsilon^x}{x}. \tag{3.3}$$

Consequently, for all $x > 0$, the integral (α) is convergent. To guarantee the convergence of (β) we observe that

$$e^{-t} t^{x-1} \leq \frac{n!}{t^{n-x+1}} \tag{3.4}$$

for all $n \in \mathbb{N}$ and $t \geq 1$. This shows us that

$$\int_1^A e^{-t} t^{x-1} dt \leq n! \int_1^A \frac{1}{t^{n-x+1}} dt = n! \left. \frac{t^{-n+x}}{x-n} \right|_1^A = \frac{n!}{x-n} \left(\frac{1}{A^{n-x}} - 1 \right) \tag{3.5}$$

provided that A is sufficiently large and n is chosen such that $n \geq x+1$. Thus, the integral (β) is convergent.

The point of departure is the following integral representation.

Lemma 3.1. *For all $x > 0$, the integral*

$$\int_0^\infty e^{-t} t^{x-1} dt \tag{3.6}$$

is convergent.

By definition we let

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \tag{3.7}$$

Definition 3.1. *The function $x \mapsto \Gamma(x)$, $x > 0$, as defined by (3.7), is called the Gamma function.*

Obviously, we have the following properties:

(i) Γ *is positive for all* $x > 0$,

(ii) $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$.

Integration by parts yields

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty e^{-t}\, t^x dt = -e^{-t} t^x \Big|_0^\infty - \int_0^\infty (-e^{-t})\, x t^{x-1} dt \\
&= x \int_0^\infty e^{-t} t^{x-1} dt = x\, \Gamma(x).
\end{aligned} \tag{3.8}$$

Lemma 3.2. *The Gamma function* Γ *satisfies the functional equation*

$$\Gamma(x+1) = x\Gamma(x), \qquad x > 0. \tag{3.9}$$

As an immediate consequence we obtain

$$\Gamma(x+n) = (x+n-1)\cdots(x+1)x\, \Gamma(x) \tag{3.10}$$

for $x > 0$ and $n \in \mathbb{N}$. This gives us

Lemma 3.3. *For* $n \in \mathbb{N}_0$,

$$\Gamma(n+1) = n!\,. \tag{3.11}$$

Proof. The assertion is clear for $n = 0, 1$. For $n \geq 2$ we have

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) \qquad (3.12)\\
&= n(n-1)\Gamma(n-1) \\
&= n \cdot \ldots \cdot 1\, \underbrace{\Gamma(1)}_{=1} \\
&= n!\,,
\end{aligned}$$

as required. □

Remark 3.1. *The Gamma function restricted to positive integers is the well known factorial function.*

Next we deal with the derivatives of the Gamma function.

Lemma 3.4. *The Gamma function* Γ *is differentiable for all* $x > 0$, *and we have*

$$\Gamma'(x) = \int_0^\infty e^{-t} (\ln(t)) t^{x-1} dt. \tag{3.13}$$

Γ is infinitely often differentiable for all $x > 0$, and we have

$$\Gamma^{(k)}(x) = \int_0^\infty e^{-t}(\ln(t))^k t^{x-1} dt, \ k \in \mathbb{N}. \tag{3.14}$$

An elementary calculation shows us that

$$\begin{aligned} (\Gamma'(x))^2 &= \left(\int_0^\infty e^{-t}(\ln(t)) t^{x-1} dt\right)^2 \\ &= \left(\int_0^\infty e^{-\frac{t}{2}} t^{\frac{x-1}{2}} (\ln(t)) e^{-\frac{t}{2}} t^{\frac{x-1}{2}} dt\right)^2. \end{aligned} \tag{3.15}$$

The Cauchy-Schwarz inequality yields

$$\begin{aligned} (\Gamma'(x))^2 &\leq \int_0^\infty \left(e^{-\frac{t}{2}} t^{\frac{x-1}{2}}\right)^2 dt \int_0^\infty \left(e^{-\frac{t}{2}} t^{\frac{x-1}{2}} (\ln(t))\right)^2 dt \\ &= \int_0^\infty e^{-t} t^{x-1} dt \int_0^\infty e^{-t} t^{x-1} (\ln(t))^2 \ dt \\ &= \Gamma(x) \ \Gamma''(x). \end{aligned} \tag{3.16}$$

Lemma 3.5. *(Gauß' Expression of the Second Order Logarithmic Derivative) For $x > 0$,*

$$(\Gamma'(x))^2 \leq \Gamma(x) \ \Gamma''(x). \tag{3.17}$$

Equivalently, we have

$$\left(\frac{d}{dx}\right)^2 \ln(\Gamma(x)) = \frac{\Gamma''(x)}{\Gamma(x)} - \left(\frac{\Gamma'(x)}{\Gamma(x)}\right)^2 > 0. \tag{3.18}$$

In other words, $x \mapsto \ln(\Gamma(x))$, $x > 0$, is a convex function.

Euler's Beta Function

Next we notice that for $\gamma > 0, \ \delta > 0$, the integral

$$\int_0^1 t^{\gamma-1}(1-t)^{\delta-1} dt \tag{3.19}$$

is convergent.

Definition 3.2. *The function $(\gamma, \delta) \mapsto B(\gamma, \delta), \ \gamma, \delta > 0$, defined by*

$$B(\gamma, \delta) = \int_0^1 t^{\gamma-1}(1-t)^{\delta-1} \ dt \tag{3.20}$$

is called the Euler Beta function.

For $\gamma, \delta > 0$ we see that

$$\begin{aligned} \Gamma(\gamma)\Gamma(\delta) &= \int_0^\infty e^{-t} t^{\gamma-1} dt \int_0^\infty e^{-s} s^{\delta-1}\, ds \\ &= \iint\limits_{\substack{0\le t<\infty \\ 0\le s<\infty}} e^{-(t+s)} t^{\gamma-1} s^{\delta-1} dt\, ds. \end{aligned} \tag{3.21}$$

Note that the transition from one-dimensional to two-dimensional integrals is permitted by Fubini's theorem.

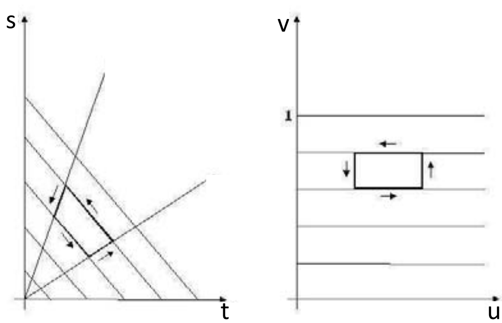

FIGURE 3.1
The illustration of the coordinate transformation relating the Beta and the Gamma functions.

We make a coordinate transformation (cf. Figure 3.1) as follows:

$$\begin{aligned} t &= u(1-v) \quad , \quad 0 \le u < \infty, \\ s &= uv \quad , \quad 0 \le v \le 1. \end{aligned} \tag{3.22}$$

It is not difficult to verify that the functional determinant of the coordinate transformation is given by

$$\frac{\partial(t,s)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u(1-v) + uv = u \ge 0 \tag{3.23}$$

Thus we find

$$\begin{aligned}\int\int_{\substack{0\le t<\infty\\0\le s<\infty}} e^{-(t+s)}t^{\gamma-1}s^{\delta-1}dt\ ds &= \int\int_{\substack{0\le v\le 1\\0\le u<\infty}} e^{-u}(u(1-v))^{\gamma-1}(uv)^{\delta-1}u\ du\ dv\\ &= \int\int_{\substack{0\le v\le 1\\0\le u<\infty}} e^{-u}u^{\gamma+\delta-2}(1-v)^{\gamma-1}v^{\delta-1}u\ du\ dv\\ &= \int_0^\infty e^{-u}u^{\gamma+\delta-1}du\int_0^1 v^{\delta-1}(1-v)^{\gamma-1}dv.\end{aligned} \tag{3.24}$$

This leads to

Theorem 3.1. *For $\gamma,\delta>0$*

$$B(\gamma,\delta)=\frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}. \tag{3.25}$$

In particular,

$$\begin{aligned}B\left(\frac{1}{2},\frac{1}{2}\right) &= \int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\ dt \\ &= 2\int_0^1 (1-u^2)^{-\frac{1}{2}}\ du\\ &= 2\arcsin(1)=2\frac{\pi}{2}=\pi.\end{aligned} \tag{3.26}$$

Therefore we have

$$\frac{\Gamma^2\left(\frac{1}{2}\right)}{\Gamma(1)}=\pi. \tag{3.27}$$

This shows that

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}=\int_0^\infty e^{-t}t^{-\frac{1}{2}}\ dt. \tag{3.28}$$

Other types of integrals can be derived from

$$\begin{aligned}\int_0^\infty e^{-t^\alpha}dt &\overset{u=t^\alpha}{=} \frac{1}{\alpha}\int_0^\infty e^{-u}u^{\frac{1}{\alpha}-1}\ du\\ &= \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right),\ \alpha>0.\end{aligned} \tag{3.29}$$

Lemma 3.6. *For $\alpha>0$,*

$$\int_0^\infty e^{-t^\alpha}dt=\Gamma\left(\frac{\alpha+1}{\alpha}\right). \tag{3.30}$$

In particular,

$$\int_0^\infty e^{-t^2} dt = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}. \tag{3.31}$$

Moreover, we have

$$\int_0^\infty t^{\gamma-1} e^{-t^\alpha} dt = \frac{1}{\alpha}\Gamma\left(\frac{\gamma}{\alpha}\right), \quad \gamma, \alpha > 0 \tag{3.32}$$

and

$$\int_0^\infty t^{\gamma-1} e^{-\alpha t^2} dt = \frac{1}{2}\alpha^{-\frac{\gamma}{2}}\Gamma\left(\frac{\gamma}{2}\right), \quad \gamma, \alpha > 0. \tag{3.33}$$

q=1	$\|\|\mathbb{S}^0\|\| = 2$
q=2	$\|\|\mathbb{S}^1\|\| = 2\pi$
q=3	$\|\|\mathbb{S}^2\|\| = 4\pi$

TABLE 3.1
The area of the unit sphere $\mathbb{S}^{q-1}$ for $q = 1, 2, 3$.

Within the notational framework of polar coordinates (2.29), (2.30) we give the well known calculation of the area $||\mathbb{S}^{q-1}||$ of the unit sphere $\mathbb{S}^{q-1}$ in $\mathbb{R}^q$: By definition (see Table 3.1), we set

$$||\mathbb{S}^0|| = 2, \tag{3.34}$$

$\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$; hence, its area is equal to

$$||\mathbb{S}^1|| = 2\pi. \tag{3.35}$$

Furthermore, $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$; hence, its area is known to be equal to

$$||\mathbb{S}^2|| = 4\pi. \tag{3.36}$$

We are interested in deriving the area of the sphere $\mathbb{S}^{q-1}$ in $\mathbb{R}^q$ $(q > 3)$:

$$||\mathbb{S}^{q-1}|| = \int_{\mathbb{S}^{q-1}} dS_{(q-1)}(\xi_{(q)}). \tag{3.37}$$