

The background of the cover features several complex, colorful mathematical surfaces. In the top left, there is a surface with concentric, rainbow-colored rings. In the top right, a more intricate surface with multiple lobes and a central vortex-like structure is shown. On the right side, a series of stacked, flared, cone-like shapes are visible. In the bottom left, a smooth, curved surface transitions from blue to pink. In the bottom right, a complex, knotted surface with a grid pattern and rainbow colors is depicted.

SECOND EDITION

CRC CONCISE
ENCYCLOPEDIA
OF
MATHEMATICS

ERIC W. WEISSTEIN



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THE COVER

The cover of this book consists of a collage of images from the second edition of Alfred Gray's "Modern Differential Geometry of Curves and Surfaces with Mathematica" published in 1998 by CRC Press LLC. Thanks go to Jonathan Pennell for his patience and help with the cover for the new edition.

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Introduction to the First Edition

The *CRC Concise Encyclopedia of Mathematics* is a compendium of mathematical definitions, formulas, figures, tabulations, and references. It is written in an informal style intended to make it accessible to a broad spectrum of readers with a wide range of mathematical backgrounds and interests. Although mathematics is a fascinating subject, it all too frequently is clothed in specialized jargon and dry formal exposition that make many interesting and useful mathematical results inaccessible to laypeople. This problem is often further compounded by the difficulty in locating concrete and easily understood examples. To give perspective to a subject, I find it helpful to learn why it is useful, how it is connected to other areas of mathematics and science, and how it is actually implemented. While a picture may be worth a thousand words, explicit examples are worth at least a few hundred! This work attempts to provide enough details to give the reader a flavor for a subject without getting lost in minutiae. While absolute rigor may suffer somewhat, I hope the improvement in usefulness and readability will more than make up for the deficiencies of this approach.

The format of this work is somewhere between a handbook, a dictionary, and an encyclopedia. It differs from existing dictionaries of mathematics in a number of important ways. The entries are extensively cross-referenced, not only to related entries but also to many external sites on the Internet. This makes locating information very convenient. It also provides a highly efficient way to “navigate” from one related concept to another. Standard mathematical references, combined with a few popular ones, are also given at the end of most entries to facilitate additional reading and exploration. In the interests of offering abundant examples, this work also contains a large number of explicit formulas and derivations, providing a ready place to locate a particular formula, as well as including the framework for understanding where it comes from.

The selection of topics in this work is more extensive than in most mathematical dictionaries (e.g., Borowski and Borwein’s *HarperCollins Dictionary of Mathematics* and Jeans and Jeans’ *Mathematics Dictionary*). At the same time, the descriptions are more accessible than in “technical” mathematical encyclopedias (e.g., Hazewinkel’s *Encyclopaedia of Mathematics* and Iyanaga’s *Encyclopedic Dictionary of Mathematics*). While the latter remain models of accuracy and rigor, they are not terribly useful to the undergraduate, research scientist, or recreational mathematician. In this work, the most useful, interesting, and entertaining (at least to my mind) aspects of topics are discussed in addition to their technical definitions. For example, in my entry for pi (π), the definition in terms of the diameter and circumference of a circle is supplemented by a great many formulas and series for pi, including some of the amazing discoveries of Ramanujan. These formulas are comprehensible to readers with only minimal mathematical background, and are interesting to both those with and without formal mathematics training. However, they have not previously been collected in a single convenient location. For this reason, I hope that, in addition to serving as a reference source, this work has some of the same flavor and appeal of Martin Gardner’s delightful *Scientific American* columns.

Everything in this work has been compiled by me alone. I am an astronomer by training, but have picked up a fair bit of mathematics along the way. It never ceases to amaze me how mathematical connections weave their way through the physical sciences. It frequently transpires that some piece of recently acquired knowledge turns out to be just what I need to solve some apparently unrelated problem. I have therefore developed the habit of picking up and storing away odd bits of information for future use. This work has provided a mechanism for organizing what has turned out to be a fairly large collection of mathematics. I have also found it very difficult to find clear yet accessible explanations of technical mathematics unless I already have some familiarity with the subject. I hope this encyclopedia will provide jumping-off points for people who are interested in the subjects listed here but who, like me, are not necessarily experts.

The encyclopedia has been compiled over the last 11 years or so, beginning in my college years and continuing during graduate school: The initial document was written in *Microsoft Word*® on a Mac Plus® computer, and had reached about 200 pages by the time I started graduate school in 1990. When Andrew Treverrow made his OzTEX program available for the Mac; I began the task of converting all my documents to TEX resulting in a vast improvement in readability. While undertaking the *Word* to TEX conversion, I also began cross-referencing entries, anticipating that eventually I would be able to convert the entire document to hypertext. This hope was realized beginning in 1995, when the Internet explosion was in full swing and I learned of Nikos Drakos’s excellent TEX to HTML converter, LATEX2HTML. After some additional effort, I was able to post an HTML version of my encyclopedia to the World Wide Web.

The selection of topics included in this compendium is not based on any fixed set of criteria, but rather reflects my own random walk through mathematics. In truth, there is no good way of selecting topics in such a work. The mathematician James Sylvester may have summed up the situation most aptly. According to

Sylvester (as quoted in the introduction to Ian Stewart's book *From Here to Infinity*), "Mathematics is not a book confined within a cover and bound between brazen clasps, whose contents it needs only patience to ransack; it is not a mine, whose treasures may take long to reduce into possession, but which fill only a limited number of veins and lodes; it is not a soil, whose fertility can be exhausted by the yield of successive harvests; it is not a continent or an ocean, whose area can be mapped out and its contour defined; it is as limitless as that space which it finds too narrow for its aspiration; its possibilities are as infinite as the worlds which are forever crowding in and multiplying upon the astronomer's gaze; it is as incapable of being restricted within assigned boundaries or being reduced to definitions of permanent validity, as the consciousness of life."

Several of Sylvester's points apply particularly to this undertaking: As he points out, mathematics itself cannot be confined to the pages of a book. The results of mathematics, however, are shared and passed on primarily through the printed (and now electronic) medium. While there is no danger of mathematical results being lost through lack of dissemination, many people miss out on fascinating and useful mathematical results simply because they are not aware of them. Not only does collecting many results in one place provide a single starting point for mathematical exploration, but it should also lessen the aggravation of encountering explanations for new concepts which themselves use unfamiliar terminology. In this work, the reader is only a cross-reference away from the necessary background material. As to Sylvester's second point, the very fact that the quantity of mathematics is so great means that any attempt to catalog it with any degree of completeness is doomed to failure. This certainly does not mean that it's not worth trying. Strangely, except for relatively small works usually on particular subjects, there do not appear to have been any substantial attempts to collect and display in a place of prominence the treasure trove of mathematical results that have been discovered (invented?) over the years (one notable exception being Sloane and Plouffe's *Encyclopedia of Integer Sequences*). This work, the product of the "gazing" of a single astronomer, attempts to fill that omission.

Finally, a few words about logistics. Because of the alphabetical listing of entries in the encyclopedia, neither table of contents nor index are included. In many cases, a particular entry of interest can be located from a cross-reference (indicated in SMALL CAPS TYPEFACE in the text) in a related article. In addition, most articles are followed by a "see also" list of related entries for quick navigation. This can be particularly useful if you are looking for a specific entry (say, "Zeno's Paradoxes"), but have forgotten the exact name. By examining the "see also" list at bottom of the entry for "Paradox," you will likely recognize Zeno's name and thus quickly locate the desired entry.

In cases where the same word is applied in different contexts, the context is indicated in parentheses or appended to the end. Examples of the first type are "Crossing Number (Graph)" and "Crossing Number (Link)." Examples of the second type are "Convergent Sequence" and "Convergent Series." In the case of an entry like "Euler Theorem," which may describe one of three or four different formulas, I have taken the liberty of adding descriptive words ("Euler's Something Theorem") to all variations, or kept the standard name for the most commonly used variant and added descriptive words for the others. In cases where specific examples are derived from a general concept, em dashes (—) are used (for example, "Fourier Series," "Fourier Series — Power Series," "Fourier Series — Square Wave," "Fourier Series — Triangle"). The decision to put a possessive 's at the end of a name or to use a lone trailing apostrophe is based on whether the final "s" is pronounced. "Gauss's Theorem" is therefore written out, whereas "Archimedes' Recurrence Formula" is not. Finally, given the absence of a definitive stylistic convention, plurals of numerals are written without an apostrophe (e.g., 1990s instead of 1990's).

In an endeavor of this magnitude, errors and typographical mistakes are inevitable. The blame for these lies with me alone. Although the current length makes extensive additions in a printed version problematic, I plan to continue updating, correcting, and improving the work..

Eric Weisstein
Charlottesville, Virginia
August 8, 1998

Preface to the New Edition

The long awaited second edition of this Encyclopedia is finished, and it is now more complete than ever. Heavily revised by the author Eric Weisstein over the past three years, it contains well over 3,000 pages. Mr. Weisstein has updated all of the original material, added approximately 3,600 new entries and many illustrations, and updated the bibliographies that follow each entry to include the most recent references. As yet another enhancement, this edition integrates the use of the Mathematica software into many of its entries, presenting the precise commands that allow you to implement the formulas presented, perform many different calculations, construct graphical displays of your results, and generate remarkable mathematical illustrations. This is a unique touch and to our knowledge, a first for an encyclopedia.

With definitions, formulas, and facts presented in clear, engaging prose along with a multitude of illustrations, extensive cross-references, and even links to the Internet, this new and improved edition remains one of the most readable and accessible references in mathematics. This is truly a unique book written by an individual who is clearly dedicated to the study and field of mathematics. Users of the first edition of the Encyclopedia have described it as “extraordinary,” “impressive,” and “fascinating,” and report spending hours browsing its pages simply for pleasure. We hope you will do the same.

Acknowledgments

Although I alone have compiled and typeset this work, many people have contributed indirectly and directly to its creation. I have not yet had the good fortune to meet Donald Knuth of Stanford University, but he is unquestionably the person most directly responsible for making this work possible. Before his mathematical typesetting program TEX, it would have been impossible for a single individual to compile such a work as this. Had Prof. Bateman owned a personal computer equipped with TEX, perhaps his shoe box of notes would not have had to await the labors of Erdelyi, Magnus, and Oberhettinger to become a three-volume work on mathematical functions. Andrew Trevorrow's shareware implementation of TEX for the Macintosh, OzTEX (www.kagi.com/authors/akt/oztex.html), was also of fundamental importance. Nikos Drakos and Ross Moore have provided another building block for this work by developing the LATEX2HTML program (www-dsed.llnl.gov/files/programs/unix/latex2html/manual/manual.html), which has allowed me to easily maintain and update an on-line version of the encyclopedia long before it existed in book form.

I would like to thank Steven Finch of MathSoft, Inc., for his interesting on-line essays about mathematical constants (www.mathsoft.com/asolve/constant/constant.html), and also for his kind permission to reproduce excerpts from some of these essays. I hope that Steven will someday publish his detailed essays in book form. Thanks also to Neil Sloane and Simon Plouffe for compiling and making available the printed and on-line (www.research.att.com/~njas/sequences/) versions of the Encyclopedia of Integer Sequences, an immensely valuable compilation of useful information which represents a truly mind-boggling investment of labor.

Thanks to Robert Dickau, Simon Plouffe, and Richard Schroepel for reading portions of the manuscript and providing a number of helpful suggestions and additions. Thanks also to algebraic topologist Ryan Budney for sharing some of his expertise, to Charles Walkden for his helpful comments about dynamical systems theory, and to Lambros Lambrou for his contributions. Thanks to David W. Wilson for a number of helpful comments and corrections. Thanks to Dale Rolfsen, compiler James Bailey, and artist Ali Roth for permission to reproduce their beautiful knot and link diagrams. Thanks to Gavin Theobald for providing diagrams of his masterful polygonal dissections. Thanks to Wolfram Research, not only for creating an indispensable mathematical tool in *Mathematica*[®], but also for permission to include figures from the *Mathematica*[®] book and *MathSource* repository for the braid, conical spiral, double helix, Enneper's surfaces, Hadamard matrix, helicoid, helix, Henneberg's minimal surface, hyperbolic polyhedra, Klein bottle, Maeder's "owl" minimal surface, Penrose tiles, polyhedron, and Scherk's minimal surfaces entries.

Sincere thanks to Judy Schroeder for her skill and diligence in the monumental task of proofreading the entire document for syntax. Thanks also to Bob Stern, my executive editor from CRC Press, for his encouragement, and to Mimi Williams of CRC Press for her careful reading of the manuscript for typographical and formatting errors. As this encyclopedia's entry on PROOFREADING MISTAKES shows, the number of mistakes that are expected to remain after three independent proofreadings is much lower than the original number, but unfortunately still nonzero. Many thanks to the library staff at the University of Virginia, who have provided invaluable assistance in tracking down many an obscure citation. Finally, I would like to thank the hundreds of people who took the time to e-mail me comments and suggestions while this work was in its formative stages. Your continued comments and feedback are very welcome.

Numerals

$(-1, 0, 1)$ -Matrix

The number of distinct $(-1, 0, 1)$ - $n \times n$ matrices (counting row and column permutations, the transpose, and multiplication by -1 as equivalent) having $2n$ different row and column sums for $n = 2, 4, 6, \dots$ are 1, 4, 39, 2260, 1338614, ... (Kleber). For example, the 2×2 matrix is given by

$$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$

To get the total number from these counts (assuming that 0 is not the missing sum, which is true for $n \leq 10$), multiply by $(2n!)^2$. In general, if an n -matrix which has different column and row sums (collectively called line sums), then

1. n is even,
2. The number in $\{-n, 1-n, 2-n, \dots, n\}$ that does not appear as a line sum is either $-n$ or n , and
3. Of the largest line sums, half are column sums and half are row sums

(Bodendiek and Burosch 1995, F. Galvin).

See also ALTERNATING SIGN MATRIX, C -MATRIX, INTEGER MATRIX

References

Bodendiek, R. and Burosch, G. "Solution to the Antimagic 0, 1, -1 Matrix Problem." Aufgabe 5.30 in *Streifzüge durch die Kombinatorik: Aufgaben und Lösungen aus dem Schatz der Mathematik-Olympiaden*. Heidelberg, Germany: Spektrum Akademischer Verlag, pp. 250–253, 1995.

$(-1, 1)$ -Matrix

See also HADAMARD MATRIX, INTEGER MATRIX

References

Kahn, J.; Komlós, J.; and Szemerédi, E. "On the Probability that a Random ± 1 Matrix is Singular." *J. Amer. Math. Soc.* **8**, 223–240, 1995.

0-Free

ZEROFREE

0

DIVISION BY ZERO, FALLACY, NAUGHT, ZERO, ZERO DIVISOR, ZERO-FORM, ZERO MATRIX, ZERO-SUM GAME, ZEROFREE

$0 = 1$

FALLACY

$(0, 1)$ -Matrix

A $(0, 1)$ -INTEGER MATRIX, i.e., a matrix each of whose elements is 0 or 1, also called a binary matrix.

The numbers of binary matrices with no adjacent 1s (in either columns or rows) for $n = 1, 2, \dots$, are given by 2, 7, 63, 1234, ... (Sloane's A006506). For example, the binary matrices with no adjacent 1s are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

These numbers are closely related to the HARD SQUARE ENTROPY CONSTANT. The numbers of binary matrices with no *three* adjacent 1s for $n = 2, \dots$, are given by 2, 16, 265, 16561, ... (Sloane's A050974).

Wilf (1997) considers the complexity of transforming an $m \times n$ binary matrix A into a TRIANGULAR MATRIX by permutations of the rows and columns of A , and concludes that the problem falls in difficulty between a known easy case and a known hard case of the general NP-COMPLETE PROBLEM.

See also ADJACENCY MATRIX, FROBENIUS-KÖNIG THEOREM, GALE-RYSER THEOREM, HADAMARD'S MAXIMUM DETERMINANT PROBLEM, HARD SQUARE ENTROPY CONSTANT, IDENTITY MATRIX, INCIDENCE MATRIX, INTEGER MATRIX, LAM'S PROBLEM, S -CLUSTER, S -RUN

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Wilf, H. "On Crossing Numbers, and Some Unsolved Problems." In *Combinatorics, Geometry, and Probability: A Tribute to Paul Erdős. Papers from the Conference in Honor of Erdős' 80th Birthday Held at Trinity College, Cambridge, March 1993* (Ed. B. Bollobás and A. Thomason). Cambridge, England: Cambridge University Press, pp. 557–562, 1997.

Williamson, J. "Determinants Whose Elements Are 0 and 1." *Amer. Math. Monthly* **53**, 427–434, 1946.

1

The number one (1), also called "unity" is the first POSITIVE INTEGER. It is an ODD NUMBER. Although the number 1 used to be considered a PRIME NUMBER, it

requires special treatment in so many definitions and applications involving primes greater than or equal to 2 that it is usually placed into a class of its own (Wells 1986, p. 31). The number 1 is sometimes also called "unity," so the th roots of 1 are often called the th ROOTS OF UNITY. FRACTIONS having 1 as a NUMERATOR are called UNIT FRACTIONS. If only one root, solution, etc., exists to a given problem, the solution is called UNIQUE.

The GENERATING FUNCTION having all COEFFICIENTS 1 is given by

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

See also FALLACY, ONE-FORM, ONE-MOUTH THEOREM, ONE-NINTH CONSTANT, ONE-SHEETED HYPERBOLOID, ONE-TO-ONE, ONE-WAY FUNCTION, 2, 3, COMPLEXITY (NUMBER), EXACTLY ONE, ROOT OF UNITY, UNIQUE, UNIT FRACTION, ZERO

References

Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers*. Middlesex, England: Penguin Books, pp. 30–32, 1986.

2

The number two (2) is the second POSITIVE INTEGER and the first PRIME NUMBER. It is EVEN, and is the only EVEN PRIME (the PRIMES other than 2 are called the ODD PRIMES). The number 2 is also equal to its FACTORIAL since $2! = 2$. A quantity taken to the POWER 2 is said to be SQUARED. The number of times k a given BINARY number $b_n \dots b_2 b_1 b_0$ is divisible by 2 is given by the position of the first $b_k = 1$, counting from the right. For example, $12 = 1100$ is divisible by 2 twice, and $13 = 1101$ is divisible by 2 zero times.

The only known solutions to the CONGRUENCE

$$2^n \equiv 3 \pmod{n}$$

are $n = 4700063497$ (Sloane's A050259; Guy 1994) and

63130707451134435989380140059866138830623361447484274774099906755

(P.-L. Montgomery 1999). In general, the least satisfying

$$2^n \equiv k \pmod{n}$$

for $k = 2, 3, \dots$ are $n = 3, 4700063497, 6, 19147, 10669, 25, 9, 2228071, \dots$ (Sloane's A036236).

See also 1, BINARY, 3, RULER FUNCTION, SQUARED, TWO-EARS THEOREM, TWO-FORM, TWO-GRAPH, TWO-SCALE EXPANSION, TWO-SHEETED HYPERBOLOID, ZERO

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2x mod 1 Map

Let x_0 be a RATIONAL NUMBER in the CLOSED INTERVAL $[0, 1]$, and generate a SEQUENCE using the MAP

$$x_{n+1} \equiv 2x_n \pmod{1}. \quad (1)$$

Then the number of periodic ORBITS of period p (for PRIME) is given by

$$N_p = \frac{2^p - 2}{p} \quad (2)$$

(i.e., the number of period- repeating bit strings, modulo shifts). Since a typical ORBIT visits each point with equal probability, the NATURAL INVARIANT is given by

$$\rho(x) = 1. \quad (3)$$

See also TENT MAP

References

- Ott, E. *Chaos in Dynamical Systems*. Cambridge, England: Cambridge University Press, pp. 26–31, 1993.

3

3 is the only INTEGER which is the sum of the preceding POSITIVE INTEGERS ($1 + 2 = 3$) and the only number which is the sum of the FACTORIALS of the preceding POSITIVE INTEGERS ($1! + 2! = 3$). It is also the first ODD PRIME. A quantity taken to the POWER 3 is said to be CUBED.

The sequence 1, 31, 331, 3331, 33331, ... (Sloane's A033175) consisting of $n = 0, 1, \dots$ 3s followed by a 1. The th tern is given by

$$a(n) = \frac{10^{n+1} - 7}{3}.$$

The result is prime for , 2, 3, 4, 5, 6, 7, 17, 39, ... (Sloane's A055520); i.e., for 3, 31, 331, 3331, 33331, 333331, 3333331, 33333331, ... (Sloane's A051200), a fact which Gardner (1997) calls "a remarkable pattern that is entirely accidental and leads nowhere."

See also 1, 2, 3X+1 MAPPING, CUBED, PERIOD THREE THEOREM, TERNARY, THREE-CHOICE POLYGON, THREE-CHOICE WALK, THREE-COLORABLE, THREE CONICS THEOREM, THREE JUG PROBLEM, THREE-VALUED LOGIC, TREFOIL KNOT, WIGNER 3J-SYMBOL, ZERO

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3x + 1 Mapping

COLLATZ PROBLEM

4

See also FOUR COINS PROBLEM, FOUR-COLOR THEOREM, FOUR CONICS THEOREM, FOUR EXPONENTIALS CONJECTURE, FOUR TRAVELERS PROBLEM, FOUR-VECTOR, FOUR-VERTEX THEOREM, LAGRANGE'S FOUR-SQUARE THEOREM

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Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers*. Middlesex, England: Penguin Books, pp. 55–58, 1986.

4-D Geometry

4-DIMENSIONAL GEOMETRY

4-Dimensional Geometry

4-dimensional geometry is Euclidean geometry extended into one additional DIMENSION. The prefix "hyper-" is usually used to refer to the 4- (and higher-) dimensional analogs of 3-dimensional objects, e.g. HYPERCUBE, HYPERPLANE, HYPERSPHERE. -dimensional POLYHEDRA are called POLYTOPES. the 4-dimensional cases of general -dimensional objects are often given special names, such as those summarized in the following table.

2-D	3-D	4-D	General
CIRCLE	SPHERE	GLOME	HYPERSPHERE
SQUARE	CUBE	TESSERACT	HYPERCUBE

EQUILATERAL TRIANGLE	TETRAHEDRON	PENTATOPE	SIMPLEX
POLYGON	POLYHEDRON	POLYCHORON	POLYTOPE
LINE SEGMENT	PLANE	HYPERPLANE	HYPERPLANE
SQUARE	OCTAHEDRON	16-CELL	CROSS POLYTOPE
EDGE	FACE	FACET	FACET
AREA	VOLUME	CONTENT	CONTENT

The SURFACE AREA of a HYPERSPHERE in -D is given by

$$S_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{1}{2}n\right)},$$

and the VOLUME by

$$V_n = \frac{\pi^{n/2}R^n}{\Gamma\left(1 + \frac{1}{2}n\right)},$$

where $\Gamma(n)$ is the GAMMA FUNCTION.

See also DIMENSION, HYPERCUBE, HYPERSPHERE

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5

See also FIVE DISKS PROBLEM, MIQUEL FIVE CIRCLES THEOREM, PENTAGON, PENTAGRAM, PENTAHEDRON, TETRAHEDRON 5-COMPOUND

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Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers*. Middlesex, England: Penguin Books, pp. 58–67, 1986.

5-Cell

PENTATOPE

6

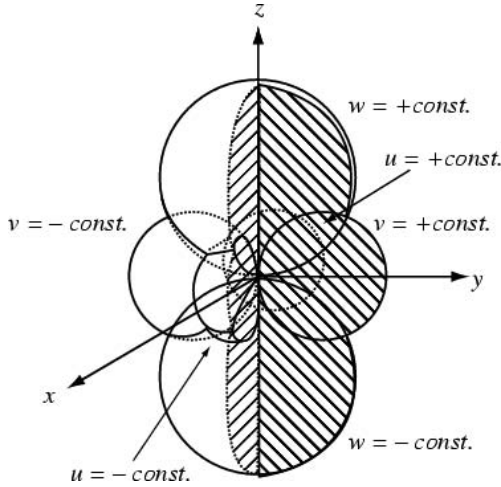
See also 6-SPHERE COORDINATES, HEXAGON, HEXAHE-

DRON, SIX CIRCLES THEOREM, SIX-COLOR THEOREM, SIX EXPONENTIALS THEOREM, WIGNER 6J-SYMBOL

References

Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers*. Middlesex, England: Penguin Books, pp. 67–69, 1986.

6-Sphere Coordinates



The coordinate system obtained by INVERSION of CARTESIAN COORDINATES, with $u, v, w \in (-\infty, \infty)$. The transformation equations are

$$x = \frac{u}{u^2 + v^2 + w^2} \quad (1)$$

$$y = \frac{v}{u^2 + v^2 + w^2} \quad (2)$$

$$z = \frac{w}{u^2 + v^2 + w^2} \quad (3)$$

The equations of the surfaces of constant coordinates are given by

$$\left(x - \frac{1}{2u}\right)^2 + y^2 + z^2 = \frac{1}{4u^2}, \quad (4)$$

which gives spheres tangent to the yz -plane at the origin for u constant,

$$x^2 + \left(y - \frac{1}{2v}\right)^2 + z^2 = \frac{1}{4v^2}, \quad (5)$$

which gives spheres tangent to xz -plane at the origin for v constant, and

$$x^2 + y^2 + \left(z - \frac{1}{2w}\right)^2 = \frac{1}{4w^2}. \quad (6)$$

which gives spheres tangent to the xy -plane at the origin for w constant.

The metric coefficients are

$$g_{uu} = g_{vv} = g_{ww} = \frac{1}{(u^2 + v^2 + w^2)^2}. \quad (7)$$

See also CARTESIAN COORDINATES, INVERSION

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7

See also SEVEN CIRCLES THEOREM

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8

See also EIGHT CURVE, EIGHT-POINT CIRCLE THEOREM, EIGHT SURFACE

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8-Cell

TESSERACT

9

See also NINE-POINT CENTER, NINE-POINT CIRCLE, NINE-POINT CONIC, WIGNER 9J-SYMBOL

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10

The number 10 (ten) is the basis for the DECIMAL system of notation. In this system, each "decimal place" consists of a DIGIT 0–9 arranged such that each DIGIT is multiplied by a POWER of 10, decreasing from left to right, and with a decimal place indicating the $10^0 = 1$ s place. For example, the number 1234.56 specifies

$$1 \times 10^3 + 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0 + 5 \times 10^{-1} \\ + 6 \times 10^{-2}.$$

The decimal places to the left of the decimal point are 1, 10, 100, 1000, 10000, 100000, 1000000, 10000000, 100000000, ... (Sloane's A011557), called one, ten, HUNDRED, THOUSAND, ten thousand, hundred thousand, MILLION, 10 million, 100 million, and so on. The names of subsequent decimal places for LARGE NUMBERS differ depending on country.

Any POWER of 10 which can be written as the PRODUCT of two numbers not containing 0s must be OF THE FORM $2^n \cdot 5^n = 10^n$ for an INTEGER such that neither 2^n nor 5^n contains any ZEROS. The largest known such number is

$$10^{23} = 2^{33} \cdot 5^{33} = 8, 589, 934, 592 \\ \cdot 116, 415, 321, 826, 934, 814, 453, 125.$$

A complete list of known such numbers is

$$\begin{aligned} 10^1 &= 2^1 \cdot 5^1 \\ 10^2 &= 2^2 \cdot 5^2 \\ 10^3 &= 2^3 \cdot 5^3 \\ 10^4 &= 2^4 \cdot 5^4 \\ 10^5 &= 2^5 \cdot 5^5 \\ 10^6 &= 2^6 \cdot 5^6 \\ 10^7 &= 2^7 \cdot 5^7 \\ 10^9 &= 2^9 \cdot 5^9 \\ 10^{18} &= 2^{18} \cdot 5^{18} \\ 10^{33} &= 2^{33} \cdot 5^{33} \end{aligned}$$

(Madachy 1979). Since all POWERS of 2 with exponents $86 < n \leq 4.6 \times 10^7$ contain at least one ZERO (M. Cook), no other POWER of ten less than 46 million can be written as the PRODUCT of two numbers not containing 0s.

See also BILLION, DECIMAL, HUNDRED, LARGE NUMBER, MILLIARD, MILLION, THOUSAND, TRILLION, ZERO

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11

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12

One DOZEN, or a twelfth of a GROSS.

See also DOZEN, GROSS

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13

A NUMBER traditionally associated with bad luck. A so-called BAKER'S DOZEN is equal to 13. Fear of the number 13 is called TRISKAIDEKAPHOBIA. There are 13 ARCHIMEDEAN SOLIDS. Mazur and Tate (1973/74) proved that there is no ELLIPTIC CURVE over the rationals $\mathbb{Q}$ having a RATIONAL POINT of order 13.

See also BAKER'S DOZEN, TRISKAIDEKAPHOBIA

References

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14

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15

See also 15 PUZZLE, FIFTEEN THEOREM

15 Puzzle

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

A puzzle introduced by Sam Loyd in 1878. It consists of 15 squares numbered from 1 to 15 which are placed in a 4×4 box leaving one position out of the 16 empty. The goal is to reposition the squares from a given arbitrary starting arrangement by sliding them one at a time into the configuration shown above. For some initial arrangements, this rearrangement is possible, but for others, it is not.

To address the solubility of a given initial arrangement, proceed as follows. If the SQUARE containing the number i appears "before" (reading the squares in the box from left to right and top to bottom) numbers which are less than i , then call it an inversion of order i , and denote it n_i . Then define

$$N \equiv \sum_{i=1}^{15} n_i = \sum_{i=2}^{15} n_i,$$

where the sum need run only from 2 to 15 rather than 1 to 15 since there are no numbers less than 1 (so n_1 must equal 0). If N is EVEN, the position is possible, otherwise it is not. This can be formally proved using ALTERNATING GROUPS. For example, in the following arrangement

2	1	3	4
5	6	7	8
9	10	11	12
13	14	15	

$n_2 = 1$ (2 precedes 1) and all other $n_i = 0$, so $N = 1$ and the puzzle cannot be solved.

Johnson (1879) proved that odd permutations of the puzzle are impossible, which Story (1879) proved that all even permutations are possible. While Herstein and Kaplansky (1978) wrote that "no really easy proof seems to be known," Archer (1999) presented a simple proof. A more general result due to Wilson (1974) showed that for any CONNECTED GRAPH on nodes, with the exception of CYCLE GRAPHS C_n and the THETA-0 GRAPH, either exactly half or all of the $n!$ possible labelings are obtainable by sliding labels, depending on whether the graph is BIPARTITE (Archer 1999). θ_0 has six inequivalent labelings, which has $(n-2)!$ inequivalent labelings.

Reversing the order of the "8 Puzzle" made on a 3×3 board can be proved to require at least 26 moves, although the best solution requires 30 moves (Gardner 1984, pp. 200 and 206–207). The number of distinct solutions in 28, 30, 32, ... moves are 0, 10, 112, 512, ... (Sloane's A046164), giving 634 solutions better than the 36-move solution given by Dudeney (1949).

References

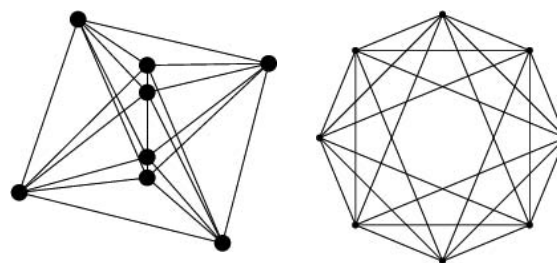
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15 Schoolgirl Problem

KIRKMAN'S SCHOOLGIRL PROBLEM

16-Cell



The finite regular 4-D CROSS POLYTOPE with SCHLÄFLI SYMBOL $\{3, 3, 4\}$ and VERTICES which are the PERMUTATIONS of $(, 0, 0, 0)$. The 16-cell is the dual of the TESSERACT. Its graph is isomorphic to the CIRCULANT GRAPH $Ci_{1, 2, 3}(8)$.

See also 24-CELL, 120-CELL, 600-CELL, CELL, CROSS POLYTOPE, HYPERCUBE, PENTATOPE, POLYCHORON, POLYTOPE, TESSERACT

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17

17 is a FERMAT PRIME which means that the 17-sided REGULAR POLYGON (the HEPTADECAGON) is CONSTRUCTIBLE using COMPASS and STRAIGHTEDGE (as proved by Gauss).

See also CONSTRUCTIBLE POLYGON, FERMAT PRIME, HEPTADECAGON

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17-gon

HEPTADECAGON

18-Point Problem

Place a point somewhere on a LINE SEGMENT. Now place a second point and number it 2 so that each of the points is in a different half of the LINE SEGMENT. Continue, placing every i th point so that all points are on different $(1/N)$ th of the LINE SEGMENT. Formally, for a given N , does there exist a sequence of real numbers $x_1, x_2, \dots, x_N$ such that for every $n \in \{1, \dots, N\}$ and every $k \in \{1, \dots, n\}$, the inequality

$$\frac{k-1}{n} \leq x_i < \frac{k}{n}$$

holds for some $i \in \{1, \dots, n\}$? Surprisingly, it is only possible to place 17 points in this manner (Berlekamp and Graham 1970, Warmus 1976).

Steinhaus (1979) gives a 14-point solution (0.06, 0.55, 0.77, 0.39, 0.96, 0.28, 0.64, 0.13, 0.88, 0.48, 0.19, 0.71, 0.35, 0.82), and Warmus (1976) gives the 17-point solution

$$\frac{4}{7} \leq x_1 < \frac{7}{12}, \frac{2}{7} \leq x_2 < \frac{5}{17}, \frac{16}{17} \leq x_3 < 1, \frac{1}{14} \leq x_4 < \frac{1}{13},$$

$$\frac{8}{11} \leq x_5 < \frac{11}{15}, \frac{5}{11} \leq x_6 < \frac{6}{13}, \frac{1}{7} \leq x_7 < \frac{2}{13}, \frac{14}{17} \leq x_8 < \frac{5}{6},$$

$$\frac{3}{8} \leq x_9 < \frac{5}{13}, \frac{11}{17} \leq x_{10} < \frac{2}{3}, \frac{3}{14} \leq x_{11} < \frac{3}{13},$$

$$\frac{15}{17} \leq x_{12} < \frac{11}{12}, \frac{1}{2} \leq x_{13} < \frac{9}{17}, 0 \leq x_{14} < \frac{1}{17},$$

$$\frac{13}{17} \leq x_{15} < \frac{4}{5}, \frac{5}{16} \leq x_{16} < \frac{6}{17}, \frac{10}{17} \leq x_{17} < \frac{11}{17},$$

Warmus (1976) states that there are 768 patterns of 17-point solutions (counting reversals as equivalent).

See also DISCREPANCY THEOREM, POINT PICKING

References

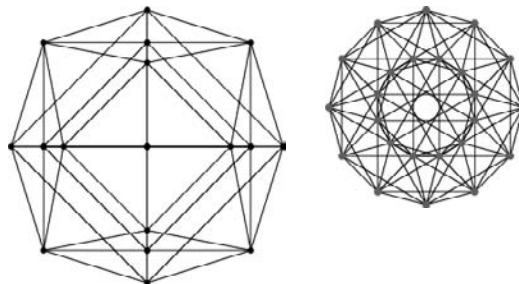
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24-Cell

A finite regular 4-D POLYTOPE with SCHLÄFLI SYMBOL $\{3, 4, 3\}$. Coxeter (1969) gives a list of the VERTEX positions. The EVEN coefficients of the D_4 lattice are 1, 24, 24, 96, ... (Sloane's A004011), and the 24 shortest vectors in this lattice form the 24-cell (Coxeter 1973, Conway and Sloane 1993, Sloane and Plouffe 1995). The 24-cell is self-dual, and is the unique regular convex POLYCHORON which has no direct 3-D analog.

One construction for the 24-cell evokes comparison with the RHOMBIC DODECAHEDRON. Given two equal cubes, we construct this dodecahedron by cutting one cube into six congruent square pyramids, and attaching these to the six squares bounding the other cube. Similarly, given two equal tesseracts, we can construct the 24-cell by cutting one tesseract into eight congruent cubic pyramids, and attaching these to the eight cubes bounding the other tesseract (Towle).

See also 16-CELL, 120-CELL, 600-CELL, CELL, HYPERCUBE, PENTATOPE, POLYCHORON, POLYTOPE

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36 Officer Problem

How can a delegation of six regiments, each of which sends a colonel, a lieutenant-colonel, and major, a captain, a lieutenant, and a sub-lieutenant be ar-

ranged in a regular 6×6 array such that no row or column duplicates a rank or a regiment? The answer is that no such arrangement is possible.

See also EULER'S GRAECO-ROMAN SQUARES CONJECTURE, LATIN SQUARE

References

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42

According to Adams (1997), 42 is the ultimate answer to life, the universe, and everything, although it is left as an exercise to the reader to determine the actual question leading to this result.

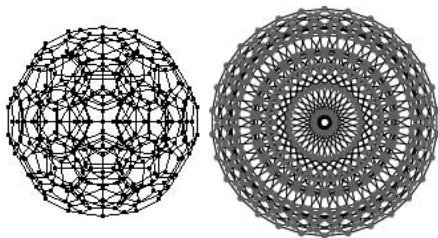
References

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72 Rule

RULE OF 72

120-Cell



A finite regular 4-D POLYTOPE with SCHLÄFLI SYMBOL $\{5, 3, 3\}$. The 120-cell has 600 vertices (Coxeter 1969), and consists of 120 DODECAHEDRA and 720 PENTAGONS (Coxeter 1973, p. 264). In the plate following p. 176, Coxeter (1973) illustrates the polytope. The dual of the 120-cell is the 600-CELL.

See also 16-CELL, 24-CELL, 600-CELL, CELL, HYPERCUBE, PENTATOPE, POLYCHORON, POLYTOPE, SIMPLEX

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144

A DOZEN DOZEN, also called a GROSS. 144 is a SQUARE NUMBER and a SUM-PRODUCT NUMBER.

See also DOZEN

163

The number 163 is very important in number theory, since $d = 163$ is the largest number such that the IMAGINARY QUADRATIC FIELD $\mathbb{Q}(-\sqrt{d})$ has CLASS NUMBER $h(-d) = 1$. It also satisfies the curious identities

$$163 \sum_{i=0}^4 \binom{8}{i} \quad (1)$$

$$\frac{1}{2} \left[4^4 + \binom{8}{4} \right] \quad (2)$$

$$\frac{1}{2} \left[4^4 + \sum_{i=0}^4 \binom{4}{i}^2 \right], \quad (3)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT (Stoschek). An approximation due to Stoschek is given by

$$\pi \approx \frac{2^9}{163} = \frac{512}{163} \approx 3.1411043, \quad (4)$$

which is good to 3 digits.

See also RAMANUJAN CONSTANT

References

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196-Algorithm

Take any POSITIVE INTEGER of two DIGITS or more, reverse the DIGITS, and add to the original number. Now repeat the procedure with the SUM so obtained. This procedure quickly produces PALINDROMIC NUMBERS for most INTEGERS. For example, starting with the number 5280 produces (5280, 6105, 11121, 23232). The end results of applying the algorithm to 1, 2, 3, ... are 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 11, 33, 44, 55, 66, 77, 88, 99, 121, ... (Sloane's A033865). The value for 89 is especially large, being 8813200023188.

The first few numbers not known to produce PALINDROMES are 196, 887, 1675, 7436, 13783, ... (Sloane's A006960), which are simply the numbers obtained by iteratively applying the algorithm to the number 196. This number therefore lends itself to the name of the ALGORITHM. In 1990, John Walker computed 2,415,836 iterations of the algorithm on 196 and obtained a number having 1,000,000 digits. This was extended in 1995 by Tim Irvin, who obtained a

number having 2,000,000 digits. The `rec.puzzles` archive states that a 3,924,257-digit nonpalindromic number is obtained after 9,480,000 iterations.

The number of terms $a(n)$ in the iteration sequence required to produce a PALINDROMIC NUMBER from (i.e., $a(n) = 1$ for a PALINDROMIC NUMBER, $a(n) = 2$ if a PALINDROMIC NUMBER is produced after a single iteration of the 196-algorithm, etc.) for $1, 2, \dots$ are 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 2, 2, 2, 2, 2, 2, 3, 2, 2, 1, ... (Sloane's A030547). The smallest numbers which require 1, 2, ... iterations to reach a palindrome are 0, 10, 19, 59, 69, 166, 79, 188, ... (Sloane's A023109).

The 196-algorithm can be implemented in *Mathematica* as

```
PalindromicQ[n_Integer?Positive]:= Module[
  {sn=ToString[n]},
  sn=StringReverse[sn]
]
Algorithm196[n_Integer?PalindromicQ,it_:0]:=
  {n}
  Algorithm196[n_Integer?Positive,
  it_:Infinity]:=
    FixedPointList[# + ToExpression[StringRe-
    verse[ToString[#]]]&,
    n, it, SameTest->(PalindromicQ[#2]&)]
```

M. Sofroniou gives an efficient *Mathematica* implementation which has complexity $\mathcal{O}(k^2)$ for steps, requiring approximately 10.6 hours on a 450 MHz Pentium II to compute 250,000 iterations. Extrapolating the timing data suggests that approximately 42 days would be needed on this same machine to match Walker's 2,415,836 iterations.

See also ADDITIVE PERSISTENCE, DIGITADDITION, MULTIPLICATIVE PERSISTENCE, PALINDROMIC NUMBER, PALINDROMIC NUMBER CONJECTURE, RATS SEQUENCE, RECURRING DIGITAL INVARIANT

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239

Some interesting properties (as well as a few arcane ones not reiterated here) of the number 239 are discussed in Beeler *et al.* (1972, Item 63). 239 appears in MACHIN'S FORMULA

$$\frac{1}{4}\pi = 4 \tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right),$$

which is related to the fact that

$$2 \cdot 13^4 - 1 = 239^2,$$

which is why 239/169 is the 7th CONVERGENT of $\sqrt{2}$. Another pair of INVERSE TANGENT FORMULAS involving 239 is

$$\tan^{-1}\left(\frac{1}{239}\right)\tan^{-1}\left(\frac{1}{70}\right) - \tan^{-1}\left(\frac{1}{99}\right) \\ \tan^{-1}\left(\frac{1}{408}\right) + \tan^{-1}\left(\frac{1}{577}\right).$$

239 needs 4 SQUARES (the maximum) to express it, 9 CUBES (the maximum, shared only with 23) to express it, and 19 fourth POWERS (the maximum) to express it (see WARING'S PROBLEM). However, 239 doesn't need the maximum number of fifth POWERS (Beeler *et al.* 1972, Item 63).

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243

Feynman (1997) noticed the curious fact that the decimal expansion

$$\frac{1}{243} = 0.004115226337448559 \dots$$

repeats pairs of the digits 0, 1, 2, 3, ... separated by the digits 4, 5, 6, 7, ... Just after this point, the pattern breaks, since the fraction is given exactly by the repeating decimal

$$\frac{1}{243} = 0.\overline{004115226337448559670781893}.$$

This pattern is related to the fact that

$$\frac{1}{9} = 0.\bar{1}$$

and

$$\frac{1}{81} = 0.0123456789.$$

References

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257-gon

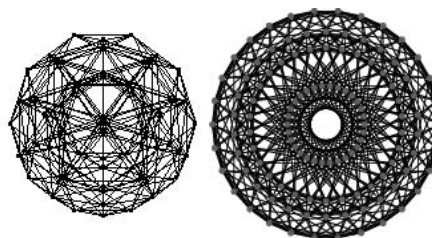
257 is a FERMAT PRIME, and the 257-gon is therefore a CONSTRUCTIBLE POLYGON using COMPASS and STRAIGHTEDGE, as proved by Gauss. An illustration of the 257-gon is not included here, since its 257 segments so closely resemble a CIRCLE. Richelot and Schwendenwein found constructions for the 257-gon in 1832 (Coxeter 1969). De Temple (1991) gives a construction using 150 CIRCLES (24 of which are CARLYLE CIRCLES) which has GEOMETROGRAPHY symbol $94S_1 + 47S_2 + 275C_1 + 0C_2 + 150C_3$ and SIMPLICITY 566.

See also 65537-GON, CONSTRUCTIBLE POLYGON, FERMAT PRIME, HEPTADECAGON, PENTAGON

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600-Cell



A finite regular 4-D POLYTOPE with SCHLÄFLI SYMBOL $\{3, 3, 5\}$. The 600-cell has 120 VERTICES (Coxeter 1969). In the plate following p. 160, Coxeter (1973) gives two illustrations of the polytope.

The dual of the 600-cell is the 120-CELL.

See also 16-CELL, 24-CELL, 120-CELL, CELL, HYPERCUBE, PENTATOPE, POLYCHORON, POLYTOPE, SIMPLEX

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666

A number known as the BEAST NUMBER appearing in the Bible and ascribed various numerological properties.

See also APOCALYPTIC NUMBER, BEAST NUMBER, LEVIATHAN NUMBER

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1729

1729 is sometimes called the HARDY-RAMANUJAN NUMBER. It is the smallest TAXICAB NUMBER, i.e., the smallest number which can be expressed as the sum of two cubes in two different ways:

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

See also HARDY-RAMANUJAN NUMBER, TAXICAB NUMBER

2187

The digits in the number 2187 form the two VAMPIRE NUMBERS: $21 \times 87 = 1827$ and $2187 = 27 \times 81$. 2187 is also given by 3^7 .

See also VAMPIRE NUMBER

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65537-gon

65537 is the largest known FERMAT PRIME, and the 65537-gon is therefore a CONSTRUCTIBLE POLYGON using COMPASS and STRAIGHTEDGE, as proved by Gauss. The 65537-gon has so many sides that it is, for all intents and purposes, indistinguishable from a CIRCLE using any reasonable printing or display methods.

Hermes spent 10 years on the construction of the 65537-gon at Königsberg around (1900). After the Second World War, his manuscripts were moved to the Mathematical Institute in Göttingen, where they can now be viewed (Coxeter 1969).

De Temple (1991) notes that a GEOMETRIC CONSTRUCTION can be done using 1332 or fewer CARLYLE CIRCLES.

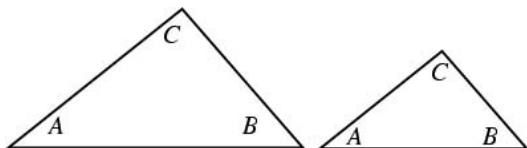
See also 257-GON, CONSTRUCTIBLE POLYGON, HEPTADECAGON, PENTAGON

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A

AAA Theorem



Specifying three ANGLES A , B , and C does not uniquely define a TRIANGLE, but any two TRIANGLES with the same ANGLES are SIMILAR. Specifying two ANGLES of a TRIANGLE automatically gives the third since the sum of ANGLES in a TRIANGLE sums to 180° (π RADIANS), i.e.,

$$C = \pi - A - B.$$

See also AAS THEOREM, ASA THEOREM, ASS THEOREM, SAS THEOREM, SSS THEOREM, TRIANGLE

AAS Theorem



Specifying two angles A and B and a side a uniquely determines a TRIANGLE with AREA

$$K = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{a^2 \sin B \sin(\pi - A - B)}{2 \sin A}. \quad (1)$$

The third angle is given by

$$C = \pi - A - B, \quad (2)$$

since the sum of angles of a TRIANGLE is 180° (π RADIANS). Solving the LAW OF SINES

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad (3)$$

for b gives

$$b = a \frac{\sin B}{\sin A}. \quad (4)$$

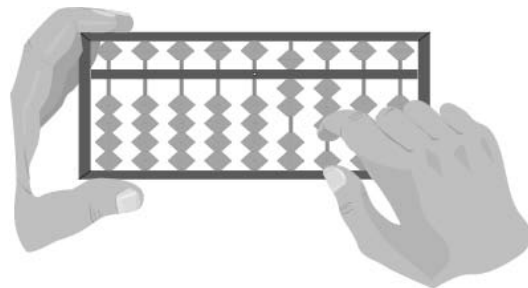
Finally,

$$c = b \cos A + a \cos B = a(\sin B \cot A + \cos B) \quad (5)$$

$$= a \sin B(\cot A + \cot B). \quad (6)$$

See also AAA THEOREM, ASA THEOREM, ASS THEOREM, SAS THEOREM, SSS THEOREM, TRIANGLE

Abacus



A mechanical counting device consisting of a frame holding a series of parallel rods on each of which beads are strung. Each bead represents a counting unit, and each rod a place value. The primary purpose of the abacus is not to perform actual computations, but to provide a quick means of storing numbers during a calculation. Abaci were used by the Japanese and Chinese, as well as the Romans.

See also ROMAN NUMERAL, SLIDE RULE

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abc Conjecture

A CONJECTURE due to J. Oesterlé and D. W. Masser. It states that, for any INFINITESIMAL $\epsilon > 0$, there exists a CONSTANT C_ϵ such that for any three RELATIVELY PRIME INTEGERS a , b , c satisfying

$$a + b = c, \quad (1)$$

the INEQUALITY

$$\max(|a|, |b|, |c|) \leq C_\epsilon \prod_{p|abc} p^{1+\epsilon} \quad (2)$$

holds, where $p|abc$ indicates that the PRODUCT is over PRIMES p which DIVIDE the PRODUCT abc . If this CONJECTURE were true, it would imply FERMAT'S LAST THEOREM for sufficiently large POWERS (Goldfeld 1996). This is related to the fact that the abc conjecture implies that there are at least $C \ln x$ WIEFERICH PRIMES $\leq x$ for some constant C (Silverman 1988, Vardi 1991).

The conjecture can also be stated by defining the height and radical of the sum $P : a + b = c$ as

$$h(P) = \max\{\ln|a|, \ln|b|, \ln|c|\} \quad (3)$$

$$r(P) = \sum_{p|abc} \ln p, \quad (4)$$

where p runs over all prime divisors of a , b , and c . Then the abc conjecture states that for all $\epsilon > 0$, there exists a constant K such that for all $P : a + b + c$,

$$h(P) \leq r(P) + \epsilon h(P) + K \quad (5)$$

(van Frankenhuysen 2000). van Frankenhuysen (2000) has shown that there exists an infinite sequence of sums $P : a + b = c$ or RATIONAL INTEGERS with large height compared to the radical,

$$h(p) \geq r(P) + 4K_l \frac{\sqrt{h(P)}}{\ln[h(P)]}, \quad (6)$$

with

$$K_l = 2^{l/2} \left(\frac{2\pi}{e} \right)^{1/4} > 1.517 \quad (7)$$

for $l = 0.5990$, improving a result of Stewart and Tijdeman (1986).

See also FERMAT'S LAST THEOREM, MASON'S THEOREM, MORDELL CONJECTURE, ROTH'S THEOREM, WIEFERICH PRIME

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Abel Polynomial

A polynomial $A_n(x; a)$ given by the associated SHEFFER SEQUENCE with

$$f(t) = te^{at}, \quad (1)$$

given by

$$A_n(x; a) = x(x - an)^{n-1}. \quad (2)$$

The GENERATING FUNCTION is

$$\sum_{k=0}^{\infty} \frac{A_k(x; a)}{k!} t^k = e^{xW(at)/a}, \quad (3)$$

where $W(x)$ is LAMBERT'S W -FUNCTION. The associated BINOMIAL IDENTITY is

$$\begin{aligned} (x+y)(x+y-an)^{n-1} \\ = \sum_{k=0}^n \binom{n}{k} xy(x-ak)^{k-1} [y-a(n-k)]^{n-k-1}, \end{aligned} \quad (4)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT, a formula originally due to Abel (Riordan 1979, p. 18; Roman 1984, pp. 30 and 73).

The first few Abel polynomials are

$$\begin{aligned} A_0(x; a) &= 1 \\ A_1(x; a) &= x \\ A_2(x; a) &= x(x-2a) \\ A_3(x; a) &= x(x-3a)^2 \\ A_4(x; a) &= x(x-4a)^3. \end{aligned}$$

References

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Abel Transform

The following INTEGRAL TRANSFORM relationship, known as the Abel transform, exists between two functions $f(x)$ and $g(t)$ for $0 < \alpha < 1$,

$$f(x) = \int_0^x \frac{g(t) dt}{(x-t)^\alpha} \quad (1)$$

$$g(t) = -\frac{\sin(\pi\alpha)}{\pi} \frac{d}{dt} \int_0^t \frac{f(x) dx}{(x-t)^{1-\alpha}} \quad (2)$$

$$= -\frac{\sin(\pi\alpha)}{\pi} \left[\int_0^t \frac{df}{dx} \frac{dx}{(t-x)^{1-\alpha}} + \frac{f(0)}{t^{1-\alpha}} \right]. \quad (3)$$

The Abel transform is used in calculating the radial

mass distribution of galaxies (Binney and Tremaine 1987) and inverting planetary radio occultation data to obtain atmospheric information as a function of height.

Bracewell (1999, p. 262) defines a slightly different form of the Abel transform given by

$$g(x) = \mathcal{A}[f(r)] = 2 \int_x^\infty \frac{f(r)r \, dr}{\sqrt{r^2 - x^2}}. \quad (4)$$

The following table gives a number of common Abel transform pairs (Bracewell 1999, p. 264). Here,

$$\Pi_a(x) \equiv \Pi\left(\frac{x}{2a} - \frac{1}{2}\right) = \begin{cases} 1 & \text{for } 0 < x < 0 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

where $\Pi(x)$ is the RECTANGLE FUNCTION, and

$$M(x) = 2\pi \left[x^{-3} \int_0^x J_0(x) \, dx - x^{-2} J_0(x) \right] \quad (6)$$

$$= \frac{\pi^2}{x^2} [J_1(x)\mathcal{H}_0(x) - J_0(x)\mathcal{H}_1(x)], \quad (7)$$

where $J_n(x)$ is a BESSEL FUNCTION OF THE FIRST KIND and $\mathcal{H}_n(x)$ is a STRUVE FUNCTION.

$f(r)$	$g(x)$	conditions
$\Pi_a(r)$	$2\sqrt{a^2 - x^2}$	$a^2 > x^2$
$(a^2 - r^2)^{-1/2} \Pi_a(r)$	π	$a^2 > x^2$
$\sqrt{a^2 - r^2} \Pi_a(r)$	$\frac{1}{2}\pi(a^2 - x^2)$	$a^2 > x^2$
$(a^2 - r^2) \Pi_a(r)$	$\frac{4}{3}(a^2 - x^2)^{3/2}$	$a^2 > x^2$
$(a^2 - r^2)^{3/2} \Pi_a(r)$	$\frac{3}{8}\pi(a^2 - x^2)^2$	$a^2 > x^2$
$(a - r) \Pi_a(r)$	$a\sqrt{a^2 - x^2} - x^2 \cosh^{-1}\left(\frac{a}{x}\right)$	
$\frac{1}{\pi} \cosh^{-1}\left(\frac{a}{r}\right)$	$a - x$	
$\delta(r - a)$	$\frac{2a}{\sqrt{a^2 - x^2}} \Pi_a(x)$	
e^{-r^2/σ^2}	$\sigma\sqrt{\pi}e^{-x^2/\sigma^2}$	$\sigma > 0$
$r^2 e^{-r^2/\sigma^2}$	$\sigma(x^2 + \frac{1}{2}\sigma^2)\sqrt{\pi}e^{-x^2/\sigma^2}$	$\sigma > 0$
$\frac{e^{-r^2/\sigma^2}}{\sigma\sqrt{\pi}}(r^2 - \frac{1}{2}\sigma^2)$	$x^2 e^{-x^2/\sigma^2}$	$\sigma > 0$
$\frac{1}{b^2 + r^2}$	$\frac{\pi}{\sqrt{b^2 + r^2}}$	$b^2 + x^2 > 0$
$J_0(\omega r)$	$\frac{2 \cos(\omega x)}{\omega}$	$\omega > 0$
$M(r)$	$\frac{8\pi^4}{\omega^2 x^2} \sin^2\left(\frac{x\omega}{2\pi}\right)$	$\omega > 0$

See also FOURIER TRANSFORM, HILBERT TRANSFORM, INTEGRAL EQUATION

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Abel's Binomial Theorem

The identity

$$\sum_{y=0}^m \binom{m}{y} (w-y)^{m-y-1} (z+y)^y = w^{-1} (z+w+m)^m$$

(Bhatnagar 1995, p. 51). There are a host of other such BINOMIAL IDENTITIES.

See also BINOMIAL IDENTITY, Q-ABEL'S THEOREM

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Abel's Convergence Theorem

Given a TAYLOR SERIES

$$f(z) = \sum_{n=0}^{\infty} C_n z^n = \sum_{n=0}^{\infty} C_n r^n e^{in\theta}, \quad (1)$$

where the COMPLEX NUMBER z has been written in the polar form $z = re^{i\theta}$, examine the REAL and IMAGINARY PARTS

$$u(r, \theta) = \sum_{n=0}^{\infty} C_n r^n \cos(n\theta) \quad (2)$$

$$v(r, \theta) = \sum_{n=0}^{\infty} C_n r^n \sin(n\theta). \quad (3)$$

Abel's theorem states that, if $u(1, \theta)$ and $v(1, \theta)$ are CONVERGENT, then

$$u(1, \theta) + iv(1, \theta) = \lim_{r \rightarrow 1} f(re^{i\theta}). \quad (4)$$

Stated in words, Abel's theorem guarantees that, if a REAL POWER SERIES CONVERGES for some POSITIVE value of the argument, the DOMAIN of UNIFORM CONVERGENCE extends at least up to and including this point. Furthermore, the continuity of the sum function extends at least up to and including this point.

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Abel's Curve Theorem

The sum of the values of an INTEGRAL of the "first" or "second" sort

$$\int_{x_0, y_0}^{x_1, y_1} \frac{P dx}{Q} + \dots + \int_{x_0, y_0}^{x_N, y_N} \frac{P dx}{Q} = F(z)$$

and

$$\frac{P(x_1, y_1) dx_1}{Q(x_1, y_1) dz} + \dots + \frac{P(x_N, y_N) dx_N}{Q(x_N, y_N) dz} = \frac{dF}{dz},$$

from a FIXED POINT to the points of intersection with a curve depending rationally upon any number of parameters is a RATIONAL FUNCTION of those parameters.

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Abel's Differential Equation

The Abel equation of the first kind is given by

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3 + \dots$$

(Murphy 1960, p. 23; Zwillinger 1997, p. 120), and the Abel equation of the second kind by

$$[g_0(x) + g_1(x)y]y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

(Murphy 1960, p. 25; Zwillinger 1997, p. 120).

References

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Zwillinger, D. *Handbook of Differential Equations*, 3rd ed. Boston, MA: Academic Press, p. 120, 1997.

Abel's Differential Equation Identity

Given a homogeneous linear SECOND-ORDER ORDINARY DIFFERENTIAL EQUATION,

$$y'' + P(x)y' + Q(x)y = 0, \quad (1)$$

call the two linearly independent solutions $y_1(x)$ and $y_2(x)$. Then

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \quad (2)$$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0. \quad (3)$$

Abel's Duplication Formula

Now, take $y_1 \times (3)$ minus $y_2 \times (2)$,

$$y_1[y_2'' + P(x)y_2' + Q(x)y_2] - y_2[y_1'' + P(x)y_1' + Q(x)y_1] = 0 \quad (4)$$

$$(y_1y_2'' - y_2y_1'') + P(y_1y_2' - y_1'y_2) + Q(y_1y_2 - y_1y_2) = 0 \quad (5)$$

$$(y_1y_2'' - y_2y_1'') + P(y_1y_2' - y_1'y_2) = 0. \quad (6)$$

Now, use the definition of the WRONSKIAN and take its DERIVATIVE,

$$W \equiv y_1y_2' - y_1'y_2 \quad (7)$$

$$W' = (y_1y_2' + y_1y_2'') - (y_1'y_2 + y_1''y_2)$$

$$y_1y_2'' - y_1''y_2. \quad (8)$$

Plugging W and W' into (6) gives

$$W' + PW = 0. \quad (9)$$

This can be rearranged to yield

$$\frac{dW}{W} = -P(x) dx \quad (10)$$

which can then be directly integrated to

$$\ln \left[\frac{W(x)}{W_0} \right] = - \int P(x) dx, \quad (11)$$

where $\ln x$ is the NATURAL LOGARITHM. Exponentiating then yields Abel's identity

$$W(x) = W_0 e^{-\int P(x) dx}, \quad (12)$$

where W_0 is a constant of integration.

See also ORDINARY DIFFERENTIAL EQUATION-SECOND-ORDER

References

Boyce, W. E. and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, 4th ed. New York: Wiley, pp. 118, 262, 277, and 355, 1986.

Abel's Duplication Formula

The duplication formula for ROGERS L -FUNCTION follows from ABEL'S FUNCTIONAL EQUATION and is given by

$$\frac{1}{2}L(x^2) = L(x) - L\left(\frac{x}{1+x}\right).$$

See also ABEL'S FUNCTIONAL EQUATION, DILOGARITHM

References

Gordon, B. and McIntosh, R. J. "Algebraic Dilogarithm Identities." *Ramanujan J.* **1**, 431-48, 1997.

Abel's Functional Equation

Let $L(x)$ denote the ROGERS L -FUNCTION defined in terms of the usual DILOGARITHM by

$$L(x) = \frac{6}{\pi^2} \left[\text{Li}_2(x) + \frac{1}{2} \ln x \ln(1-x) \right] \\ = \frac{6}{\pi^2} \left[\sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2} \ln x \ln(1-x) \right],$$

then $L(x)$ satisfies the functional equation

$$L(x) + L(y) = L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right).$$

ABEL'S DUPLICATION FORMULA follows from this identity.

See also ABEL'S DUPLICATION FORMULA, DILOGARITHM, FUNCTIONAL EQUATION, POLYLOGARITHM, RIEMANN ZETA FUNCTION, ROGERS L -FUNCTION

References

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Abel's Impossibility Theorem

In general, POLYNOMIAL equations higher than fourth degree are incapable of algebraic solution in terms of a finite number of ADDITIONS, SUBTRACTIONS, MULTIPLICATIONS, DIVISIONS, and ROOT EXTRACTIONS. This was also shown by Ruffini in 1813 (Wells 1986, p. 59).

See also CUBIC EQUATION, GALOIS'S THEOREM, POLYNOMIAL, QUADRATIC EQUATION, QUARTIC EQUATION, QUINTIC EQUATION

References

- Abel, N. H. "Beweis der Unmöglichkeit, algebraische Gleichungen von höheren Graden als dem vierten allgemein aufzulösen." *J. reine angew. Math.* 1, 65, 1826. Reprinted in Abel, N. H. *Oeuvres Completes* (Ed. L. Sylow and S. Lie). New York: Johnson Reprint Corp., pp. 66–7, 1988.
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- Herstein, I. N. *Topics in Algebra*, 2nd ed. New York: Wiley, 1975.

Hungerford, T. W. *Algebra*. New York: Springer-Verlag, 1980.

van der Waerden, B. L. *A History of Algebra: From al-Khwarizmi to Emmy Noether*. New York: Springer-Verlag, pp. 85–8, 1985.

Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers*. Middlesex, England: Penguin Books, p. 59, 1986.

Abel's Inequality

Let $\{f_n\}$ and $\{a_n\}$ be SEQUENCES with $f_n \geq f_{n+1} > 0$ for $n = 1, 2, \dots$, then

$$\left| \sum_{n=1}^m a_n f_n \right| \leq A f_1,$$

where

$$A = \max\{|a_1|, |a_1 + a_2|, \dots, |a_1 + a_2 + \dots + a_m|\}.$$

Abel's Irreducibility Theorem

If one ROOT of the equation $f(x) = 0$, which is irreducible over a FIELD K , is also a ROOT of the equation $F(x) = 0$ in K , then all the ROOTS of the irreducible equation $f(x) = 0$ are ROOTS of $F(x) = 0$. Equivalently, $F(x)$ can be divided by $f(x)$ without a REMAINDER,

$$F(x) = f(x)F_1(x),$$

where $F_1(x)$ is also a POLYNOMIAL over K .

See also ABEL'S LEMMA, KRONECKER'S POLYNOMIAL THEOREM, SCHÖNEMANN'S THEOREM

References

- Abel, N. H. "Mémoire sur une classe particulière d'équations résolubles algébriquement." *J. reine angew. Math.* 4, 1829.
- Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions*. New York: Dover, p. 120, 1965.

Abel's Lemma

The pure equation

$$x^p = C$$

of PRIME degree p is irreducible over a FIELD when C is a number of the FIELD but not the p th POWER of an element of the FIELD.

Jeffreys and Jeffreys (1988) use the term "Abel's lemma" for another LEMMA related to ABEL'S UNIFORM CONVERGENCE TEST.

See also ABEL'S IRREDUCIBILITY THEOREM, GAUSS'S POLYNOMIAL THEOREM, KRONECKER'S POLYNOMIAL THEOREM, SCHÖNEMANN'S THEOREM

References

- Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions*. New York: Dover, p. 118, 1965.

Jeffreys, H. and Jeffreys, B. S. "Abel's Lemma." §1.1153 in *Methods of Mathematical Physics*, 3rd ed. Cambridge, England: Cambridge University Press, pp. 41–2, 1988.

Abel's Test

ABEL'S UNIFORM CONVERGENCE TEST

Abel's Theorem

ABEL'S BINOMIAL THEOREM, ABEL'S CONVERGENCE THEOREM, ABEL'S CURVE THEOREM, ABEL'S IMPOSSIBILITY THEOREM, ABEL'S IRREDUCIBILITY THEOREM, ABELIAN THEOREM, Q -ABEL'S THEOREM

Abel's Uniform Convergence Test

Let $\{u_n(x)\}$ be a SEQUENCE of functions. If

1. $u_n(x)$ can be written $u_n(x) = a_n f_n(x)$,
2. Σa_n is CONVERGENT,
3. $f_n(x)$ is a MONOTONIC DECREASING SEQUENCE (i.e., $f_{n+1}(x) \leq f_n(x)$) for all n , and
4. $f_n(x)$ is BOUNDED in some region (i.e., $0 \leq f_n(x) \leq M$ for all $x \in [a, b]$)

then, for all $x \in [a, b]$, the SERIES $\Sigma u_n(x)$ CONVERGES UNIFORMLY.

See also CONVERGENCE TESTS, CONVERGENT SERIES, UNIFORM CONVERGENCE

References

- Bromwich, T. J. Fa. and MacRobert, T. M. *An Introduction to the Theory of Infinite Series*, 3rd ed. New York: Chelsea, p. 59, 1991.
- Jeffreys, H. and Jeffreys, B. S. "Abel's Lemma" and "Abel's Test." §1.1153–.1154 in *Methods of Mathematical Physics*, 3rd ed. Cambridge, England: Cambridge University Press, pp. 41–2, 1988.
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Abelian

A group or other algebraic object is said to be Abelian if the law of commutativity always holds. If an algebraic object is not Abelian, it is said to be NON-ABELIAN.

See also ABELIAN CATEGORY, ABELIAN DIFFERENTIAL, ABELIAN FUNCTION, ABELIAN GROUP, ABELIAN INTEGRAL, ABELIAN VARIETY, COMMUTATIVE, NON-ABELIAN

Abelian Category

An Abelian category is an abstract mathematical CATEGORY which displays some of the characteristic properties of the CATEGORY of all ABELIAN GROUPS.

See also ABELIAN GROUP, CATEGORY

References

- Freyd, P. *Abelian Categories: An Introduction to the Theory of Functors*. New York: Harper & Row, 1964.
- Grothendieck, A. "Sur quelques points d'algèbre homologique." *Tôhoku Math. J.* **9**, 119–21, 1957.
- Mac Lane, S. and Gehring, F. W. *Categories for the Working Mathematician*, 2nd ed. New York: Springer-Verlag, 1998.

Abelian Differential

An Abelian differential is an ANALYTIC or MEROMORPHIC DIFFERENTIAL on a COMPACT or closed RIEMANN SURFACE.

Abelian Extension

This entry contributed by NICOLAS BRAY

If F is an ALGEBRAIC GALOIS EXTENSION of K such that the GALOIS GROUP of the extension is ABELIAN, then F is said to be an Abelian extension of K .

See also ALGEBRAIC EXTENSION, GALOIS EXTENSION, GALOIS GROUP

Abelian Function

An INVERSE FUNCTION of an ABELIAN INTEGRAL. Abelian functions have two variables and four periods, and can be defined by

$$\Theta\left(\nu, \tau; \frac{q'}{q}\right) = \sum_{\lambda=-\infty}^{\infty} 2^{2\pi i \nu(\lambda+q') + \pi i \tau(\lambda+q')^2 + 2\pi i q(\lambda+q')}$$

Baker (1907, p. 21). Abelian functions are a generalization of ELLIPTIC FUNCTIONS, and are also called hyperelliptic functions.

See also ABELIAN INTEGRAL, ELLIPTIC FUNCTION, THETA FUNCTIONS

References

- Baker, H. F. *Abelian Functions: Abel's Theorem and the Allied Theory, Including the Theory of the Theta Functions*. New York: Cambridge University Press, 1995.
- Baker, H. F. *An Introduction to the Theory of Multiply Periodic Functions*. London: Cambridge University Press, 1907.
- Weisstein, E. W. "Books about Abelian Functions." <http://www.treasure-troves.com/books/AbelianFunctions.html>.

Abelian Group

N.B. A detailed online essay by S. Finch was the starting point for this entry.

A GROUP for which the elements COMMUTE (i.e., $AB = BA$ for all elements A and B) is called an Abelian group. All CYCLIC GROUPS are Abelian, but an Abelian group is not necessarily CYCLIC. All SUBGROUPS of an Abelian group are NORMAL. In an Abelian group, each element is in a CONJUGACY CLASS by itself, and the CHARACTER TABLE involves POWERS of a single element known as a GENERATOR.

No general formula is known for giving the number of nonisomorphic FINITE GROUPS of a given ORDER. However, the number of nonisomorphic Abelian FINITE GROUPS $a(n)$ of any given ORDER n is given by writing n as

$$n = \prod_i p_i^{\alpha_i}, \quad (1)$$

where the p_i are distinct PRIME FACTORS, then

$$a(n) = \prod_i P(\alpha_i), \quad (2)$$

where $P(k)$ is the PARTITION FUNCTION. This gives 1, 1, 1, 2, 1, 1, 1, 3, 2, ... (Sloane's A000688). The smallest orders for which $n = 1, 2, 3, \dots$ nonisomorphic Abelian groups exist are 1, 4, 8, 36, 16, 72, 32, 900, 216, 144, 64, 1800, 0, 288, 128, ... (Sloane's A046056), where 0 denotes an impossible number (i.e., not a product of partition numbers) of nonisomorphic Abelian, groups. The "missing" values are 13, 17, 19, 23, 26, 29, 31, 34, 37, 38, 39, 41, 43, 46, ... (Sloane's A046064). The incrementally largest numbers of Abelian groups as a function of order are 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, ... (Sloane's A046054), which occur for orders 1, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, ... (Sloane's A046055).

The KRONECKER DECOMPOSITION THEOREM states that every FINITE Abelian group can be written as a GROUP DIRECT PRODUCT of CYCLIC GROUPS of PRIME POWER ORDER. If the ORDER of a FINITE GROUP is a PRIME p , then there exists a single Abelian group of order p (denoted $\mathbb{Z}_p$) and no non-Abelian groups. If the ORDER is a prime squared p^2 then there are two Abelian groups (denoted $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \times \mathbb{Z}_p$). If the ORDER is a prime cubed p^3 , then there are three Abelian groups (denoted $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$, and $\mathbb{Z}_{p^3}$), and five groups total. If the order is a PRODUCT of two primes p and q , then there exists exactly one Abelian group of ORDER pq (denoted $\mathbb{Z}_p \times \mathbb{Z}_q$).

Another interesting result is that if $a(n)$ denotes the number of nonisomorphic Abelian groups of ORDER n , then

$$\sum_{n=1}^{\infty} a(n) n^{-s} = \zeta(s) \zeta(2s) \zeta(3s) \cdots, \quad (3)$$

where $\zeta(s)$ is the RIEMANN ZETA FUNCTION. Srinivasan (1973) has also shown that

$$\sum_{n=1}^N a(n) = A_1 N + A_2 N^{1/2} + A_3 N^{1/3} + \mathcal{O}[x^{105/407} (\ln x)^2], \quad (4)$$

where

$$A_k \equiv \prod_{\substack{j=1 \\ j \neq k}} \zeta\left(\frac{j}{k}\right) = \begin{cases} 2.294856591 \dots & \text{for } k=1 \\ -14.6475663 \dots & \text{for } k=2 \\ 118.6924619 \dots & \text{for } k=3, \end{cases} \quad (5)$$

and $\zeta(s)$ is again the RIEMANN ZETA FUNCTION. [Richert (1952) incorrectly gave $A_3 = 114$.] DeKoninck and Ivic (1980) showed that

$$\sum_{n=1}^N \frac{1}{a(n)} = BN + \mathcal{O}[\sqrt{N} (\ln N)^{-1/2}], \quad (6)$$

where

$$B \equiv \prod \left\{ 1 - \sum_{k=2}^{\infty} \left[\frac{1}{P(k-2)} - \frac{1}{P(k)} \right] \frac{1}{p^k} \right\} = 0.752 \dots \quad (7)$$

is a product over PRIMES. Bounds for the number of nonisomorphic non-Abelian groups are given by Neumann (1969) and Pyber (1993).

See also FINITE GROUP, GROUP THEORY, KRONECKER DECOMPOSITION THEOREM, PARTITION FUNCTION P , RING

References

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- DeKoninck, J.-M. and Ivic, A. *Topics in Arithmetical Functions: Asymptotic Formulae for Sums of Reciprocals of Arithmetical Functions and Related Fields*. Amsterdam, Netherlands: North-Holland, 1980.
- Erdos, P. and Szekeres, G. "Über die Anzahl abelscher Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem." *Acta Sci. Math. (Szeged)* **7**, 95–02, 1935.
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- Pyber, L. "Enumerating Finite Groups of Given Order." *Ann. Math.* **137**, 203–20, 1993.
- Richert, H.-E. "Über die Anzahl abelscher Gruppen gegebener Ordnung I." *Math. Zeitschr.* **56**, 21–2, 1952.
- Sloane, N. J. A. Sequences A000688/M0064 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.
- Srinivasan, B. R. "On the Number of Abelian Groups of a Given Order." *Acta Arith.* **23**, 195–05, 1973.

Abelian Integral

AN INTEGRAL OF THE FORM

$$\int_0^x \frac{dt}{\sqrt{R(t)}},$$

where $R(t)$ is a POLYNOMIAL of degree > 4 . They are also called HYPERELLIPTIC INTEGRALS.

See also ABELIAN FUNCTION, ELLIPTIC INTEGRAL

References

Siegel, C. L. *Topics in Complex Function Theory, Vol. 2: Automorphic Functions and Abelian Integrals*. New York: Wiley, 1988.

Abelian Theorem

A theorem which asserts that if a sequence or function behaves regularly, then some average of it behaves regularly. For example,

$$A(x) \sim x$$

implies

$$A_1(x) = \int_0^x A(t) dt \sim \frac{1}{2}x^2$$

for any $A(x)$. The converse is false, but can be made into a correct TAUBERIAN THEOREM if $A(x)$ is subjected to an appropriate additional condition (Hardy 1999, p. 46).

See also TAUBERIAN THEOREM

References

Hardy, G. H. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed.* New York: Chelsea, p. 46, 1999.

Abelian Variety

An Abelian variety is an algebraic GROUP which is a complete ALGEBRAIC VARIETY. An Abelian variety of DIMENSION 1 is an ELLIPTIC CURVE.

See also ALBANESE VARIETY

References

Murty, V. K. *Introduction to Abelian Varieties*. Providence, RI: Amer. Math. Soc., 1993.
Shimura, G. *Abelian Varieties With Complex Multiplication and Modular Functions*. Princeton, NJ: Princeton University Press, 1999.
Shimura, G. and Taniyama, Y. *Complex Multiplication of Abelian Varieties and Its Applications to Number Theory*. Tokyo: Mathematical Society of Japan, 1961.

Abelianization

In general, groups are not ABELIAN. However, there is always a GROUP HOMOMORPHISM $h: G \rightarrow G'$ to an ABELIAN GROUP, and this homomorphism is called Abelianization. The homomorphism is abstractly described by its kernel, the COMMUTATOR SUBGROUP $[G, G]$. So $G' = G/[G, G]$. Roughly speaking, in any expression, every product becomes commutative after Abelianization. As a consequence, some previously unequal expressions may become equal, or even represent the IDENTITY ELEMENT.

For example, in the eight-element QUATERNION GROUP $G = \{\pm 1, \pm i, \pm j, \pm k\}$, the COMMUTATOR SUBGROUP is $\{\pm 1\}$. The Abelianization of G is a copy of

$\mathbb{Z}_2 \times \mathbb{Z}_2$, and for instance, $i'j' = j'i'$ in the Abelianization.

See also ABELIAN, GROUP, HOMOMORPHISM

Abel-Plana Formula

This entry contributed by DAVID ANDERSON

The Abel-Plana formula gives an expression for the difference between a discrete sum and the corresponding integral. The formula can be derived from the ARGUMENT PRINCIPLE

$$\oint_{\gamma} f(z) \frac{g'(z)}{g(z)} dz = \sum_n f(\mu_n) - \sum_m f(\nu_m), \quad (1)$$

where μ_n are the zeros of $g(z)$ and ν_m are the poles contained within the CONTOUR γ . An appropriate choice of g and γ then yields

$$\begin{aligned} \sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(x) dx \\ = \frac{1}{2}f(0) - \frac{1}{2} \int_0^{\infty} [f(it) - f(-it)] [\cot(\pi it) + i] dt, \end{aligned} \quad (2)$$

or equivalently

$$\begin{aligned} \sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(x) dx \\ = \frac{1}{2}f(0) + i \int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt. \end{aligned} \quad (3)$$

The formula is particularly useful in Casimir effect calculations involving differences between quantized modes and free modes.

See also ARGUMENT PRINCIPLE

References

Mostepanenko, V. M. and Trunov, N. N. §2.2 in *The Casimir Effect and Its Applications*. Oxford, England: Clarendon Press, 1997.
Saharian, A. A. "The Generalized Abel-Plana Formula. Applications to Bessel Functions and Casimir Effect." http://www.ictp.trieste.it/~pub_off/preprints-sources/2000/IC2000014P.pdf.

Abhyankar's Conjecture

For a FINITE GROUP G , let $p(G)$ be the SUBGROUP generated by all the SYLOW p -SUBGROUPS of G . If X is a projective curve in characteristic $p > 0$, and if $x_0, \dots, x_t$ are points of X (for $t > 0$), then a NECESSARY and SUFFICIENT condition that G occur as the GALOIS GROUP of a finite covering Y of X , branched only at the points $x_0, \dots, x_t$, is that the QUOTIENT GROUP $G/p(G)$ has $2g + t$ generators.

Raynaud (1994) solved the Abhyankar problem in the crucial case of the affine line (i.e., the projective line with a point deleted), and Harbater (1994) proved the full Abhyankar conjecture by building upon this special solution.

See also FINITE GROUP, GALOIS GROUP, QUOTIENT GROUP, SYLOW P -SUBGROUP

References

- Abhyankar, S. "Coverings of Algebraic Curves." *Amer. J. Math.* **79**, 825–56, 1957.
 American Mathematical Society. "Notices of the AMS, April 1995, 1995 Frank Nelson Cole Prize in Algebra." <http://www.ams.org/notices/199504/prize-cole.pdf>.
 Harbater, D. "Abhyankar's Conjecture on Galois Groups Over Curves." *Invent. Math.* **117**, 1–5, 1994.
 Raynaud, M. "Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d'Abhyankar." *Invent. Math.* **116**, 425–62, 1994.

Ablowitz-Ramani-Segur Conjecture

The Ablowitz-Ramani-Segur conjecture states that a nonlinear PARTIAL DIFFERENTIAL EQUATION is solvable by the INVERSE SCATTERING METHOD only if every nonlinear ORDINARY DIFFERENTIAL EQUATION obtained by exact reduction has the PAINLEVÉ PROPERTY.

See also INVERSE SCATTERING METHOD

References

- Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An Introduction*. New York: Wiley, p. 351, 1989.

Abnormal Number

A hypothetical number which can be factored into primes in more than one way. Hardy and Wright (1979) prove the FUNDAMENTAL THEOREM OF ARITHMETIC by showing that no abnormal numbers exist.

See also FUNDAMENTAL THEOREM OF ARITHMETIC

References

- Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers*, 5th ed. Oxford, England: Clarendon Press, p. 21, 1979.

Abs

ABSOLUTE VALUE

Abscissa

The x - (horizontal) coordinate of a point in a two dimensional coordinate system. Physicists and astronomers sometimes use the term to refer to the axis itself instead of the distance along it.

See also AXIS, ORDINATE, REAL LINE, x -AXIS, y -AXIS, z -AXIS

Absolute Convergence

A SERIES $\sum_n u_n$ is said to CONVERGE absolutely if the SERIES $\sum_n |u_n|$ CONVERGES, where $|u_n|$ denotes the ABSOLUTE VALUE. If a SERIES is absolutely convergent, then the sum is independent of the order in which terms are summed. Furthermore, if the SERIES is

multiplied by another absolutely convergent series, the product series will also converge absolutely.

See also CONDITIONAL CONVERGENCE, CONVERGENT SERIES, RIEMANN SERIES THEOREM

References

- Bromwich, T. J. P. and MacRobert, T. M. "Absolute Convergence." Ch. 4 in *An Introduction to the Theory of Infinite Series*, 3rd ed. New York: Chelsea, pp. 69–7, 1991.
 Jeffreys, H. and Jeffreys, B. S. "Absolute Convergence." §1.051 in *Methods of Mathematical Physics*, 3rd ed. Cambridge, England: Cambridge University Press, p. 16, 1988.

Absolute Deviation

Let $\bar{u}$ denote the MEAN of a SET of quantities u_i , then the absolute deviation is defined by

$$\Delta u_i \equiv |u_i - \bar{u}|.$$

See also DEVIATION, MEAN DEVIATION, SIGNED DEVIATION, STANDARD DEVIATION

Absolute Error

The DIFFERENCE between the measured or inferred value of a quantity x_0 and its actual value x , given by

$$\Delta x \equiv x_0 - x$$

(sometimes with the ABSOLUTE VALUE taken) is called the absolute error. The absolute error of the SUM or DIFFERENCE of a number of quantities is less than or equal to the SUM of their absolute errors.

See also ERROR PROPAGATION, PERCENTAGE ERROR, RELATIVE ERROR

References

- Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover, p. 14, 1972.

Absolute Frequency

The number of data points which fall within a given CLASS in a FREQUENCY DISTRIBUTION.

See also CUMULATIVE FREQUENCY, FREQUENCY DISTRIBUTION, RELATIVE FREQUENCY, RELATIVE CUMULATIVE FREQUENCY

References

- Kenney, J. F. and Keeping, E. S. "Frequency Distributions." §1.8 in *Mathematics of Statistics*, Pt. 1, 3rd ed. Princeton, NJ: Van Nostrand, pp. 12–9, 1962.

Absolute Geometry

GEOMETRY which depends only on the first four of EUCLID'S POSTULATES and not on the PARALLEL POSTULATE. Euclid himself used only the first four

postulates for the first 28 propositions of the *ELEMENTS*, but was forced to invoke the PARALLEL POSTULATE on the 29th.

See also AFFINE GEOMETRY, *ELEMENTS*, EUCLID'S POSTULATES, GEOMETRY, ORDERED GEOMETRY, PARALLEL POSTULATE

References

Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid*. New York: Vintage Books, pp. 90–1, 1989.

Absolute Moment

The absolute moment of M_n of a probability function $P(x)$ taken about a point a is defined by

$$M_n = \int |x - a|^n P(x) dx.$$

See also CENTRAL MOMENT, MOMENT, RAW MOMENT

References

Papoulis, A. *Probability, Random Variables, and Stochastic Processes*, 2nd ed. New York: McGraw-Hill, p. 146, 1984.

Absolute Monotonic Sequence

See also ABSOLUTELY MONOTONIC SEQUENCE

References

Feller, W. *An Introduction to Probability Theory and Its Applications*, Vol. 2, 3rd ed. New York: Wiley, p. 224, 1971.

Absolute Pseudoprime

CARMICHAEL NUMBER

Absolute Square

Also known as the squared norm. The absolute square of a COMPLEX NUMBER z is written $|z|^2$, where $|z|$ is the MODULUS and is defined as

$$|z|^2 \equiv z\bar{z}, \quad (1)$$

where $\bar{z}$ denotes the COMPLEX CONJUGATE of z . For a REAL NUMBER, (1) simplifies to

$$|z|^2 = z^2. \quad (2)$$

If the COMPLEX NUMBER is written $z = x + iy$, then the absolute square can be written

$$|x + iy|^2 = x^2 + y^2. \quad (3)$$

An absolute square can be computed in terms of x and y using the *Mathematica* command `ComplexExpand[Abs[z]^2, TargetFunctions -> {Conjugate}]`.

An important identity involving the absolute square is given by

$$\begin{aligned} |a \pm be^{-i\delta}|^2 &= (a \pm be^{-i\delta})(a \pm be^{i\delta}) \\ &= a^2 + b^2 \pm ab(e^{i\delta} + e^{-i\delta}) = a^2 + b^2 \pm 2ab \cos \delta. \end{aligned} \quad (4)$$

If $a = 1$, then (4) becomes

$$\begin{aligned} |1 \pm be^{-i\delta}|^2 &= 1 + b^2 \pm 2b \cos \delta \\ &= (1 \pm b)^2 \mp 4b \sin^2(\tfrac{1}{2}\delta). \end{aligned} \quad (5)$$

If $a = 1$, and $b = 1$, then

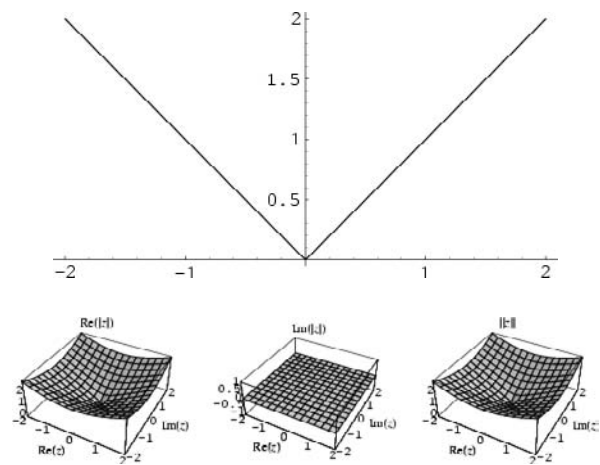
$$|1 - e^{-i\delta}|^2 = 4 \sin^2(\tfrac{1}{2}\delta). \quad (6)$$

Finally,

$$\begin{aligned} |e^{i\phi_1} + e^{i\phi_2}|^2 &= (e^{i\phi_1} + e^{i\phi_2})(e^{-i\phi_1} + e^{-i\phi_2}) \\ &= 2[1 + \cos(\phi_2 - \phi_1)] \\ &= 4 \cos^2[\tfrac{1}{2}(\phi_2 - \phi_1)]. \end{aligned} \quad (7)$$

See also ARGUMENT (COMPLEX NUMBER), COMPLEX NUMBER, MODULUS (COMPLEX NUMBER)

Absolute Value



The absolute value of a REAL NUMBER x is denoted $|x|$ and given by the "unsigned" portion of x ,

$$|x| = x \operatorname{sgn}(x) = \begin{cases} -x & \text{for } x \leq 0 \\ x & \text{for } x \geq 0, \end{cases}$$

where $\operatorname{sgn} x$ is the sign function *SGN*. The absolute value is therefore always greater than or equal to 0. The same notation is used to denote the MODULUS of a COMPLEX NUMBER $z = x + iy$, $|z| \equiv \sqrt{x^2 + y^2}$, a *P-ADIC* NORM, or a general VALUATION. The NORM of a VECTOR $\mathbf{x}$ is also denoted $|\mathbf{x}|$, although $|\mathbf{x}|$ is more commonly used.

Other NOTATIONS similar to the absolute value are the FLOOR FUNCTION $\lfloor x \rfloor$, NINT function $[x]$, and CEILING FUNCTION $\lceil x \rceil$.

The integral of the absolute value of the different of two variables is given by

$$\int_0^1 \int_0^1 |x-y|^n dx dy = \frac{2}{(n+1)(n+2)},$$

which has values 1/3, 1/6, 1/10, 1/15, 1/21, ... for $n = 1, 2, \dots$, i.e., the inverses of the TRIANGULAR NUMBERS (Sloane's A000217).

See also ABSOLUTE SQUARE, CEILING FUNCTION, FLOOR FUNCTION, MODULUS (COMPLEX NUMBER), NINT, RECTANGLE FUNCTION, SGN, TRIANGLE FUNCTION, VALUATION

References

Sloane, N. J. A. Sequences A000217/M2535 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.

Absolutely Continuous

A MEASURE λ is absolutely continuous with respect to another measure μ if $\lambda(E) = 0$ for every set with $\mu(E) = 0$. This makes sense as long as μ is a POSITIVE MEASURE, such as LEBESGUE MEASURE, but λ can be any measure, possibly a COMPLEX MEASURE.

By the RADON-NIKODYM THEOREM, this is equivalent to saying that

$$\lambda(E) = \int_E f d\mu$$

where the integral is the LEBESGUE INTEGRAL, for some INTEGRABLE function f . The function f is like a derivative, and is called the RADON-NIKODYM DERIVATIVE $d\lambda/d\mu$.

The measure supported at 0 ($\mu^{(E)} = 1$ iff $0 \in E$) is not absolutely continuous with respect to LEBESGUE MEASURE, and is a SINGULAR MEASURE.

See also COMPLEX MEASURE, CONCENTRATED, HAAR MEASURE, LEBESGUE DECOMPOSITION (MEASURE), LEBESGUE MEASURE, MUTUALLY SINGULAR, POLAR REPRESENTATION (MEASURE), SINGULAR MEASURE

References

Rudin, W. *Functional Analysis*, 2nd ed. New York: McGraw-Hill, pp. 121–25, 1991.

Absolutely Fair

A sequence of random variates $X_0, X_1, \dots$ is called absolutely fair if for $n = 1, 2, \dots$,

$$(X_1) = 0$$

and

$$(X_{n+1} | X_1, \dots, X_n) = 0$$

(Feller 1971, p. 210).

See also MARTINGALE

References

Feller, W. *An Introduction to Probability Theory and Its Applications*, Vol. 2, 3rd ed. New York: Wiley, 1971.

Absolutely Monotonic Function

This entry contributed by RONALD M. AARTS

A function $f(x)$ is absolutely monotonic in the interval $a < x < b$ if it has nonnegative derivatives of all orders in the region, i.e.,

$$f^{(k)}(x) \geq 0 \quad (1)$$

for $a < x < b$ and $k = 0, 1, 2, \dots$. For example, the functions

$$f(x) = -\ln(-x) \quad (-1 \leq x < 0) \quad (2)$$

and

$$f(x) = \sin^{-1} x \quad (0 \leq x \leq 1) \quad (3)$$

are absolutely monotonic functions (Widder 1941).

See also ABSOLUTELY MONOTONIC SEQUENCE

References

Widder, D. V. Ch. 4 in *The Laplace Transform*. Princeton, NJ: Princeton University Press, 1941.

Absolutely Monotonic Sequence

See also ABSOLUTE MONOTONIC SEQUENCE, ABSOLUTELY MONOTONIC FUNCTION

References

Feller, W. *An Introduction to Probability Theory and Its Applications*, Vol. 2, 3rd ed. New York: Wiley, p. 224, 1971.

Absorption Law

The law appearing in the definition of a BOOLEAN ALGEBRA which states

$$a \wedge (a \vee b) = a \vee (a \wedge b) = a$$

for binary operators $\vee$ and $\wedge$ (which most commonly are logical OR and logical AND).

See also BOOLEAN ALGEBRA, LATTICE

References

Birkhoff, G. and Mac Lane, S. *A Survey of Modern Algebra*, 5th ed. New York: Macmillan, p. 317, 1996.

Abstract Algebra

That portion of ALGEBRA dealing with theoretical as opposed to applied topics. Ash (1998) includes the following areas in his definite of abstract algebra: logic and foundations, counting, elementary NUMBER THEORY, informal SET THEORY, LINEAR ALGEBRA, and the theory of linear operators.

See also ALGEBRA

References

Ash, R. B. *A Primer of Abstract Mathematics*. Washington, DC: Math. Assoc. Amer., 1998.

Abstract Manifold

An abstract manifold is a MANIFOLD in the context of an abstract space with no particular embedding, or representation in mind. It is a TOPOLOGICAL SPACE with an ATLAS of COORDINATE CHARTS.

For example, the SPHERE $\mathbb{S}^2$ can be considered a SUBMANIFOLD of $\mathbb{R}^3$ or a QUOTIENT SPACE $O(3)/O(2)$. But as an abstract manifold, it is just a MANIFOLD, which can be covered by two coordinate charts $\phi_1 : \mathbb{R}^2 \rightarrow \mathbb{S}^2$ and $\phi_2 : \mathbb{R}^2 \rightarrow \mathbb{S}^2$, with the single TRANSITION FUNCTION,

$$\phi_2^{-1} \circ \phi_1 : \mathbb{R}^2 - (0, 0) \rightarrow \mathbb{R}^2 - (0, 0)$$

defined by

$$\phi_2^{-1} \circ \phi_1(x, y) = (x/r^2, y/r^2)$$

where $r^2 = x^2 + y^2$. It can also be thought of as two disks glued together at their boundary.

See also ALGEBRAIC MANIFOLD, HOMOGENEOUS SPACE, MANIFOLD, SUBMANIFOLD, TOPOLOGICAL SPACE

Abstract Mathematics

ABSTRACT ALGEBRA

Abstract Simplicial Complex

An abstract simplicial complex is a collection S of finite nonempty sets such that if A is an element of S , then so is every nonempty subset of A (Munkres 1993, p. 15).

See also SIMPLICIAL COMPLEX

References

Munkres, J. R. *Elements of Algebraic Topology*. Perseus Press, 1993.

Abstract Vector Space

See also QUOTIENT VECTOR SPACE, VECTOR SPACE

Abstraction Operator

LAMBDA CALCULUS

Abundance

The abundance of a number n is the quantity

$$A(n) \equiv \sigma(n) - 2n,$$

where $\sigma(n)$ is the DIVISOR FUNCTION. Kravitz has conjectured that no numbers exist whose abundance is an ODD SQUARE (Guy 1994).

The following table lists special classifications given to a number n based on the value of $A(n)$.

$A(n)$ Number

< 0 DEFICIENT NUMBER

-1 ALMOST PERFECT NUMBER

0 PERFECT NUMBER

1 QUASIPERFECT NUMBER

> 0 ABUNDANT NUMBER

See also ABUNDANCY, DEFICIENCY

References

Guy, R. K. *Unsolved Problems in Number Theory*, 2nd ed. New York: Springer-Verlag, pp. 45–6, 1994.

Abundancy

The ratio $\sigma(n)/n$, where $\sigma(n)$ is the DIVISOR FUNCTION.

See also ABUNDANCE, ABUNDANT NUMBER

References

Guy, R. K. "The Second Strong Law of Small Numbers." *Math. Mag.* **63**, 3–0, 1990.

Abundant Number

An abundant number is an INTEGER n which is not a PERFECT NUMBER and for which

$$s(n) \equiv \sigma(n) - n > n, \quad (1)$$

where $\sigma(n)$ is the DIVISOR FUNCTION. The quantity $\sigma(n) - 2n$ is sometimes called the ABUNDANCE. The first few abundant numbers are 12, 18, 20, 24, 30, 36, ... (Sloane's A005101). Abundant numbers are sometimes called EXCESSIVE NUMBERS.

There are only 21 abundant numbers less than 100, and they are all EVEN. The first ODD abundant number is

$$945 = 3^3 \cdot 7 \cdot 5. \quad (2)$$

That 945 is abundant can be seen by computing

$$s(945) = 975 > 945. \quad (3)$$

Any multiple of a PERFECT NUMBER or an abundant number is also abundant. Every number greater than 20161 can be expressed as a sum of two abundant numbers.

Define the density function

$$A(x) \equiv \lim_{n \rightarrow \infty} \frac{|\{n : \sigma(n) \geq xn\}|}{n} \quad (4)$$

for a POSITIVE REAL NUMBER x , then Davenport (1933) proved that $A(x)$ exists and is continuous for all x , and Erdos (1934) gave a simplified proof (Finch). Wall (1971) and Wall *et al.* (1977) showed that

$$0.2441 < A(2) < 0.2909, \quad (5)$$

and Deléglise (1998) showed that

$$0.2474 < A(2) < 0.2480. \quad (6)$$

A number which is abundant but for which all its PROPER DIVISORS are DEFICIENT is called a PRIMITIVE ABUNDANT NUMBER (Guy 1994, p. 46).

See also ALIQUOT SEQUENCE, DEFICIENT NUMBER, HIGHLY ABUNDANT NUMBER, MULTIAMICABLE NUMBERS, PERFECT NUMBER, PRACTICAL NUMBER, PRIMITIVE ABUNDANT NUMBER, WEIRD NUMBER

References

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Acceleration

Let a particle travel a distance $s(t)$ as a function of time t (here, s can be thought of as the ARC LENGTH of the curve traced out by the particle). The SPEED (the SCALAR NORM of the VECTOR VELOCITY) is then given by

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}. \quad (1)$$

The acceleration is defined as the time DERIVATIVE of the VELOCITY, so the SCALAR acceleration is given by

$$a \equiv \frac{dv}{dt} \quad (2)$$

$$= \frac{d^2s}{dt^2} \quad (3)$$

$$= \frac{\frac{dx}{dt} \frac{d^2x}{dt^2} + \frac{dy}{dt} \frac{d^2y}{dt^2} + \frac{dz}{dt} \frac{d^2z}{dt^2}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}} \quad (4)$$

$$= \frac{dx}{ds} \frac{d^2x}{dt^2} + \frac{dy}{ds} \frac{d^2y}{dt^2} + \frac{dz}{ds} \frac{d^2z}{dt^2} \quad (5)$$

$$= \frac{d\mathbf{r}}{ds} \frac{d^2\mathbf{r}}{dt^2}. \quad (6)$$

The VECTOR acceleration is given by

$$\mathbf{a} \equiv \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2s}{dt^2} \hat{\mathbf{T}} + \kappa \left(\frac{ds}{dt}\right)^2 \hat{\mathbf{N}}, \quad (7)$$

where $\hat{\mathbf{T}}$ is the UNIT TANGENT VECTOR, κ the CURVATURE, s the ARC LENGTH, and $\hat{\mathbf{N}}$ the UNIT NORMAL VECTOR.

Let a particle move along a straight LINE so that the positions at times t_1 , t_2 , and t_3 are s_1 , s_2 , and s_3 , respectively. Then the particle is uniformly accelerated with acceleration a IFF

$$a \equiv 2 \left[\frac{(s_2 - s_3)t_1 + (s_3 - s_1)t_2 + (s_1 - s_2)t_3}{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)} \right] \quad (8)$$

is a constant (Klamkin 1995, 1996).

Consider the measurement of acceleration in a rotating reference frame. Apply the ROTATION OPERATOR

$$\tilde{R} \equiv \left(\frac{d}{dt} \right)_{\text{body}} + \omega \times \quad (9)$$

twice to the RADIUS VECTOR $\mathbf{r}$ and suppress the *body* notation,

$$\begin{aligned} \mathbf{a}_{\text{space}} &= \tilde{R}^2 \mathbf{r} = \left(\frac{d}{dt} + \omega \times \right)^2 \mathbf{r} \\ &= \left(\frac{d}{dt} + \omega \times \right) \left(\frac{d\mathbf{r}}{dt} + \omega \times \mathbf{r} \right) \\ &= \frac{d^2\mathbf{r}}{dt^2} + \frac{d}{dt}(\omega \times \mathbf{r}) + \omega \times \frac{d\mathbf{r}}{dt} + \omega \times (\omega \times \mathbf{r}) \\ &= \frac{d^2\mathbf{r}}{dt^2} + \omega \times \frac{d\mathbf{r}}{dt} + \mathbf{r} \times \frac{d\omega}{dt} + \omega \times \frac{d\mathbf{r}}{dt} \\ &\quad + \omega \times (\omega \times \mathbf{r}). \end{aligned} \quad (10)$$

Grouping terms and using the definitions of the VELOCITY $\mathbf{v} \equiv d\mathbf{r}/dt$ and ANGULAR VELOCITY $\boldsymbol{\alpha} \equiv d\omega/dt$ give the expression

$$\mathbf{a}_{\text{space}} = \frac{d^2 \mathbf{r}}{dt^2} + 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r} \times \boldsymbol{\alpha}. \quad (11)$$

Now, we can identify the expression as consisting of three terms

$$\mathbf{a}_{\text{body}} \equiv \frac{d^2 \mathbf{r}}{dt^2}, \quad (12)$$

$$\mathbf{a}_{\text{Coriolis}} \equiv 2\boldsymbol{\omega} \times \mathbf{v}, \quad (13)$$

$$\mathbf{a}_{\text{centrifugal}} \equiv \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \quad (14)$$

a "body" acceleration, centrifugal acceleration, and Coriolis acceleration. Using these definitions finally gives

$$\mathbf{a}_{\text{space}} = \mathbf{a}_{\text{body}} + \mathbf{a}_{\text{Coriolis}} + \mathbf{a}_{\text{centrifugal}} + \mathbf{r} \times \boldsymbol{\alpha}, \quad (15)$$

where the fourth term will vanish in a uniformly rotating frame of reference (i.e., $\boldsymbol{\alpha} = \mathbf{0}$). The centrifugal acceleration is familiar to riders of merry-go-rounds, and the Coriolis acceleration is responsible for the motions of hurricanes on Earth and necessitates large trajectory corrections for intercontinental ballistic missiles.

See also ANGULAR ACCELERATION, ARC LENGTH, JERK, VELOCITY

References

Klamkin, M. S. "Problem 1481." *Math. Mag.* **68**, 307, 1995.
Klamkin, M. S. "A Characteristic of Constant Acceleration."
Solution to Problem 1481. *Math. Mag.* **69**, 308, 1996.

Accidental Cancellation

ANOMALOUS CANCELLATION

Accretion

CUMULATION

Accumulation Point

An accumulation point is a POINT which is the limit of a SEQUENCE, also called a LIMIT POINT. For some MAPS, periodic orbits give way to CHAOTIC ones beyond a point known as the accumulation point.

See also BOLZANO-WEIERSTRASS THEOREM Bolzano-Weierstrass Theorem, CANTOR'S INTERSECTION THEOREM, CHAOS, FRACTIONAL PART, HEINE-BOREL THEOREM, LIMIT POINT, LOGISTIC MAP, MODE LOCKING, PERIOD DOUBLING, PISOT-VIJAYARAGHAVAN CONSTANT

Achilles and the Tortoise Paradox

ZENO'S PARADOXES

Achiral

AMPHICHIRAL

Ackermann Function

The Ackermann function is the simplest example of a WELL DEFINED TOTAL FUNCTION which is COMPUTABLE but not PRIMITIVE RECURSIVE, providing a counterexample to the belief in the early 1900s that every COMPUTABLE FUNCTION was also PRIMITIVE RECURSIVE (Dötzel 1991). It grows faster than an exponential function, or even a multiple exponential function. The Ackermann function $A(x, y)$ is defined by

$$A(x, y) \equiv \begin{cases} y + 1 & \text{if } x = 0 \\ A(x - 1, 1) & \text{if } y = 0 \\ A(x - 1, A(x, y - 1)) & \text{otherwise.} \end{cases} \quad (1)$$

Special values for INTEGER x include

$$A(0, y) = y + 1 \quad (2)$$

$$A(1, y) = y + 2 \quad (3)$$

$$A(2, y) = 2y + 3 \quad (4)$$

$$A(3, y) = 2^{y+3} - 3 \quad (5)$$

$$A(4, y) = \underbrace{2^{2^{\cdot^{\cdot^2}}}}_{y+3} - 3. \quad (6)$$

Expressions of the latter form are sometimes called POWER TOWERS. $A(0, y)$ follows trivially from the definition. $A(1, y)$ can be derived as follows,

$$\begin{aligned} A(1, y) &= A(0, A(1, y - 1)) = A(1, y - 1) + 1 \\ &= A(0, A(1, y - 2)) + 1 = A(1, y - 2) + 2 \\ &= \dots = A(1, 0) + y = A(0, 1) + y = y + 2. \end{aligned} \quad (7)$$

$A(2, y)$ has a similar derivation,

$$\begin{aligned} A(2, y) &= A(1, A(2, y - 1)) = A(2, y - 1) + 2 \\ &= A(1, A(2, y - 2)) + 2 = A(2, y - 2) + 4 = \dots \\ &= A(2, 0) + 2y = A(1, 1) + 2y = 2y + 3. \end{aligned} \quad (8)$$

Buck (1963) defines a related function using the same fundamental RECURRENCE RELATION (with arguments flipped from Buck's convention)

$$F(x, y) = F(x - 1, F(x, y - 1)), \quad (9)$$

but with the slightly different boundary values

$$F(0, y) = y + 1 \quad (10)$$

$$F(1, 0) = 2 \quad (11)$$

$$F(2, 0) = 2 \quad (12)$$

$$F(x, 0) = 1 \quad \text{for } x = 3, 4, \dots \quad (13)$$

Buck's recurrence gives

$$F(1, y) = 2 + y \quad (14)$$

$$F(2, y) = 2y \quad (15)$$

$$F(3, y) = 2^y \quad (16)$$

$$F(4, y) = \underbrace{2^{2^{\cdot^{\cdot^2}}}}_y. \quad (17)$$

Taking $F(4, n)$ gives the sequence 1, 2, 4, 16, 65536, 2^{65536} , ... (Sloane's A006263). Defining $ah(x) = F(x, x)$

for $x = 0, 1, \dots$ then gives 1, 3, 4, 8, 65536, $\underbrace{2^{2^{\cdot^{\cdot^2}}}}_m$, ...

(Sloane's A001695), where $m = \underbrace{2^{2^{\cdot^{\cdot^2}}}}_{65536}$, a truly huge

number!

See also ACKERMANN NUMBER, COMPUTABLE FUNCTION, GOODSTEIN SEQUENCE, POWER TOWER, PRIMITIVE RECURSIVE FUNCTION, TAK FUNCTION, TOTAL FUNCTION

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Ackermann Number

A number OF THE FORM $\underbrace{n \uparrow \cdots \uparrow n}_n$, where ARROW

NOTATION has been used. The first few Ackermann numbers are $1 \uparrow 1 = 1$, $2 \uparrow \uparrow 2 = 4$, and

$$3 \uparrow \uparrow \uparrow 3 = \underbrace{3^{3^{3^{\cdot^{\cdot^3}}}}}_{7,625,507,484,987}.$$

See also ACKERMANN FUNCTION, ARROW NOTATION, POWER TOWER

References

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Acnode

Another name for an ISOLATED POINT.

See also CRUNODE, SPINODE, TACNODE

Acoptic Polyhedron

A term invented by B. Grünbaum in an attempt to promote concrete and precise POLYHEDRON terminology. The word "coptic" derives from the Greek for "to cut," and acoptic polyhedra are defined as POLYHEDRA for which the FACES do not intersect (cut) themselves, making them 2-MANIFOLDS.

See also HONEYCOMB, NOLID, POLYHEDRON, SPONGE

Action

Let $M(X)$ denote the GROUP of all invertible MAPS $X \rightarrow X$ and let G be any GROUP. A HOMOMORPHISM $\theta : G \rightarrow M(X)$ is called an action of G on X . Therefore, θ satisfies

1. For each $g \in G$, $\theta(g)$ is a MAP $X \rightarrow X : x \mapsto \theta(g)x$,
2. $\theta(gh)x = \theta(g)(\theta(h)x)$,
3. $\theta(e)x = x$, where e is the group identity in G ,
4. $\theta(g^{-1})x = \theta(g)^{-1}x$.

See also CASCADE, FLOW, SEMIDIRECT PRODUCT, SEMIFLOW

Actuarial Polynomial

The polynomials $a_n^{(\beta)}(x)$ given by the SHEFFER SEQUENCE with

$$g(t) = (1 - t)^{-\beta} \quad (1)$$

$$f(t) = \ln(1 - t), \quad (2)$$

giving GENERATING FUNCTION

$$\sum_{k=0}^{\infty} \frac{a_n^{(\beta)}}{k!} t^k = e^{x(1-e^t)+\beta t}. \quad (3)$$

The Sheffer identity is

$$a_n^{(\beta)}(x+y) = \sum_{k=0}^n \binom{n}{k} a_k^{(\beta)}(y) \phi_{n-k}(-x), \quad (4)$$

where $\phi_n(x)$ is an EXPONENTIAL POLYNOMIAL. The actuarial polynomials are given in terms of the EXPONENTIAL POLYNOMIALS $\phi_n(x)$ by

$$a_n^{(\beta)}(x) = (1 - t)^\beta \phi_n(-x) \quad (5)$$

$$= \sum_{k=0}^n \binom{\beta}{k} \phi_n^{(k)}(-x). \quad (6)$$

They are related to the STIRLING NUMBERS OF THE SECOND KIND $S(n, m)$ by

$$a_n^{(\beta)}(x) = \sum_{k=0}^n \binom{\beta}{k} \sum_{j=k}^n S(n, j) (j)_k (-x)^{j-k}, \quad (7)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT and $(x)_n$ is a FALLING FACTORIAL. The actuarial polynomials also satisfy the identity

$$\alpha_n^{(\beta)}(-x) = e^{-x} \sum_{k=0}^{\infty} \frac{(k+\beta)^n}{k!} x^k \quad (8)$$

(Roman 1984, p. 125; Whittaker and Watson 1990, p. 336).

The first few polynomials are

$$\alpha_0^{(\beta)}(x) = 1$$

$$\alpha_1^{(\beta)}(x) = -x + \beta$$

$$\alpha_2^{(\beta)}(x) = x^2 - x(1 + 2\beta) + \beta^2$$

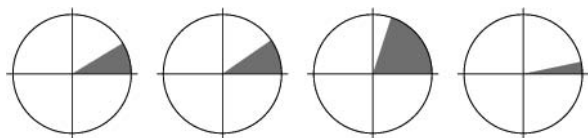
$$\alpha_3^{(\beta)}(x) = -x^3 + 3x^2(\beta + 1) - x(3\beta^2 + 3\beta + 1) + \beta^3.$$

See also SHEFFER SEQUENCE

References

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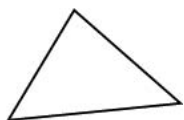
Acute Angle



An ANGLE of less than $\pi/2$ RADIANS (90°) is called an acute angle.

See also ACUTE TRIANGLE, ANGLE, FULL ANGLE, OBTUSE ANGLE, REFLEX ANGLE, RIGHT ANGLE, STRAIGHT ANGLE

Acute Triangle



A TRIANGLE in which all three ANGLES are ACUTE ANGLES. A TRIANGLE which is neither acute nor a RIGHT TRIANGLE (i.e., it has an OBTUSE ANGLE) is called an OBTUSE TRIANGLE. From the LAW OF CO-

SINES, for a triangle with side lengths a , b , and c ,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

with C the angle opposite side C . For an angle to be acute, $\cos C > 0$. Therefore, an acute triangle satisfies $a^2 + b^2 > c^2$, $b^2 + c^2 > a^2$, and $c^2 + a^2 > b^2$.

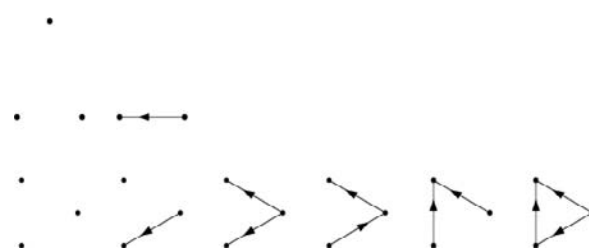
The smallest number of acute triangles into which an arbitrary OBTUSE TRIANGLE can be dissected is seven if $B > 90^\circ$, $B - A$, $B - C < 90^\circ$, and otherwise eight (Manheimer 1960, Gardner 1981, Wells 1991). A SQUARE can be dissected into as few as 9 acute triangles (Gardner 1981, Wells 1991).

See also OBTUSE TRIANGLE, ONO INEQUALITY, RIGHT TRIANGLE

References

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- Gardner, M. "Mathematical Games: The Games and Puzzles of Lewis Carroll and the Answers to February's Problems." *Sci. Amer.* **202**, 172–82, Mar. 1960.
- Gardner, M. "Mathematical Games: The Inspired Geometrical Symmetries of Scott Kim." *Sci. Amer.* **244**, 22–1, Jun. 1981.
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- Johnson, R. S. "Problem 256 [1977: 155]." *Crux Math.* **4**, 53–4, 1978.
- Nelson, H. L. "Solution to Problem 256." *Crux Math.* **4**, 102–04, 1978.
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Acyclic Digraph



An acyclic digraph is a DIRECTED GRAPH containing no directed cycles, also known as a directed acyclic graph or a "DAG." Every acyclic digraph has at least one node of OUTDEGREE 0. The numbers of acyclic digraphs on $n = 1, 2, \dots$ vertices are 1, 2, 6, 31, 302, 5984, ... (Sloane's A003087).

See also DIRECTED GRAPH, FOREST

References

- Harary, F. *Graph Theory*. Reading, MA: Addison-Wesley, p. 200, 1994.
- Robinson, R. W. "Counting Unlabeled Acyclic Digraphs." In *Combinatorial Mathematics V (Melbourne 1976)*. Providence, RI: Amer. Math. Soc., pp. 28–3, 1976.

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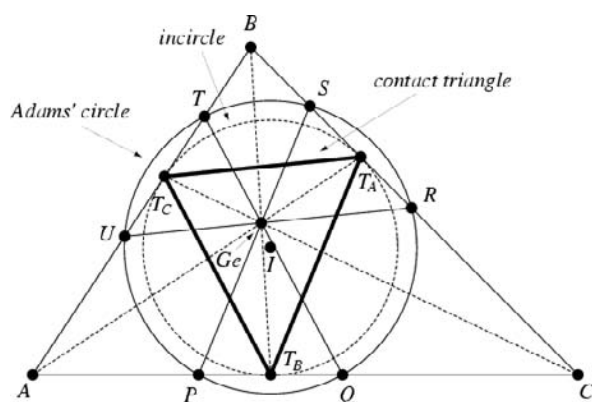
Acyclic Graph

FOREST

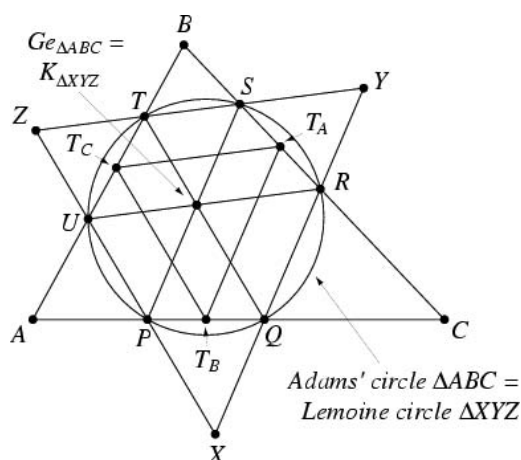
Ad

ADJOINT REPRESENTATION, ADJOINT REPRESENTATION (LIE GROUP)

Adams' Circle



Given a TRIANGLE ΔABC , construct the CONTACT TRIANGLE $\Delta T_A T_B T_C$. Now extend lines parallel to the sides of the CONTACT TRIANGLE from the GERGONNE POINT. These intersect the triangle ΔABC in the six points P, Q, R, S, T , and U . As C. Adams proved in 1843, these points are CONCYCLIC in a CIRCLE now known as Adams' circle. Moreover, Adams' circle is concentric with the INCIRCLE of ΔABC (Honsberger 1995, pp. 62–4).



Extend the segments UP, TS , and RQ to form a TRIANGLE ΔXYZ . Then the GERGONNE POINT of ΔABC is the SYMMEDIAN POINT of ΔXYZ , and Adams' circle of

ΔABC is the LEMOINE CIRCLE of ΔXYZ (Honsberger 1995, p. 98).

See also CONTACT TRIANGLE, GERGONNE POINT

References

Honsberger, R. "A Real Gem." §7.4 (v) in *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*. Washington, DC: Math. Assoc. Amer., pp. 62–4 and 98, 1995.

Adams' Method

Adams' method is a numerical METHOD for solving linear FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS OF THE FORM

$$\frac{dy}{dx} = f(x, y). \quad (1)$$

Let

$$h = x_{n+1} - x_n \quad (2)$$

be the step interval, and consider the MACLAURIN SERIES of y about x_n ,

$$y_{n+1} = y_n + \left(\frac{dy}{dx}\right)_n (x - x_n) + \frac{1}{2} \left(\frac{d^2y}{dx^2}\right)_n (x - x_n)^2 + \dots \quad (3)$$

$$\left(\frac{dy}{dx}\right)_{n+1} = \left(\frac{dy}{dx}\right)_n + \left(\frac{d^2y}{dx^2}\right)_n (x - x_n) + \dots \quad (4)$$

Here, the DERIVATIVES of y are given by the BACKWARD DIFFERENCES

$$q_n \equiv \left(\frac{dy}{dx}\right)_n = \frac{\Delta y_n}{x_{n+1} - x_n} = \frac{y_{n+1} - y_n}{h} \quad (5)$$

$$\nabla q_n \equiv \left(\frac{d^2y}{dx^2}\right)_n = q_n - q_{n-1} \quad (6)$$

$$\nabla^2 q_n \equiv \left(\frac{d^3y}{dx^3}\right)_n = \nabla q_n - \nabla q_{n-1}, \quad (7)$$

etc. Note that by (1), q_n is just the value of $f(x_n, y_n)$.

For first-order interpolation, the method proceeds by iterating the expression

$$y_{n+1} = y_n + q_n h \quad (8)$$

where $q_n \equiv f(x_n, y_n)$. The method can then be extended to arbitrary order using the finite difference integration formula from Beyer (1987)

$$\int_0^1 f_p dp = \left(1 + \frac{1}{2} \nabla + \frac{5}{12} \nabla^2 + \frac{3}{8} \nabla^3 + \frac{251}{720} \nabla^4 + \frac{95}{288} \nabla^5 + \frac{19087}{60480} \nabla^6 + \dots\right) f_p \quad (9)$$

to obtain

$$y_{n+1} - y_n = h(q_n + \frac{1}{2} \nabla q_{n-1} + \frac{5}{12} \nabla^2 q_{n-2} + \frac{3}{8} \nabla^3 q_{n-3} + \frac{251}{720} \nabla^4 q_{n-4} + \frac{95}{288} \nabla^5 q_{n-5} + \dots). \quad (10)$$

Note that von Kármán and Biot (1940) confusingly use the symbol normally used for FORWARD DIFFERENCES δ to denote BACKWARD DIFFERENCES ∇ .

See also GILL'S METHOD, MILNE'S METHOD, PREDICTOR-CORRECTOR METHODS, RUNGE-KUTTA METHOD

References

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- Kármán, T. von and Biot, M. A. *Mathematical Methods in Engineering: An Introduction to the Mathematical Treatment of Engineering Problems*. New York: McGraw-Hill, pp. 14–0, 1940.
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Adams-Bashforth-Moulton Method

ADAMS' METHOD

Addend

A quantity to be ADDED to another, also called a SUMMAND. For example, in the expression $a + b + c$, a , b , and c are all addends. The first of several addends, or "the one to which the others are added" (a in the previous example), is sometimes called the AUGEND.

See also ADDITION, AUGEND, PLUS, RADICAND

Addition

$$\begin{array}{r} \text{1 1} \quad \leftarrow \text{carries} \\ \text{1 5 8} \quad \leftarrow \text{addend 1} \\ + \text{2 4 9} \quad \leftarrow \text{addend 2} \\ \hline \text{4 0 7} \quad \leftarrow \text{sum} \end{array}$$

The combining of two or more quantities using the PLUS operator. The individual numbers being combined are called ADDENDS, and the total is called the SUM. The first of several ADDENDS, or "the one to which the others are added," is sometimes called the AUGEND. The opposite of addition is SUBTRACTION. While the usual form of adding two n -digit INTEGERS (which consists of summing over the columns right to left and "CARRYING" a 1 to the next column if the sum exceeds 9) requires n operations (plus carries), two n -

digit INTEGERS can be added in about $2 \lg n$ steps by n processors using carry-lookahead addition (McGeoch 1993). Here, $\lg x$ is the LG function, the LOGARITHM to the base 2.

See also ADDEND, AMENABLE NUMBER, AUGEND, CARRY, DIFFERENCE, DIVISION, MULTIPLICATION, PLUS, SUBTRACTION, SUM

References

- McGeoch, C. C. "Parallel Addition." *Amer. Math. Monthly* **100**, 867–71, 1993.

Addition Chain

An addition chain for a number n is a SEQUENCE $1 = a_0 < a_1 < \dots < a_r = n$, such that each member after a_0 is the SUM of two earlier (not necessarily distinct) ones. The number r is called the length of the addition chain. For example,

1, 1 + 1 = 2, 2 + 2 = 4, 4 + 2 = 6, 6 + 2 = 8, 8 + 6 = 14 is an addition chain for 14 of length $r = 5$ (Guy 1994).

See also BRAUER CHAIN, HANSEN CHAIN, SCHOLZ CONJECTURE

References

- Guy, R. K. "Addition Chains. Brauer Chains. Hansen Chains." §C6 in *Unsolved Problems in Number Theory*, 2nd ed. New York: Springer-Verlag, pp. 111–13, 1994.

Addition-Multiplication Magic Square

46	81	117	102	15	76	200	203
19	60	232	175	54	69	153	78
216	161	17	52	171	90	58	75
135	114	50	87	184	189	13	68
150	261	45	38	91	136	92	27
119	104	108	23	174	225	57	30
116	25	133	120	51	26	162	207
39	34	138	243	100	29	105	152

200	87	95	42	99	1	46	108	170
14	44	10	184	81	85	150	261	19
138	243	17	50	116	190	56	33	5
57	125	232	9	7	66	68	230	54
4	70	22	51	115	216	171	25	174
153	23	162	76	250	58	3	35	88
145	152	75	11	6	63	270	34	92
110	2	28	135	136	69	29	114	225
27	102	207	290	38	100	55	8	21

17	171	126	54	230	100	93	264	145
124	66	290	85	57	168	162	23	225
216	115	75	279	198	29	170	76	42
261	186	33	210	68	38	200	135	69
50	270	92	87	248	165	21	153	114
105	51	152	150	27	207	116	62	330
138	25	243	132	58	310	95	63	136
190	84	34	184	125	81	297	174	31
99	232	155	19	189	102	46	250	108

A square which is simultaneously a MAGIC SQUARE and MULTIPLICATION MAGIC SQUARE. The top square shown above has order eight, with addition MAGIC CONSTANT 840 and multiplicative magic constant 2,058,068,231,856,000 (Horner 1955, Hunter and Madachy 1975). The bottom two squares have order nine with addition MAGIC CONSTANTS 848 and

1200 and multiplicative magic constants 5,804,807,833,440,000 and 1,619,541,385,529,760,000, respectively (Hunter and Madachy 1975, Madachy 1979).

1	-3	2
-4	12	-8
3	-9	6

L. Sallows has constructed an interesting 3×3 magic square in which the products of corresponding pairs of 2×2 diagonals are 12, 24, 36, and 72, while the products of the numbers in the pair of 3×3 diagonals also give 72.

See also MAGIC SQUARE

References

- Horner, W. W. "Addition-Multiplication Magic Square of Order 8." *Scripta Math.* **21**, 23–7, 1955.
- Hunter, J. A. H. and Madachy, J. S. "Mystic Arrays." Ch. 3 in *Mathematical Diversions*. New York: Dover, pp. 30–1, 1975.
- Madachy, J. S. *Madachy's Mathematical Recreations*. New York: Dover, pp. 89–1, 1979.

Additive Number Theory

The portion of NUMBER THEORY concerned with expressing an integer as a sum of integers from some given set.

See also CIRCLE METHOD, MULTIPLICATIVE NUMBER THEORY, NUMBER THEORY

Additive Persistence

Consider the process of taking a number, adding its DIGITS, then adding the DIGITS of the number derived from it, etc., until the remaining number has only one DIGIT. The number of additions required to obtain a single DIGIT from a number n is called the additive persistence of n , and the DIGIT obtained is called the DIGITAL ROOT of n .

For example, the sequence obtained from the starting number 9876 is (9876, 30, 3), so 9876 has an additive persistence of 2 and a DIGITAL ROOT of 3. The additive persistences of the first few positive integers are 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, ... (Sloane's A031286). The smallest numbers of additive persistence n for $n=0, 1, \dots$ are 0, 10, 19, 199, 1999999999999999999999, ... (Sloane's A006050).

See also ADDITIVE PERSISTENCE, DIGITADDITION, DIGITAL ROOT, MULTIPLICATIVE PERSISTENCE, NARCISSISTIC NUMBER, RECURRING DIGITAL INVARIANT

References

- Hinden, H. J. "The Additive Persistence of a Number." *J. Recr. Math.* **7**, 134–35, 1974.
- Sloane, N. J. A. Sequences A006050/M4683 and A031286 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.
- Sloane, N. J. A. "The Persistence of a Number." *J. Recr. Math.* **6**, 97–8, 1973.
- Weisstein, E. W. "Integer Sequences." MATHEMATICA NOTEBOOK INTEGERSEQUENCES.M.

Adèle

An element of an ADÉLE GROUP, sometimes called a REPARTITION in older literature (e.g., Chevalley 1951, p. 25). Adèles arise in both NUMBER FIELDS and FUNCTION FIELDS. The adèles of a NUMBER FIELD are the additive SUBGROUPS of all elements in $\prod k_v$, where v is the PLACE, whose ABSOLUTE VALUE is < 1 at all but finitely many vs .

Let F be a FUNCTION FIELD of algebraic functions of one variable. Then a MAP r which assigns to every PLACE P of F an element $r(P)$ of F such that there are only a finite number of PLACES P for which $vp(r(P)) < 0$ is called an adèle (Chevalley 1951, p. 1951).

See also FUNCTION FIELD, IDELE

References

- Chevalley, C. C. *Introduction to the Theory of Algebraic Functions of One Variable*. Providence, RI: Amer. Math. Soc., p. 25, 1951.
- Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–49, 1996.

Adèle Group

The restricted topological GROUP DIRECT PRODUCT of the GROUP G_{k_v} with distinct invariant open subgroups G_0 .

References

- Weil, A. *Adèles and Algebraic Groups*. Princeton, NJ: Princeton University Press, 1961.

Adem Relations

Relations in the definition of a STEENROD ALGEBRA which state that, for $i < 2j$,

$$Sq^i \circ Sq^j(x) = \sum_{k=0}^{\lfloor i \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} \circ Sq^k(x),$$

where $f \circ g$ denotes function COMPOSITION and $\lfloor i \rfloor$ is the FLOOR FUNCTION.

See also STEENROD ALGEBRA

Adequate Knot

A class of KNOTS containing the class of ALTERNATING KNOTS. Let $c(K)$ be the CROSSING NUMBER. Then for KNOT SUM $K_1 \# K_2$ which is an adequate knot,

$$c(K_1 \# K_2) = c(K_1) + c(K_2).$$

This relationship is postulated to hold true for all KNOTS.

See also ALTERNATING KNOT, CROSSING NUMBER (LINK)

Adiabatic Invariant

A property of motion which is conserved to exponential accuracy in the small parameter representing the typical rate of change of the gross properties of the body.

See also ALGEBRAIC INVARIANT, LYAPUNOV CHARACTERISTIC NUMBER

Adjacency List

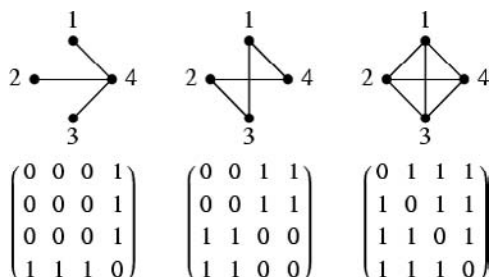
The adjacency list representation of a GRAPH consists of n lists one for each vertex v_i , $1 \leq i \leq n$, which gives the vertices to which v_i is adjacent. The adjacency lists of a graph g may be computed using `ToAdjacencyLists[g]` in the *Mathematica* add-on package `DiscreteMath`Combinatorica`` (which can be loaded with the command `<<DiscreteMath``). A graph may be constructed from adjacency lists using `FromAdjacencyLists[e]`.

See also ADJACENCY MATRIX

References

Skiena, S. "Adjacency Lists." §3.1.2 in *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*. Reading, MA: Addison-Wesley, pp. 86–7, 1990.

Adjacency Matrix



The adjacency matrix of a simple GRAPH is a MATRIX with rows and columns labeled by VERTICES, with a 1 or 0 in position (v_i, v_j) according to whether v_i and v_j are ADJACENT or not. For a simple graph with no self-loops, the adjacency matrix must have 0s on the diagonal. For an undirected graph, the adjacency matrix is symmetrical. The adjacency matrix of a graph can be computed using `Edges[g]` in the *Mathematica* add-on package `DiscreteMath`Combinatorica`` (which can be loaded with the command `<<DiscreteMath``).

See also ADJACENCY LIST, INCIDENCE MATRIX, INTEGER MATRIX

References

Chartrand, G. *Introductory Graph Theory*. New York: Dover, p. 218, 1985.
Skiena, S. "Adjacency Matrices." §3.1.1 in *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*. Reading, MA: Addison-Wesley, pp. 81–5, 1990.

Adjacency Relation

The SET E of EDGES of a GRAPH (V, E) , being a set of unordered pairs of elements of V , constitutes a RELATION on V . Formally, an adjacency relation is any RELATION which is IRREFLEXIVE and SYMMETRIC.

See also IRREFLEXIVE, RELATION, SYMMETRIC

Adjacent Fraction

Two FRACTIONS are said to be adjacent if their difference has a unit NUMERATOR. For example, $1/3$ and $1/4$ are adjacent since $1/3 - 1/4 = 1/12$, but $1/2$ and $1/5$ are not since $1/2 - 1/5 = 3/10$. Adjacent fractions can be adjacent in a FAREY SEQUENCE.

See also FAREY SEQUENCE, FORD CIRCLE, FRACTION, NUMERATOR

References

Pickover, C. A. *Keys to Infinity*. New York: Wiley, p. 119, 1995.

Adjacent Value

The value nearest to but still inside an inner FENCE.

References

Tukey, J. W. *Explanatory Data Analysis*. Reading, MA: Addison-Wesley, p. 667, 1977.

Adjacent Vertices

In a GRAPH G , two VERTICES are adjacent if they are joined by an EDGE.

See also EDGE (GRAPH), GRAPH, VERTEX (GRAPH)

Adjoint

Given a SECOND-ORDER ORDINARY DIFFERENTIAL EQUATION

$$\mathcal{L}u(x) \equiv p_0 \frac{d^2u}{dx^2} + p_1 \frac{du}{dx} + p_2u, \quad (1)$$

where $p_i \equiv p_i(x)$ and $u \equiv u(x)$, the adjoint operator $\mathcal{L}^*$ is defined by

$$\begin{aligned} \mathcal{L}^*u &\equiv \frac{d}{dx^2}(p_0u) - \frac{d}{dx}(p_1u) + p_2u \\ &= p_0 \frac{d^2u}{dx^2} + (2p_0' - p_1) \frac{du}{dx} + (p_0'' - p_1' + p_2)u. \end{aligned} \quad (2)$$

Write the two LINEARLY INDEPENDENT solutions as $y_1(x)$ and $y_2(x)$. Then the adjoint operator can also be

written

$$\mathcal{L}^*u = \int (y_2 \mathcal{L}y_1 - y_1 \mathcal{L}y_2) dx = \left[\frac{p_1}{p_0} (y_2' y_2 - y_1 y_2') \right]. \quad (3)$$

In general, given two adjoint operators $\tilde{A}$ and $\tilde{B}$,

$$(\tilde{A}\tilde{B})^* = \tilde{B}^* \tilde{A}^*, \quad (4)$$

which can be generalized to

$$(\tilde{A}\tilde{B} \dots \tilde{Z})^* = \tilde{Z}^* \dots \tilde{B}^* \tilde{A}^*. \quad (5)$$

Note that many older physics text use the a DAGGER notation $A^\dagger$ to denote the adjoint (Arfken 1985). For example, (Dirac 1982, p. 26) denotes the adjoint of the BRA vector $\langle P|\alpha$ as $\alpha^\dagger|P\rangle$, or $\tilde{\alpha}|P\rangle$. The term Hermitian conjugate is sometimes also used instead of adjoint (Griffiths 1987, p. 22)

See also ADJOINT CURVE, ADJOINT MATRIX, DAGGER, HERMITIAN OPERATOR, SELF-ADJOINT, STURM-LIOUVILLE THEORY

References

- Arfken, G. *Mathematical Methods for Physicists*, 3rd ed. Orlando, FL: Academic Press, 1985.
 Dirac, P. A. M. "Conjugate Relations." §8 in *Principles of Quantum Mechanics*, 4th ed. Oxford, England: Oxford University Press, pp. 26–9, 1982.
 Griffiths, D. J. *Introduction to Elementary Particles*. New York: Wiley, p. 220, 1987.

Adjoint Curve

A curve which has at least multiplicity $r_i - 1$ at each point where a given curve (having only ordinary singular points and cusps) has a multiplicity r_i is called the adjoint to the given curve. When the adjoint curve is of order $n - 3$, it is called a special adjoint curve.

References

- Coolidge, J. L. *A Treatise on Algebraic Plane Curves*. New York: Dover, p. 30, 1959.

Adjoint Matrix

The adjoint matrix, sometimes also called the adjugate matrix or conjugate transpose (Golub and van Loan 1996, p. 14), of an $m \times n$ MATRIX A is the $n \times m$ matrix defined by

$$A^* \equiv \tilde{A}^T, \quad (1)$$

where the ADJOINT operator is denoted with a star, T denotes the TRANSPOSE, and $\tilde{A}$ denotes the CONJUGATE MATRIX. Unfortunately, several different notations are in use. Older physics text commonly use $A^\dagger$ (Arfken 1985, p. 210), mathematicians commonly use A^* (Courant and Hilbert 1989, p. 9), and computer scientists sometimes use A^H (Golub and van Loan 1996, p. 14). In this work, a star is used to denote the adjoint operator, so care must be taken not to confuse

this with the star used in older physics and engineering texts to denote the COMPLEX CONJUGATE.

If a MATRIX is SELF-ADJOINT, it is said to be HERMITIAN. The adjoint matrix of a MATRIX product is given by

$$(ab)^*_{ij} \equiv [\overline{(ab)^T}]_{ij}. \quad (2)$$

Using the identity for the product of TRANSPOSE gives

$$\begin{aligned} [\overline{(ab)^T}]_{ij} &= [\overline{b^T a^T}]_{ij} = \overline{b^T a^T}_{ik} = [\overline{b^T}]_{ik} [\overline{a^T}]_{kj} = b^*_{ik} a^*_{kj} \\ &= [b^* a^*]_{ij}, \end{aligned} \quad (3)$$

where EINSTEIN SUMMATION has been used here to sum over repeated indices, it follows that

$$(AB)^* = B^* A^*. \quad (4)$$

See also ADJOINT, COMPLEX CONJUGATE, DAGGER, HERMITIAN MATRIX, SCHUR DECOMPOSITION, TRANSPOSE

References

- Arfken, G. *Mathematical Methods for Physicists*, 3rd ed. Orlando, FL: Academic Press, p. 210, 1985.
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 Courant, R. and Hilbert, D. *Methods of Mathematical Physics, Vol. 1*. New York: Wiley, 1989.
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Adjoint Operator

Given a SECOND-ORDER ORDINARY DIFFERENTIAL EQUATION

$$p_i \equiv p_i(x) \quad (1)$$

where $u \equiv u(x)$ and $\mathcal{L}^*$, the adjoint operator $\mathcal{L}^*u$ (denoted by a DAGGER), is defined by

$$\begin{aligned} \frac{d}{dx^2} (p_0 u) - \frac{d}{dx} (p_1 u) + p_2 u (y_1 y_2'' - y_2 y_1'') + P (y_1 y_2' - y_1' y_2) \\ + Q (y_1 y_2 - y_1 y_2) = 0^{p0} \\ p_0 \frac{d^2 u}{dx^2} + (2p_0' - p_1) \frac{du}{dx} + (p_0'' - p_1' + p_2) u = \mathcal{L}^* u \\ = \int (y_2 \mathcal{L}y_1 - y_1 \mathcal{L}y_2) dx = \left[\frac{p_1}{p_0} (y_1' y_2 - y_1 y_2') \right]. \end{aligned} \quad (2)$$

Write the two LINEARLY INDEPENDENT solutions as $y' = f_0(x) + f_1(x)y + f_0(x)y^2 + f_3(x)y^3 + \dots$ and $[g_0(x) + g_1(x)y]y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$. Then the adjoint operator can also be written

$$\tilde{A}. \quad (3)$$

In general, given two adjoint operators $\tilde{B}$ and $(\tilde{A}\tilde{B})^* = \tilde{B}^* \tilde{A}^*$,

$$(\tilde{A}\tilde{B}\dots\tilde{Z})^* = \tilde{Z}^* \dots \tilde{B}^* \tilde{A}^*, \quad (4)$$

which can be generalized to

$$A^\dagger. \quad (5)$$

The adjoint of the BRA vector $\langle P|\alpha$ is denoted $\alpha^\dagger|P\rangle$, or $\bar{\alpha}|P\rangle$ (Dirac 1982, p. 26). The term Hermitian conjugate is sometimes also used (Griffiths 1987, p. 22)

See also ADJOINT MATRIX, DAGGER, HERMITIAN OPERATOR, SELF-ADJOINT OPERATOR, STURM-LIOUVILLE THEORY

References

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Adjoint Representation

A LIE ALGEBRA is a VECTOR SPACE g with a LIE BRACKET $[X, Y]$, satisfying the JACOBI IDENTITY. Hence any element X gives a linear transformation given by

$$\text{ad}(X)(Y) = [X, Y], \quad (1)$$

which is called the adjoint representation of g . It is a LIE ALGEBRA REPRESENTATION because of the JACOBI IDENTITY,

$$\begin{aligned} [\text{ad}(X_1), \text{ad}(X_2)](Y) &= [X_1, [X_2, Y]] - [X_2, [X_1, Y]] \\ &= [[X_1, X_2], Y] = \text{ad}([X_1, X_2])(Y). \end{aligned} \quad (2)$$

A REPRESENTATION is given by matrices. The simplest LIE ALGEBRA is gl_n the set of matrices. Consider the adjoint representation of gl_2 , which has four dimensions and so will be a four dimensional representation. The matrices

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (3)$$

$$e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (4)$$

$$e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad (5)$$

$$e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (6)$$

give a basis for gl_2 . Using this basis, the adjoint representation is described by the following matrices,

$$\text{ad } e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (7)$$

$$\text{ad } e_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad (8)$$

$$\text{ad } e_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (9)$$

$$\text{ad } e_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (10)$$

The following *Mathematica* function gives the adjoint representation of the matrix m in the Lie algebra, given by a basis, the list of matrices g .

```
ad[g_List, m_List?MatrixQ] := Transpose[LinearSolve[Transpose[Flatten/@g], Flatten[m.#1-#1.m]]&/@g]
```

See also COMMUTATOR, LIE ALGEBRA, LIE GROUP, LIE BRACKET, NILPOTENT LIE ALGEBRA, REPRESENTATION, SEMISIMPLE LIE ALGEBRA

References

- Fulton, W. and Harris, J. *Representation Theory*. New York: Springer-Verlag, 1991.
Jacobson, N. *Lie Algebras*. New York: Dover, 1979.
Knapp, A. *Lie Groups Beyond an Introduction*. Boston, MA: Birkhäuser, 1996.

Adjugate Matrix

ADJOINT MATRIX

Adjunction

If a is an element of a FIELD F over the PRIME FIELD P , then the set of all RATIONAL FUNCTIONS of a with COEFFICIENTS in P is a FIELD derived from P by adjunction of a .

Adleman-Pomerance-Rumely Primality Test

A modified MILLER'S PRIMALITY TEST which gives a guarantee of PRIMALITY or COMPOSITENESS. The ALGORITHM's running time for a number n has been proved to be as $\mathcal{O}((\ln n)^{c \ln \ln \ln n})$ for some $c > 0$. It was simplified by Cohen and Lenstra (1984), implemented by Cohen and Lenstra (1987), and subsequently optimized by Bosma and van der Hulst (1990).

References

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Adleman-Rumely Primality Test

ADLEMAN-POMERANCE-RUMELY PRIMALITY TEST

Admissible

A string or word is said to be admissible if that word appears in a given SEQUENCE. For example, in the SEQUENCE *aabaabaabaabaab . . . , a , aa , baab* are all admissible, but *bb* is inadmissible.

See also BLOCK GROWTH

Ado's Theorem

Every finite-dimensional LIE ALGEBRA of characteristic $p = 0$ has a FAITHFUL finite-dimensional representation.

See also IWASAWA'S THEOREM, LIE ALGEBRA

References

- Jacobson, N. *Lie Algebras*. New York: Dover, pp. 202–03, 1979.

Affine Complex Plane

The set $\mathbb{A}^2$ of all ORDERED PAIRS of COMPLEX NUMBERS.

See also AFFINE CONNECTION, AFFINE EQUATION, AFFINE GEOMETRY, AFFINE GROUP, AFFINE HULL, AFFINE PLANE, AFFINE SPACE, AFFINE TRANSFORMATION, AFFINITY, COMPLEX PLANE, COMPLEX PROJECTIVE PLANE

Affine Connection

CONNECTION COEFFICIENT

Affine Equation

A nonhomogeneous LINEAR EQUATION or system of nonhomogeneous LINEAR EQUATIONS is said to be affine.

See also AFFINE COMPLEX PLANE, AFFINE CONNECTION, AFFINE GEOMETRY, AFFINE GROUP, AFFINE HULL, AFFINE PLANE, AFFINE SPACE, AFFINE TRANSFORMATION, AFFINITY

Affine Geometry

A GEOMETRY in which properties are preserved by PARALLEL PROJECTION from one PLANE to another. In an affine geometry, the third and fourth of EUCLID'S POSTULATES become meaningless. This type of GEOMETRY was first studied by Euler.

See also ABSOLUTE GEOMETRY, AFFINE COMPLEX PLANE, AFFINE CONNECTION, AFFINE EQUATION, AFFINE GROUP, AFFINE HULL, AFFINE PLANE, AFFINE SPACE, AFFINE TRANSFORMATION, AFFINITY, ORDERED GEOMETRY

References

- Birkhoff, G. and Mac Lane, S. "Affine Geometry." §9.13 in *A Survey of Modern Algebra*, 5th ed. New York: Macmillan, pp. 268–75, 1996.
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Affine Group

The set of all nonsingular AFFINE TRANSFORMATIONS of a TRANSLATION in SPACE constitutes a GROUP known as the affine group. The affine group contains the full linear group and the group of TRANSLATIONS as SUBGROUPS.

See also AFFINE COMPLEX PLANE, AFFINE CONNECTION, AFFINE EQUATION, AFFINE GEOMETRY, AFFINE HULL, AFFINE PLANE, AFFINE SPACE, AFFINE TRANSFORMATION, AFFINITY

References

- Birkhoff, G. and Mac Lane, S. *A Survey of Modern Algebra*, 5th ed. New York: Macmillan, p. 237, 1996.

Affine Hull

The IDEAL generated by a SET in a VECTOR SPACE.

See also AFFINE COMPLEX PLANE, AFFINE CONNECTION, AFFINE EQUATION, AFFINE GEOMETRY, AFFINE GROUP, AFFINE PLANE, AFFINE SPACE, AFFINE TRANSFORMATION, AFFINITY, CONVEX HULL, HULL

Affine Plane

A 2-D AFFINE GEOMETRY constructed over a FINITE FIELD. For a FIELD F of size n , the affine plane consists of the set of points which are ordered pairs of elements in F and a set of lines which are themselves a set of points. Adding a POINT AT INFINITY and LINE AT INFINITY allows a PROJECTIVE PLANE to be constructed from an affine plane. An affine plane of order n is a BLOCK DESIGN OF THE FORM $(n^2, n, 1)$. An affine plane of order n exists IFF a PROJECTIVE PLANE of order n exists.

See also AFFINE COMPLEX PLANE, AFFINE CONNECTION, AFFINE EQUATION, AFFINE GEOMETRY, AFFINE

GROUP, AFFINE HULL, AFFINE SPACE, AFFINE TRANSFORMATION, AFFINITY, PROJECTIVE PLANE

References

Lindner, C. C. and Rodger, C. A. *Design Theory*. Boca Raton, FL: CRC Press, 1997.

Affine Scheme

Let P be the set of PRIME IDEALS of a COMMUTATIVE RING A . Then an affine scheme is a technical mathematical object defined as the SPECTRUM $\sigma(A)$ of P , regarded as a local-ringed space with a structure sheaf. A local-ringed space that is locally isomorphic to an affine scheme is called a SCHEME (Itô 1986, p. 69).

See also PRIME IDEAL, SCHEME, SPECTRUM (RING)

References

Itô, K. (Ed.). "Schemes." §16D in *Encyclopedic Dictionary of Mathematics, 2nd ed., Vol. 1*. Cambridge, MA: MIT Press, p. 69, 1986.

Affine Space

Let V be a VECTOR SPACE over a FIELD K , and let A be a nonempty SET. Now define addition $p + \mathbf{a} \in A$ for any VECTOR $\mathbf{a} \in V$ and element $p \in A$ subject to the conditions

1. $p + \mathbf{0} = p$,
2. $(p + \mathbf{a}) + \mathbf{b} = p + (\mathbf{a} + \mathbf{b})$,
3. For any $q \in A$, there EXISTS a unique VECTOR $\mathbf{a} \in V$ such that $q = p + \mathbf{a}$.

Here, $\mathbf{a}, \mathbf{b} \in V$. Note that (1) is implied by (2) and (3). Then A is an affine space and K is called the COEFFICIENT FIELD.

In an affine space, it is possible to fix a point and coordinate axis such that every point in the SPACE can be REPRESENTED AS an n -tuple of its coordinates. Every ordered pair of points A and B in an affine space is then associated with a VECTOR $\overrightarrow{AB}$.

See also AFFINE COMPLEX PLANE, AFFINE CONNECTION, AFFINE EQUATION, AFFINE GEOMETRY, AFFINE GROUP, AFFINE HULL, AFFINE PLANE, AFFINE SPACE, AFFINE TRANSFORMATION, AFFINITY

Affine Transformation

Any TRANSFORMATION preserving COLLINEARITY (i.e., all points lying on a LINE initially still lie on a LINE after TRANSFORMATION) and ratios of distances (e.g., the midpoint of a line segment remains the midpoint after transformation). An affine transformation may also be thought of as a shearing transformation (Croft *et al.* 1991). An affine transformation is also called an AFFINITY.

An affine transformation of $\mathbb{R}^n$ is a MAP $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ OF THE FORM

Affine Transformation

$$F(\mathbf{p}) = A\mathbf{p} + \mathbf{q} \quad (1)$$

for all $\mathbf{p} \in \mathbb{R}^n$, where A is a linear transformation of $\mathbb{R}^n$. If $\det(A) = 1$, the transformation is ORIENTATION-PRESERVING; if $\det(A) = -1$, it is ORIENTATION-REVERSING.

CONTRACTION, EXPANSION, DILATION, REFLECTION, SIMILARITY TRANSFORMATIONS, SPIRAL SIMILARITIES, ROTATION, and TRANSLATION are all affine transformations, as are their combinations. A particular example combining ROTATION and EXPANSION is the rotation-enlargement transformation

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= s \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \\ &= s \begin{bmatrix} \cos \alpha (x - x_0) + \sin \alpha (y - y_0) \\ -\sin \alpha (x - x_0) + \cos \alpha (y - y_0) \end{bmatrix}. \end{aligned} \quad (2)$$

Separating the equations,

$$x' = (s \cos \alpha)x + (s \sin \alpha)y - s(x_0 \cos \alpha + y_0 \sin \alpha) \quad (3)$$

$$y' = (-s \sin \alpha)x + (s \cos \alpha)y + s(x_0 \sin \alpha - y_0 \cos \alpha). \quad (4)$$

This can be also written as

$$x' = ax + by + c \quad (5)$$

$$y' = bx + ay + d, \quad (6)$$

where

$$a = s \cos \alpha \quad (7)$$

$$b = -s \sin \alpha. \quad (8)$$

The scale factor s is then defined by

$$s \equiv \sqrt{a^2 + b^2}, \quad (9)$$

and the rotation ANGLE by

$$\alpha = \tan^{-1} \left(-\frac{b}{a} \right). \quad (10)$$

See also AFFINE COMPLEX PLANE, AFFINE CONNECTION, AFFINE EQUATION, AFFINE GEOMETRY, AFFINE GROUP, AFFINE HULL, AFFINE PLANE, AFFINE SPACE, AFFINE TRANSFORMATION, AFFINITY, EQUIAFFINITY, EUCLIDEAN MOTION

References

- Croft, H. T.; Falconer, K. J.; and Guy, R. K. *Unsolved Problems in Geometry*. New York: Springer-Verlag, p. 3, 1991.
- Gray, A. *Modern Differential Geometry of Curves and Surfaces with Mathematics*, 2nd ed. Boca Raton, FL: CRC Press, p. 130, 1997.
- Zwillinger, D. (Ed.). "Affine Transformations." §4.3.2 in *CRC Standard Mathematical Tables and Formulae*. Boca Raton, FL: CRC Press, pp. 265–66, 1995.

Affine Variety

An affine variety V is a VARIETY contained in AFFINE SPACE. For example,

$$\{(x, y, z) : x^2 + y^2 - z^2 = 0\} \quad (1)$$

is the CONE, and

$$\{(x, y, z) : x^2 + y^2 - z^2 = 0, ax + by + cz = 0\} \quad (2)$$

is a CONIC SECTION, which is a SUBVARIETY of the cone. The cone can be written $V(x^2 + y^2 - z^2)$ to indicate that it is the variety corresponding to $x^2 + y^2 - z^2 = 0$. Naturally, many other polynomials vanish on $V(x^2 + y^2 - z^2)$, in fact all polynomials in $I(C) = \{x^2 + y^2 - z^2\}$. The set $I(C)$ is an IDEAL in the POLYNOMIAL RING $\mathbb{C}[x, y, z]$. Note also, that the ideal of polynomials vanishing on the conic section is the IDEAL generated by $x^2 + y^2 - z^2$ and $ax + by + cz$.

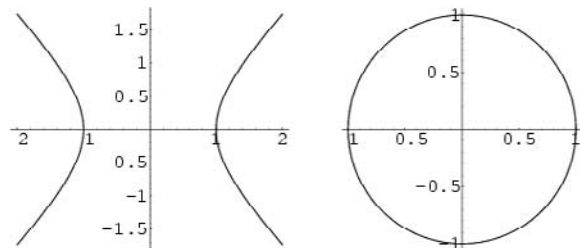
A MORPHISM between two affine varieties is given by polynomial coordinate functions. For example, the map $\phi(x, y, z) = (x^2, y^2, z^2)$ is a MORPHISM from $X = V(x^2 + y^2 + z^2)$ to $Y = V(x + y + z)$. Two affine varieties are ISOMORPHIC if there is a MORPHISM which has an inverse morphism. For example, the affine variety $V(x^2 + y^2 + z^2)$ is isomorphic to the cone $V(x^2 + y^2 - z^2)$ via the coordinate change $\phi(x, y, z) = (x, y, iz)$.

Many polynomials f may be factored, for instance $f = x^2 + y^2 = (x + iy)(x - iy)$, and then $V(f) = V(x + iy) \cup V(x - iy)$. Consequently, only IRREDUCIBLE POLYNOMIALS, and more generally only PRIME IDEALS p are used in the definition of a variety. An affine variety V is the set of common zeros of a collection of polynomials $p_1, \dots, p_k$, i.e.,

$$V = \{x = (x_1, \dots, x_n) : p_1(x) = \dots = p_k(x) = 0\} \quad (3)$$

as long as the IDEAL $I = (p_1, \dots, p_k)$ is a PRIME IDEAL. More classically, an affine variety is defined by any set of polynomials, i.e., what is now called an ALGEBRAIC SET. Most points in V will have dimension $n - k$, but V may have singular points like the origin in the cone.

When V is one-dimensional generically (at almost all points), which typically occurs when $k = n - 1$, then V is called a curve. When V is two-dimensional, it is called a surface. In the case of COMPLEX affine space, a curve is a RIEMANN SURFACE, possibly with some singularities.



Mathematica has a built-in function `ImplicitPlot` in the *Mathematica* add-on package `Graphics`Im-`

`plicitPlot`` (which can be loaded with the command `<<Graphics``) that will graph affine varieties in the real affine plane. For example, the following graphs a hyperbola and a circle.

```
<<Graphics`;
Show[GraphicsArray[{
  ImplicitPlot[x^2 - y^2 == 1, {x, -2, 2},
    DisplayFunction -> Identity],
  ImplicitPlot[x^2 + y^2 == 1, {x, -2, 2},
    DisplayFunction -> Identity]
}]]
```

An extension to this function called `ImplicitPlot3D` can be downloaded from MathSource and used to plot affine varieties in three-dimensional space.

See also ALGEBRAIC SET, CATEGORY THEORY, COMMUTATIVE ALGEBRA, CONIC SECTION, GROEBNER BASIS, PROJECTIVE VARIETY, SCHEME, STACK (MODULI SPACE), INTRINSIC VARIETY, ZARISKI TOPOLOGY

References

- Bump, D. *Algebraic Geometry*. Singapore: World Scientific, pp. 1–, 1998.
 Cox, D.; Little, J.; and O’Shea, D. *Ideals, Varieties, and Algorithms*. New York: Springer-Verlag, pp. 5–9, 1997.
 Hartshorne, R. *Algebraic Geometry*. New York: Springer-Verlag, 1977.

Affinity

AFFINE TRANSFORMATION

Affix

In the archaic terminology of Whittaker and Watson (1990), the COMPLEX NUMBER z representing $x + iy$.

References

- Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, 1990.

Aggregate

An archaic word for infinite SETS such as those considered by Georg Cantor.

See also CLASS (SET), SET

AGM

ARITHMETIC-GEOMETRIC MEAN

Agnesi’s Witch

WITCH OF AGNESI

Agnésienne

WITCH OF AGNESI

Agonic Lines

SKEW LINES

Ahlfors Five Island Theorem

Let $f(z)$ be a TRANSCENDENTAL MEROMORPHIC FUNCTION, and let $D_1, D_2, \dots, D_5$ be five SIMPLY CONNECTED domains in $\mathbb{C}$ with disjoint closures (Ahlfors 1932). Then there exists $j \in \{1, 2, \dots, 5\}$ and, for any $R > 0$, a SIMPLY CONNECTED domain $G \subset \{z \in \mathbb{C} : |z| > R\}$ such that $f(z)$ is a CONFORMAL MAP of G onto D_j . If $f(z)$ has only finitely many POLES, then "five" may be replaced by "three" (Ahlfors 1933).

See also MEROMORPHIC FUNCTION, TRANSCENDENTAL FUNCTION

References

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- Nevanlinna, R. *Analytic Functions*. New York: Springer-Verlag, 1970.

Ahlfors-Bers Theorem

The RIEMANN'S MODULI SPACE gives the solution to RIEMANN'S MODULI PROBLEM, which requires an ANALYTIC parameterization of the compact RIEMANN SURFACES in a fixed HOMEOMORPHISM.

A-Integrable

A generalization of the LEBESGUE INTEGRAL. A MEASURABLE FUNCTION $f(x)$ is called A-integrable over the CLOSED INTERVAL $[a, b]$ if

$$m\{x : |f(x)| > n\} = \mathcal{O}(n^{-1}), \quad (1)$$

where m is the LEBESGUE MEASURE, and

$$I = \lim_{n \rightarrow \infty} \int_a^b [f(x)]_n dx \quad (2)$$

exists, where

$$[f(x)]_n = \begin{cases} f(x) & \text{if } |f(x)| \leq n \\ 0 & \text{if } |f(x)| > n. \end{cases} \quad (3)$$

References

- Titchmarsh, E. C. "On Conjugate Functions." *Proc. London Math. Soc.* **29**, 49–0, 1928.

Airy Differential Equation

Some authors define a general Airy differential equation as

Airy Differential Equation

$$y'' \pm k^2 xy = 0. \quad (1)$$

This equation can be solved by series solution using the expansions

$$y = \sum_{n=0}^{\infty} a_n x^n \quad (2)$$

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \end{aligned} \quad (3)$$

$$\begin{aligned} y^n &= \sum_{n=0}^{\infty} (n+1) n a_{n+1} x^{n-1} = \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^{n-1} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n. \end{aligned} \quad (4)$$

Specializing to the "conventional" Airy differential equation occurs by taking the MINUS SIGN and setting $k^2 = 1$. Then plug (4) into

$$y'' - xy = 0 \quad (5)$$

to obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} a_n x^n = 0 \quad (6)$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \quad (7)$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \quad (8)$$

$$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - a_{n-1}] x^n = 0. \quad (9)$$

In order for this equality to hold for all x , each term must separately be 0. Therefore,

$$a_2 = 0 \quad (10)$$

$$(n+2)(n+1) a_{n+2} = a_{n-1}. \quad (11)$$

Starting with the $n = 3$ term and using the above RECURRENCE RELATION, we obtain

$$5 \cdot 4a_5 = 20a_5 = a_2 = 0. \quad (12)$$

Continuing, it follows by INDUCTION that

$$a_2 = a_5 = a_8 = a_{11} = \dots a_{3n-1} = 0 \quad (13)$$

for $n = 1, 2, \dots$. Now examine terms OF THE FORM a_{3n} .

$$a_3 = \frac{a_0}{3 \cdot 2} \quad (14)$$

$$a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{(6 \cdot 5)(3 \cdot 2)} \quad (15)$$

$$a_9 = \frac{a_6}{9 \cdot 8} = \frac{a_0}{(9 \cdot 8)(6 \cdot 5)(3 \cdot 2)}. \quad (16)$$

Again by INDUCTION,

$$a_{3n} = \frac{a_0}{[(3n)(3n-1)][(3n-3)(3n-4)] \cdots [6 \cdot 5][3 \cdot 2]} \quad (17)$$

for $n = 1, 2, \dots$. Finally, look at terms OF THE FORM a_{3n+1} ,

$$a_4 = \frac{a_1}{4 \cdot 3} \quad (18)$$

$$a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{(7 \cdot 6)(4 \cdot 3)} \quad (19)$$

$$a_{10} = \frac{a_7}{10 \cdot 9} = \frac{a_1}{(10 \cdot 9)(7 \cdot 6)(4 \cdot 3)}. \quad (20)$$

By INDUCTION,

$$a_{3n+1} = \frac{a_1}{[(3n+1)(3n)][(3n-2)(3n-3)] \cdots [7 \cdot 6][4 \cdot 3]} \quad (21)$$

for $n = 1, 2, \dots$. The general solution is therefore

$$y = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{(3n)(3n-1)(3n-3)(3n-4) \cdots 3 \cdot 2} \right] + a_1 \left[x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{(3n+1)(3n)(3n-2)(3n-3) \cdots 4 \cdot 3} \right]. \quad (22)$$

For a general k^2 with a MINUS SIGN, equation (1) is

$$y'' - k^2 xy = 0, \quad (23)$$

and the solution is

$$y(x) = \frac{1}{3} \sqrt{x} [AI_{-1/3}(\frac{2}{3} kx^{3/2}) - BI_{1/3}(\frac{2}{3} kx^{3/2})], \quad (24)$$

where I is a MODIFIED BESSEL FUNCTION OF THE FIRST KIND. This is usually expressed in terms of the AIRY FUNCTIONS $Ai(x)$ and $Bi(x)$

$$y(x) = A' Ai(k^{2/3}x) + B' Bi(k^{2/3}x). \quad (25)$$

If the PLUS SIGN is present instead, then

$$y'' + k^2 xy = 0 \quad (26)$$

and the solutions are

$$y(x) = \frac{1}{3} \sqrt{x} [AJ_{-1/3}(\frac{2}{3} kx^{3/2}) + BJ_{1/3}(\frac{2}{3} kx^{3/2})], \quad (27)$$

where $J(z)$ is a BESSEL FUNCTION OF THE FIRST KIND.

A generalization of the Airy differential equation is given by

$$y''' - 4xy' - 2y = 0, \quad (28)$$

which has solutions

$$y = C_1 [Ai(x)]^2 + C_2 Ai(x) Bi(x) + C_3 [Bi(x)]^2 \quad (29)$$

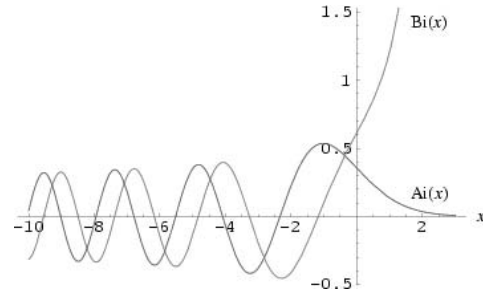
(Abramowitz and Stegun 1972, p. 448; Zwillinger 1997, p. 128).

See also AIRY-FOCK FUNCTIONS, AIRY FUNCTIONS, BESSEL FUNCTION OF THE FIRST KIND, MODIFIED BESSEL FUNCTION OF THE FIRST KIND

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Airy Functions." §10.4.1 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover, pp. 446–52, 1972.
Zwillinger, D. (Ed.). *CRC Standard Mathematical Tables and Formulae*. Boca Raton, FL: CRC Press, p. 413, 1995.
Zwillinger, D. *Handbook of Differential Equations*, 3rd ed. Boston, MA: Academic Press, p. 121, 1997.

Airy Functions



The $Ai(x)$ and $Bi(x)$ functions are defined as the two LINEARLY INDEPENDENT solutions to

$$y'' - yz = 0. \quad (1)$$

(Abramowitz and Stegun 1972, pp. 446–47; illustrated above), written in the form

$$y(z) = A Ai(z) + B Bi(z), \quad (2)$$

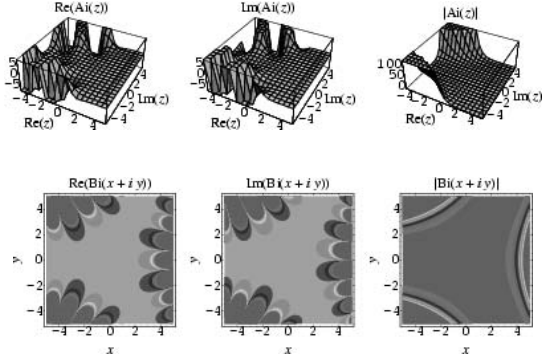
where

$$Ai(z) = \frac{1}{3} \sqrt{x} \left[I_{-1/3} \left(\frac{2}{3} z^{3/2} \right) - I_{1/3} \left(\frac{2}{3} z^{3/2} \right) \right] = \sqrt{\frac{z}{3\pi}} K_{1/3} \left(\frac{2}{3} z^{3/2} \right) \quad (3)$$

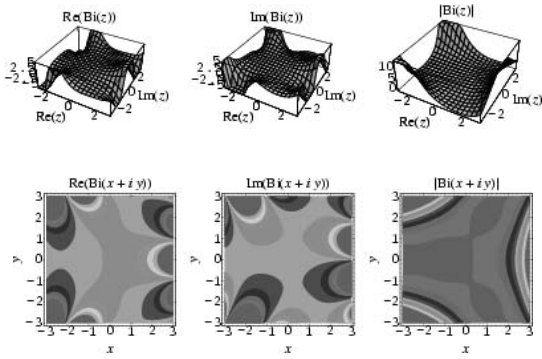
$$Bi(z) = \sqrt{\frac{z}{3}} \left[I_{-1/3} \left(\frac{2}{3} z^{3/2} \right) + I_{1/3} \left(\frac{2}{3} z^{3/2} \right) \right], \quad (4)$$

where $I(z)$ is a MODIFIED BESSEL FUNCTION OF THE FIRST KIND and $K(z)$ is a MODIFIED BESSEL FUNCTION OF THE SECOND KIND. The functions are implemented in *Mathematica* as `AiryAi[z]` and `AiryBi[z]`. Their derivatives are implemented as `AiryAiPrime[z]` and

AiryBiPrime[z].



Plots of $\text{Ai}(z)$ in the COMPLEX PLANE are illustrated above, and $\text{Bi}(z)$ is illustrated below.



The Airy $\text{Ai}(x)$ function is given by the integral

$$\text{Ai}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(z t + t^3/3)} dt \quad (5)$$

and the INFINITE SERIES

$$\begin{aligned} \text{Ai}(x) = & \frac{1}{3^{2/3}\pi} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{3}(n+1)\right)}{n!} \\ & \times (3^{1/3}x)^n \sin\left[\frac{2(n+1)\pi}{3}\right] \end{aligned} \quad (6)$$

(Banderier *et al.*). A generalization of the Airy function has been constructed by Hardy.

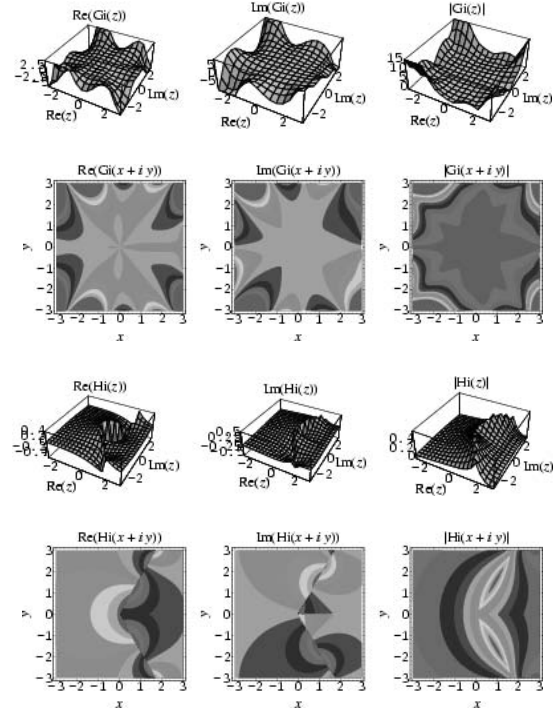
For $z = 0$,

$$\text{Ai}(0) = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})} \quad (7)$$

$$\text{Bi}(0) = \frac{1}{3^{1/6}\Gamma(\frac{2}{3})}, \quad (8)$$

where $\Gamma(z)$ is the GAMMA FUNCTION.

The ASYMPTOTIC SERIES of $\text{Ai}(z)$ has a different form in different QUADRANTS of the COMPLEX PLANE, a fact known as the STOKES PHENOMENON.



Functions related to the Airy functions have been defined as

$$\text{Gi}(z) \equiv \frac{1}{\pi} \int_0^{\infty} \sin\left(\frac{1}{3}t^3 + zt\right) dt \quad (9)$$

$$\text{Hi}(z) \equiv \frac{1}{\pi} \int_0^{\infty} \exp\left(-\frac{1}{3}t^3 + zt\right) dt, \quad (10)$$

where $\text{Gi}(z)$ is defined for $\Im[z] \neq 0$ and $\text{Hi}(z)$ for $\Re[z] \geq 0$. The can be expressed in terms of the Airy functions by

$$\begin{aligned} \text{Gi}(z) = & -\frac{z^2}{2\pi} {}_1F_4\left(1 : \frac{2}{3}, \frac{5}{6}, \frac{7}{6}, \frac{4}{3}; \frac{1}{1296} z^6\right) \\ & + \frac{[\text{sgn}(z)]^6}{360\pi z^6} {}_1F_4\left(1 : \frac{7}{6}, \frac{4}{3}, \frac{5}{3}, \frac{11}{6}; \frac{1}{1296} z^6\right) + \frac{z^6}{6|z|^6} \\ & \times [\text{Bi}(-|z|) + \text{Bi}(|z|)] - \frac{i\sqrt{3}|z|^3}{6z^4} [\text{Ai}(-|z|) - \text{Ai}(|z|)] \\ & + \frac{1}{6z^4|z|^6} \{\Im[z] + \Re[z][\text{Bi}(|z|) - \text{Bi}(-|z|)]\} \end{aligned} \quad (11)$$

$$\begin{aligned} \text{Hi}(z) = & \frac{2}{3}\sqrt{\frac{2}{3}} \left[J_{-1/3}\left(\frac{2}{3}(-z)^{3/2}\right) - J_{1/3}\left(\frac{2}{3}(-z)^{3/2}\right) \right] \\ & + \frac{z^2}{2\pi} {}_1F_2\left(1 : \frac{4}{3}, \frac{5}{3}; \frac{1}{9} z^3\right), \end{aligned} \quad (12)$$

where ${}_pF_q$ is a GENERALIZED HYPERGEOMETRIC FUNCTION, SGN is the sign function, $|z|$ is the MODULUS of z , $\Re[z]$ is the REAL PART, $\Im[z]$ is the IMAGINARY PART, and $J_n(z)$ is a BESSEL FUNCTION OF THE FIRST KIND.

Watson (1966, pp. 188–90) gives a slightly more general definition of the Airy function as the solution to the AIRY DIFFERENTIAL EQUATION

$$\Phi'' \pm k^2 \Phi x = 0 \quad (13)$$

which is FINITE at the ORIGIN, where Φ' denotes the DERIVATIVE $d\Phi/dx$, $k^2 = 1/3$, and either SIGN is permitted. Call these solutions $(1/\pi)\Phi(\pm k^2, x)$, then

$$\frac{1}{\pi}\Phi\left(\frac{1}{3}, x\right) \equiv \int_0^\infty \cos(t^3 \pm xt) dt \quad (14)$$

$$\Phi\left(\frac{1}{3}, x\right) = \frac{1}{3}\pi \sqrt{\frac{x}{3}} \left[J_{-1/3}\left(\frac{2x^{3/2}}{3^{3/2}}\right) + J_{1/3}\left(\frac{2x^{3/2}}{3^{3/2}}\right) \right] \quad (15)$$

$$\Phi\left(-\frac{1}{3}, x\right) = \frac{1}{3}\pi \sqrt{\frac{x}{3}} \left[I_{-1/3}\left(\frac{2x^{3/2}}{3^{3/2}}\right) - I_{1/3}\left(\frac{2x^{3/2}}{3^{3/2}}\right) \right], \quad (16)$$

where $J(z)$ is a BESSEL FUNCTION OF THE FIRST KIND. Using the identity

$$K_n(x) = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin(n\pi)}, \quad (17)$$

where $K(z)$ is a MODIFIED BESSEL FUNCTION OF THE SECOND KIND, the second case can be re-expressed

$$\Phi\left(-\frac{1}{3}, x\right) = \frac{1}{3}\pi \sqrt{\frac{x}{3}} \frac{2}{\pi} \sin\left(\frac{1}{3}\pi\right) K_{1/3}\left(\frac{2x^{3/2}}{3^{3/2}}\right) \quad (18)$$

$$= \frac{\pi}{3} \sqrt{\frac{x}{3}} \frac{2}{\pi} \frac{\sqrt{3}}{2} K_{1/3}\left(\frac{2x^{3/2}}{3^{3/2}}\right) \quad (19)$$

$$= \frac{1}{3} \sqrt{x} K_{1/3}\left(\frac{2x^{3/2}}{3^{3/2}}\right). \quad (20)$$

See also AIRY-FOCK FUNCTIONS, BESSEL FUNCTION OF THE FIRST KIND, MAP-AIRY DISTRIBUTION, MODIFIED BESSEL FUNCTION OF THE FIRST KIND, MODIFIED BESSEL FUNCTION OF THE SECOND KIND

References

- Abramowitz, M. and Stegun, C. A. (Eds.). "Airy Functions." §10.4 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover, pp. 446–52, 1972.
- Banderier, C.; Flajolet, P.; Schaeffer, G.; and Soria, M. "Planar Maps and Airy Phenomena." Preprint.
- Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Bessel Functions of Fractional Order, Airy Functions, Spherical Bessel Functions." §6.7 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing*, 2nd ed. Cambridge, England: Cambridge University Press, pp. 234–45, 1992.

Spanier, J. and Oldham, K. B. "The Airy Functions $\text{Ai}(x)$ and $\text{Bi}(x)$." Ch. 56 in *An Atlas of Functions*. Washington, DC: Hemisphere, pp. 555–62, 1987.

Watson, G. N. *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge, England: Cambridge University Press, 1966.

Airy Projection

A MAP PROJECTION. The inverse equations for ϕ are computed by iteration. Let the ANGLE of the projection plane be θ_b . Define

$$a = \begin{cases} 0 & \text{for } \theta_b = \frac{1}{2}\pi \\ \frac{\ln[\frac{1}{2} \cos(\frac{1}{2}\pi - \theta_b)]}{\tan[\frac{1}{2}(\frac{1}{2}\pi - \theta_b)]} & \text{otherwise.} \end{cases} \quad (1)$$

For proper convergence, let $x_i = \pi/6$ and compute the initial point by checking

$$x_i = |\exp[-(\sqrt{x^2 + y^2} + a \tan x_i) \tan x_i]|. \quad (2)$$

As long as $x_i > 1$, take $x_{i+1} = x_i/2$ and iterate again. The first value for which $x_i < 1$ is then the starting point. Then compute

$$x_i = \cos^{-1}\{\exp[-(\sqrt{x^2 + y^2} + a \tan x_i) \tan x_i]\} \quad (3)$$

until the change in x_i between evaluations is smaller than the acceptable tolerance. The (inverse) equations are then given by

$$\phi = \frac{1}{2}\pi - 2x_i \quad (4)$$

$$\lambda = \tan^{-1}\left(-\frac{x}{y}\right). \quad (5)$$

AiryAi

AIRY FUNCTIONS

AiryAiPrime

AIRY FUNCTIONS

AiryBi

AIRY FUNCTIONS

AiryBiPrime

AIRY FUNCTIONS

Airy-Fock Functions

The three Airy-Fock functions are

$$v(z) = \frac{1}{2}\sqrt{\pi} \text{Ai}(z) \quad (1)$$

$$w_1(z) = 2e^{i\pi/6}v(\omega z) \quad (2)$$

$$w_2(z) = 2e^{-i\pi/6}v(\omega^{-1}z), \quad (3)$$

where $\text{Ai}(z)$ is an AIRY FUNCTION. These functions satisfy

$$v(z) = \frac{\omega_1(z) - \omega_2(z)}{2i} \quad (4)$$

$$\overline{w_1(z)} = w_2(\bar{z}), \quad (5)$$

where $\bar{z}$ is the COMPLEX CONJUGATE of z .

See also AIRY FUNCTIONS

References

Hazewinkel, M. (Managing Ed.). *Encyclopaedia of Mathematics: An Updated and Annotated Translation of the Soviet "Mathematical Encyclopaedia."* Dordrecht, Netherlands: Reidel, p. 65, 1988.

Aitken Interpolation

An algorithm similar to NEVILLE'S ALGORITHM for constructing the LAGRANGE INTERPOLATING POLYNOMIAL. Let $f(x|x_0, x_1, \dots, x_k)$ be the unique POLYNOMIAL of k th ORDER coinciding with $f(x)$ at $x_0, \dots, x_k$. Then

$$\begin{aligned} f(x|x_0, x_1) &= \frac{1}{x_1 - x_0} \left| \begin{array}{c} f_0 x_0 - x \\ f_1 x_1 - x \end{array} \right| \\ f(x|x_0, x_2) &= \frac{1}{x_2 - x_0} \left| \begin{array}{c} f_0 x_0 - x \\ f_2 x_2 - x \end{array} \right| \\ f(x|x_0, x_1, x_2) &= \frac{1}{x_2 - x_1} \left| \begin{array}{c} f(x|x_0, x_1)x_1 - x \\ f(x|x_0, x_2)x_2 - x \end{array} \right| \\ f(x|x_0, x_1, x_2, x_3) &= \frac{1}{x_3 - x_2} \left| \begin{array}{c} f(x|x_0, x_1)x_2 - x \\ f(x|x_0, x_1)x_3 - x \end{array} \right|. \end{aligned}$$

See also LAGRANGE INTERPOLATING POLYNOMIAL

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover, p. 879, 1972.
 Acton, F. S. *Numerical Methods That Work*, 2nd printing. Washington, DC: Math. Assoc. Amer., pp. 93–4, 1990.
 Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing*, 2nd ed. Cambridge, England: Cambridge University Press, p. 102, 1992.

Aitken's Delta Squared Process

An ALGORITHM which extrapolates the partial sums s_n of a SERIES $\sum_n a_n$ whose CONVERGENCE is approximately geometric and accelerates its rate of CONVERGENCE. The extrapolated partial sum is given by

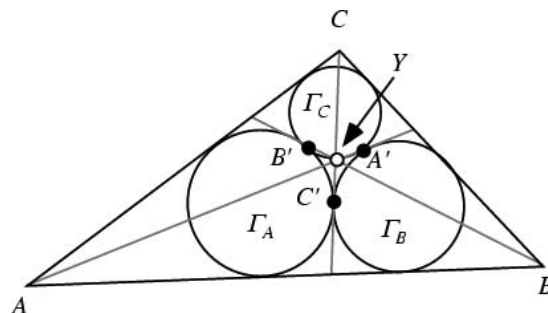
$$s'_n \equiv s_{n+1} - \frac{(s_{n+1} - s_n)^2}{s_{n+1} - 2s_n + s_{n-1}}.$$

See also EULER'S SERIES TRANSFORMATION

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover, p. 18, 1972.
 Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing*, 2nd ed. Cambridge, England: Cambridge University Press, p. 160, 1992.

Ajima-Malfatti Points



The lines connecting the vertices and corresponding circle-circle intersections in MALFATTI'S TANGENT TRIANGLE PROBLEM coincide in a point Y called the first Ajima-Malfatti point (Kimberling and MacDonald 1990, Kimberling 1994). Similarly, letting A' , B' , and C' be the excenters of ABC , then the lines $A'A''$, $B'B''$, and $C'C''$ are coincident in another point called the second Ajima-Malfatti point. The points are sometimes simply called the malfatti points (Kimberling 1994).

References

Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–87, 1994.
 Kimberling, C. "1st and 2nd Ajima-Malfatti Points." <http://cedar.evansville.edu/~ck6/tcenters/recent/ajmalf.html>.
 Kimberling, C. and MacDonald, I. G. "Problem E 3251 and Solution." *Amer. Math. Monthly* **97**, 612–13, 1990.

Akinetor

Moon, P. and Spencer, D. E. *Theory of Holors: A Generalization of Tensors*. Cambridge, England: Cambridge University Press, 1986.

Akisation

CUMULATION

Albanese Variety

An ABELIAN VARIETY which is canonically attached to an ALGEBRAIC VARIETY which is the solution to a certain universal problem. The Albanese variety is dual to the PICARD VARIETY.

References

Hazewinkel, M. (Managing Ed.). *Encyclopaedia of Mathematics: An Updated and Annotated Translation of the Soviet "Mathematical Encyclopaedia."* Dordrecht, Netherlands: Reidel, pp. 67–8, 1988.

Albers Conic Projection

ALBERS EQUAL-AREA CONIC PROJECTION

Albers Equal-Area Conic Projection



AN EQUAL-AREA PROJECTION. Let ϕ_0 be the LATITUDE for the origin of the CARTESIAN COORDINATES and λ_0 its LONGITUDE. Let ϕ_1 and ϕ_2 be the standard parallels. Then

$$x = \rho \sin \theta \quad (1)$$

$$y = \rho_0 - \rho \cos \theta, \quad (2)$$

where

$$\rho = \frac{\sqrt{C - 2n \sin \phi}}{n} \quad (3)$$

$$\theta = n(\lambda - \lambda_0) \quad (4)$$

$$\rho_0 = \frac{\sqrt{C - 2n \sin \phi_0}}{n} \quad (5)$$

$$C = \cos^2 \phi_1 + 2n \sin \phi_1 \quad (6)$$

$$n = \frac{1}{2}(\sin \phi_1 + \sin \phi_2). \quad (7)$$

The inverse FORMULAS are

$$\phi = \sin^{-1} \left(\frac{C - \rho^2 n^2}{2n} \right) \quad (8)$$

$$\lambda = \lambda_0 + \frac{\theta}{n}, \quad (9)$$

where

$$\rho = \sqrt{x^2 + (\rho_0 - y)^2} \quad (10)$$

$$\theta = \tan^{-1} \left(\frac{x}{\rho_0 - y} \right). \quad (11)$$

See also EQUAL-AREA PROJECTION

References

Snyder, J. P. *Map Projections--A Working Manual*. U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 98–03, 1987.

Alcuin's Sequence

THE INTEGER SEQUENCE 1, 0, 1, 1, 2, 1, 3, 2, 4, 3, 5, 4, 7, 5, 8, 7, 10, 8, 12, 10, 14, 12, 16, 14, 19, 16, 21, 19, ... (Sloane's A005044) given by the COEFFICIENTS of the MACLAURIN SERIES for $1/(1-x^2)(1-x^3)(1-x^4)$. The number of different TRIANGLES which have INTEGRAL sides and PERIMETER n is given by

$$T(n) = P_3(n) = \sum_{1 \leq j \leq \lfloor n/2 \rfloor} P_2(j) \quad (1)$$

$$= \left\lfloor \frac{n^2}{12} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n+2}{4} \right\rfloor \quad (2)$$

$$= \begin{cases} \frac{n^2}{48} & \text{for } n \text{ even} \\ \left\lfloor \frac{(n+3)^2}{48} \right\rfloor & \text{for } n \text{ odd.} \end{cases} \quad (3)$$

where $P_2(n)$ and $P_3(n)$ are PARTITION FUNCTIONS, with $P_k(n)$ giving the number of ways of writing n as a sum of k terms, $[x]$ is the NINT function, and $\lfloor x \rfloor$ is the FLOOR FUNCTION (Jordan *et al.* 1979, Andrews 1979, Honsberger 1985). Strangely enough, $T(n)$ for $n = 3, 4, \dots$ is precisely Alcuin's sequence.

See also PARTITION FUNCTION P , TRIANGLE

References

Andrews, G. "A Note on Partitions and Triangles with Integer Sides." *Amer. Math. Monthly* **86**, 477, 1979.
Honsberger, R. *Mathematical Gems III*. Washington, DC: Math. Assoc. Amer., pp. 39–7, 1985.
Jordan, J. H.; Walch, R.; and Wisner, R. J. "Triangles with Integer Sides." *Amer. Math. Monthly* **86**, 686–89, 1979.
Sloane, N. J. A. Sequences A005044/M0146 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.

Aleksandrov's Uniqueness Theorem

A CONVEX BODY in EUCLIDEAN n -SPACE that is centrally symmetric with center at the ORIGIN is determined among all such bodies by its brightness function (the VOLUME of each projection).

See also TOMOGRAPHY

References

Gardner, R. J. "Geometric Tomography." *Not. Amer. Math. Soc.* **42**, 422–29, 1995.

Aleksandrov-Cech Cohomology

A THEORY which satisfies all the EILENBERG-STEENROD AXIOMS with the possible exception of the LONG EXACT SEQUENCE OF A PAIR AXIOM, as well as a certain additional continuity CONDITION.

References

Hazewinkel, M. (Managing Ed.). *Encyclopaedia of Mathematics: An Updated and Annotated Translation of the*

Soviet "Mathematical Encyclopaedia." Dordrecht, Netherlands: Reidel, p. 68, 1988.

Aleph

The SET THEORY symbol ($\aleph$) for the CARDINALITY of an INFINITE SET.

See also ALEPH-0, ALEPH-1, COUNTABLE SET, COUNTABLY INFINITE, FINITE, INFINITE, TRANSFINITE NUMBER, UNCOUNTABLY INFINITE

Aleph-0

The SET THEORY symbol $\aleph_0$ for a SET having the same CARDINAL NUMBER as the "small" INFINITE SET of INTEGERS. The ALGEBRAIC NUMBERS also belong to $\aleph_0$. Rather surprising properties satisfied by $\aleph_0$ include

$$\aleph_0^r = \aleph_0 \quad (1)$$

$$r\aleph_0 = \aleph_0 \quad (2)$$

$$\aleph_0 + f = \aleph_0, \quad (3)$$

where f is any FINITE SET. However,

$$\aleph_0^{\aleph_0} = C, \quad (4)$$

where C is the CONTINUUM.

See also ALEPH-1, CARDINAL NUMBER, CONTINUUM, CONTINUUM HYPOTHESIS, COUNTABLY INFINITE, FINITE, INFINITE, TRANSFINITE NUMBER, UNCOUNTABLY INFINITE

Aleph-1

The SET THEORY symbol $\aleph_1$ for the smallest INFINITE SET larger than ALEPH-0, and equal to the CARDINALITY of the set of countable ORDINAL NUMBERS.

The CONTINUUM HYPOTHESIS asserts that $\aleph_1 = c$, where c is the CARDINALITY of the "large" INFINITE SET of REAL NUMBERS (called the CONTINUUM in SET THEORY). However, the truth of the CONTINUUM HYPOTHESIS depends on the version of SET THEORY you are using and so is UNDECIDABLE.

Curiously enough, n -D SPACE has the same number of points (c) as 1-D SPACE, or any FINITE INTERVAL of 1-D SPACE (a LINE SEGMENT), as was first recognized by Georg Cantor.

See also ALEPH-0, CARDINALITY, CONTINUUM, CONTINUUM HYPOTHESIS, COUNTABLY INFINITE, FINITE, INFINITE, ORDINAL NUMBER, TRANSFINITE NUMBER, UNCOUNTABLY INFINITE

Alethic

A term in LOGIC meaning pertaining to TRUTH and FALSEHOOD.

See also FALSE, PREDICATE, TRUE

Alexander Ideal

The order IDEAL in Λ , the RING of integral LAURENT POLYNOMIALS, associated with an ALEXANDER MATRIX for a KNOT K . Any generator of a principal Alexander ideal is called an ALEXANDER POLYNOMIAL. Because the ALEXANDER INVARIANT of a TAME KNOT in $\mathbb{S}^3$ has a SQUARE presentation MATRIX, its Alexander ideal is PRINCIPAL and it has an ALEXANDER POLYNOMIAL $\Delta(t)$.

See also ALEXANDER INVARIANT, ALEXANDER MATRIX, ALEXANDER POLYNOMIAL

References

Rolfsen, D. *Knots and Links*. Wilmington, DE: Publish or Perish Press, pp. 206–07, 1976.

Alexander Invariant

The Alexander invariant $H_*(\hat{X})$ of a KNOT K is the HOMOLOGY of the INFINITE cyclic cover of the complement of K , considered as a MODULE over Λ , the RING of integral LAURENT POLYNOMIALS. The Alexander invariant for a classical TAME KNOT is finitely presentable, and only H_1 is significant.

For any KNOT K^n in $\mathbb{S}^{n+2}$ whose complement has the homotopy type of a FINITE COMPLEX, the Alexander invariant is finitely generated and therefore finitely presentable. Because the Alexander invariant of a TAME KNOT in $\mathbb{S}^3$ has a SQUARE presentation MATRIX, its ALEXANDER IDEAL is PRINCIPAL and it has an ALEXANDER POLYNOMIAL denoted $\Delta(t)$.

See also ALEXANDER IDEAL, ALEXANDER MATRIX, ALEXANDER POLYNOMIAL

References

Rolfsen, D. *Knots and Links*. Wilmington, DE: Publish or Perish Press, pp. 206–07, 1976.

Alexander Matrix

A presentation matrix for the ALEXANDER INVARIANT $H_1(\hat{X})$ of a KNOT K . If V is a SEIFERT MATRIX for a TAME KNOT K in $\mathbb{S}^3$, then $V^T - tV$ and $V^T - tV^T$ are Alexander matrices for K , where V^T denotes the MATRIX TRANSPOSE.

See also ALEXANDER IDEAL, ALEXANDER INVARIANT, ALEXANDER POLYNOMIAL, SEIFERT MATRIX

References

Rolfsen, D. *Knots and Links*. Wilmington, DE: Publish or Perish Press, pp. 206–07, 1976.

Alexander Polynomial

A POLYNOMIAL invariant of a KNOT discovered in 1923 by J. W. Alexander (Alexander 1928). In technical language, the Alexander polynomial arises from the HOMOLOGY of the infinitely cyclic cover of a KNOT's complement. Any generator of a PRINCIPAL ALEXANDER IDEAL is called an Alexander polynomial (Rolfsen

1976). Because the ALEXANDER INVARIANT of a TAME KNOT in $\mathbb{S}^3$ has a SQUARE presentation MATRIX, its ALEXANDER IDEAL IS PRINCIPAL and it has an Alexander polynomial denoted $\Delta(t)$.

Let Ψ be the MATRIX PRODUCT of BRAID WORDS of a KNOT, then

$$\frac{\det(1 - \Psi)}{1 + t + \dots + t^{n-1}} = \Delta_L, \quad (1)$$

where Δ_L is the Alexander polynomial and $\det$ is the DETERMINANT. The Alexander polynomial of a TAME KNOT in $\mathbb{S}^3$ satisfies

$$\Delta(t) = \det(V^T - tV). \quad (2)$$

where V is a SEIFERT MATRIX, $\det$ is the DETERMINANT, and V^T denotes the MATRIX TRANSPOSE. The Alexander polynomial also satisfies

$$\Delta(1) = \pm 1. \quad (3)$$

The Alexander polynomial of a splittable link is always 0. Surprisingly, there are known examples of nontrivial KNOTS with Alexander polynomial 1. An example is the $(-3, 5, 7)$ PRETZEL KNOT.

The Alexander polynomial remained the *only* known KNOT POLYNOMIAL until the JONES POLYNOMIAL was discovered in 1984. Unlike the Alexander polynomial, the more powerful JONES POLYNOMIAL *does*, in most cases, distinguish HANDEDNESS. A normalized form of the Alexander polynomial symmetric in t and t^{-1} and satisfying

$$\Delta(\text{unknot}) = 1 \quad (4)$$

was formulated by J. H. Conway and is sometimes denoted V_L . The NOTATION $[a + b + c + \dots]$ is an abbreviation for the Conway-normalized Alexander polynomial of a KNOT

$$a + b(x + x^{-1}) + c(x^2 + x^{-2}) + \dots \quad (5)$$

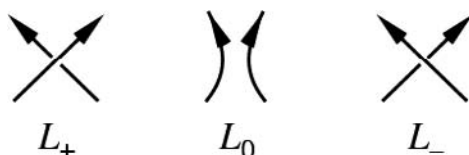
For a description of the NOTATION for LINKS, see Rolfsen (1976, p. 389). Examples of the Conway-Alexander polynomials for common KNOTS include

$$V_{\text{TK}} = [1 - 1 = -x^{-1} + 1 - x] \quad (6)$$

$$V_{\text{FEK}} = [3 - 1 = -x^{-1} + 3 - x] \quad (7)$$

$$V_{\text{SSK}} = [1 - 1 + 1 = x^{-2} - x^{-1} + 1 - x + x^2] \quad (8)$$

for the TREFOIL KNOT, FIGURE-OF-EIGHT KNOT, and SOLOMON'S SEAL KNOT, respectively. Multiplying through to clear the NEGATIVE POWERS gives the usual Alexander polynomial, where the final SIGN is determined by convention.



Let an Alexander polynomial be denoted Δ , then there exists a SKEIN RELATIONSHIP (discovered by J. H. Conway)

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) + (t^{-1/2} - t^{1/2})\Delta_{L_0}(t) = 0 \quad (9)$$

corresponding to the above LINK DIAGRAMS (Adams 1994). A slightly different SKEIN RELATIONSHIP convention used by Doll and Hoste (1991) is

$$\nabla_{L_+} - \nabla_{L_-} = z\nabla_{L_0}. \quad (10)$$

These relations allow Alexander polynomials to be constructed for arbitrary knots by building them up as a sequence of over- and undercrossings.

For a KNOT,

$$\Delta_K(-1) \equiv \begin{cases} 1 \pmod{8} & \text{if } \text{Arf}(K) = 0, \\ 5 \pmod{8} & \text{if } \text{Arf}(K) = 1, \end{cases} \quad (11)$$

where Arf is the ARF INVARIANT (Jones 1985). If K is a KNOT and

$$|\Delta_K(i)| > 3. \quad (12)$$

then K cannot be REPRESENTED AS a closed 3-BRAID. Also, if

$$\Delta_K(e^{2\pi i/5}) > \frac{13}{2}, \quad (13)$$

then K cannot be REPRESENTED AS a closed 4-braid (Jones 1985).

The HOMFLY POLYNOMIAL $P(a, z)$ generalizes the Alexander polynomial (as well as the JONES POLYNOMIAL) with

$$\nabla(z) = P(1, z) \quad (14)$$

(Doll and Hoste 1991).

Rolfsen (1976) gives a tabulation of Alexander polynomials for KNOTS up to 10 CROSSINGS and LINKS up to 9 CROSSINGS.

See also BRAID GROUP, JONES POLYNOMIAL, KNOT, KNOT DETERMINANT, LINK, SKEIN RELATIONSHIP

References

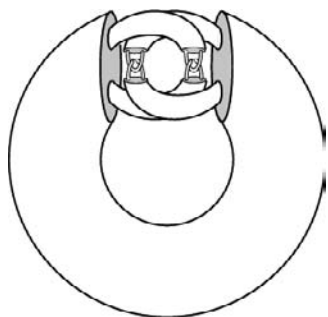
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Stoimenow, A. "Conway Polynomials." <http://guests.mpim-bonn.mpg.de/alex/ptab/c10.html>.

Alexander's Horned Sphere



The above solid, composed of a countable UNION of COMPACT SETS, is called Alexander's horned sphere. It is HOMEOMORPHIC with the BALL $\mathbb{B}^3$, and its boundary is therefore a SPHERE. It is therefore an example of a wild embedding in $\mathbb{E}^3$. The outer complement of the solid is not SIMPLY CONNECTED, and its fundamental GROUP is not finitely generated. Furthermore, the set of nonlocally flat ("bad") points of Alexander's horned sphere is a CANTOR SET.

The complement in $\mathbb{R}^3$ of the bad points for Alexander's horned sphere is SIMPLY CONNECTED, making it inequivalent to ANTOINE'S HORNED SPHERE. Alexander's horned sphere has an uncountable infinity of WILD POINTS, which are the limits of the sequences of the horned sphere's branch points (roughly, the "ends" of the horns), since any NEIGHBORHOOD of a limit contains a horned complex.

A humorous drawing by Simon Frazer (Guy 1983, Schroeder 1991, Albers 1994) depicts mathematician John H. Conway with Alexander's horned sphere growing from his head.



See also ANTOINE'S HORNED SPHERE

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Alexander's Theorem

Any LINK can be represented by a closed BRAID.

Alexander-Conway Polynomial

CONWAY POLYNOMIAL

Alexander-Spanier Cohomology

A fundamental result of DE RHAM COHOMOLOGY is that the k th DE RHAM COHOMOLOGY VECTOR SPACE of a MANIFOLD M is canonically isomorphic to the Alexander-Spanier cohomology VECTOR SPACE $H^k(M; \mathbb{R})$ (also called cohomology with compact support). In the case that M is COMPACT, Alexander-Spanier cohomology is exactly "singular" COHOMOLOGY.

Algebra

The branch of mathematics dealing with such topics as GROUP THEORY, invariant theory, and COHOMOLOGY which studies number systems and operations within them. The word "algebra" is a distortion of the Arabic title of a treatise by al-Khwarizmi about algebraic methods. Note that mathematicians refer to the "school algebra" generally taught in middle and high school as "ARITHMETIC," reserving the word "algebra" for the more advanced aspects of the subject.

Formally, an algebra is a VECTOR SPACE V , over a FIELD F with a MULTIPLICATION which turns it into a RING defined such that, if $f \in F$ and $x, y \in V$, then

$$f(xy) = (fx)y = x(fy).$$

In addition to the usual algebra of REAL NUMBERS, there are ≈ 1151 additional CONSISTENT algebras which can be formulated by weakening the FIELD AXIOMS, at least 200 of which have been *rigorously* proven to be self-CONSISTENT (Bell 1945).

Algebras which have been investigated and found to be of interest are usually named after one or more of their investigators. This practice leads to exotic-sounding (but unenlightening) names which algebraists frequently use with minimal or nonexistent explanation.

See also ABSTRACT ALGEBRA, ALTERNATIVE ALGEBRA, ASSOCIATIVE ALGEBRA, B*-ALGEBRA, BANACH ALGEBRA, BOOLEAN ALGEBRA, BOREL SIGMA ALGEBRA, C*-

ALGEBRA, CAYLEY ALGEBRA, CLIFFORD ALGEBRA, COMMUTATIVE ALGEBRA, DERIVATION ALGEBRA, EXTERIOR ALGEBRA, FUNDAMENTAL THEOREM OF ALGEBRA, GRADED ALGEBRA, GRASSMANN ALGEBRA, HECKE ALGEBRA, HEYTING ALGEBRA, HOMOLOGICAL ALGEBRA, HOPF ALGEBRA, JORDAN ALGEBRA, LIE ALGEBRA, LINEAR ALGEBRA, MEASURE ALGEBRA, NONASSOCIATIVE ALGEBRA, POWER ASSOCIATIVE ALGEBRA, QUATERNION, ROBBINS ALGEBRA, SCHUR ALGEBRA, SEMISIMPLE ALGEBRA, SIGMA ALGEBRA, SIMPLE ALGEBRA, STEENROD ALGEBRA, UMBRAL ALGEBRA, VON NEUMANN ALGEBRA

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Algebraic Closure

The FIELD $\bar{F}$ is called an algebraic closure of F if $\bar{F}$ is algebraic over F and if every polynomial $f(x) \in F[x]$ SPLITS completely over $\bar{F}$, so that $\bar{F}$ can be said to contain all the elements that are algebraic over F .

For example, the FIELD of COMPLEX NUMBERS $\mathbb{C}$ is the algebraic closure of the FIELD of REALS $\mathbb{R}$.

See also ALGEBRAICALLY CLOSED, SPLITTING FIELD

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Algebraic Coding Theory

CODING THEORY

Algebraic Combinatorics

The use of techniques from algebra, topology, and geometry in the solution of combinatorial problems, or the use of combinatorial methods to attack problems in these areas (Billera *et al.* 1999, p. ix).

See also COMBINATORICS

References

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Algebraic Congruence

A CONGRUENCE OF THE FORM

$$f(x) \equiv 0 \pmod{n}$$

where $f(x)$ is an INTEGER POLYNOMIAL (Nagell 1951, p. 73).

See also CONGRUENCE, FUNCTIONAL CONGRUENCE

References

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Algebraic Connectivity

The second smallest EIGENVALUE of the LAPLACIAN MATRIX of a graph G . This eigenvalue is greater than 0 IFF G is a CONNECTED GRAPH.

See also CONNECTED GRAPH, FIEDLER VECTOR, LAPLACIAN MATRIX

References

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Algebraic Curve

An algebraic curve over a FIELD K is an equation $f(X, Y) = 0$, where $f(X, Y)$ is a POLYNOMIAL in X and Y with COEFFICIENTS in K . A nonsingular algebraic curve is an algebraic curve over K which has no SINGULAR POINTS over K . A point on an algebraic curve is simply a solution of the equation of the curve. A K -RATIONAL POINT is a point (X, Y) on the curve, where X and Y are in the FIELD K .

See also ALGEBRAIC GEOMETRY, ALGEBRAIC VARIETY, CURVE

References

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Algebraic Expression

An algebraic expression in variables $\{x_1, \dots, x_n\}$ is an expression constructed with the variables and ALGEBRAIC NUMBERS using addition, multiplication, and rational powers.

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Algebraic Extension

This entry contributed by NICOLAS BRAY

An extension F of a FIELD K is said to be algebraic if every element of F is algebraic over K (i.e., is the root of a nonzero polynomial with coefficients in K).

See also GALOIS EXTENSION

Algebraic Function

A function which can be constructed using only a finite number of ELEMENTARY OPERATIONS together with the INVERSES of functions capable of being so constructed. Nonalgebraic functions are called TRANSCENDENTAL FUNCTIONS.

See also ELEMENTARY FUNCTION, ELEMENTARY OPERATION, TRANSCENDENTAL FUNCTION

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Algebraic Function Field

FUNCTION FIELD

Algebraic Geometry

Algebraic geometry is the study of geometries that come from algebra, in particular, from RINGS. In CLASSICAL ALGEBRAIC GEOMETRY, the algebra is the RING of POLYNOMIALS, and the geometry is the set of zeros of polynomials, called an ALGEBRAIC VARIETY. For instance, the UNIT CIRCLE is the set of zeros of $x^2 + y^2 = 1$ and is an ALGEBRAIC VARIETY, as are all of the CONIC SECTIONS.

In the twentieth century, it was discovered that the basic ideas of classical algebraic geometry can be applied to any COMMUTATIVE RING with a unit, such as the INTEGERS. The geometry of such a ring is determined by its algebraic structure, in particular its PRIME IDEALS. Grothendieck defined SCHEMES as the basic geometric objects, which have the same relationship to the geometry of a ring as a MANIFOLD to a COORDINATE CHART. The language of CATEGORY THEORY evolved at around the same time, largely in response to the needs of the increasing abstraction in algebraic geometry.

As a consequence, algebraic geometry became very useful in other areas of mathematics, most notably in ALGEBRAIC NUMBER THEORY. For instance, Deligne used it to prove a variant of the RIEMANN HYPOTHESIS. Also, Andrew Wiles' proof of FERMAT'S LAST THEOREM used the tools developed in algebraic geometry.

In the latter part of the twentieth century, researchers have tried to extend the relationship between algebra and geometry to arbitrary NONCOMMUTATIVE RINGS. The study of geometries associated to noncommutative rings is called NONCOMMUTATIVE GEOMETRY.

See also ALGEBRAIC CURVE, ALGEBRAIC NUMBER THEORY, ALGEBRAIC VARIETY, CATEGORY THEORY, COMMUTATIVE ALGEBRA, CONIC SECTION, DIFFERENTIAL GEOMETRY, GEOMETRY, NONCOMMUTATIVE GEOMETRY, PLANE CURVE, SCHEME, SPACE CURVE, ZARISKI TOPOLOGY

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Algebraic Integer

If r is a ROOT of the POLYNOMIAL equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0,$$

where the a_i s are INTEGERS and r satisfies no similar equation of degree $< n$, then r is called an algebraic integer of degree n . An algebraic integer is a special case of an ALGEBRAIC NUMBER (for which the leading COEFFICIENT a_n need not equal 1). RADICAL INTEGERS are a SUBRING of the algebraic integers.

A SUM or PRODUCT of algebraic integers is again an algebraic integer. However, ABEL'S IMPOSSIBILITY THEOREM shows that there are algebraic integers of degree ≥ 5 which are not expressible in terms of ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION, and ROOT EXTRACTION (the ELEMENTARY OPERATIONS)

on COMPLEX NUMBERS. In fact, if ELEMENTARY OPERATIONS are allowed on real numbers only, then there are real numbers which are algebraic integers of degree 3 which cannot be so expressed.

The GAUSSIAN INTEGERS are algebraic integers of $\mathbb{Q}(\sqrt{-1})$, since $a + bi$ are roots of

$$z^2 - 2az + a^2 + b^2 = 0.$$

See also ALGEBRAIC NUMBER, CASUS IRREDUCIBILIS, ELEMENTARY OPERATION, EUCLIDEAN NUMBER, RADICAL INTEGER

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Algebraic Invariant

A quantity such as a DISCRIMINANT which remains unchanged under a given class of algebraic transformations. Such invariants were originally called HYPERDETERMINANTS by Cayley.

See also DISCRIMINANT (POLYNOMIAL), INVARIANT, QUADRATIC INVARIANT

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Algebraic Knot

A single component ALGEBRAIC LINK. Most knots up to 11 crossings are algebraic, but they quickly become outnumbered by nonalgebraic knots for more crossings (Hoste *et al.* 1998).

See also ALGEBRAIC LINK, KNOT, LINK

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Algebraic K-Theory

K-THEORY

Algebraic Language

Let X be an alphabet (i.e., a finite and nonempty set), and call its member letters. A word on X is a finite sequence of letters $a_1 \dots a_n$, where $a_1, \dots, a_n \in X$. Denote the empty word by e , and the set of all words in X by X^* . Define the concatenation (also called product) of a word $u = a_1 \dots a_n$ with a word $v = b_1 \dots b_m$ as $uv = a_1 \dots a_n b_1 \dots b_m$. In general, concatenation is not commutative. Use the notation $|u|_a$ to mean the number of letters a in the word u . A language $\mathfrak{L}$ is then a subset of X^* , and $\mathfrak{L}$ is said to be algebraic when a set of rewriting rules, applied recursively, forms all the words of $\mathfrak{L}$ and no others.

See also DYCK LANGUAGE

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Algebraic Link

A class of fibered knots and links which arises in ALGEBRAIC GEOMETRY. An algebraic link is formed by connecting the NW and NE strings and the SW and SE strings of an ALGEBRAIC TANGLE (Adams 1994).

See also ALGEBRAIC KNOT, ALGEBRAIC TANGLE, FIBRATION, TANGLE

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Algebraic Manifold

An algebraic manifold is another name for a smooth ALGEBRAIC VARIETY. It can be covered by COORDINATE CHARTS so that the TRANSITION FUNCTIONS are given by RATIONAL FUNCTIONS. Technically speaking, the coordinate charts should be to all of affine space $\mathbb{C}^n$.

For example, the SPHERE is an algebraic manifold, with a chart given by STEREOGRAPHIC PROJECTION to $\mathbb{C}$, and another chart at ∞ , with the TRANSITION FUNCTION given by $1/z$. In this setting, it is called the

RIEMANN SPHERE. The TORUS is also an algebraic manifold, in this setting called an ELLIPTIC CURVE, with charts given by ELLIPTIC FUNCTIONS such as the WEIERSTRASS ELLIPTIC FUNCTION.

See also ABSTRACT MANIFOLD, ALGEBRAIC GEOMETRY, ALGEBRAIC VARIETY, ELLIPTIC CURVE, MANIFOLD

Algebraic Number

If r is a ROOT of the POLYNOMIAL equation

$$a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0, \quad (1)$$

where the a_i s are INTEGERS and r satisfies no similar equation of degree $< n$, then r is an algebraic number of degree n . If r is an algebraic number and $a_0 = 1$, then it is called an ALGEBRAIC INTEGER. It is also true that if the c_i s in

$$a_0 x^n + c_1 x^{n-1} + \dots + c_{n-1} x + c_n = 0 \quad (2)$$

are algebraic numbers, then any ROOT of this equation is also an algebraic number.

If α is an algebraic number of degree n satisfying the POLYNOMIAL

$$a(x - \alpha)(x - \beta)(x - \gamma) \dots, \quad (3)$$

then there are $n - 1$ other algebraic numbers $\beta, \gamma, \dots$ called the conjugates of α . Furthermore, if α satisfies any other algebraic equation, then its conjugates also satisfy the same equation (Conway and Guy 1996).

Any number which is not algebraic is said to be TRANSCENDENTAL. The set of algebraic numbers is denoted $\mathbb{A}$ (*Mathematica*), or sometimes $\mathbb{Q}$ (Nesterenko 1999), and is implemented in *Mathematica* as *Algebraics*. A number x can then be tested to see if it is algebraic using the command `Element[x, Algebraics]`.

See also ALGEBRAIC INTEGER, EUCLIDEAN NUMBER, HERMITE-LINDEMANN THEOREM, RADICAL INTEGER, Q-BAR, TRANSCENDENTAL NUMBER

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Algebraic Number Field

NUMBER FIELD

Algebraic Number Theory

NUMBER THEORY

Algebraic Projective Geometry

PROJECTIVE GEOMETRY

Algebraic Set

An algebraic set is the locus of zeros of a collection of POLYNOMIALS. For example, the circle is the set of zeros of $x^2 + y^2 - 1$ and the point at (a, b) is the set of zeros of x and y . The algebraic set $\{(x, 0)\} \cup \{(0, y)\}$ is the set of solutions to $xy = 0$. It decomposes into two irreducible algebraic sets, called ALGEBRAIC VARIETIES. In general, an algebraic set can be written uniquely as the finite union of ALGEBRAIC VARIETIES. The intersection of two algebraic sets is an algebraic set corresponding to the union of the polynomials. For example, $x = 0$ and $y = 0$ intersect at $(0, 0)$, i.e., where $x = 0$ and $y = 0$. In fact, the intersection of an arbitrary number of algebraic sets is itself an algebraic set. However, only a finite union of algebraic sets is algebraic. If X is the set of solutions to $f_i = 0$ and Y is the set of solutions to $g_j = 0$, then $X \cup Y$ is the set of solutions to $f_i g_j = 0$. Consequently, the algebraic sets are the closed sets in a TOPOLOGY, called the ZARISKI TOPOLOGY.

The set of polynomials vanishing on an algebraic set X is an IDEAL in the POLYNOMIAL RING. Conversely, any IDEAL defines an algebraic set since it is a collection of polynomials. HILBERT'S NULLSTELLENSATZ describes the precise relationship between IDEALS and algebraic sets.

See also ALGEBRAIC VARIETY, CATEGORY THEORY, COMMUTATIVE ALGEBRA, CONIC SECTION, HILBERT'S NULLSTELLENSATZ, IDEAL, PRIME IDEAL, PROJECTIVE VARIETY, SCHEME, ZARISKI TOPOLOGY

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Algebraic Surface

The set of ROOTS of a POLYNOMIAL $f(x, y, z) = 0$. An algebraic surface is said to be of degree $n = \max(i + j + k)$, where n is the maximum sum of powers of all terms $a_m x^i y^j z^k$. The following table lists the names of algebraic surfaces of a given degree.

Order	Surface
3	CUBIC SURFACE
4	QUARTIC SURFACE
5	QUINTIC SURFACE
6	SEXTIC SURFACE
7	HEPTIC SURFACE
8	OCTIC SURFACE
9	NONIC SURFACE
10	DECIC SURFACE
12	DODECIC SURFACE

See also BARTH DECIC, BARTH SEXTIC, BOY SURFACE, CAYLEY CUBIC, CHAIR, CLEBSCH DIAGONAL CUBIC, CUSHION, DERVISH, ENDRAß OCTIC, HEART SURFACE, HENNEBERG'S MINIMAL SURFACE, KUMMER SURFACE, ORDER (ALGEBRAIC SURFACE), ROMAN SURFACE, SARTI DODECIC, SURFACE, TOGLIATTI SURFACE

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Algebraic Tangle

Any TANGLE obtained by additions and multiplications of rational TANGLES (Adams 1994).

See also ALGEBRAIC LINK, TANGLE

References

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Algebraic Topology

The study of intrinsic qualitative aspects of spatial objects (e.g., SURFACES, SPHERES, TORI, CIRCLES, KNOTS, LINKS, configuration spaces, etc.) that remain invariant under both-directions continuous ONE-TO-ONE (HOMEOMORPHIC) transformations. The discipline of algebraic topology is popularly known as "RUBBER-SHEET GEOMETRY" and can also be viewed as the study of DISCONNECTIVITIES. Algebraic topology has a great deal of mathematical machinery for

studying different kinds of HOLE structures, and it gets the prefix "algebraic" since many HOLE structures are represented best by algebraic objects like GROUPS and RINGS.

A technical way of saying this is that algebraic topology is concerned with FUNCTORS from the topological CATEGORY of GROUPS and HOMOMORPHISMS. Here, the FUNCTORS are a kind of filter, and given an "input" SPACE, they spit out something else in return. The returned object (usually a GROUP or RING) is then a representation of the HOLE structure of the SPACE, in the sense that this algebraic object is a vestige of what the original SPACE was like (i.e., much information is lost, but some sort of "shadow" of the SPACE is retained—just enough of a shadow to understand some aspect of its HOLE-structure, but no more). The idea is that FUNCTORS give much simpler objects to deal with. Because SPACES by themselves are very complicated, they are unmanageable without looking at particular aspects.

COMBINATORIAL TOPOLOGY is a special type of algebraic topology that uses COMBINATORIAL methods.

See also CATEGORY, COMBINATORIAL TOPOLOGY, DIFFERENTIAL TOPOLOGY, FUNCTOR, HOMOTOPY THEORY, TOPOLOGY

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Algebraic Unknotting Number

The algebraic unknotting number of a knot K in $\mathbb{S}^3$ is defined as the algebraic unknotting number of the S -equivalence class of a SEIFERT MATRIX of K . The algebraic unknotting number of an element in an S -equivalent class is defined as the minimum number of algebraic unknotting operations necessary to transform the element to the S -equivalence class of the zero matrix (Saeki 1999).

See also SEIFERT MATRIX, UNKNOTTING NUMBER

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Algebraic Variety

A generalization to n -D of ALGEBRAIC CURVES. More technically, an algebraic variety is a reduced SCHEME of FINITE type over a FIELD K . An algebraic variety V is defined as the SET of points in the REALS $\mathbb{R}^n$ (or the COMPLEX NUMBERS $\mathbb{C}^n$) satisfying a system of POLYNOMIAL equations $f_i(x_1, \dots, x_n) = 0$ for $i = 1, 2, \dots$. According to the HILBERT BASIS THEOREM, a FINITE number of equations suffices.

A variety is the set of common zeros to a collection of POLYNOMIALS. In classical algebraic geometry, the polynomials have COMPLEX NUMBERS for coefficients. Because of the FUNDAMENTAL THEOREM OF ALGEBRA, such polynomials always have zeros. For example,

$$\{(x, y, z) : x^2 + y^2 - z^2\}$$

is the CONE, and

$$\{(x, y, z) : x^2 + y^2 - z^2, ax + by + cz = 0\}$$

is a CONIC SECTION, which is a SUBVARIETY of the cone.

Actually, the cone and the conic section are examples of AFFINE VARIETIES because they are in AFFINE SPACE. A general variety is comprised of affine varieties glued together, like the COORDINATE CHARTS of a MANIFOLD. The FIELD of coefficients can be any ALGEBRAICALLY CLOSED field. When a variety is embedded in projective space, it is a PROJECTIVE ALGEBRAIC VARIETY. Also, an INTRINSIC VARIETY can be thought of as an abstract object, like a MANIFOLD, independent of any particular embedding. A SCHEME is a generalization of a variety, which includes the possibility of replacing $\mathbb{C}[x, y, z]$ by any COMMUTATIVE RING with a unit. A further generalization is a STACK.

See also ABELIAN VARIETY, AFFINE VARIETY, ALBANESE VARIETY, ALGEBRAIC NUMBER THEORY, BRAUER-SEVERI VARIETY, CATEGORY THEORY, CHOW VARIETY, COMMUTATIVE ALGEBRA, CONIC SECTION, INTRINSIC VARIETY, PICARD VARIETY, PROJECTIVE ALGEBRAIC VARIETY, SCHEME, STACK (MODULI SPACE), ZARISKI TOPOLOGY

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Algebraically Closed

A FIELD K is said to be algebraically closed if every POLYNOMIAL with coefficients in K has a ROOT in K .

See also ALGEBRAIC CLOSURE, FIELD

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Algebraically Independent

This entry contributed by JOHNNY CHEN

Let K be a FIELD, and A a K -algebra. Elements $y_1, \dots, y_n$ are algebraically independent over K if the natural surjection $K[Y_1, \dots, Y_n] \rightarrow K[y_1, \dots, y_n]$ is an isomorphism. In other words, there are no polynomial relations $F(y_1, \dots, y_n) = 0$ with coefficients in K .

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See also IRRATIONAL NUMBER, LINDEMANN-WEIERSTRASS THEOREM, SCHANUEL'S CONJECTURE, SHIDLOVSKII THEOREM, TRANSCENDENTAL NUMBER

Algebraics

ALGEBRAIC NUMBER

Algebroidal Function

AN ANALYTIC FUNCTION $f(z)$ satisfying the irreducible algebraic equation

$$A_0(z)f^k + A_1(z)f^{k-1} + \dots + A_k(z) = 0$$

with single-valued MEROMORPHIC FUNCTIONS $A_j(z)$ in a COMPLEX DOMAIN G is called a k -algebroidal function in G .

See also MEROMORPHIC FUNCTION

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Algorithm

A specific set of instructions for carrying out a procedure or solving a problem, usually with the requirement that the procedure terminate at some point. Specific algorithms sometimes also go by the name METHOD, PROCEDURE, or TECHNIQUE. The word "algorithm" is a distortion of al-Khwarizmi, an Arab mathematician who wrote an influential treatise about algebraic methods.

See also 196-ALGORITHM, ALGORITHMIC COMPLEXITY, ARCHIMEDES ALGORITHM, BHASKARA-BROUCKNER ALGORITHM, BORCHARDT-PFAFF ALGORITHM, BRELAZ'S HEURISTIC ALGORITHM, BUCHBERGER'S ALGORITHM,

BULIRSCH-STOER ALGORITHM, BUMPING ALGORITHM, COMPUTABLE FUNCTION, CONTINUED FRACTION FACTORIZATION ALGORITHM, DECISION PROBLEM, DIJKSTRA'S ALGORITHM, EUCLIDEAN ALGORITHM, FERGUSON-FORCADE ALGORITHM, FERMAT'S ALGORITHM, FLOYD'S ALGORITHM, GAUSSIAN APPROXIMATION ALGORITHM, GENETIC ALGORITHM, GOSPER'S ALGORITHM, GREEDY ALGORITHM, HASSE'S ALGORITHM, HJLS ALGORITHM, JACOBI ALGORITHM, KRUSKAL'S ALGORITHM, LEVINE-O'SULLIVAN GREEDY ALGORITHM, LLL ALGORITHM, MARKOV ALGORITHM, MILLER'S ALGORITHM, NEVILLE'S ALGORITHM, NEWTON'S METHOD, PRIME FACTORIZATION ALGORITHMS, PRIMITIVE RECURSIVE FUNCTION, PROGRAM, PSLQ ALGORITHM, PSOS ALGORITHM, QUOTIENT-DIFFERENCE ALGORITHM, RISC ALGORITHM, SCHRAGE'S ALGORITHM, SHANKS' ALGORITHM, SPIGOT ALGORITHM, SYRACUSE ALGORITHM, TOTAL FUNCTION, TURING MACHINE, ZASSENHAUS-BERLEKAMP ALGORITHM, ZEILBERGER'S ALGORITHM

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Algorithmic Complexity

BIT COMPLEXITY, KOLMOGOROV COMPLEXITY

Alhazen's Billiard Problem

In a given CIRCLE, find an ISOSCELES TRIANGLE whose LEGS pass through two given POINTS inside the CIRCLE. This can be restated as: from two POINTS in the PLANE of a CIRCLE, draw LINES meeting at the POINT of the CIRCUMFERENCE and making equal ANGLES with the NORMAL at that POINT.

The problem is called the billiard problem because it corresponds to finding the POINT on the edge of a circular "BILLIARD" table at which a cue ball at a given POINT must be aimed in order to carom once off the edge of the table and strike another ball at a second given POINT. The solution leads to a BIQUADRATIC EQUATION OF THE FORM

$$H(x^2 - y^2) - 2Kxy + (x^2 + y^2)(hy - kx) = 0.$$

The problem is equivalent to the determination of the point on a spherical mirror where a ray of light will reflect in order to pass from a given source to an observer. It is also equivalent to the problem of finding, given two points and a CIRCLE such that the points are both inside or outside the CIRCLE, the ELLIPSE whose FOCI are the two points and which is tangent to the given CIRCLE.

The problem was first formulated by Ptolemy in 150 AD, and was named after the Arab scholar Alhazen, who discussed it in his work on optics. It was not until 1997 that Neumann proved the problem to be insoluble using a COMPASS and RULER construction because the solution requires extraction of a CUBE ROOT (Neumann 1998). This is the same reason that the CUBE DUPLICATION problem is insoluble.

See also BILLIARDS, BILLIARD TABLE PROBLEM, CUBE DUPLICATION

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Alhazen's Problem

ALHAZEN'S BILLIARD PROBLEM

Alias Transformation

A transformation in which the coordinate system is changed, leaving vectors in the original coordinate system "fixed" while changing their representation in the new coordinate system. In contrast, a transformation in which vectors are transformed in a fixed coordinate system is called an ALIBI TRANSFORMATION.

See also ALIBI TRANSFORMATION, ROTATION FORMULA

Aliasing

Given a power spectrum (a plot of power vs. frequency), aliasing is a false translation of power falling in some frequency range ($-f_c$, f_c) outside the range. Aliasing can be caused by discrete sampling below the NYQUIST FREQUENCY. The sidelobes of any INSTRUMENT FUNCTION (including the simple SINC SQUARED function obtained simply from FINITE sampling) are also a form of aliasing. Although sidelobe contribution at large offsets can be minimized with the use of an APODIZATION FUNCTION, the tradeoff is a widening of the response (i.e., a lowering of the resolution).

See also APODIZATION FUNCTION, NYQUIST FREQUENCY

Alibi Transformation

A transformation in which vectors are transformed in a fixed coordinate system. In contrast, a transformation in which the coordinate system is changed, leaving vectors in the original coordinate system "fixed" while changing their representation in the new coordinate system, is called an ALIAS TRANSFORMATION.

See also ALIAS TRANSFORMATION, ROTATION FORMULA

Aliquant Divisor

A number which does not DIVIDE another exactly. For instance, 4 and 5 are aliquant divisors of 6. A number which is not an aliquant divisor (i.e., one that *does* DIVIDE another exactly) is said to be an ALIQUOT DIVISOR.

See also ALIQUOT DIVISOR, DIVISOR, PROPER DIVISOR

Aliquot Cycle

ALIQUOT SEQUENCE, SOCIABLE NUMBERS

Aliquot Divisor

A number which DIVIDES another exactly. For instance, 1, 2, 3, and 6 are aliquot divisors of 6. A number which is not an aliquot divisor is said to be an ALIQUANT DIVISOR. The term "aliquot" is frequently used to specifically mean a PROPER DIVISOR, i.e., a DIVISOR of a number other than the number itself.

See also ALIQUANT DIVISOR, DIVISOR, PROPER DIVISOR

Aliquot Sequence

Let

$$s(n) \equiv \sigma(n) - n$$

where $\sigma(n)$ is the DIVISOR FUNCTION and $s(n)$ is the RESTRICTED DIVISOR FUNCTION. Then the SEQUENCE of numbers

$$s^0(n) \equiv n, s^1(n) = s(n), s^2(n) = s(s(n)), \dots$$

is called an aliquot sequence. If the SEQUENCE for a given n is bounded, it either ends at $s(1) = 0$ or becomes periodic.

1. If the SEQUENCE reaches a constant, the constant is known as a PERFECT NUMBER.
2. If the SEQUENCE reaches an alternating pair, it is called an AMICABLE PAIR.
3. If, after k iterations, the SEQUENCE yields a cycle of minimum length t OF THE FORM $s^{k+1}(n), s^{k+2}(n), \dots, s^{k+t}(n)$, then these numbers form a group of SOCIABLE NUMBERS of order t .

It has not been proven that all aliquot sequences eventually terminate and become period. The smallest number whose fate is not known is 276, which has been computed up to $s^{628}(276)$ (Guy 1994). There are five such sequences less than 1000, namely 276, 552, 564, 660, and 966, sometimes called the "Lehmer five." Furthermore, there are 934 open sequences $\leq 10^5$, and 9710 open sequences $\leq 10^6$ (Creyaufmüller).

See also 196-ALGORITHM, ADDITIVE PERSISTENCE, AMICABLE NUMBERS, CATALAN'S ALIQUOT SEQUENCE CONJECTURE, MULTIAMICABLE NUMBERS, MULTIPERFECT NUMBER, MULTIPLICATIVE PERSISTENCE, PERFECT NUMBER, SOCIABLE NUMBERS, UNITARY ALIQUOT SEQUENCE

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Alladi-Grinstead Constant

N.B. A detailed online essay by S. Finch was the starting point for this entry.

Let $N(n)$ be the number of ways in which the FACTORIAL $n!$ can be decomposed into n FACTORS of the form $P_k^{b_k}$ arranged in nondecreasing order. Also define

$$m(n) \equiv \max(p_1^{b_1}), \quad (1)$$

i.e., $m(n)$ is the LEAST PRIME FACTOR raised to its appropriate POWER in the factorization. Then define

$$\alpha(n) \equiv \frac{\ln m(n)}{\ln n} \quad (2)$$

where $\ln(x)$ is the NATURAL LOGARITHM. For instance,

$$\begin{aligned} 9! &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2^2 \cdot 5 \cdot 7 \cdot 3^4 \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 2^3 \cdot 3^3 \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 7 \cdot 2^3 \cdot 3^2 \cdot 3^2 \\ &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 2^2 \cdot 2^2 \cdot 5 \cdot 7 \cdot 3^3 \\ &= 2 \cdot 2 \cdot 2 \cdot 2^2 \cdot 2^2 \cdot 5 \cdot 7 \cdot 3^2 \cdot 3^2 \\ &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 3^2 \cdot 2^4 \\ &= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 2^2 \cdot 5 \cdot 7 \cdot 2^3 \cdot 3^2 \\ &= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 2^5 \\ &= 2 \cdot 3 \cdot 3 \cdot 2^2 \cdot 2^2 \cdot 2^2 \cdot 5 \cdot 7 \cdot 3^2 \\ &= 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 2^2 \cdot 5 \cdot 7 \cdot 2^4 \\ &= 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 2^3 \cdot 2^3 \\ &= 3 \cdot 3 \cdot 3 \cdot 3 \cdot 2^2 \cdot 2^2 \cdot 5 \cdot 7 \cdot 2^3, \end{aligned} \quad (3)$$

so

$$\alpha(9) = \frac{\ln 3}{\ln 9} = \frac{\ln 3}{2 \ln 3} = \frac{1}{2}. \quad (4)$$

For large n ,

$$\lim_{n \rightarrow \infty} \alpha(n) = e^c - 1 = 0.809394020534\dots, \quad (5)$$

where

$$c \equiv \sum_{k=2}^{\infty} \frac{1}{k} \ln \left(\frac{k}{k-1} \right). \quad (6)$$

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Allais Paradox

Choose between the following two alternatives:

1. 90% chance of an unknown amount x and a 10% chance of \$1 million, or
2. 89% chance of the same unknown amount x , 10% chance of \$2.5 million, and 1% chance of nothing.

The PARADOX is to determine which choice has the larger EXPECTATION VALUE, $0.9x + \$100,000$ or $0.89x + \$250,000$. However, the best choice depends on the unknown amount, even though it is the same in both cases! This appears to violate the INDEPENDENCE AXIOM.

See also INDEPENDENCE AXIOM, MONTY HALL PROBLEM, NEWCOMB'S PARADOX

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Allegory

A technical mathematical object which bears the same resemblance to binary relations as CATEGORIES do to FUNCTIONS and SETS.

See also CATEGORY

References

- Freyd, P. J. and Scedrov, A. *Categories, Allegories*. Amsterdam, Netherlands: North-Holland, 1990.

Allometric

Mathematical growth in which one population grows at a rate PROPORTIONAL to the POWER of another population.

References

- Coffey, W. J. *Geography Towards a General Spatial Systems Approach*. London: Routledge, Chapman & Hall, 1981.

All-Pairs Shortest Path

The shortest distance between any pair of vertices in the shortest-path spanning tree, as long as the path giving the shortest path does not pass through the root of the spanning tree (Skiena 1990, p. 228). The problem can be solved using n applications of DIJKSTRA'S ALGORITHM or FLOYD'S ALGORITHM. The latter also works in the case of a weighted graph where the edges have negative weights.

See also FLOYD'S ALGORITHM, DIJKSTRA'S ALGORITHM, GRAPH GEODESIC

References

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All-Poles Model

MAXIMUM ENTROPY METHOD

All-to-All Communication

GOSSIPING

Almost All

Given a property P , if $P(x) \sim x$ as $x \rightarrow \infty$ (so the number of numbers less than x not satisfying the property P is $\sigma(x)$), then P is said to hold true for almost all numbers. For example, almost all positive integers are COMPOSITE NUMBERS (which is not in conflict with the second of EUCLID'S THEOREMS that there are an infinite number of PRIMES).

See also FOR ALL, NORMAL ORDER

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Almost Alternating Knot

An ALMOST ALTERNATING LINK with a single component.

See also ALMOST ALTERNATING LINK

Almost Alternating Link

Call a projection of a LINK an almost alternating projection if one crossing change in the projection makes it an alternating projection. Then an almost alternating link is a LINK with an almost alternating projection, but no alternating projection. Every ALTERNATING KNOT has an almost alternating projection. A PRIME KNOT which is almost alternating is either a TORUS KNOT or a HYPERBOLIC KNOT. Therefore, no SATELLITE KNOT is an almost alternating knot.

All nonalternating 9-crossing PRIME KNOTS are almost alternating. Of the 393 nonalternating knots and links with 11 or fewer crossings, all but five are known to be almost alternating (and 3 of these have 11 crossings). The fate of the remaining five is not known. The $(q, 2)$, $(4, 3)$, and $(5, 3)$ -TORUS KNOTS are almost alternating (Adams 1994, p. 142).

See also ALTERNATING KNOT, LINK

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. New York: W. H. Freeman, pp. 139–46, 1994.

Almost Everywhere

A property of X is said to hold almost everywhere if the SET of points in X where this property fails has MEASURE ZERO.

See also ALMOST EVERYWHERE CONVERGENCE, MEASURE ZERO

References

Jeffreys, H. and Jeffreys, B. S. "Measure Zero": 'Almost Everywhere'." §1.1013 in *Methods of Mathematical Physics, 3rd ed.* Cambridge, England: Cambridge University Press, pp. 29–0, 1988.

Sansone, G. *Orthogonal Functions, rev. English ed.* New York: Dover, p. 1, 1991.

Almost Everywhere Convergence

A weakened version of POINTWISE CONVERGENCE hypothesis which states that, for X a MEASURE SPACE, $f_n(x) \rightarrow f(x)$ for all $x \in Y$, where Y is a measurable subset of X such that $\mu(X \setminus Y) = 0$.

See also POINTWISE CONVERGENCE

References

Browder, A. *Mathematical Analysis: An Introduction*. New York: Springer-Verlag, 1996.

Almost Integer

A number which is very close to an INTEGER. One surprising example involving both E and π is

$$e^\pi - \pi = 19.999099979 \dots \quad (1)$$

which can also be written as

$$(\pi + 20)^i = -0.9999999992 - 0.0000388927i \approx -1 \quad (2)$$

$$\cos(\ln(\pi + 20)) \approx -0.9999999992. \quad (3)$$

Applying COSINE a few more times gives

$$\begin{aligned} &\cos(\pi \cos(\pi \cos(\ln(\pi + 20)))) \\ &\approx -1 + 3.9321609261 \times 10^{-35}. \end{aligned} \quad (4)$$

This curious near-identity was apparently noticed almost simultaneously around 1988 by N. J. A. - Sloane, J. H. Conway, and S. Plouffe, but no satisfying explanation as to "why" it has been true has yet been discovered.

An interesting near-identity is given by

$$\begin{aligned} &\frac{1}{4} \left[\cos\left(\frac{1}{10}\right) + \cosh\left(\frac{1}{10}\right) + 2\cos\left(\frac{1}{20}\sqrt{2}\right) \cosh\left(\frac{1}{20}\sqrt{2}\right) \right] \\ &= 1 + 2.480 \dots \times 10^{-13} \end{aligned} \quad (5)$$

(W. Dubuque). Other remarkable near-identities are given by

$$\frac{5(1 + \sqrt{5})[\Gamma(\frac{3}{4})]^2}{e^{5x/6}\sqrt{\pi}} = 1 + 4.5422 \dots \times 10^{-14} \quad (6)$$

where $\Gamma(z)$ is the GAMMA FUNCTION (S. Plouffe),

$$e^6 - \pi^4 - \pi^5 = 0.000017673 \dots \quad (7)$$

(D. Wilson),

$$r \left(\frac{160}{\pi} \right)^{1/13} \approx 0.9999996766, \quad (8)$$

where $r \approx 0.739085$ is the root of $x = \cos x$ (L. A. Broukhis),

$$\ln 2 + \log_{10} 2 = 0.994177 \dots \quad (9)$$

(D. Davis),

$$\frac{163}{\ln 163} = 31.9999983738 \dots \quad (10)$$

(posted to sci.math; origin unknown),

$$eK^{5/7-\gamma}\pi^{-(2/7+\gamma)} \approx 1.00014678 \quad (11)$$

$$\frac{K^{\gamma-19/7}\pi^{2/7+\gamma}}{2\phi} \approx 1.00105 \quad (12)$$

$$e\gamma\phi(K\pi)^{-(2/7+\gamma)} \approx 1.01979, \quad (13)$$

where K is CATALAN'S CONSTANT, γ is the EULER-MASCHERONI CONSTANT, and ϕ is the GOLDEN RATIO (D. Barron), and

$$163(\pi - e) = 68.999664 \dots \quad (14)$$

$$\frac{53453}{\ln 53453} = 4910.00000122 \dots \quad (15)$$

$$\begin{aligned} &\left[(2-1)^2 + \frac{(5^2-1)^2}{6^2+1} \right] e - \left[(2+1)^2 + \frac{(5^2+1)^2}{6^2-1} \right]^{-1} \\ &= \frac{613}{37} e - \frac{35}{991} = 44.9999999993962 \dots \end{aligned} \quad (16)$$

(Stoschek). Stoschek also gives an interesting near-identity involving the fine structure constant α and FEIGENBAUM CONSTANT δ ,

$$(28 - \delta^{-1})(\alpha^{-1} - 137) \approx 0.999998. \quad (17)$$

The near identity

$$3\sqrt{2}(\sqrt{5}-2) = 1.0015516 \dots \quad (18)$$

arises by noting that the stellation ratio $3(\sqrt{5}-2)$ in the CUMULATION of the DODECAHEDRON to form the GREAT DODECAHEDRON is approximately equal to $\sqrt{2}$.

A set of almost integers due to D. Hickerson are those OF THE FORM

$$h_n = \frac{n!}{2(\ln 2)^{n+1}}. \quad (19)$$

for $1 \leq n \leq 15$, as summarized in the following table.

n	h_n
0	0.72135
1	1.04068
2	3.00278
3	12.99629
4	74.99874
5	541.00152
6	4683.00125
7	47292.99873
8	545834.99791
9	7087261.00162
10	102247563.00527
11	1622632572.99755
12	28091567594.98157
13	526858348381.00125
14	10641342970443.08453
15	230283190977853.03744
16	5315654681981354.51308
17	130370767029135900.45799

These numbers are close to integers due to the fact that the quotient is the dominant term in an infinite series for the number of possible outcomes of a race between n people (with ties are allowed). Calling this number $f(n)$, it follows that

$$f(n) = \sum_{k=1}^n \binom{n}{k} f(n-k) \quad (20)$$

for $n \geq 1$, where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. From this, we obtain the exponential generating function for f

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!} z^n = \frac{1}{2 - e^z}, \quad (21)$$

and then by CONTOUR INTEGRATION it can be shown that

$$f(n) = \frac{1}{2} n! \sum_{k=-\infty}^{\infty} \frac{1}{(\ln 2 + 2\pi i k)^{n+1}} \quad (22)$$

for $n \geq 1$, where i is the square root of -1 and the sum is over all integers k (here, the imaginary parts of the terms for k and $-k$ cancel each other, so this sum is real.) The $k = 0$ term dominates, so $f(n)$ is asymptotic to $n!/(2(\ln 2)^{n+1})$. In fact, the other terms are quite

small for n from 1 to 15, so $f(n)$ is the nearest integer to $n!/(2(\ln 2)^{n+1})$ for these values (Hickerson), given by the sequence 1, 3, 13 75, 541, 4683, ... (Sloane's A034172).

A large class of IRRATIONAL "almost integers" can be found using the theory of MODULAR FUNCTIONS, and a few rather spectacular examples are given by Ramanujan (1913–4). Such approximations were also studied by Hermite (1859), Kronecker (1863), and Smith (1965). They can be generated using some amazing (and very deep) properties of the J -FUNCTION. Some of the numbers which are closest approximations to INTEGERS are $e^{\pi\sqrt{163}}$ (sometimes known as the RAMANUJAN CONSTANT and which corresponds to the field $\mathbb{Q}(\sqrt{-163})$ which has CLASS NUMBER 1 and is the IMAGINARY QUADRATIC FIELD of maximal discriminant), $e^{\pi\sqrt{22}}$, $e^{\pi\sqrt{37}}$, and $e^{\pi\sqrt{58}}$, the last three of which have CLASS NUMBER 2 and are due to Ramanujan (Berndt 1994, Waldschmidt 1988).

The properties of the J -FUNCTION also give rise to the spectacular identity

$$\left[\frac{\ln(640320^3 + 744)}{\pi} \right]^2 = 163 + 2.32167 \dots \times 10^{-29} \quad (23)$$

(Le Lionnais 1983, p. 152).

The list below gives numbers OF THE FORM $x \equiv e^{\pi\sqrt{n}}$ for $n \leq 1000$ for which $[x] - x \leq 0.01$.

$e^{\pi\sqrt{6}} = 2, 197.990869543 \dots$
 $e^{\pi\sqrt{17}} = 422, 150.997675680 \dots$
 $e^{\pi\sqrt{18}} = 614, 551.992885619 \dots$
 $e^{\pi\sqrt{22}} = 2, 508, 951.998257424 \dots$
 $e^{\pi\sqrt{25}} = 6, 635, 623.999341134 \dots$
 $e^{\pi\sqrt{37}} = 199, 148, 647.999978046551 \dots$
 $e^{\pi\sqrt{43}} = 884, 736, 743.999777466 \dots$
 $e^{\pi\sqrt{55}} = 24, 591, 257, 751.999999822213 \dots$
 $e^{\pi\sqrt{59}} = 30, 197, 683, 486.993182260 \dots$
 $e^{\pi\sqrt{67}} = 147, 197, 952, 743.999998662454 \dots$
 $e^{\pi\sqrt{74}} = 545, 518, 122, 089.999174678853 \dots$
 $e^{\pi\sqrt{149}} = 45, 116, 546, 012, 289, 599.991830287 \dots$
 $e^{\pi\sqrt{163}} = 262, 537, 412, 640, 768, 743.99999999999250072 \dots$
 $e^{\pi\sqrt{177}} = 1, 418, 556, 986, 635, 586, 485.996179355 \dots$
 $e^{\pi\sqrt{232}} = 604, 729, 957, 825, 300, 084, 759.999992171526 \dots$
 $e^{\pi\sqrt{267}} = 19, 683, 091, 854, 079, 461, 001, 445.992737040 \dots$
 $e^{\pi\sqrt{326}} = 4, 309, 793, 301, 730, 386, 363, 005, 719.996011651 \dots$
 $e^{\pi\sqrt{386}} = 639, 355, 180, 631, 208, 421, 212, 174, 016.997669832 \dots$
 $e^{\pi\sqrt{522}} = 14, 871, 070, 263, 238, 043, 663, 567, \dots$
 $\dots 627, 879, 007.999848726 \dots$
 $e^{\pi\sqrt{566}} = 288, 099, 755, 064, 053, 264, 917, 867, \dots$
 $\dots 975, 825, 573.993898311 \dots$
 $e^{\pi\sqrt{638}} = 28, 994, 858, 898, 043, 231, 996, 779, \dots$
 $\dots 771, 804, 797, 161.992372939 \dots$
 $e^{\pi\sqrt{719}} = 3, 842, 614, 373, 539, 548, 891, 490, \dots$
 $\dots 294, 277, 805, 829, 192.999987249 \dots$

$$\begin{aligned}
e^{\pi\sqrt{790}} &= 223, 070, 667, 213, 077, 889, 794, 379, \dots \\
&\dots 623, 183, 838, 336, 437.992055117\dots \\
e^{\pi\sqrt{792}} &= 249, 433, 117, 287, 892, 229, 255, 125, \dots \\
&\dots 388, 685, 911, 710, 805.996097323\dots \\
e^{\pi\sqrt{928}} &= 365, 698, 321, 891, 389, 219, 219, 142, \dots \\
&\dots 531, 076, 638, 716, 362, 775.998259747\dots \\
e^{\pi\sqrt{986}} &= 6, 954, 830, 200, 814, 801, 770, 418, 837, \dots \\
&\dots 940, 281, 460, 320, 666, 108.994649611\dots
\end{aligned}$$

Gosper noted that the expression

$$\begin{aligned}
&1 - 262537412640768744e^{-\pi\sqrt{163}} - 196884e^{-2\pi\sqrt{163}} \\
&\quad + 103378831900730205293632e^{-3\pi\sqrt{163}}. \quad (24)
\end{aligned}$$

differs from an INTEGER by a mere 10^{-59} .

See also CLASS NUMBER, J -FUNCTION, PI, PISOT-VIJAYARAGHAVAN CONSTANT

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Almost Perfect Number

A number n for which the DIVISOR FUNCTION satisfies $\sigma(n) = 2n - 1$ is called almost perfect. The only known almost perfect numbers are the POWERS of 2, namely 1, 2, 4, 8, 16, 32, ... (Sloane's A000079). Singh (1997) calls almost perfect numbers SLIGHTLY DEFECTIVE.

See also QUASIPERFECT NUMBER

References

- Guy, R. K. "Almost Perfect, Quasi-Perfect, Pseudoperfect, Harmonic, Weird, Multiperfect and Hyperperfect Numbers." §B2 in *Unsolved Problems in Number Theory*, 2nd ed. New York: Springer-Verlag, pp. 16 and 45–3, 1994.
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Almost Periodic Function

This entry contributed by RONALD M. AARTS

A function representable as a generalized Fourier series. Let $\mathcal{R}$ be a METRIC SPACE with metric $\rho(x, y)$. Following Bohr (1947), a CONTINUOUS FUNCTION $x(t)$ for $(-\infty < t < \infty)$ with values in $\mathcal{R}$ is called an almost periodic function if, for every $\epsilon > 0$, there exists $\ell = \ell(\epsilon) > 0$ such that every interval $[t_0, t_0 + \ell(\epsilon)]$ contains at least one number τ for which

$$\rho[x(t), x(t + \tau)] < \epsilon \quad (-\infty < t < \infty). \quad (1)$$

Another formal description can be found in Krasnosel'skii *et al.* (1973).

Every almost periodic function is bounded and uniformly continuous on the entire REAL LINE. In addition, the range of an almost period function is compact in $\mathcal{R}$.

See also FOURIER SERIES, PERIODIC FUNCTION

References

- Bohr, H. *Almost Periodic Functions*. New York: Chelsea, 1947.
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Almost Prime

A number n with prime factorization

$$n = \prod_{i=1}^r p_i^{a_i}$$

is called k -almost prime when the sum of the POWERS $\sum_{i=1}^r a_i = k$. The set of k -almost primes is denoted P_k . The PRIMES correspond to the "1-almost prime" numbers 2, 3, 5, 7, 11, ... (Sloane's A000040). The 2-almost prime numbers correspond to SEMIPRIMES 4, 6, 9, 10, 14, 15, 21, 22, ... (Sloane's A001358). The first few 3-almost primes are 8, 12, 18, 20, 27, 28, 30, 42, 44, 45, 50, 52, 63, 66, 68, 70, 75, 76, 78, 92, 98, 99, ... (Sloane's A014612). The first few 4-almost primes are 16, 24, 36, 40, 54, 56, 60, 81, 84, 88, 90, 100, ... (Sloane's A014613). The first few 5-almost primes are 32, 48, 72, 80, ... (Sloane's A014614).

See also CHEN'S THEOREM, PRIME NUMBER, SEMI-PRIME

References

Sloane, N. J. A. Sequences A000040/M0652, A001358/M3274, A014612, A014613, and A014614 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.

Almost Unit

An almost unit is a nonunit in the INTEGRAL DOMAIN of FORMAL POWER SERIES with a nonzero first coefficient, $P = a_1x + z_2x^2 + \dots$, where $a_1 \neq 0$. Under the operation of composition, the almost units in the INTEGRAL DOMAIN of FORMAL POWER SERIES over a FIELD F form a GROUP (Henrici 1988, p. 45).

See also SCHUR-JABOTINSKY THEOREM

References

Henrici, P. *Applied and Computational Complex Analysis, Vol. 1: Power Series-Integration-Conformal Mapping-Location of Zeros*. New York: Wiley, p. 45, 1988.

Alon-Tarsi Conjecture

See also LATIN SQUARE

References

Drisko, A. A. "Proof of the Alon-Tarsi Conjecture for $n = 2^r p$." *Electronic J. Combinatorics* **5**, No. 1, R28, 1–, 1998. http://www.combinatorics.org/Volume_5/v5iltoc.html.

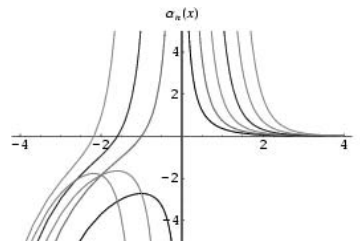
Alpha

Alpha is the name for the first letter in the Greek alphabet: α .

In finance, alpha is a financial measure giving the difference between a fund's actual return and its expected level of performance, given its level of risk (as measured by BETA). A POSITIVE alpha indicates that a fund has performed better than expected based on its BETA, whereas a NEGATIVE alpha indicates poorer performance.

See also ALPHA FUNCTION, ALPHA-TEST, ALPHA VALUE, BETA, SHARPE RATIO

Alpha Function



$$\alpha_n(z) \equiv \int_1^\infty t^n e^{-zt} dt = n! z^{-(n+1)} e^{-z} \sum_{k=0}^n \frac{z^k}{k!}.$$

It is equivalent to

$$\alpha_n(z) = \mathbf{E}_{-n}(z),$$

where $\mathbf{E}_n(z)$ is the $\mathbf{E}_N$ -FUNCTION.

See also BETA EXPONENTIAL FUNCTION, $\mathbf{E}_N$ -FUNCTION

Alpha Value

An alpha value is a number $0 \leq \alpha \leq 1$ such that $P(z \geq z_{\text{observed}}) \leq \alpha$ is considered "SIGNIFICANT," where P is a P -VALUE.

See also CONFIDENCE INTERVAL, P -VALUE, SIGNIFICANCE

Alphabet

A SET (usually of letters) from which a SUBSET is drawn. A sequence of letters is called a WORD, and a set of WORDS is called a CODE.

See also CODE, STRING, WORD

Alpha-Beta Conjecture

MANNN'S THEOREM

Alphamagic Square

A MAGIC SQUARE for which the number of letters in the word for each number generates another MAGIC SQUARE. This definition depends, of course, on the language being used. In English, for example,

5	22	18	4	9	8
28	15	2	11	7	3
12	8	25	6	5	10

where the MAGIC SQUARE on the right corresponds to the number of letters in

five	twenty-two	eighteen
twenty-eight	fifteen	two
twelve	eight	twenty-five

References

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 Sallows, L. C. F. "Alpha Magic Squares." In *The Lighter Side of Mathematics* (Ed. R. K. Guy and R. E. Woodrow). Washington, DC: Math. Assoc. Amer., 1994.

Alphametic

A CRYPTARITHM in which the letters used to represent distinct DIGITS are derived from related words or meaningful phrases. The term was coined by Hunter in 1955 (Madachy 1979, p. 178).

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Alpha-Test

For some constant α_0 , $\alpha(f, z) < \alpha_0$ implies that z is an APPROXIMATE ZERO of f , where

$$\alpha(f, z) = \frac{|f(z)|}{|f'(z)|} \sup_{k \geq 1} \left| \frac{f^{(k)}(z)}{k! f'(z)} \right|^{1/(k-1)}$$

Smale (1986) found a constant $\alpha \approx 0.130707$ for the test, and this value was subsequently improved to $\alpha_0 = 3 - 2\sqrt{2} \approx 0.171573$ by Wang and Han (1989), and further improved by Wang and Zhao (1995; Petkovic *et al.* 1997, p. 2).

See also APPROXIMATE ZERO, NEWTON'S METHOD, POINT ESTIMATION THEORY

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Alternating Algebra

EXTERIOR ALGEBRA

Alternating Group

A PERMUTATION GROUP of an even number of permutations on a set of length n , denoted A_n or $\text{Alt}(n)$ (Scott 1987, p. 267). An alternating group is a NORMAL SUBGROUP of the PERMUTATION GROUP, and has ORDER $n!/2$, the first few values of which for $n = 2, 3, \dots$ are 1, 3, 12, 60, 360, 2520, ... (Sloane's A001710). Alternating groups are FINITE analogs of the families of simple LIE GROUPS.

Alternating groups with $n \geq 5$ are non-ABELIAN SIMPLE GROUPS (Scott 1987, p. 295). The number of conjugacy classes in the alternating groups A_n for $n = 2, 3, \dots$ are 1, 3, 4, 5, 7, 9, ... (Sloane's A000702).

See also 15 PUZZLE, FINITE GROUP, GROUP, JORDAN'S SYMMETRIC GROUP THEOREM, LIE GROUP, PERMUTATION GROUP, SIMPLE GROUP, SYMMETRIC GROUP

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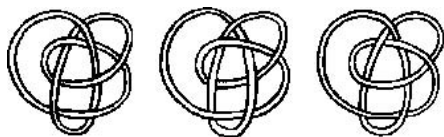
Alternating Knot

An alternating knot is a KNOT which possesses a knot diagram in which crossings alternate between under- and overpasses. Not all knot diagrams of alternating knots need be alternating diagrams.

The TREFOIL KNOT and FIGURE-OF-EIGHT KNOT are alternating knots. The number of PRIME alternating and nonalternating knots of n crossings are summarized in the following table.

type	Sloane	counts
alternating	A002864	0, 0, 1, 1, 2, 3, 7, 18, 41, 123, 367, 1288, 4878, 19536, 85263, 379799, ...
nonalternating	A051763	0, 0, 0, 0, 0, 0, 0, 3, 8, 42, 185, 888, 5110, 27436, 168030, 1008906, ...

The 3 nonalternating knots of eight crossings are 08–19, 08–20, and 08–21, illustrated below (Wells 1991).



One of TAIT'S KNOT CONJECTURES states that the number of crossings is the same for any diagram of a reduced alternating knot. Furthermore, a reduced alternating projection of a knot has the least number of crossings for any projection of that knot. Both of these facts were proved true by Kauffman (1988), Thistlethwaite (1987), and Murasugi (1987). FLYPE moves are sufficient to pass between all minimal diagrams of a given alternating knot (Hoste *et al.* 1998).

If K has a reduced alternating projection of n crossings, then the SPAN of K is An . Let $c(K)$ be the CROSSING NUMBER. Then an alternating knot $K_1 \# K_2$ (a KNOT SUM) satisfies

$$c(K_1 \# K_2) = c(K_1) + c(K_2).$$

In fact, this is true as well for the larger class of ADEQUATE KNOTS and postulated for all KNOTS.

It is conjectured that the proportion of knots which are alternating tends exponentially to zero with increasing crossing number (Hoste *et al.* 1998), a statement which has been proved true for alternating links.

See also ADEQUATE KNOT, ALMOST ALTERNATING LINK, ALTERNATING LINK, FLYPING CONJECTURE, TAIT'S KNOT CONJECTURES

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Wells, D. *The Penguin Dictionary of Curious and Interesting Geometry*. London: Penguin, p. 160, 1991.

Alternating Knot Diagram

A KNOT DIAGRAM which has alternating under- and overcrossings as the KNOT projection is traversed. The first KNOT which does not have an alternating diagram has 8 crossings.

Alternating Link

A LINK which has a LINK DIAGRAM with alternating underpasses and overpasses.

The proportion of links which are alternating tends exponentially to zero with increasing crossing number (Sundberg and Thistlethwaite 1998, Thistlethwaite 1998).

See also ALMOST ALTERNATING LINK, ALTERNATING KNOT

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Alternating Multilinear Form

An alternating multilinear form on a REAL VECTOR SPACE V is a MULTILINEAR FORM

$$F : V \otimes \cdots \otimes V \rightarrow \mathbb{R} \quad (1)$$

such that

$$\begin{aligned} F(x_1, \dots, x_i, x_{i+1}, \dots, x_n) \\ = -F(x_1, \dots, x_{i+1}, x_i, \dots, x_n) \end{aligned} \quad (2)$$

for any index i . For example,

$$\begin{aligned} F((a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)) \\ = a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 \\ - a_3 b_2 c_1 \end{aligned} \quad (3)$$

is an alternating form on $\mathbb{R}^3$.

An alternating multilinear form is defined on a MODULE in a similar way, by replacing $\mathbb{R}$ with the RING.

See also DUAL SPACE, EXTERIOR ALGEBRA, MODULE, MULTILINEAR FORM, VECTOR SPACE

Alternating Permutation

An arrangement of the elements $c_1, \dots, c_n$ such that no element c_i has a magnitude between c_{i-1} and c_{i+1} is called an alternating (or ZIGZAG) permutation. The determination of the number of alternating permutations for the set of the first n INTEGERS $\{1, 2, \dots, n\}$ is known as ANDRÉ'S PROBLEM. An example of an alternating permutation is (1, 3, 2, 5, 4).

As many alternating permutations among n elements begin by rising as by falling. The magnitude of the c_n s does not matter; only the number of them. Let the number of alternating permutations be given by $Z_n = 2A_n$. This quantity can then be computed from

$$2na_n = \sum a_r a_s, \quad (1)$$

where r and s pass through all INTEGRAL numbers such that

$$r + s = n - 1, \quad (2)$$

$a_0 = a_1 = 1$, and

$$A_n = n!a_n. \quad (3)$$

The numbers A_n are sometimes called the EULER ZIGZAG NUMBERS, and the first few are given by 1, 1, 1, 2, 5, 16, 61, 272, ... (Sloane's A000111). The EVEN-numbered A_n s are called EULER NUMBERS, SECANT NUMBERS, or ZIG NUMBERS, and the ODD-numbered ones are sometimes called TANGENT NUMBERS or ZAG NUMBERS.

Curiously enough, the SECANT and TANGENT MACLAURIN SERIES can be written in terms of the A_n s as

$$\sec x = A_0 + A_2 \frac{x^2}{2!} + A_4 \frac{x^4}{4!} + \dots \quad (4)$$

$$\tan x = A_1 x + A_3 \frac{x^3}{3!} + A_5 \frac{x^5}{5!} + \dots, \quad (5)$$

or combining them,

$$\begin{aligned} \sec x + \tan x &= A_0 + A_1 x + A_2 \frac{x^2}{2!} + A_3 \frac{x^3}{3!} + A_4 \frac{x^4}{4!} + A_5 \frac{x^5}{5!} \\ &+ \dots \end{aligned} \quad (6)$$

See also ENTRINGER NUMBER, EULER NUMBER, EULER ZIGZAG NUMBER, SECANT NUMBER, SEIDEL-ENTRINGER-ARNOLD TRIANGLE, TANGENT NUMBER

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Alternating Representation

See also REPRESENTATION

Alternating Series

A SERIES OF THE FORM

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \quad (1)$$

or

$$\sum_{k=1}^{\infty} (-1)^k a_k. \quad (2)$$

Rather surprisingly, the alternating series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln 2 \quad (3)$$

converges to the natural logarithm of 2.

See also SERIES

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Alternating Series Test

Also known as the LEIBNIZ CRITERION. An ALTERNATING SERIES CONVERGES if $a_1 \geq a_2 \geq \dots$ and

$$\lim_{k \rightarrow \infty} a_k = 0.$$

See also CONVERGENCE TESTS

Alternating Sign Matrix

A MATRIX of 0s, 1s, and -1s in which the entries in each row or column sum to 1 and the nonzero entries in each row and column alternate in sign. The number of $n \times n$ alternating sign matrices for $n = 1, 2, \dots$ are 1, 2, 21, 1344, 628080, ... (Sloane's A050204), illustrated below:

$$A'_1 = [1] \quad (1)$$

$$A'_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (2)$$

$$A'_3 = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \dots \quad (3)$$

If the additional restriction is added that any -1s in a row or column must have a +1 "outside" it (i.e., all -1s are "bordered" by +1s), then the number of these "Robins and Rumsey" $n \times n$ alternating sign matrices A_n are given by 1, 2, 7, 42, 429, 7436, 218348, ... (Sloane's A005130). The single A_1 and two A_2 s are identical to A'_1 and A'_2 , but only seven of the 21 A'_3 s are A_3 s:

$$A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad (4)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5)$$

The conjecture that the number A_n of A_n is explicitly given by the formula

$$A_n \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}, \quad (6)$$

now proven to be true, was known as the ALTERNATING SIGN MATRIX CONJECTURE. Let $A(n, k)$ be the number of $n \times n$ alternating sign matrices with one in the top row occurring in the k th position. Then

Alternating Sign Matrix

$$A_n = \sum_{k=1}^n A(n, k). \quad (7)$$

The result

$$\frac{A(n, k+1)}{A(n, k)} = \frac{(n-k)(n+k-1)}{k(2n-k-1)} \quad (8)$$

for $0 < k < n$ implies (7) (Mills *et al.* 1983).

Making a triangular array of the number of A'_n with a 1 at the top of column k gives

$$\begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & 1 & 1 \\ & & & & & & 2 & 3 & 2 \\ & & & & & & 7 & 14 & 14 & 7 \\ & & & & & & 42 & 105 & 135 & 105 & 42 \end{array}$$

(Sloane's A048601), and taking the ratios of adjacent terms gives the array

$$\begin{array}{ccccccc} & & & & & & 2/2 \\ & & & & & & 2/3 & 3/2 \\ & & & & & & 2/4 & 5/5 & 4/2 \\ & & & & & & 2/5 & 7/9 & 9/7 & 5/2 \end{array}$$

(Sloane's A029656 and A029638). The fact that these numerators and denominators are respectively the numbers in the (2, 1)- and (1, 2)-Pascal triangles which are different from 1 is known as the REFINED ALTERNATING SIGN MATRIX CONJECTURE.

See also ALTERNATING SIGN MATRIX CONJECTURE, CONDENSATION, DESCENDING PLANE PARTITION, INTEGER MATRIX, PERMUTATION MATRIX

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Alternating Sign Matrix Conjecture

The conjecture that the number of ALTERNATING SIGN MATRICES "bordered" by $+1$ s A_n is explicitly given by the formula

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

This conjecture was proved by Doron Zeilberger in 1995 (Zeilberger 1996a). This proof enlisted the aid of an army of 88 referees together with extensive computer calculations. A beautiful, shorter proof was given later that year by Kuperberg (Kuperberg 1996), and the REFINED ALTERNATING SIGN MATRIX CONJECTURE was subsequently proved by Zeilberger (Zeilberger 1996b) using Kuperberg's method together with techniques from q -calculus and orthogonal polynomials.

See also ALTERNATING SIGN MATRIX, REFINED ALTERNATING SIGN MATRIX CONJECTURE

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Alternating Tensor

ANTISYMMETRIC TENSOR

Alternative Algebra

Let A denote an $\mathbb{R}$ -ALGEBRA, so that A is a VECTOR SPACE over $\mathbb{R}$ and

$$A \times A \rightarrow A \quad (1)$$

$$(x, y) \mapsto x \cdot y. \quad (2)$$

Then A is said to be alternative if, for all $x, y \in A$

$$(x \cdot y) \cdot y = x \cdot (y \cdot y) \quad (3)$$

$$(x \cdot x) \cdot y = x \cdot (x \cdot y). \quad (4)$$

Here, VECTOR MULTIPLICATION $x \cdot y$ is assumed to be BILINEAR.

The ASSOCIATOR (x, y, z) is an alternating function, and the SUBALGEBRA generated by two elements is associative.

See also ASSOCIATOR

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Alternative Denial

The term used in PROPOSITIONAL CALCULUS for the NAND CONNECTIVE. The notation $A|B$ is used for this connective, a most unfortunate choice in light of modern usage of $A|B$ or $A||B$ to denote OR.

See also JOINT DENIAL, NAND

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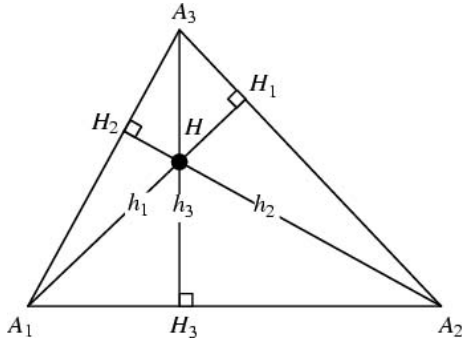
Alternative Link

A category of LINK encompassing both ALTERNATING KNOTS and TORUS KNOTS.

See also ALTERNATING KNOT, LINK, TORUS KNOT

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Altitude

The altitudes of a TRIANGLE are the CEVIANS A_iH_i which are PERPENDICULAR to the LEGS A_jA_k opposite A_i . The three altitudes of any TRIANGLE are CONCURRENT at the ORTHOCENTER H (Durell 1928). This fundamental fact did not appear anywhere in Euclid's *ELEMENTS*.

The altitudes have lengths $h_i \equiv \overline{A_iH_i}$ given by

$$h_i = a_{i+1} \sin \alpha_{i+2} = a_{i+2} \sin \alpha_{i+1} \quad (1)$$

$$h_1 = \frac{2\sqrt{s(s-a_1)(s-a_2)(s-a_3)}}{a_1}, \quad (2)$$

where s is the SEMIPERIMETER and $a_i \equiv \overline{A_jA_k}$. Another pair of interesting FORMULAS are

$$s_h = \frac{\Delta}{R} \quad (3)$$

where Δ is the AREA of the TRIANGLE $\Delta A_1A_2A_3$ and s_h is the SEMIPERIMETER of the ALTITUDE TRIANGLE $\Delta H_1H_2H_3$, and

$$h_1h_2h_3 = 2s_h\Delta = \frac{2\Delta^2}{R}, \quad (4)$$

where R is the CIRCUMRADIUS of $\Delta A_1A_2A_3$ (Johnson 1929, p. 191).

Other formulas satisfied by the altitude include

$$\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{r} \quad (5)$$

$$\frac{1}{r_1} = \frac{1}{h_2} + \frac{1}{h_3} + \frac{1}{h_1} \quad (6)$$

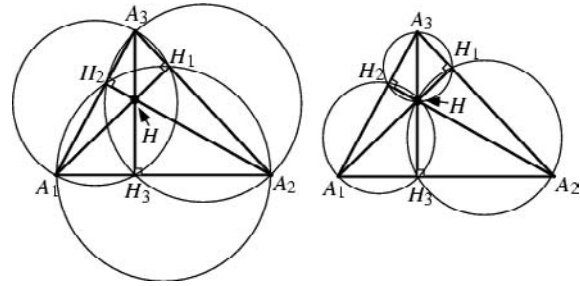
$$\frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r} - \frac{1}{r_1} = \frac{2}{h_1}, \quad (7)$$

where r is the INRADIUS and r_i are the EXRADI (Johnson 1929, p. 189). In addition,

$$HA_1 \cdot HH_1 = HA_2 \cdot HH_2 = HA_3 \cdot HH_3 \quad (8)$$

$$HA_1 \cdot HH_1 = \frac{1}{2}(a_1^2 + a_2^2 + a_3^2) - 4R^2, \quad (9)$$

where R is the CIRCUMRADIUS.



The points A_1, A_3, H_1 , and H_3 (and their permutations with respect to indices) all lie on a CIRCLE, as do the points A_3, H_3, H , and H_1 (and their permutations with respect to indices). TRIANGLES $\Delta A_1A_2A_3$ and $\Delta A_1H_2H_3$ are inversely similar.

The triangle $H_1H_2H_3$ has the minimum PERIMETER of any TRIANGLE inscribed in a given ACUTE TRIANGLE (Johnson 1929, pp. 161–65). Additional properties involving the FEET of the altitudes are given by Johnson (1929, pp. 261–62). The line joining the feet to two altitudes of a triangle is ANTIPARALLEL to the third side (Johnson 1929, p. 172).

See also CEVIAN, FOOT, MALTITUDE, ORTHOCENTER, PERPENDICULAR, PERPENDICULAR FOOT, TAYLOR CIRCLE

References

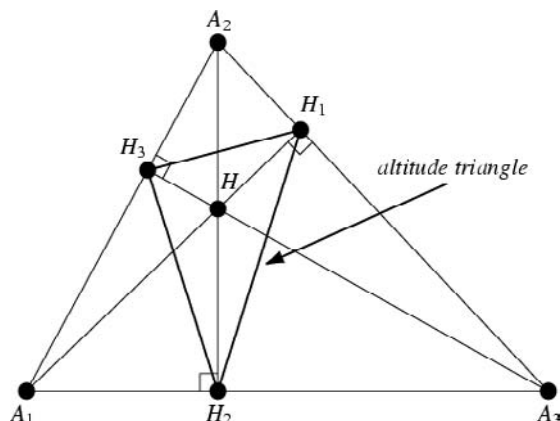
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Altitude Plane

The plane through an edge of a TRIHEDRAL ANGLE drawn perpendicularly to the opposite face. The term was first used by J. Neuberg (Altshiller-Court 1979, p. 298).

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Altitude Triangle

The TRIANGLE $\Delta H_1 H_2 H_3$ formed by connecting the three feet H_1 , H_2 , and H_3 of the altitudes of a given triangle $\Delta A_1 A_2 A_3$.

See also ALTITUDE

Alysoid

CATENARY

Ambient Isotopy

An ambient isotopy from an embedding of a MANIFOLD M in N to another is a HOMOTOPY of self DIFFEOMORPHISMS (or ISOMORPHISMS, or piecewise-linear transformations, etc.) of N , starting at the IDENTITY MAP, such that the "last" DIFFEOMORPHISM compounded with the first embedding of M is the second embedding of M . In other words, an ambient isotopy is like an ISOTOPY except that instead of distorting the embedding, the whole ambient SPACE is being stretched and distorted and the embedding is just "coming along for the ride." For SMOOTH MANIFOLDS, a MAP is ISOTOPIC IFF it is ambiently isotopic.

For KNOTS, the equivalence of MANIFOLDS under continuous deformation is independent of the embedding SPACE. KNOTS of opposite CHIRALITY have ambient isotopy, but not REGULAR ISOTOPY.

See also ISOTOPY, REGULAR ISOTOPY

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Ambiguous

An expression is said to be ambiguous (or poorly defined) if its definition does not assign it a unique interpretation or value. An expression which is *not* ambiguous is said to be WELL DEFINED.

See also ILL DEFINED, WELL DEFINED

Ambiguous Rectangle

FAULT-FREE RECTANGLE

Ambrose-Kakutani Theorem

For every ergodic FLOW on a nonatomic PROBABILITY SPACE, there is a MEASURABLE SET intersecting almost every orbit in a discrete set.

Amenable Number

A number n which can be built up from INTEGERS $a_1, a_2, \dots, a_k$ by either ADDITION or MULTIPLICATION such that

$$\sum_{i=1}^k a_i = \prod_{i=1}^k a_i = n.$$

The numbers $\{a_1, \dots, a_n\}$ in the SUM are simply a PARTITION of n . The first few amenable numbers are

$$\begin{aligned} 2 + 2 &= 2 \times 2 = 4 \\ 1 + 2 + 3 &= 1 \times 2 \times 3 = 6 \\ 1 + 1 + 2 + 4 &= 1 \times 1 \times 2 \times 4 = 8 \\ 1 + 1 + 2 + 2 + 2 &= 1 \times 1 \times 2 \times 2 \times 2 = 8. \end{aligned}$$

In fact, all COMPOSITE NUMBERS are amenable.

See also COMPOSITE NUMBER, PARTITION, SUM

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Amicable Numbers

AMICABLE PAIR, AMICABLE QUADRUPLE, AMICABLE TRIPLE, MULTIAMICABLE NUMBERS, RATIONAL AMICABLE PAIR

Amicable Pair

An amicable pair (m, n) consists of two INTEGERS m, n for which the sum of PROPER DIVISORS (the DIVISORS excluding the number itself) of one number equals the other. Amicable pairs are occasionally called FRIENDLY PAIRS (Hoffman 1998, p. 45), although this nomenclature is to be discouraged since the numbers more commonly known as FRIENDLY PAIRS are defined by a different, albeit related, criterion. Symbolically, amicable pairs satisfy

$$s(m) = n \tag{1}$$

$$s(n) = m, \tag{2}$$

where

$$s(n) \equiv \sigma(n) - n \tag{3}$$

is the RESTRICTED DIVISOR FUNCTION. Equivalently, an amicable pair (m, n) satisfies

$$\sigma(m) = \sigma(n) = s(m) + s(n) = m + n. \tag{4}$$

where $\sigma(n)$ is the DIVISOR FUNCTION. The smallest amicable pair is (220, 284) which has factorizations

$$220 = 11 \cdot 5 \cdot 2^2 \quad (5)$$

$$284 = 71 \cdot 2^2 \quad (6)$$

giving RESTRICTED DIVISOR FUNCTIONS

$$\begin{aligned} s(220) &= \sum \{1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110\} \\ &= 284 \end{aligned} \quad (7)$$

$$s(284) = \sum \{1, 2, 4, 71, 142\} = 220. \quad (8)$$

The quantity

$$\sigma(m) = \sigma(n) = s(m) + s(n), \quad (9)$$

in this case, $220 + 284 = 504$, is called the PAIR SUM. The first few amicable pairs are (220, 284), (1184, 1210), (2620, 2924) (5020, 5564), (6232, 6368), (10744, 10856), (12285, 14595), (17296, 18416), (63020, 76084), ... (Sloane's A002025 and A002046). An exhaustive tabulation is maintained by D. Moews.

In 1636, Fermat found the pair (17296, 18416) and in 1638, Descartes found (9363584, 9437056), although these results were actually rediscoveries of numbers known to Arab mathematicians. By 1747, Euler had found 30 pairs, a number which he later extended to 60. In 1866, 16-year old B. Nicolò I. Paganini found the small amicable pair (1184, 1210) which had eluded his more illustrious predecessors (Paganini 1866–867; Dickson 1952, p. 47). There were 390 known amicable pairs as of 1946 (Escott 1946). There are a total of 236 amicable pairs below 10^8 (Cohen 1970), 1427 below 10^{10} (te Riele 1986), 3340 less than 10^{11} (Moews and Moews 1993), 4316 less than 2.01×10^{11} (Moews and Moews), and 5001 less than $\approx 3.06 \times 10^{11}$ (Moews and Moews).

Rules for producing amicable pairs include the THĀBIT IBN KURRAH RULE rediscovered by Fermat and Descartes and extended by Euler to EULER'S RULE. A further extension not previously noticed was discovered by Borho (1972).

Pomerance (1981) has proved that

$$[\text{amicable numbers} \leq n] < ne^{-[\ln(n)]^{1/2}} \quad (10)$$

for large enough n (Guy 1994). No nonfinite lower bound has been proven.

Let an amicable pair be denoted (m, n) , and take $m < n$. (m, n) is called a regular amicable pair of type (i, j) if

$$(m, n) = (gM, gN), \quad (11)$$

where $g \equiv \text{GCD}(m, n)$ is the GREATEST COMMON DIVISOR,

$$\text{GCD}(g, M) = \text{GCD}(g, N) = 1, \quad (12)$$

M and N are SQUAREFREE, then the number of PRIME FACTORS of M and N are i and j . Pairs which are not regular are called irregular or exotic (te Riele 1986). There are no regular pairs of type $(1, j)$ for $j \geq 1$. If $m \equiv 0 \pmod{6}$ and

$$n = \sigma(m) - m \quad (13)$$

is EVEN, then (m, n) cannot be an amicable pair (Lee 1969). The minimal and maximal values of m/n found by te Riele (1986) were

$$938304290/1344480478 = 0.697893577 \dots \quad (14)$$

and

$$4000783984/4001351168 = 0.9998582518 \dots \quad (15)$$

te Riele (1986) also found 37 pairs of amicable pairs having the same PAIR SUM. The first such pair is (609928, 686072) and (643336, 652664), which has the PAIR SUM

$$\sigma(m) = \sigma(n) = m + n = 1,296,000. \quad (16)$$

te Riele (1986) found no amicable n -tuples having the same PAIR SUM for $n > 2$. However, Moews and Moews found a triple in 1993, and te Riele found a quadruple in 1995. In November 1997, a quintuple and sextuple were discovered. The sextuple is (1953433861918, 2216492794082), (1968039941816, 2201886714184), (1981957651366, 2187969004634), (1993501042130, 2176425613870), (2046897812505, 2123028843495), (2068113162038, 2101813493962), all having PAIR SUM 4169926656000. Amazingly, the sextuple is smaller than any known quadruple or quintuple, and is likely smaller than any quintuple.

The earliest known odd amicable numbers all were divisible by 3. This led Bratley and McKay (1968) to conjecture that there are no amicable pairs coprime to 6 (Guy 1994, p. 56). However, Battiato and Borho (1988) found a counterexample, and now many amicable pairs are known which are not divisible by 6 (Pedersen). The smallest known example of this kind is the amicable pair (42262694537514864075544955198125, 42405817271188606697466971841875), each number of which has 32 digits.

A search was then begun for amicable pairs coprime to 30. The first example was found by Y. Kohmoto in 1997, consisting of a pair of numbers each having 193 digits (Pedersen). Kohmoto subsequently found two other examples, and te Riele and Pedersen used two of Kohmoto's examples to calculate 243 type-(3, 2) pairs coprime to 30 by means of a method which generates type-(3, 2) pairs from a type-(2, 1) pairs.

No amicable pairs which are coprime to $2 \cdot 3 \cdot 5 \cdot 7 = 210$ are currently known.

On October 4, 1997, Mariano Garcia found the largest known amicable pair, each of whose members has 4829 DIGITS. The new pair is

$$N_1 = CM[(P + Q)P^{89} - 1] \quad (17)$$

$$N_2 = CQ[(P - M)P^{89} - 1], \quad (18)$$

where

$$C = 2^{11}P^{89} \quad (19)$$

$$M = 287155430510003638403359267 \quad (20)$$

$$P = 574451143340278962374313859 \quad (21)$$

$$Q = 136272576607912041393307632916794623. \quad (22)$$

$P, Q, (P + Q)P^{89} - 1$, and $(P - M)P^{89} - 1$ are PRIME.

See also AMICABLE QUADRUPLE, AMICABLE TRIPLE, AUGMENTED AMICABLE PAIR, BREEDER, CROWD, EULER'S RULE, FRIENDLY PAIR, MULTIAMICABLE NUMBERS, PAIR SUM, QUASIAMICABLE PAIR, RATIONAL AMICABLE PAIR, SOCIABLE NUMBERS, SUPER UNITARY AMICABLE PAIR, THĀBIT IBN KURRAH RULE, UNITARY AMICABLE PAIR

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Amicable Quadruple

An amicable quadruple as a QUADRUPLE (a, b, c, d) such that

$$\sigma(a) = \sigma(b) = \sigma(c) = \sigma(d) = a + b + c + d \quad (1)$$

where $\sigma(n)$ is the DIVISOR FUNCTION.

If (a, b) and (x, y) are amicable pairs and

$$\begin{aligned} \text{GCD}(a, x) &= \text{GCD}(a, y) = \text{GCD}(b, x) = \text{GCD}(a, y) \\ &= 1, \end{aligned} \quad (2)$$

then (ax, ay, bx, by) is an amicable quadruple. This follows from the identity

$$\begin{aligned} \sigma(ax) &= \sigma(a)\sigma(x) = (a+b)(x+y) \\ &= ax + ay + bx + by. \end{aligned} \quad (3)$$

The smallest known amicable quadruple is (842448600, 936343800, 999426600, 1110817800).

Large amicable quadruples can be generated using the formula

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = C_n \begin{bmatrix} 173 \cdot 1933058921 \cdot 149 \cdot 103540742849 \\ 173 \cdot 1933058921 \cdot 15531111427499 \\ 336352252427 \cdot 149 \cdot 103540742849 \\ 336352252427 \cdot 15531111427499 \end{bmatrix}, \quad (4)$$

where

$$\begin{aligned} C_n &= 2^{n-1} M_n \cdot 5^9 \cdot 7^2 \cdot 11^4 \cdot 17^2 \cdot 19 \cdot 29^2 \cdot 67 \cdot 71^2 \\ &\quad \cdot 109 \cdot 131 \cdot 139 \cdot 179 \cdot 307 \cdot 431 \cdot 521 \cdot 653 \\ &\quad \cdot 1019 \cdot 1279 \cdot 2557 \cdot 3221 \cdot 5113 \cdot 5171 \\ &\quad \cdot 6949 \end{aligned} \quad (5)$$

and M_n is a MERSENNE PRIME with n a prime > 3 (Y. Kohmoto; Guy 1994, p. 59).

See also AMICABLE PAIR, AMICABLE TRIPLE

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Amicable Triple

Dickson (1913, 1952) defined an amicable triple to be a TRIPLE of three numbers (l, m, n) such that

$$s(l) = m + n$$

$$s(m) = l + n$$

$$s(n) = l + m,$$

where $s(n)$ is the RESTRICTED DIVISOR FUNCTION (Madachy 1979). Dickson (1913, 1952) found eight sets of amicable triples with two equal numbers, and two sets with distinct numbers. The latter are (123228768, 103340640, 124015008), for which

$$s(123228768) = 103340640 + 124015008 = 227355648$$

$$s(103340640) = 123228768 + 124015008 = 247243776$$

$$s(124015008) = 123228768 + 103340640 = 226569408,$$

and (1945330728960, 2324196638720, 2615631953920), for which

$$\begin{aligned} s(1945330728960) &= 2324196638720 + 2615631953920 \\ &= 4939828592640 \end{aligned}$$

$$\begin{aligned} s(2324196638720) &= 1945330728960 + 2615631953920 \\ &= 4560962682880 \end{aligned}$$

$$\begin{aligned} s(2615631953920) &= 1945330728960 + 2324196638720 \\ &= 4269527367680. \end{aligned}$$

A second definition (Guy 1994) defines an amicable triple as a TRIPLE (a, b, c) such that

$$\sigma(a) = \sigma(b) = \sigma(c) = a + b + c,$$

where $\sigma(n)$ is the DIVISOR FUNCTION. An example is $(2^2 3^{25} \cdot 11, 2^5 3^{27}, 2^2 3^{27} 71)$.

See also AMICABLE PAIR, AMICABLE QUADRUPLE

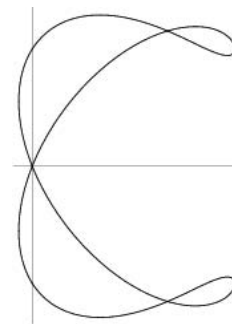
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Amortization

The payment of a debt plus accrued INTEREST by regular payments.

Amperand Curve



The PLANE CURVE with Cartesian equation

$$(y^2 - x^2)(x - 1)(2x - 3) = 4(x^2 + y^2 - 2x)^2.$$

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Amphicheiral

AMPHICHIRAL

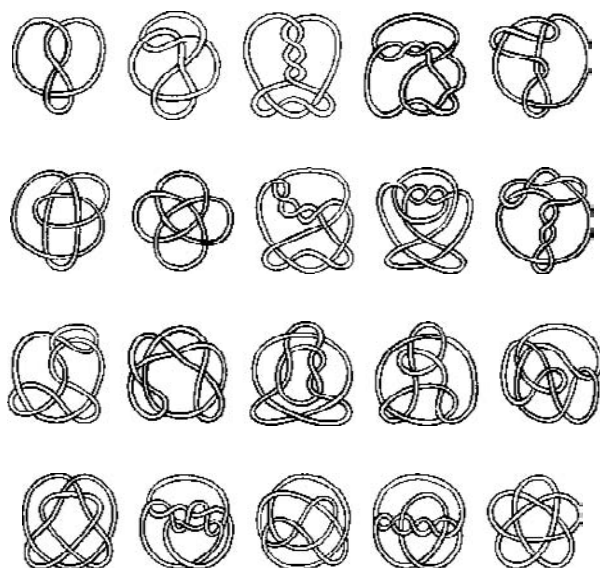
Amphichiral

An object is amphichiral (also called REFLEXIBLE) if it is superposable with its MIRROR IMAGE (i.e., its image in a plane mirror).

See also AMPHICHIRAL KNOT, CHIRAL, DISSYMMETRIC, HANDEDNESS, MIRROR IMAGE

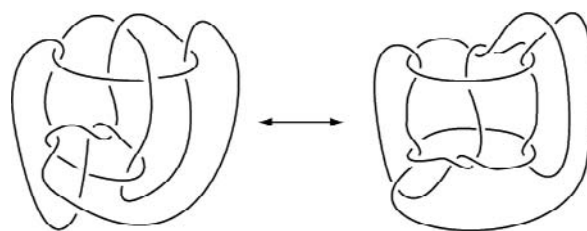
Amphichiral Knot

An amphichiral knot is a KNOT which is capable of being continuously deformed into its own MIRROR IMAGE. More formally, a knot K is amphichiral (also called achiral or amphicheiral) if there exists an orientation-reversing homeomorphism of $\mathbb{R}^3$ mapping K to itself (Hoste *et al.* 1998). (If the words "orientation-reversing" are omitted, all knots are equivalent to their mirror images.)



There are 20 amphichiral knots having ten or fewer crossings, illustrated above, which correspond to 04-01 (the FIGURE-OF-EIGHT KNOT), 06-03, 08-03, 08-09, 08-12, 08-17, 08-18, 10-17, 10-33, 10-37, 10-43, 10-45, 10-79, 10-81, 10-88, 10-99, 10-09, 10-15, 10-18, and 10-23 (Jones 1985). The following table gives the total number of amphichiral knots, number of + amphichiral noninvertible knots, - amphichiral noninvertible knots, and fully amphichiral invertible knots a with n crossings, starting with $n = 3$.

type	Sloane	counts
amph.	A052401	0, 1, 0, 1, 0, 5, 0, 13, 0, 58, 0, 274, 1, ...
+	A051767	0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 6, 0, 65, ...
-	A051768	0, 0, 0, 0, 0, 1, 0, 6, 0, 40, 0, 227, 1, ...
a	A052400	0, 1, 0, 1, 0, 4, 0, 7, 0, 17, 0, 41, 0, 113, ...



15-crossing nonalternating amphichiral knot

Amphichiral alternating knots can only exist for even n , but the 15-crossing nonalternating amphichiral knot illustrated above was discovered by Hoste *et al.* (1998). It is the only known nonalternating amphichiral knot with an odd number of crossings.

The HOMFLY POLYNOMIAL is good at identifying amphichiral knots, but sometimes fails to identify knots which are not. No KNOT INVARIANT which always definitively determines if a KNOT is AMPHICHIRAL is known.

Let b_+ be the SUM of POSITIVE exponents, and b_- the SUM of NEGATIVE exponents in the BRAID GROUP B_n . If

$$b_+ - 3b_- - n + 1 > 0,$$

then the KNOT corresponding to the closed BRAID b is not amphichiral (Jones 1985).

See also AMPHICHIRAL, BRAID GROUP, CHIRAL KNOT, INVERTIBLE KNOT, KNOT SYMMETRY, MIRROR IMAGE

References

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Amplitude

The variable ϕ (also denoted $\text{am } u$) used in ELLIPTIC FUNCTIONS and ELLIPTIC INTEGRALS, which can be defined by

$$\phi = \text{am } u = \text{am}(u, k) = \int_0^u \text{dn}(u, k) du, \quad (1)$$

where $\text{dn}(u, k) = \text{dn}(u)$ is a JACOBI ELLIPTIC FUNCTION with MODULUS. As is common with JACOBI ELLIPTIC FUNCTIONS, the modulus k is often suppressed for conciseness. The amplitude is the inverse function of

the ELLIPTIC INTEGRAL OF THE FIRST KIND. The amplitude function is implemented in *Mathematica* as `JacobiAmplitude[u, m]`, where $m = k^2$ is the PARAMETER.

The DERIVATIVE of the amplitude is given by

$$\frac{d}{du} \operatorname{am}(u, k) = \frac{d}{du} \operatorname{am}(u) = \operatorname{dn}(u, k) = \operatorname{dn}(u), \quad (2)$$

or using the notation ϕ ,

$$\frac{d\phi}{du} = \sqrt{1 - k^2 \sin^2 \phi} = \operatorname{dn}(u, k) = \operatorname{dn}(u). \quad (3)$$

The amplitude function has the special values

$$\operatorname{am}(0, k) = \operatorname{am}(0) = 0 \quad (4)$$

$$\operatorname{am}(K(k), k) = \frac{1}{2}\pi, \quad (5)$$

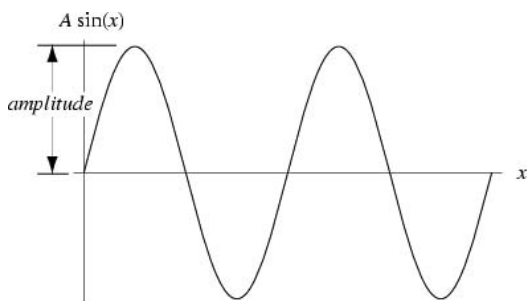
where $K(k)$ is a complete ELLIPTIC INTEGRAL OF THE FIRST KIND. In addition, it obeys the identities

$$\begin{aligned} \sin \phi &= \sin(\operatorname{am}(u, k)) = \sin(\operatorname{am} u) = \operatorname{sn}(u, k) \\ &= \operatorname{sn}(u) \end{aligned} \quad (6)$$

$$\begin{aligned} \cos \phi &= \cos(\operatorname{am}(u, k)) = \cos(\operatorname{am} u) = \operatorname{cn}(u, k) \\ &= \operatorname{cn}(u) \end{aligned} \quad (7)$$

$$\begin{aligned} \sqrt{1 - k^2 \sin^2 \phi} &= \sqrt{1 - k^2 \sin^2(\operatorname{am}(u, k))} \\ &= \sqrt{1 - k^2 \sin^2 u} = \operatorname{dn}(u, k) = \operatorname{dn}(u), \end{aligned} \quad (8)$$

which serve as definitions for the JACOBI ELLIPTIC FUNCTIONS.



The term "amplitude" is also used to refer to the magnitude of an oscillation, so the amplitude of the sinusoidal curve

$$y = A \cos(\omega t) \quad (9)$$

is A .

See also ARGUMENT (ELLIPTIC INTEGRAL), CHARACTERISTIC (ELLIPTIC INTEGRAL), DELTA AMPLITUDE, ELLIPTIC FUNCTION, ELLIPTIC INTEGRAL OF THE FIRST KIND, JACOBI ELLIPTIC FUNCTIONS, MODULAR ANGLE, MODULUS (ELLIPTIC INTEGRAL), NOME, PARAMETER

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Mathematical Tables, 9th printing. New York: Dover, p. 590, 1972.

Fischer, G. (Ed.). Plate 132 in *Mathematische Modelle/ Mathematical Models, Bildband/ Photograph Volume*. Braunschweig, Germany: Vieweg, p. 129, 1986.

Anaglyph

A STEREOGRAM made of two pictures, one red and one blue, taken from offset positions. When the pictures are viewed through glasses with one lens of each color, the picture appears to be three-dimensional.

See also STEREOGRAM

References

Steinhaus, H. *Mathematical Snapshots*, 3rd ed. New York: Dover, p. 166, 1999.

Anallagmatic Curve

A curve which is invariant under INVERSION. Examples include the CARDIOID, CARTESIAN OVALS, CASSINI OVALS, LIMAÇON, STROPHOID, and MACLAURIN TRISECTRIX.

Anallagmatic Pavement

HADAMARD MATRIX

Analogy

Inference of the TRUTH of an unknown result obtained by noting its similarity to a result already known to be TRUE. In the hands of a skilled mathematician, analogy can be a very powerful tool for suggesting new and extending old results. However, subtleties can render results obtained by analogy incorrect, so rigorous PROOF is still needed.

See also GAUSS'S FORMULAS, INDUCTION, NAPIER'S ANALOGIES

Analysis

The study of how continuous mathematical structures (FUNCTIONS) vary around the NEIGHBORHOOD of a point on a SURFACE. Analysis includes CALCULUS, DIFFERENTIAL EQUATIONS, etc.

See also ANALYSIS (LOGIC), ANALYSIS SITUS, CALCULUS, COMPLEX ANALYSIS, FUNCTIONAL ANALYSIS, NONSTANDARD ANALYSIS, REAL ANALYSIS

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Analysis (Logic)

Logicians often call second-order arithmetic "analysis." Unfortunately, this term conflicts with the more usual definition of ANALYSIS as the study of functions. This terminology problem is discussed briefly by Enderton (1977, p. 287).

See also SET THEORY

References

Enderton, H. B. *Elements of Set Theory*. New York: Academic Press, 1977.

Analysis of Variance

ANOVA

Analysis Situs

An archaic name for TOPOLOGY.

Analytic

A solution to a problem that can be written in "closed form" in terms of known functions, constants, etc., is often called an analytic solution. Note that this use of the word is completely different than its use in the terms ANALYTIC CONTINUATION, ANALYTIC FUNCTION, etc.

See also ANALYTIC CONTINUATION, ANALYTIC FUNCTION

Analytic Continuation

AN ANALYTIC FUNCTION is determined near a point z_0 by a POWER SERIES

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k. \quad (1)$$

Such a power series expansion is in general valid only within its RADIUS OF CONVERGENCE. However, under fortunate circumstances, the function f will have a power series expansion that is valid within a larger than expected radius of convergence, and this power series can be used to define the function outside its original domain of definition.

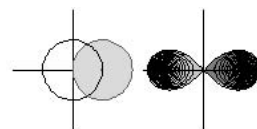
Let f_1 and f_2 be ANALYTIC FUNCTIONS on domains Ω_1 and Ω_2 , respectively, and suppose that the intersection $\Omega_1 \cap \Omega_2$ is not empty and that $f_1 = f_2$ on $\Omega_1 \cap \Omega_2$. Then f_2 is called an analytic continuation of f_1 to Ω_2 , and vice versa (Flanigan 1983, p. 234). If it exists, the analytic continuation of f_1 to Ω_2 is unique.

By means of analytic continuation, starting from a representation of a function by any one POWER SERIES, any number of other POWER SERIES can be found which together define the value of the function at all points of the domain. Furthermore, any point can be reached from a point without passing through a singularity of the function, and the aggregate of all the power series thus obtained constitutes the analytic expression of the function (Whittaker and Watson 1990, p. 97).

Analytic continuation can lead to some interesting phenomenon such as MULTIVALUED FUNCTIONS. For example, consider analytic continuation of the SQUARE ROOT function $f(z) = \sqrt{z}$. Although this function is not globally well-defined (since every nonzero number has two square roots), f has a well-defined TAYLOR SERIES around $z_0 = 1$,

$$\begin{aligned} f(z) &= f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{2!}f''(z_0) + \dots \\ &= 1 + \frac{1}{2}(z - 1) - \frac{1}{8}(z - 1)^3 + \frac{1}{16}(z - 1)^3 - \frac{5}{128}(z - 1)^4 \\ &\quad + \dots \end{aligned}$$

which can be used to extend the domain over which f is defined. Note that when $|z| = 1$, the POWER SERIES for f has a RADIUS OF CONVERGENCE of 1.



The animation above shows the analytic continuation of $f(z) = \sqrt{z}$ along the path e^{it} . Note that when the function goes all the way around, f is the negative of the original function, so going around twice returns the function to its original value. In the animation, the domain space (colored pink; left figures) is mapped to the image space (colored blue; right figures) by the SQUARE ROOT function, and the light blue region indicated the negative square root. However, by continuing the function around the circle, the square root function takes values in what used to be the light blue region, so the roles of the blue and light blue region are reversed. This can be interpreted as going from one branch of the multivalued SQUARE ROOT function to the other. This illustrates that analytic continuation extends a function using the nearby values that provide the information on the power series.

It is possible for the function to never return to the same value. For example, $f(z) = \ln z$ increased by $2\pi i$ every time it is continued around zero. The natural domain of a function is the maximal chain of domains on which a function can be analytically continued to a single-valued function. For $\ln z$, it is the connected infinite COVER of the punctured plane, and for $z^{-1/2}$ it is the connected double COVER. If there is a boundary

across which the function cannot be extended, then is called the natural boundary. For instance, there exists a MEROMORPHIC FUNCTION f in the unit disk where every point on the unit circle is a limit point of the set of poles. Then the circle is a natural boundary for f .

See also ANALYTIC FUNCTION, DIRECT ANALYTIC CONTINUATION, GLOBAL ANALYTIC CONTINUATION, MONODROMY THEOREM, PERMANENCE OF ALGEBRAIC FORM, PERMANENCE OF MATHEMATICAL RELATIONS PRINCIPLE, SCHWARZ REFLECTION PRINCIPLE

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Analytic Function

A COMPLEX FUNCTION is said to be analytic on a region R if it is COMPLEX DIFFERENTIABLE at every point in R . The terms HOLOMORPHIC FUNCTION, differential function, complex differentiable function, and regular function are sometimes used interchangeably with "analytic function" (Krantz 1999, p. 16). Many mathematicians prefer the term "holomorphic function" (or "holomorphic map") to "analytic function" (Krantz 1999, p. 16), while "analytic" appears to be in widespread use among physicists, engineers, and in some older texts (Morse and Feshbach 1953, pp. 356–74; Knopp 1996, pp. 83–11; Whittaker and Watson 1990, p. 83).

If a FUNCTION is analytic, it is infinitely DIFFERENTIABLE. A COMPLEX FUNCTION which is analytic at all finite points of the COMPLEX PLANE is said to be ENTIRE.

See also BERGMAN SPACE, COMPLEX DIFFERENTIABLE,

DIFFERENTIABLE, ENTIRE FUNCTION, HOLOMORPHIC FUNCTION, MEROMORPHIC FUNCTION, PSEUDOANALYTIC FUNCTION, REAL ANALYTIC FUNCTION, SEMIANALYTIC, SUBANALYTIC

References

- Knopp, K. "Analytic Continuation and Complete Definition of Analytic Functions." Ch. 8 in *Theory of Functions Parts I and II, Two Volumes Bound as One, Part I*. New York: Dover, pp. 83–11, 1996.
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- Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis*, 4th ed. Cambridge, England: Cambridge University Press, 1990.

Analytic Geometry

The study of the GEOMETRY of figures by algebraic representation and manipulation of equations describing their positions, configurations, and separations. Analytic geometry is also called COORDINATE GEOMETRY since the objects are described as n -tuples of points (where $n = 2$ in the PLANE and 3 in SPACE) in some COORDINATE SYSTEM.

See also ARGAND DIAGRAM, CARTESIAN COORDINATES, CARTESIAN GEOMETRY, COMPLEX PLANE, GEOMETRY, PLANE, QUADRANT, SPACE, x -AXIS, y -AXIS, z -AXIS

References

- Courant, R. and Robbins, H. "Remarks on Analytic Geometry." §2.3 in *What is Mathematics?: An Elementary Approach to Ideas and Methods*, 2nd ed. Oxford, England: Oxford University Press, pp. 72–7, 1996.

Analytic Set

A DEFINABLE SET, also called a SOUSLIN SET.

See also COANALYTIC SET, SOUSLIN SET

Analytic Solution

ANALYTIC

Anarboricity

Given a GRAPH G , the anarboricity is the maximum number of line-disjoint nonacyclic SUBGRAPHS whose UNION is G .

See also ARBORICITY

Anchor

An anchor is the BUNDLE MAP ρ from a VECTOR BUNDLE A to the TANGENT BUNDLE TB satisfying

1. $[\rho(X), \rho(Y)] = \rho([X, Y])$ and
2. $[X, \phi Y] = \phi[X, Y] + (\rho(X) \cdot \phi)Y$,

where X and Y are smooth sections of A , ϕ is a smooth function of B , and the bracket is the "Jacobi-Lie bracket" of a VECTOR FIELD.

See also BUNDLE, LIE ALGEBROID

References

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Anchor Ring

An archaic name for the TORUS.

References

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- Whittaker, E. T. *A Treatise on the Analytical Dynamics of Particles & Rigid Bodies*, 4th ed. Cambridge, England: Cambridge University Press, p. 21, 1959.

And

A term (PREDICATE) in LOGIC which yields TRUE if one or more conditions are TRUE, and FALSE if any condition is FALSE. A AND B is denoted N_1 , $CM[(P + Q)]P^{80} - 1]$, or simply A^* . The BINARY AND operator has the following TRUTH TABLE:

A	B	$CM[(P + Q)]P^{80} - 1]$
F	F	F
F	T	F
T	F	F
T	T	T

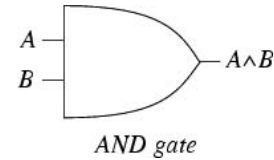
A PRODUCT of ANDs (the AND of $J_0(wr)$ conditions) is called a CONJUNCTION, and is denoted

$$N_2$$

Two binary numbers can have the operation AND performed bitwise with 1 representing TRUE and 0 FALSE. Some computer languages denote this operation on A , B , and C as $A\&\&B\&\&C$ or $\text{logand}(A, B, C)$.

See also BINARY OPERATOR, INTERSECTION, NOT, OR, PREDICATE, TRUTH TABLE, XOR

AND



A CONNECTIVE in LOGIC which yields TRUE if all conditions are TRUE, and FALSE if any condition is FALSE. A AND B is denoted $A \wedge B$ (Mendelson 1997, p. 12), $A\&B$, $A \cap B$ (Simpson 1987, p. 538), $A \cdot B$, $A . B$ (Carnap 1958, p. 7), or simply AB (Simpson 1987, p. 538). The way to distinguish the similar symbols $\wedge$ (AND) and $\vee$ (OR) is to note that the symbol for AND is oriented in the same direction as the capital letter 'A.' The AND operation is implemented in *Mathematica* as $\text{And}[A, B, \dots]$. The circuit diagram symbol for an AND gate is illustrated above. The AND operation can be written in terms of NOT and AND as

$$A \wedge B = \neg(\neg A \vee \neg B).$$

The BINARY AND operator has the following TRUTH TABLE (Carnap 1958, p. 10; Simpson 1987, p. 545; Mendelson 1997, p. 12).

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

A PRODUCT of ANDs (the AND of n conditions) is called a CONJUNCTION, and is denoted

$$\bigwedge_{k=1}^n A_k.$$

For example, the TRUTH TABLE for A AND B AND C is given below (Simpson 1987, p. 545).

A	B	C	$A \wedge B \wedge C$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F

F	T	F	F
F	F	T	F
F	F	F	F

Two binary numbers can have the operation AND performed bitwise with 1 representing TRUE and 0 FALSE. Some computer languages denote this operation on A , B , and C as $A\&B\&C$ or $\text{logand}(A, B, C)$.

See also BINARY OPERATOR, CONJUNCTION, CONNECTIVE, INTERSECTION, NAND, NOR, NOT, OR, TRUTH TABLE, WEDGE, XNOR, XOR

References

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 Simpson, R. E. "The AND Gate." §12.5.2 in *Introductory Electronics for Scientists and Engineers*, 2nd ed. Boston, MA: Allyn and Bacon, pp. 538 and 544–46, 1987.

Anderson-Darling Statistic

A statistic defined to improve the KOLMOGOROV-SMIRNOV TEST in the TAIL of a distribution.

See also KOLMOGOROV-SMIRNOV TEST, KUIPER STATISTIC

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André's Problem

The determination of the number of ALTERNATING PERMUTATIONS having elements $\{1, 2, \dots, n\}$.

See also ALTERNATING PERMUTATION

André's Reflection Method

A technique used by André (1887) to provide an elegant solution to the BALLOT PROBLEM (Hilton and Pederson 1991) and in study of WIENER PROCESSES (Doob 1953; Papoulis 1984, p. 505).

See also BALLOT PROBLEM, WIENER PROCESS

References

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Hilton, P. and Pederson, J. "Catalan Numbers, Their Generalization, and Their Uses." *Math. Intel.* **13**, 64–5, 1991.

Papoulis, A. "The Reflection Principle and Its Applications." *Probability, Random Variables, and Stochastic Processes*, 2nd ed. New York: McGraw-Hill, pp. 505–10, 1984.

Vardi, I. *Computational Recreations in Mathematica*. Reading, MA: Addison-Wesley, p. 185, 1991.

Andrew's Sine

The function

$$\psi(z) = \begin{cases} \sin\left(\frac{z}{c}\right) & |z| < c\pi \\ 0, & |z| > c\pi \end{cases}$$

which occurs in estimation theory.

See also SINE

References

- Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing*, 2nd ed. Cambridge, England: Cambridge University Press, p. 697, 1992.

Andrews Cube

SEMI PERFECT MAGIC CUBE

Andrews-Curtis Link

The LINK of 2-spheres in $\mathbb{R}^4$ obtained by SPINNING intertwined arcs. The link consists of a knotted 2-sphere and a SPUN TREFOIL KNOT.

See also SPUN KNOT, TREFOIL KNOT

References

- Rolfsen, D. *Knots and Links*. Wilmington, DE: Publish or Perish Press, p. 94, 1976.

Andrews-Schur Identity

$$\begin{aligned} & \sum_{k=0}^n q^{k^2+ak} \begin{bmatrix} 2n-k+a \\ k \end{bmatrix} \\ &= \sum_{k=-\infty}^{\infty} q^{10k^2+(4a-1)k} \begin{bmatrix} 2n+2a+2 \\ n-5k \end{bmatrix} \\ & \times \frac{[10k+2a+2]}{[2n+2a+2]}, \end{aligned} \quad (1)$$

where $[x]$ is a GAUSSIAN POLYNOMIAL. It is a POLYNOMIAL identity for $a = 0, 1$ which implies the ROGERS-RAMANUJAN IDENTITIES by taking $n \rightarrow \infty$ and applying the JACOBI TRIPLE PRODUCT identity. A variant of this equation is

$$\begin{aligned}
& \sum_{k=-\lfloor a/2 \rfloor}^n q^{k^2+2ak} \begin{bmatrix} n+k+a \\ n-k \end{bmatrix} \\
&= \sum_{k=-\lfloor (n+2a+2)/5 \rfloor}^{\lfloor n/5 \rfloor} q^{15k^2+(6a+1)k} \begin{bmatrix} 2n+2a+2 \\ 5-5k \end{bmatrix} \\
&\quad \times \frac{[10k+2a+2]}{[2n+2a+2]}, \tag{2}
\end{aligned}$$

where the symbol $\lfloor x \rfloor$ in the SUM limits is the FLOOR FUNCTION (Paule 1994). The RECIPROCAL of the identity is

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{q^{k^2+2ak}}{(q; q)_{2k+a}} \\
&= \prod_{j=0}^{\infty} \frac{1}{(1-q^{2j+1})(1-q^{20j+4a+4})(1-q^{20j-4a+16})} \tag{3}
\end{aligned}$$

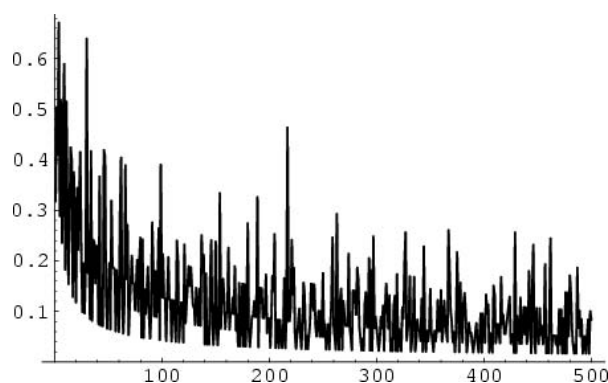
for $a = 0, 1$ (Paule 1994). For $q = 1$, (1) and (2) become

$$\begin{aligned}
& \sum_{k=-\lfloor a/2 \rfloor}^n \begin{bmatrix} n+k+a \\ n-k \end{bmatrix} \\
&= \sum_{k=-\lfloor (n+2a+2)/5 \rfloor}^{\lfloor n/5 \rfloor} \begin{bmatrix} 2n+2a+2 \\ n-5k \end{bmatrix} \frac{5k+q+1}{n+a+1}. \tag{4}
\end{aligned}$$

References

- Andrews, G. E. "A Polynomial Identity which Implies the Rogers-Ramanujan Identities." *Scripta Math.* **28**, 297–05, 1970.
- Paule, P. "Short and Easy Computer Proofs of the Rogers-Ramanujan Identities and of Identities of Similar Type." *Electronic J. Combinatorics* **1**, R10 1–, 1994. http://www.combinatorics.org/Volume_1/volume1.html#R10.

Andrica's Conjecture

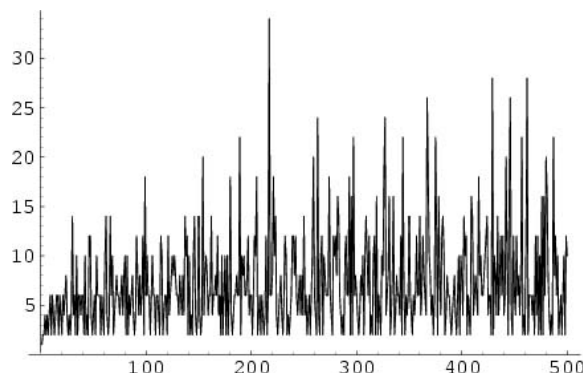


Andrica's conjecture states that, for p_n the n th PRIME NUMBER, the INEQUALITY

$$A_n \equiv \sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

holds, where the discrete function A_n is plotted above. The largest value among the first 1000 PRIMES is for

$n = 4$, giving $\sqrt{11} - \sqrt{7} \approx 0.670873$. Since the Andrica function falls asymptotically as n increases so a PRIME GAP of increasing size is needed at large n , it seems likely the CONJECTURE is true. However, it has not yet been proven.



A_n bears a strong resemblance to the PRIME DIFFERENCE FUNCTION, plotted above, the first few values of which are 1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, ... (Sloane's A001223).

A generalization of Andrica's conjecture considers the equation

$$p_{n+1}^x - p_n^x = 1$$

and solves for x . The smallest such x is $x \approx 0.567148$ (Sloane's A038458), known as the SMARANDACHE CONSTANT, which occurs for $p_n = 113$ and $p_{n+1} = 127$ (Perez).

See also BROCARD'S CONJECTURE, GOOD PRIME, FORTUNATE PRIME, PÓLYA CONJECTURE, PRIME DIFFERENCE FUNCTION, SMARANDACHE CONSTANTS, TWIN PEAKS

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Anger Differential Equation

The second-order ORDINARY DIFFERENTIAL EQUATION

$$y'' + \frac{y'}{x} + \left(1 - \frac{v^2}{x^2}\right)y = \frac{x-v}{\pi x^2} \sin(vx)$$

whose solutions are ANGER FUNCTIONS.

See also ANGER FUNCTION

References

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Anger Function

A generalization of the BESSEL FUNCTION OF THE FIRST KIND defined by

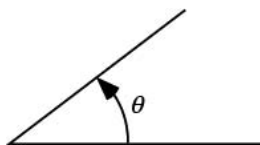
$$\mathcal{J}_v(z) \equiv \frac{1}{\pi} \int_0^\pi \cos(v\theta - z \sin \theta) d\theta.$$

If v is an INTEGER n , then $\mathcal{J}_n(z) = J_n(z)$, where $J_n(z)$ is a BESSEL FUNCTION OF THE FIRST KIND. Anger's original function had an upper limit of 2π , but the current NOTATION was standardized by Watson (1966).

See also ANGER DIFFERENTIAL EQUATION, BESSEL FUNCTION, MODIFIED STRUVE FUNCTION, PARABOLIC CYLINDER FUNCTION, STRUVE FUNCTION, WEBER FUNCTIONS

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Angle

Given two intersecting LINES or LINE SEGMENTS, the amount of ROTATION about the point of intersection (the VERTEX) required to bring one into correspondence with the other is called the angle θ between them. Angles are usually measured in DEGREES (denoted $^\circ$), RADIANS (denoted rad, or without a unit), or sometimes GRADIANS (denoted grad).

One full rotation in these three measures corresponds to 360° , 2π rad, or 400 grad. Half a full rotation is called a STRAIGHT ANGLE, and a QUARTER of a full rotation is called a RIGHT ANGLE. An angle less than a RIGHT ANGLE is called an ACUTE ANGLE, and an angle

greater than a RIGHT ANGLE is called an OBTUSE ANGLE.

The use of DEGREES to measure angles harks back to the Babylonians, whose SEXAGESIMAL number system was based on the number 60. 360° likely arises from the Babylonian year, which was composed of 360 days (12 months of 30 days each). The DEGREE is further divided into 60 ARC MINUTES, and an ARC MINUTE into 60 ARC SECONDS. A more natural measure of an angle is the RADIAN. It has the property that the ARC LENGTH around a CIRCLE is simply given by the radian angle measure times the CIRCLE RADIUS. The RADIAN is also the most useful angle measure in CALCULUS because the DERIVATIVE of TRIGONOMETRIC functions such as

$$\frac{d}{dx} \sin x = \cos x$$

does not require the insertion of multiplicative constants like $\pi/180$. GRADIANS are sometimes used in surveying (they have the nice property that a RIGHT ANGLE is exactly 100 GRADIANS), but are encountered infrequently, if at all, in mathematics.

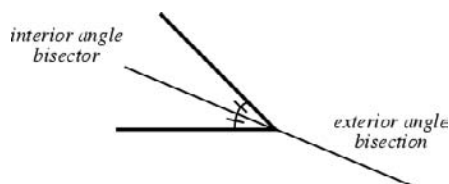
The concept of an angle can be generalized from the CIRCLE to the SPHERE. The fraction of a SPHERE subtended by an object is measured in STERADIANS, with the entire SPHERE corresponding to 4π STERADIANS.

A ruled SEMICIRCLE used for measuring and drawing angles is called a PROTRACTOR. A COMPASS can also be used to draw circular ARCS of some angular extent.

See also ACUTE ANGLE, ARC MINUTE, ARC SECOND, CENTRAL ANGLE, COMPLEMENTARY ANGLE, DEGREE, DIHEDRAL ANGLE, DIRECTED ANGLE, EULER ANGLES, EXTERIOR ANGLE, FULL ANGLE, GRADIAN, HORN ANGLE, INSCRIBED ANGLE, OBLIQUE ANGLE, OBTUSE ANGLE, PERIGON, PROTRACTOR, RADIAN, REFLEX ANGLE, RIGHT ANGLE, SOLID ANGLE, STERADIAN, STRAIGHT ANGLE, SUBTEND, SUPPLEMENTARY ANGLE, VERTEX ANGLE

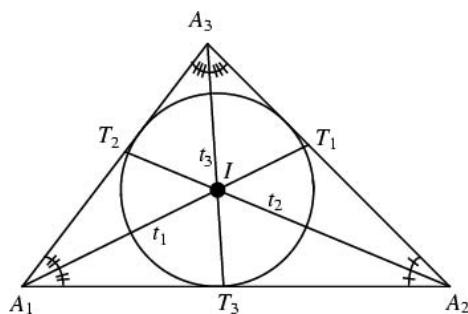
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- Harris, J. W. and Stocker, H. "Angle." §3.3 in *Handbook of Mathematics and Computational Science*. New York: Springer-Verlag, pp. 62–4, 1998.

Angle Bisector

The (interior) bisector of an ANGLE is the LINE or LINE

SEGMENT which cuts it into two equal ANGLES on the same "side" as the ANGLE.



The length of the bisector of ANGLE A_1 in the above TRIANGLE $\Delta A_1 A_2 A_3$ is given by

$$t_1^2 = a_2 a_3 \left[1 - \frac{a_1^2}{(a_2 + a_3)^2} \right],$$

where $t_i \equiv \overline{A_i T_i}$ and $a_i \equiv \overline{A_i A_k}$. The angle bisectors meet at the INCENTER I , which has TRILINEAR COORDINATES 1:1:1.

See also ANGLE BISECTOR THEOREM, CYCLIC QUADRANGLE, EXTERIOR ANGLE BISECTOR, ISODYNAMIC POINTS, ORTHOCENTRIC SYSTEM, STEINER-LEHMUS THEOREM, TRISECTION

References

- Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited*. Washington, DC: Math. Assoc. Amer., pp. 9–0, 1967.
 Dixon, R. *Mathographics*. New York: Dover, p. 19, 1991.
 Mackay, J. S. "Properties Concerned with the Angular Bisectors of a Triangle." *Proc. Edinburgh Math. Soc.* **13**, 37–02, 1895.

Angle Bisector Theorem

The ANGLE BISECTOR of an ANGLE in a TRIANGLE divides the opposite side in the same RATIO as the sides adjacent to the ANGLE.

Angle Bracket

The combination of a BRA and KET (bra + ket = bracket) which represents the INNER PRODUCT of two functions or vectors,

$$\langle f|g \rangle = \int f(x)g(x) dx$$

$$\langle \mathbf{v}|\mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w}.$$

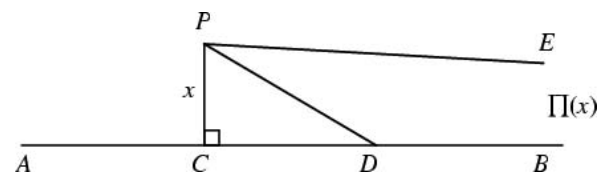
By itself, the BRA is a COVARIANT 1-VECTOR, and the KET is a CONTRAVARIANT ONE-FORM. These terms are commonly used in quantum mechanics.

See also BRA, BRACE, DIFFERENTIAL K -FORM, KET, ONE-FORM, PARENTHESIS, SQUARE BRACKET

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Angle of Parallelism



Given a point P and a LINE AB , draw the PERPENDICULAR through P and call it PC . Let PD be any other line from P which meets CB in D . In a HYPERBOLIC GEOMETRY, as D moves off to infinity along CB , then the line PD approaches the limiting line PE , which is said to be parallel to CB at P . The angle $\angle CPE$ which PE makes with PC is then called the angle of parallelism for perpendicular distance x , and is given by

$$\Pi(x) = 2 \tan^{-1}(e^{-x}).$$

This is known as LOBACHEVSKY'S FORMULA.

See also HYPERBOLIC GEOMETRY, LOBACHEVSKY'S FORMULA

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Angle Trisection

TRISECTION

Angle-Preserving Transformation

CONFORMAL MAPPING

Angular Acceleration

The angular acceleration α is defined as the time DERIVATIVE of the ANGULAR VELOCITY ω ,

$$\alpha \equiv \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \hat{\mathbf{z}} = \frac{\mathbf{a}}{r}.$$

See also ACCELERATION, ANGULAR DISTANCE, ANGULAR VELOCITY

Angular Defect

The DIFFERENCE between the SUM of face ANGLES A_i at a VERTEX of a POLYHEDRON and 2π ,

$$\delta = 2\pi - \sum_i A_i.$$

See also DESCARTES TOTAL ANGULAR DEFECT, JUMP ANGLE, SPHERICAL DEFECT

Angular Distance

The angular distance traveled around a CIRCLE is the number of RADIANS the path subtends,

$$\theta \equiv \frac{\ell}{2\pi r} 2\pi = \frac{\ell}{r}.$$

See also ANGULAR ACCELERATION, ANGULAR VELOCITY

Angular Velocity

The angular velocity ω is the time DERIVATIVE of the ANGULAR DISTANCE θ with direction $\hat{z}$ PERPENDICULAR to the plane of angular motion,

$$\omega \equiv \frac{d\theta}{dt} \hat{z} = \frac{\mathbf{v}}{r}.$$

See also ANGULAR ACCELERATION, ANGULAR DISTANCE

Anharmonic Ratio

CROSS-RATIO

Animal

1. A FIXED POLYOMINO.
2. The set of points obtained by taking the centers of a FIXED POLYOMINO.

See also POLYOMINO

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Anisohedral Tiling

A k -anisohedral tiling is a tiling which permits no n -ISOHEDRAL TILING with $n < k$.

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Annealing

SIMULATED ANNEALING

Annihilator

The term annihilator is used in several different ways in various aspects of mathematics. It is most com-

monly used to mean the SET of all functions satisfying a given set of conditions which is zero on every member of a given SET.

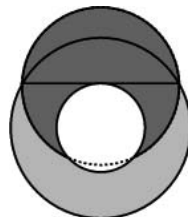
Annuity

PRESENT VALUE

Annulus

The region in common to two concentric CIRCLES of RADII a and b . The AREA of an annulus is

$$A_{\text{annulus}} = \pi(b^2 - a^2).$$



In the above figure, the area of the circle whose diameter is tangent to the inner circle and has endpoints at the outer circle is equal to the area of the annulus.

See also ANNULUS THEOREM, BULLSEYE ILLUSION, CHORD, CIRCLE, CONCENTRIC CIRCLES, LUNE, SPHERICAL SHELL

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Annulus Conjecture

ANNULUS THEOREM

Annulus Theorem

Let K_1^n and K_2^n be disjoint bicollared KNOTS in $\mathbb{R}^{n+1}$ or $\mathbb{S}^{n+1}$ and let U denote the open region between them. Then the closure of U is a closed annulus $\mathbb{S}^n \times [0, 1]$. Except for the case $n = 3$, the theorem was proved by Kirby (1969).

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- Rolfsen, D. *Knots and Links*. Wilmington, DE: Publish or Perish Press, p. 38, 1976.

Anomalous Cancellation

The simplification of a FRACTION a/b which gives a correct answer by "canceling" DIGITS of a and b . There are only four such cases for NUMERATOR and DENOMINATORS of two DIGITS in base 10: $64/16 =$

$4/1 = 4$, $98/49 = 8/4 = 2$, $95/19 = 5/1 = 5$, and $65/26 = 5/2$ (Boas 1979).

The concept of anomalous cancellation can be extended to arbitrary bases. PRIME bases have no solutions, but there is a solution corresponding to each PROPER DIVISOR of a COMPOSITE b . When $b - 1$ is PRIME, this type of solution is the only one. For base 4, for example, the only solution is $32_4/13_4 = 2_4$. Boas gives a table of solutions for $b \leq 39$. The number of solutions is EVEN unless b is an EVEN SQUARE.

b	N	b	N
4	1	26	4
6	2	27	6
8	2	28	10
9	2	30	6
10	4	32	4
12	4	34	6
14	2	35	6
15	6	36	21
16	7	38	2
18	4	39	6
20	4		
21	10		
22	6		
24	6		

See also FRACTION, PRINTER'S ERRORS, REDUCED FRACTION

References

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- Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers*. Middlesex, England: Penguin Books, pp. 26–7, 1986.

Anomalous Number

BENFORD'S LAW

Anonymous

A term in SOCIAL CHOICE THEORY meaning invariance of a result under permutation of voters.

See also DUAL VOTING, MONOTONIC VOTING

Anosov Automorphism

A HYPERBOLIC linear map $\mathbb{R}^n \rightarrow \mathbb{R}^n$ with INTEGER entries in the transformation MATRIX and DETERMINANT ± 1 is an ANOSOV DIFFEOMORPHISM of the n -TORUS, called an Anosov automorphism (or HYPERBOLIC AUTOMORPHISM). Here, the term automorphism is used in the GROUP THEORY sense.

Anosov Diffeomorphism

An Anosov diffeomorphism is a C^1 DIFFEOMORPHISM ϕ such that the MANIFOLD M is HYPERBOLIC with respect to ϕ . Very few classes of Anosov diffeomorphisms are known. The best known is ARNOLD'S CAT MAP.

A HYPERBOLIC linear map $\mathbb{R}^n \rightarrow \mathbb{R}^n$ with INTEGER entries in the transformation MATRIX and DETERMINANT ± 1 is an Anosov diffeomorphism of the n -TORUS. Not every MANIFOLD admits an Anosov diffeomorphism. Anosov diffeomorphisms are EXPANSIVE, and there are no Anosov diffeomorphisms on the CIRCLE.

It is conjectured that if $\phi: M \rightarrow M$ is an Anosov diffeomorphism on a COMPACT RIEMANNIAN MANIFOLD and the NONWANDERING SET $\Omega(\phi)$ of ϕ is M , then ϕ is TOPOLOGICALLY CONJUGATE to a FINITE-TO-ONE FACTOR of an ANOSOV AUTOMORPHISM of a NILMANIFOLD. It has been proved that any Anosov diffeomorphism on the n -TORUS is TOPOLOGICALLY CONJUGATE to an ANOSOV AUTOMORPHISM, and also that Anosov diffeomorphisms are C^1 STRUCTURALLY STABLE.

See also ANOSOV AUTOMORPHISM, AXIOM A DIFFEOMORPHISM, DYNAMICAL SYSTEM

References

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Anosov Flow

A FLOW defined analogously to the ANOSOV DIFFEOMORPHISM, except that instead of splitting the TANGENT BUNDLE into two invariant sub-BUNDLES, they are split into three (one exponentially contracting, one expanding, and one which is 1-dimensional and tangential to the flow direction).

See also DYNAMICAL SYSTEM

Anosov Map

An important example of a ANOSOV DIFFEOMORPHISM.

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix},$$

where x_{n+1} , y_{n+1} are computed mod 1.

See also ARNOLD'S CAT MAP

ANOVA

"Analysis of Variance." A STATISTICAL TEST for heterogeneity of MEANS by analysis of group VARIANCES. To apply the test, assume random sampling of a variate y with equal VARIANCES, independent errors, and a NORMAL DISTRIBUTION. Let n be the number of REPLICATES (sets of identical observations) within each of K FACTOR LEVELS (treatment groups), and y_{ij} be the j th observation within FACTOR LEVEL i . Also assume that the ANOVA is "balanced" by restricting n to be the same for each FACTOR LEVEL.

Now define the sum of square terms

$$SST \equiv \sum_{i=1}^k \sum_{j=1}^n (y_{ij} - \bar{y})^2 \quad (1)$$

$$= \sum_{i=1}^k \sum_{j=1}^n y_{ij}^2 - \frac{\left(\sum_{i=1}^k \sum_{j=1}^n y_{ij} \right)^2}{Kn} \quad (2)$$

$$SSA \equiv \frac{1}{n} \sum_{i=1}^k \left(\sum_{j=1}^n y_{ij} \right)^2 - \frac{1}{Kn} \left(\sum_{i=1}^k \sum_{j=1}^n y_{ij} \right)^2 \quad (3)$$

$$SSE \equiv \sum_{i=1}^k \sum_{j=1}^n (y_{ij} - \bar{y}_i)^2 \quad (4)$$

$$= SST - SSA, \quad (5)$$

which are the total, treatment, and error sums of squares. Here, $\bar{y}_i$ is the mean of observations within FACTOR LEVEL i , and $\bar{y}$ is the "group" mean (i.e., mean of means). Compute the entries in the following table, obtaining the P -VALUE corresponding to the calculated F -RATIO of the mean squared values

$$F = \frac{MSA}{MSE}. \quad (6)$$

Category	SS	°Freedom	Mean Squared	F -RATIO
Treatment	SSA	$K - 1$	$MSA \equiv \frac{SSA}{K - 1}$	$\frac{MSA}{MSE}$
Error	SSE	$K(n - 1)$	$MSE \equiv \frac{SSE}{K(n - 1)}$	
Total	SST	$Kn - 1$	$MST \equiv \frac{SST}{Kn - 1}$	

If the P -VALUE is small, reject the NULL HYPOTHESIS that all MEANS are the same for the different groups.

See also FACTOR LEVEL, MANOVA, REPLICATE, VARIANCE

References

Miller, R. G. *Beyond ANOVA: Basics of Applied Statistics*. Boca Raton, FL: Chapman & Hall, 1997.

Anthropomorphic Polygon

A SIMPLE POLYGON with precisely two EARS and one MOUTH.

References

Toussaint, G. "Anthropomorphic Polygons." *Amer. Math. Monthly* **122**, 31–5, 1991.

Anthyphairetic Ratio

An archaic term for a CONTINUED FRACTION.

References

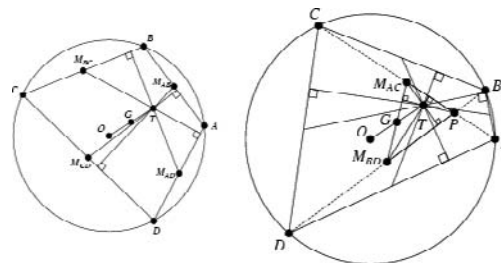
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Antiautomorphism

If a MAP $f : G \rightarrow G'$ from a GROUP G to a GROUP G' satisfies $f(ab) = f(a)f(b)$ for all $a, b \in G$, then f is said to be an antiautomorphism.

See also AUTOMORPHISM

Anticenter



The point of concurrence of the three MALTITUDES of a CYCLIC QUADRILATERAL. Let M_{AC} and M_{BD} be the MIDPOINTS of the diagonals of a CYCLIC QUADRILATERAL $ABCD$, and let P be the intersection of the diagonals. Then the ORTHOCENTER of TRIANGLE $\triangle P M_{AC} M_{BD}$ is the anticenter T of $ABCD$ (Honsberger 1995, p. 39).

See also CYCLIC QUADRILATERAL, MALTITUDE

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Anticevian Triangle

Given a center $\alpha : \beta : \gamma$, the anticevian triangle is defined as the TRIANGLE with VERTICES $-\alpha : \beta : \gamma$, $\alpha : -\beta : \gamma$, and $\alpha : \beta : -\gamma$. If $A'B'C'$ is the CEVIAN TRIANGLE of X and $A''B''C''$ is an anticevian triangle, then X and A'' are HARMONIC CONJUGATE POINTS with respect to A and A' .

See also CEVIAN TRIANGLE

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Antichain

Let P be a finite PARTIALLY ORDERED SET. An antichain in P is a set of pairwise incomparable elements (e.g., a family of SUBSETS such that, for any two of them, neither is a SUBSET of the other). Antichains are also called Sperner systems in older literature (Comtet 1974).

The following table gives the antichains on n -set $\{1, 2, \dots, n\}$ for small n .

n	antichains
1	$\emptyset, \{1\}$
2	$\emptyset, \{1\}, \{2\}, \{1, 2\}$
3	$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$
4	$\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$

The number of antichains on the n -set $\{1, 2, \dots, n\}$ for $n = 1, 2, \dots$, are 1, 2, 5, 19, 167, ... (Sloane's A014466). If the EMPTY SET is not considered a valid antichain, then these reduce to 0, 1, 4, 18, 166, ... (Sloane's A007153; Comtet 1974, p. 273). The numbers obtained by adding one to Sloane's A014466, 2, 3, 6, 20, 168, 7581, 7828354, ... (Sloane's A000372), are also frequently encountered (Speciner 1972).

The number of antichains on the n -set are equal to the number of monotonic increasing Boolean functions of n variables, and also the number of free distributive lattices with n generators (Comtet 1974, p. 273). Determining these numbers is known as DEDEKIND'S PROBLEM, and the numbers in each of these sequences are sometimes called Dedekind numbers (Sloane).

The WIDTH of P is the maximum CARDINALITY of an ANTICHAIN in P . For a PARTIAL ORDER, the size of the longest ANTICHAIN is called the WIDTH $w(P)$. Sperner (1928) proved that the maximum width of an antichain containing n elements is

$$w_{\max(n)} = \binom{n}{\lfloor n/2 \rfloor},$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT and $\lfloor n \rfloor$ is the FLOOR FUNCTION.

See also BOOLEAN FUNCTION, CHAIN, DILWORTH'S LEMMA, PARTIALLY ORDERED SET, WIDTH (PARTIAL ORDER)

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Anticlastic

When the GAUSSIAN CURVATURE K is everywhere NEGATIVE, a SURFACE is called anticlastic and is saddle-shaped. A SURFACE on which K is everywhere POSITIVE is called SYNCLASTIC. A point at which the GAUSSIAN CURVATURE is NEGATIVE is called a HYPERBOLIC POINT.

See also ELLIPTIC POINT, GAUSSIAN QUADRATURE, HYPERBOLIC POINT, PARABOLIC POINT, PLANAR POINT, SYNCLASTIC

Anticommutative

An OPERATOR $*$ for which $a * b = -b * a$ is said to be anticommutative.

See also COMMUTATIVE

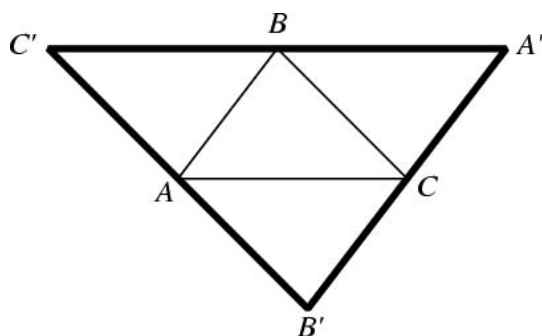
Anticommutator

For OPERATORS $\hat{A}$ and $\hat{B}$, the anticommutator is defined by

$$\{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}.$$

See also COMMUTATOR, JORDAN ALGEBRA, JORDAN PRODUCT

Anticomplementary Triangle



A TRIANGLE $\triangle A'B'C'$ which has a given TRIANGLE $\triangle ABC$ as its MEDIAL TRIANGLE. The TRILINEAR CO-

Antihomologous Points

ORDINATES of the anticomplementary triangle are

$$A' = -a^{-1} : b^{-1} : c^{-1}$$

$$B' = a^{-1} : -b^{-1} : c^{-1}$$

$$C' = a^{-1} : b^{-1} : -c^{-1}.$$

See also MEDIAL TRIANGLE

Anticross-Stitch Curve

BOX FRACTAL

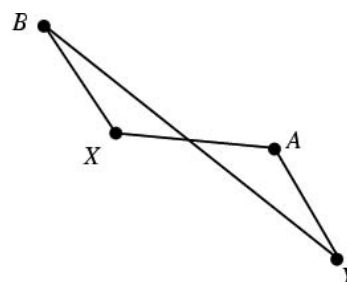
Antiderivative

INTEGRAL

Antidifferentiation

INTEGRATION

Antigonal Points



Given $\angle AXB + \angle AYB = \pi$ RADIANS in the above figure, then X and Y are said to be antigonal points with respect to A and B .

Antihomography

A CIRCLE-preserving TRANSFORMATION composed of an ODD number of INVERSIONS.

See also HOMOGRAPHY

Antihomologous Points

Two points which are COLLINEAR with respect to a SIMILITUDE CENTER but are not HOMOLOGOUS POINTS. Four interesting theorems from Johnson (1929) follow.

1. Two pairs of antihomologous points form inversely similar triangles with the HOMOTHETIC CENTER.
2. The PRODUCT of distances from a HOMOTHETIC CENTER to two antihomologous points is a constant.
3. Any two pairs of points which are antihomologous with respect to a SIMILITUDE CENTER lie on a CIRCLE.
4. The tangents to two CIRCLES at antihomologous points make equal ANGLES with the LINE through the points.

See also HOMOLOGOUS POINTS, HOMOTHETIC CENTER, SIMILITUDE CENTER

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*. Boston, MA: Houghton Mifflin, pp. 19–1, 1929.

Antilaplacian

The antilaplacian of u with respect to x is a function whose LAPLACIAN with respect to x equals u . The antilaplacian is never unique.

See also LAPLACIAN

Antilinear

An antilinear OPERATOR $\tilde{A}$ satisfies the following two properties:

$$\tilde{A}[f_1(x) + f_2(x)] = \tilde{A}f_1(x) + \tilde{A}f_2(x)$$

$$\tilde{A}cf(x) = \tilde{c}\tilde{A}f(x),$$

where $\tilde{c}$ is the COMPLEX CONJUGATE of c .

See also ANTIUNITARY, LINEAR OPERATOR

References

Sakurai, J. J. *Modern Quantum Mechanics*. Menlo Park, CA: Benjamin/Cummings, 1985.

Antilinear Operator

An antilinear OPERATOR

$$\tilde{\mathcal{L}}^*u = \int (y_2\tilde{\mathcal{L}}y_1 - y_1\tilde{\mathcal{L}}y_2) dx = \left[\frac{p_1}{p_0}(y_1'y_2 - y_1y_2') \right]$$

satisfies the following two properties:

$$PD = CB$$

$$D = PE$$

where $\angle CPE$ is the COMPLEX CONJUGATE of C_e .

See also ANTIUNITARY OPERATOR, LINEAR OPERATOR

References

Sakurai, J. J. *Modern Quantum Mechanics*. Menlo Park, CA: Benjamin/Cummings, 1985.

Antilogarithm

The INVERSE FUNCTION of the LOGARITHM, defined such that

$$\log_b(\text{antilog}_b z) = z = \text{antilog}_b(\log_b z).$$

The antilogarithm in base b of z is therefore b^z .

See also COLOGARITHM, LOGARITHM, POWER

Antimagic Graph

A GRAPH with e EDGES labeled with distinct elements $\{1, 2, \dots, c\}$ so that the SUM of the EDGE labels at each VERTEX differ.

See also LABELED GRAPH, MAGIC GRAPH

References

Hartsfield, N. and Ringel, G. *Pearls in Graph Theory: A Comprehensive Introduction*. San Diego, CA: Academic Press, 1990.

Antimagic Square

15	2	12	4
1	14	10	5
8	9	3	16
11	13	6	7

21	18	6	17	4
7	3	13	16	24
5	20	23	11	1
15	8	19	2	25
14	12	9	22	10

10	25	32	13	16	9
22	7	3	24	21	30
20	27	18	26	11	6
1	31	23	33	17	8
19	5	36	12	15	29
34	14	2	4	35	28

14	3	34	21	47	29	22
43	16	13	25	6	26	44
30	40	24	8	12	9	45
10	5	11	38	49	46	19
4	41	37	36	33	27	1
39	17	40	20	7	35	23
31	42	18	32	28	2	15

49	16	50	10	19	28	24	56
42	43	11	15	44	38	55	5
25	21	48	46	9	37	6	63
29	47	8	40	51	30	52	1
45	22	54	23	20	34	2	62
14	59	18	33	41	26	61	13
36	12	58	32	27	64	3	35
17	39	7	57	53	4	60	31

52	19	8	122	29	15	42	31	76
61	10	67	23	54	79	25	33	16
57	9	71	24	30	1	51	47	75
26	78	7	69	66	77	13	27	12
192	174	20	17	17	49	55	64	
8	65	4	62	50	34	73	41	40
56	68	2	63	14	72	35	44	6
53	30	60	32	36	3	16	43	58
117	0	5	59	48	80	28	45	18

An antimagic square is an $n \times n$ ARRAY of integers from 1 to n^2 such that each row, column, and main diagonal produces a different sum such that these sums form a SEQUENCE of consecutive integers. It is therefore a special case of a HETEROSQUARE. Antimagic squares of orders 4– are illustrated above (Madachy 1979). For the 4×4 square, the sums are 30, 31, 32, ..., 39; for the 5×5 square they are 59, 60, 61, ..., 70; and so on.

Let an antimagic square of order n have entries 0, 1, ..., $n^2 - 2$, $n^2 - 1$, and let

$$M(n) \equiv \frac{1}{2} n(n^2 + 1)$$

be the magic constant. Then if an antimagic square of order n exists, it is either positive with sums $[M(n) - n, M(n) + n + 1]$, or negative with sums $[M(n) - n - 1, M(n) + n]$ (Madachy 1979).

Antimagic squares of orders one, two, and three are impossible. In the case of the 3×3 square, there is no known method of proof of this fact except by case analysis or enumeration by computer. There are 18 families of antimagic squares of order four. The total

number of antimagic squares of orders 1, 2, ... modulo the full group of symmetries (reflection, rotation, complementation, and exchanges) are 0, 0, 0, 299710, ... (Sloane's A050257; Cormie).

Abe (1994) and Madachy (1979) ask for methods of constructing antimagic squares of every order. Recently, J. Cormie and V. Linek have developed general constructions for squares of order n for all $n > 3$, as well as for bordering antimagic squares.

See also HETEROSQUARE, MAGIC SQUARE, TALISMAN SQUARE

References

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Antimorph

A number which can be represented both in the form $x_0^2 - Dy_0^2$ and in the form $Dx_1^2 - y_1^2$. This is only possible when the PELL EQUATION

$$x^2 - Dy^2 = -1$$

is solvable. Then

$$\begin{aligned} x^2 - Dy^2 &= -(x_0 - Dy_0^2)(x_n^2 - Dy_n^2) \\ &= D(x_0y_n - y_0x_n)^2 - (x_0x_n - Dy_0y_n)^2. \end{aligned}$$

See also IDONEAL NUMBER, POLYMORPH

References

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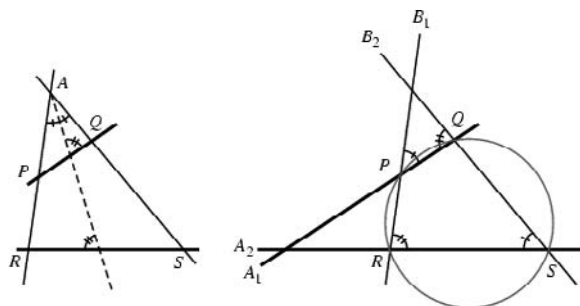
Antimorphic Number

ANTIMORPH

Antinomy

A PARADOX or contradiction.

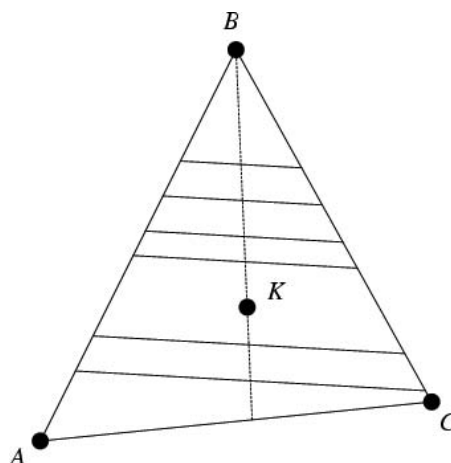
Antiparallel



Two lines PQ and RS are said to be antiparallel with respect to the sides of an ANGLE A if they make the same angle in the opposite senses with the BISECTOR of that angle. If PQ and RS are antiparallel with respect to PR and QS , then the latter are also antiparallel with respect to the former. Furthermore, if PQ and RS are antiparallel, then the points P , Q , R , and S are CONCYCLIC (Johnson 1929, p. 172; Honsberger 1995, pp. 87–8).

There are a number of fundamental relationships involving a triangle and antiparallel lines (Johnson 1929, pp. 172–73).

1. The line joining the feet to two ALTITUDES of a triangle is antiparallel to the third side.
2. The tangent to a triangle's CIRCUMCIRCLE at a vertex is antiparallel to the opposite side.
3. The radius of the CIRCUMCIRCLE at a vertex is perpendicular to all lines antiparallel to the opposite sides.



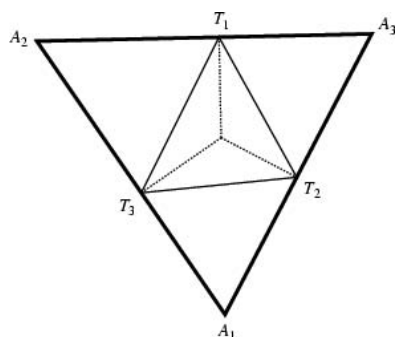
In a TRIANGLE $\triangle ABC$, a SYMMEDIAN BK bisects all segments antiparallel to a given side AC (Honsberger 1995, p. 88). Furthermore, every antiparallel to BC in $\triangle ABC$ is PARALLEL to the tangent to the CIRCUMCIRCLE of $\triangle ABC$ at A (Honsberger 1995, p. 98).

See also ANGLE, CONCYCLIC, COSINE CIRCLE, COSINE HEXAGON, HYPERPARALLEL, LEMOINE CIRCLE, LEMOINE HEXAGON, PARALLEL, TUCKER CIRCLES, TUCKER HEXAGON

References

- Casey, J. "Theory of Isogonal and Isotomic Points, and of Antiparallel and Symmedian Lines." Supp. Ch. §1 in *A Sequel to the First Six Books of the Elements of Euclid, Containing an Easy Introduction to Modern Geometry with Numerous Examples*, 5th ed., rev. enl. Dublin: Hodges, Figgis, & Co., pp. 165–73, 1888.
- Coolidge, J. L. *A Treatise on the Geometry of the Circle and Sphere*. New York: Chelsea, p. 65, 1971.
- Honsberger, R. "Parallels and Antiparallels." §9.1 in *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*. Washington, DC: Math. Assoc. Amer., pp. 87–8, 1995.
- Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*. Boston, MA: Houghton Mifflin, p. 172, 1929.
- Lachlan, R. §113 in *An Elementary Treatise on Modern Pure Geometry*. London: Macmillan, p. 63, 1893.
- Phillips, A. W. and Fisher, I. *Elements of Geometry*. New York: American Book Co., 1896.

Antipedal Triangle



The antipedal triangle A of a given triangle T is the triangle of which T is the pedal triangle. For a triangle with trilinear coordinates $\alpha : \beta : \gamma$ and angles A , B , and C , the antipedal triangle has vertices with trilinear coordinates

$$\begin{aligned}
 &-(\beta + \alpha \cos C)(\gamma + \alpha \cos B) : (\gamma + \alpha \cos B)(\alpha + \beta \cos C) : \\
 &\quad (\beta + \alpha \cos C)(\alpha + \gamma \cos B) \\
 &(\gamma + \beta \cos A)(\beta + \alpha \cos C) : -(\gamma + \beta \cos A)(\alpha + \beta \cos C) : \\
 &\quad (\alpha + \beta \cos C)(\beta + \gamma \cos A) \\
 &(\beta + \gamma \cos A)(\gamma + \alpha \cos B) : (\alpha + \gamma \cos B)(\gamma + \beta \cos A) : \\
 &\quad -(\alpha + \gamma \cos B)(\beta + \gamma \cos A) :
 \end{aligned}$$

The ISOGONAL CONJUGATE of the ANTIPEDAL TRIANGLE of a given TRIANGLE is HOMOTHETIC with the original TRIANGLE. Furthermore, the PRODUCT of their AREAS

equals the SQUARE of the AREA of the original TRIANGLE (Gallatly 1913).

See also PEDAL TRIANGLE

References

- Gallatly, W. *The Modern Geometry of the Triangle*, 2nd ed. London: Hodgson, pp. 56–8, 1913.

Antipersistent Process

A FRACTAL PROCESS for which $H < 1/2$, so $r < 0$.

See also PERSISTENT PROCESS

Antipodal Map

The MAP which takes points on the surface of a SPHERE $\mathbb{S}^2$ to their ANTIPODAL POINTS.

Antipodal Points

Two points are antipodal (i.e., each is the ANTIPODE of the other) if they are diametrically opposite. Examples include endpoints of a LINE SEGMENT, or poles of a SPHERE. Given a point on a SPHERE with LATITUDE δ and LONGITUDE λ , the antipodal point has LATITUDE $-\delta$ and LONGITUDE $\lambda \pm 180^\circ$ (where the sign is taken so that the result is between -180° and $+180^\circ$).

See also ANTIPODE, BORSUK-ULAM THEOREM, DIAMETER, GREAT CIRCLE, LYUSTERNIK-SCHNIRELMANN THEOREM, METEOROLOGY THEOREM, SPHERE

Antipode

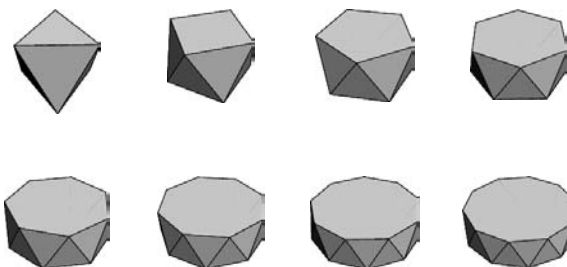
Given a point A , the point B which is the ANTIPODAL POINT of A is said to be the antipode of A .

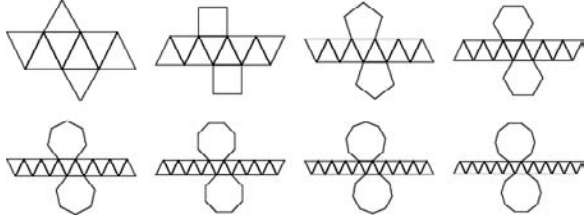
See also ANTIPODAL POINTS

References

- Tietze, H. *Famous Problems of Mathematics: Solved and Unsolved Mathematics Problems from Antiquity to Modern Times*. New York: Graylock Press, p. 25, 1965.

Antiprism





A SEMIREGULAR POLYHEDRON constructed with 2 n -gons and $2n$ TRIANGLES. The nets are particularly simple, consisting of two n -gons on top and bottom, separated by a ribbon of $2n$ triangles, with the two n -gons being offset by one ribbon segment.

The SAGITTA of a regular n -gon of side length a has length

$$s = \frac{1}{2} a \tan\left(\frac{\pi}{2n}\right) \quad (1)$$

Let d be the length of a lateral edge when the top and bottom bases separated by a distance h , then

$$s^2 + \left(\frac{1}{2}a\right)^2 + h^2 = d^2, \quad (2)$$

so

$$d = \frac{1}{2} \sqrt{4h^2 + a^2 \sec^2\left(\frac{\pi}{2n}\right)}. \quad (3)$$

For an antiprism of side lengths 1, $a = d = 1$, and solving for h gives

$$h = \sqrt{1 - \frac{1}{4} \sec^2\left(\frac{\pi}{2n}\right)}. \quad (4)$$

The CIRCUMRADIUS R_{circ} of an antiprism is given by

$$R_{\text{circ}} = \sqrt{\left(\frac{1}{2}h\right)^2 + R^2} = \frac{1}{4} \sqrt{4 \csc^2\left(\frac{\pi}{2n}\right)}, \quad (5)$$

where

$$R = \frac{1}{2} \csc\left(\frac{\pi}{n}\right) \quad (6)$$

is the CIRCUMRADIUS of one of the bases.

The TETRAHEDRON can be considered a degenerate 2-antiprism and the 3-antiprism of height $\sqrt{6}a/3$ (for side length a) is simply the OCTAHEDRON. The first few heights h_n producing unit antiprisms for $a = 1$ are

$$h_3 = \frac{1}{2} \sqrt{6} \quad (7)$$

$$h_4 = 2^{1/4} \quad (8)$$

$$h_5 = \sqrt{\frac{1}{10}(5 + \sqrt{5})} \quad (9)$$

$$h_6 = \sqrt{\sqrt{3} - 1} \quad (10)$$

$$h_8 = \sqrt{\sqrt{5 + \frac{7}{2}\sqrt{2}} - 1 - \sqrt{2}}. \quad (11)$$

The DUALS are the TRAPEZOHEDRA. The SURFACE AREA of a n -gonal antiprism is

$$\begin{aligned} S &= 2A_{n\text{-gon}} + 2nA_{\Delta} \\ &= 2 \left[\frac{1}{4} na^2 \cot\left(\frac{\pi}{n}\right) \right] + 2n \left(\frac{1}{2} a \right) \sqrt{s^2 + h^2} \\ &= \frac{1}{2} na \left[a \cot\left(\frac{\pi}{n}\right) + 2 \sqrt{h^2 + \frac{1}{4} a^2 \tan^2\left(\frac{\pi}{2n}\right)} \right]. \end{aligned} \quad (12)$$

If $h = a$, this simplifies to

$$S = \frac{1}{2} na^2 \left[\cot\left(\frac{\pi}{n}\right) + \sqrt{3} \right]. \quad (13)$$

The first few are

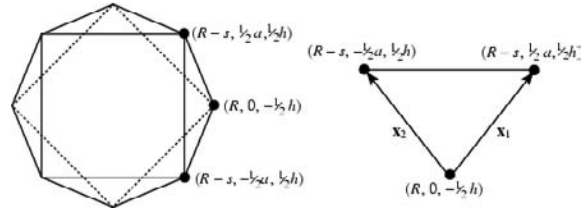
$$S_3 = 2\sqrt{3} \quad (14)$$

$$S_4 = 2(1 + \sqrt{3}) \quad (15)$$

$$S_5 = \frac{1}{2} \left(5\sqrt{3} + \sqrt{25 + 10\sqrt{5}} \right) \quad (16)$$

$$S_6 = 6\sqrt{3} \quad (17)$$

$$S_8 = 4(1 + \sqrt{2} + \sqrt{3}). \quad (18)$$



To find the volume, label vertices as in the above figure. Then the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are given by

$$\mathbf{v}_1 = (-s, \frac{1}{2}a, h) \quad (19)$$

$$\mathbf{v}_2 = (-s, -\frac{1}{2}a, h), \quad (20)$$

so the normal to one of the lateral facial planes is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (ah, 0, as), \quad (21)$$

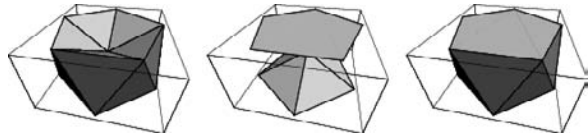
and the unit normal is

$$\begin{aligned} \hat{\mathbf{n}} &= \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|} \\ &= \left(\frac{ah}{\sqrt{a^2(h^2 + s^2)}}, 0, \frac{as}{\sqrt{a^2(h^2 + s^2)}} \right). \end{aligned} \quad (22)$$

The height of a pyramid with apex at the center and having the triangle determined by $\mathbf{x}_1$ and $\mathbf{x}_2$ as the base is then given by the projection of a vector from the origin to a point on the plane onto the normal,

$$\begin{aligned} h_{\text{pyr}} &= \hat{u} \cdot (R - s, -\frac{1}{2}a, \frac{1}{2}h) = \hat{u} \cdot (R - s, -\frac{1}{2}a, \frac{1}{2}h) \\ &= \hat{u} \cdot (R, 0, \frac{1}{2}h) \end{aligned} \quad (23)$$

$$\begin{aligned} &= \frac{a^2 h \cot\left(\frac{\pi}{2n}\right)}{4\sqrt{a^2\left[h^2 + \frac{1}{4}a^2 \tan^2\left(\frac{\pi}{2n}\right)\right]}}. \end{aligned} \quad (24)$$



The total volume of the $2n$ pyramids having the lateral faces as bases is therefore

$$\begin{aligned} V_{\text{pyr}} &= (2n) \left[\frac{1}{3} h_{\text{pyr}} \left(\frac{1}{2} a \sqrt{s^2 + h^2} \right) \right] \\ &= \frac{1}{12} a^2 h \cot\left(\frac{\pi}{2n}\right) \end{aligned} \quad (25)$$

Plugging in h and setting $a = 1$ gives

$$V_{\text{pyr}} = \frac{1}{12} n \cot\left(\frac{\pi}{2n}\right) \sqrt{1 - \frac{1}{4} \sec^2\left(\frac{\pi}{2n}\right)}. \quad (26)$$

The two pyramids having the upper and lower surfaces as bases contribute a volume

$$\begin{aligned} V_{\text{hase}} &= 2\left(\frac{1}{2}\right) \left(\frac{1}{2}h\right) \left[\frac{1}{4} na^2 \cot\left(\frac{\pi}{n}\right) \right] \\ &= \frac{1}{12} na^2 h \cot\left(\frac{\pi}{n}\right). \end{aligned} \quad (27)$$

Combining the two, setting $a = 1$, and plugging in the height h to get unit lateral edges gives the total volume as the somewhat complicated expression

$$V = \frac{1}{12} n \left[\cot\left(\frac{\pi}{2n}\right) + \cot\left(\frac{\pi}{n}\right) \right] \sqrt{1 - \frac{1}{4} \sec^2\left(\frac{\pi}{2n}\right)}. \quad (28)$$

The volumes of the first few unit antiprisms are therefore given by

$$V_3 = \frac{1}{3} \sqrt{2} \quad (29)$$

$$V_4 = \frac{1}{3} \sqrt{4 + 3\sqrt{2}} \quad (30)$$

$$V_5 = \frac{1}{6}(5 + 2\sqrt{5}) \quad (31)$$

$$v_6 = \sqrt{2(1 + \sqrt{3})} \quad (32)$$

See also GYROELONGATED PYRAMID, OCTAHEDRON, PRISM, PRISMOID, TRAPEZOHEDRON

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Antiquity

GEOMETRIC PROBLEMS OF ANTIQUITY

Antiset

A SET which transforms via converse functions. Antisets usually arise in the context of CHU SPACES.

See also CHU SPACE, SET

References

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Antisnowflake

KOCH ANTISNOWFLAKE

Antisphere

PSEUDOSPHERE

Antisquare Number

A number OF THE FORM $p^a \cdot A$ is said to be an antisquare if it fails to be a SQUARE NUMBER for the two reasons that a is ODD and A is a nonsquare modulo p .

See also SQUARE NUMBER, SQUAREFREE, SQUAREFUL

Antisymmetric

A quantity which changes SIGN when indices are reversed. For example, $A_{ij} \equiv a_i - a_j$ is antisymmetric since $A_{ij} = -A_{ji}$.

See also ANTISYMMETRIC MATRIX, ANTISYMMETRIC TENSOR, SYMMETRIC

Antisymmetric Matrix

An antisymmetric matrix is a MATRIX which satisfies the identity

$$A = -A^T \quad (1)$$

where A^T is the matrix TRANSPOSE. A matrix m may

be tested to see if it is antisymmetric using the *Mathematica* function

```
AntisymmetricQ[m_List?MatrixQ] := (m == -
Transpose[m])
```

In component notation, this becomes

$$a_{ij} = -a_{ji}. \quad (2)$$

Letting $k = i = j$, the requirement becomes

$$a_{kk} = -a_{kk}, \quad (3)$$

so an antisymmetric matrix must have zeros on its diagonal. The general 3×3 antisymmetric matrix is OF THE FORM

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix}. \quad (4)$$

Applying A^{-1} to both sides of the antisymmetry condition gives

$$-A^{-1}A^T = 1. \quad (5)$$

Any SQUARE MATRIX can be expressed as the sum of symmetric and antisymmetric parts. Write

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T). \quad (6)$$

But

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad (7)$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}, \quad (8)$$

so

$$A + A^T = \begin{bmatrix} 2a_{11} & a_{12} + a_{21} & \cdots & a_{1n} + a_{n1} \\ a_{12} + a_{21} & 2a_{22} & \cdots & a_{2n} + a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} + a_{n1} & a_{2n} + a_{n2} & \cdots & 2a_{nn} \end{bmatrix}, \quad (9)$$

which is symmetric, and

$$A - A^T = \begin{bmatrix} 0 & a_{12} - a_{21} & \cdots & a_{1n} - a_{n1} \\ -(a_{12} - a_{21}) & 0 & \cdots & a_{2n} - a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -(a_{1n} - a_{n1}) & -(a_{2n} - a_{n2}) & \cdots & 0 \end{bmatrix}, \quad (10)$$

which is antisymmetric.

See also SKEW SYMMETRIC MATRIX, SYMMETRIC MATRIX

Antisymmetric Relation

A RELATION R on a SET S is antisymmetric provided that distinct elements are never both related to one

another. In other words xRy and yRx together imply that $x = y$.

Antisymmetric Tensor

An antisymmetric (also called alternating) tensor is a TENSOR which changes sign when two indices are switched. For example, a tensor $A^{x_1, \dots, x_n}$ such that

$$A^{x_1, \dots, x_i, \dots, x_j, \dots, x_n} = -A^{x_1, \dots, x_j, \dots, x_i, \dots, x_n} \quad (1)$$

is antisymmetric.

The simplest nontrivial antisymmetric tensor is therefore an antisymmetric rank-2 tensor, which satisfies

$$A^{mn} = -A^{nm}. \quad (2)$$

Furthermore, any rank-2 TENSOR can be written as a sum of SYMMETRIC and antisymmetric parts as

$$A^{mn} = \frac{1}{2}(A^{mn} + A^{nm}) + \frac{1}{2}(A^{mn} - A^{nm}). \quad (3)$$

The antisymmetric part of a tensor A^{ab} is sometimes denoted using the special notation

$$A^{[ab]} = \frac{1}{2}(A^{ab} - A^{ba}). \quad (4)$$

For a general rank- n TENSOR,

$$A^{[a_1 \dots a_n]} \equiv \frac{1}{n!} \epsilon_{a_1 \dots a_n} \sum_{\text{permutations}} A^{a_1 \dots a_n}, \quad (5)$$

where $\epsilon_{a_1 \dots a_n}$ is the PERMUTATION SYMBOL. Symbols for the symmetric and antisymmetric parts of tensors can be combined, for example

$$T_{[d]}^{(ab)c} = \frac{1}{4}(T_{de}^{abc} + T_{de}^{bac} - T_{ed}^{abc} - T_{ed}^{bac}). \quad (6)$$

(Wald 1984, p. 26).

See also ALTERNATING MULTILINEAR FORM, EXTERIOR ALGEBRA, SYMMETRIC TENSOR, WEDGE PRODUCT

References

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Antiunitary

An operator $\hat{A}$ which satisfies:

$$\langle \hat{A}f_1 | \hat{A}f_2 \rangle = \overline{\langle f_1 | f_2 \rangle}$$

$$\hat{A}[f_1(x) + f_2(x)] = \hat{A}f_1(x) + \hat{A}f_2(x)$$

$$\hat{A}cf(x) = \tilde{c}\hat{A}f(x),$$

where $\langle f | g \rangle$ is the INNER PRODUCT and $\tilde{c}$ is the COMPLEX CONJUGATE of c .

See also ANTILINEAR, UNITARY

References

Sakurai, J. J. *Modern Quantum Mechanics*. Menlo Park, CA: Benjamin/Cummings, 1985.

Antiunitary Operator

An operator $\bar{B}$ which satisfies:

$$2\sqrt{3} = S_4$$

$$\pm 1 = C^1$$

$$\phi : M \rightarrow M = \Omega(\phi)$$

where $2(1 + \sqrt{3})$ is the INNER PRODUCT and $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ is the COMPLEX CONJUGATE of C_ϵ .

See also ANTILINEAR OPERATOR, UNITARY OPERATOR

References

Sakurai, J. J. *Modern Quantum Mechanics*. Menlo Park, CA: Benjamin/Cummings, 1985.

Antoine's Horned Sphere

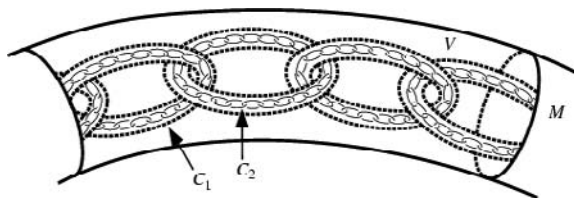
A topological 2-sphere in 3-space whose exterior is not SIMPLY CONNECTED. The outer complement of Antoine's horned sphere is not SIMPLY CONNECTED. Furthermore, the group of the outer complement is not even finitely generated. Antoine's horned sphere is inequivalent to ALEXANDER'S HORNED SPHERE since the complement in $\mathbb{R}^3$ of the bad points for ALEXANDER'S HORNED SPHERE is SIMPLY CONNECTED.

See also ALEXANDER'S HORNED SPHERE

References

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Antoine's Necklace



Construct a chain C of $2n$ components in a solid TORUS V . Now form a chain C_1 of $2n$ solid tori in V , where

$$\pi_1(V - C_1) \cong \pi_1(V - C)$$

via inclusion. In each component of C_1 , construct a smaller chain of solid tori embedded in that component. Denote the union of these smaller solid tori C_2 . Continue this process a countable number of times,

then the intersection

$$A = \bigcap_{i=1}^{\infty} C_i$$

which is a nonempty compact SUBSET of $\mathbb{R}^3$ is called Antoine's necklace. Antoine's necklace is HOMEOMORPHIC with the CANTOR SET.

See also ALEXANDER'S HORNED SPHERE, NECKLACE

References

Rolfsen, D. *Knots and Links*. Wilmington, DE: Publish or Perish Press, pp. 73–4, 1976.

Apairogon

The REGULAR POLYGON essentially equivalent to the CIRCLE having an infinite number of sides and denoted with SCHLÄFLI SYMBOL $\{\infty\}$.

See also CIRCLE, REGULAR POLYGON

References

Coxeter, H. S. M. *Regular Polytopes*, 3rd ed. New York: Dover, 1973.
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Apéry Number

The numbers defined by

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \frac{[(n+k)!]^2}{(k!)^4 [(n-k)!]^2}, \quad (1)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. The first few for $n = 0, 1, 2, \dots$ are 1, 5, 73, 1445, 33001, 819005, ... (Sloane's A005259). They are also given by the RECURRENCE RELATION

$$a_n = \frac{(34n^3 - 51n^2 + 27n - 5)a_{n-1} + (n-1)^3 a_{n-2}}{n^3} \quad (2)$$

(Beukers 1987). There is also an associated set of numbers

$$B_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k} \quad (3)$$

(Beukers 1987). The values for $n = 0, 1, \dots$ are 1, 3, 19, 147, 1251, 11253, 104959, ... (Sloane's A005258).

Both A_n and B_n arose in Apéry's irrationality proof of $\zeta(2)$ and $\zeta(3)$ (van der Poorten 1979, Beukers 1987). They satisfy some surprising congruence properties,

$$A_{mp^r-1} \equiv A_{mp^{r-1}-1} \pmod{p^{3r}} \quad (4)$$

$$B_{mp^r-1} \equiv B_{mp^{r-1}-1} \pmod{p^{3r}} \quad (5)$$

for p a PRIME ≥ 5 and $m, r \in \mathbb{N}$ (Beukers 1985, 1987), as well as

$$B_{(p-1)/2} \equiv \begin{cases} 4a^2 - 2p \pmod{p} & \text{if } p = a^2 + b^2, a \text{ odd} \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

(Stienstra and Beukers 1985, Beukers 1987). Defining γ_n from the GENERATING FUNCTION

$$\sum_{n=1}^{\infty} \gamma_n q^n = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 \quad (6)$$

gives γ_n of 1, -4, -2, 24, -11, -44, ... (Sloane's A030211; Koike 1984) for $n = 1, 3, 5, \dots$, and

$$A_{(p-1)/2} \equiv \gamma_p \pmod{p} \quad (7)$$

for p an ODD PRIME (Beukers 1987). Furthermore, for p an ODD PRIME and $m, r \in \mathbb{N}$,

$$A_{(mp^r-1)/2} - \gamma_p A_{(mp^{r-1}-1)/2} + p^3 A_{(mp^{r-2}-1)/2} \equiv 0 \pmod{p^r} \quad (8)$$

(Beukers 1987). The Apéry numbers are given by the diagonal elements $A_n = A_{nn}$ in the identity

$$\begin{aligned} A_{mn} &= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \binom{m}{k}^2 \binom{n}{j}^2 \binom{2m+n-j-k}{2m} \\ &= \sum_{k=-\infty}^{\infty} \binom{m+n-k}{k}^2 \binom{m+n-2k}{m-k}^2 \\ &= \sum_{k=-\infty}^{\infty} \binom{m}{k} \binom{n}{k} \binom{m+k}{k} \binom{n+k}{k} \end{aligned} \quad (9)$$

(Koepf 1998, p. 119).

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Apéry's Constant

N.B. A detailed online essay by S. Finch was the starting point for this entry. Apéry's constant is defined by

$$\zeta(3) = 1.2020569\dots, \quad (1)$$

(Sloane's A002117) where $\zeta(z)$ is the RIEMANN ZETA FUNCTION. Apéry (1979) proved that $\zeta(3)$ is IRRATIONAL, although it is not known if it is TRANSCENDENTAL. Sorokin (1994) and Nesterenko (1996) subsequently constructed independent proofs for the irrationality of $\zeta(3)$ (Hata 2000). $\zeta(3)$ arises naturally in a number of physical problems, including in the second- and third-order terms of the electron's gyro-magnetic ratio, computed using quantum electrodynamics.

The CONTINUED FRACTION for $\zeta(3)$ is [1, 4, 1, 18, 1, 1, 1, 4, 1, ...] (Sloane's A013631). The positions at which the numbers 1, 2, ... occur in the continued fraction are 1, 12, 25, 2, 64, 27, 17, 140, 10, ... (Sloane's A033165). The incrementally maximal terms are 1, 4, 18, 30, 428, 458, 527, ... (Sloane's A033166), which occur at positions 1, 2, 4, 29, 63, 572, ... (Sloane's A033167).

The following table summarized progress in computing upper bounds on the IRRATIONALITY MEASURE for $\zeta(3)$. Here, the exact values for two of the numerical bounds are given by

$$\mu_1 = 1 + \frac{6 \ln c_0 + d_0}{6 \ln c_0 - d_0} \approx 7.377956 \quad (2)$$

$$\mu_4 = 1 + \frac{4 \ln(\sqrt{2} + 1) + 3}{4 \ln(\sqrt{2} + 1) - 3} \approx 13.4178202, \quad (3)$$

where

$$c_0 = \frac{1}{9}(362 + 133\sqrt{7}) \quad (4)$$

$$d_0 = 26 + \pi \left[\sqrt{3} - \cot(\frac{1}{9}\pi) - \cot(\frac{2}{9}\pi) \right] \quad (5)$$

(Hata 2000).

index	upper bound	reference
1	7.377956	Hata (2000)
2	8.830284	Hata (1990)
3	12.74359	Dvornicich and Viola (1987)
4	13.41782	Sorokin (1994), Nesterenko (1996), Prévost (1996)

Beukers (1979) reproduced Apéry's rational approximation to $\zeta(3)$ using the triple integral of the form

$$\int_0^1 \int_0^1 \int_0^1 \frac{L_n(x)L_n(y)}{1 - (1 - xy)u} dx dy du, \quad (6)$$

where $L_n(x)$ is a LEGENDRE POLYNOMIAL. This integral is closely related to $\zeta(3)$ using the curious identity

$$\begin{aligned} & \int_0^1 \int_0^1 \int_0^1 \frac{x^r y^s}{1 - (1 - xy)u} dx dy du \\ &= \begin{cases} 2\zeta(3) - \sum_{l=1}^r \frac{2}{l^3} & \text{for } r=s \\ \frac{1}{\sum_{l=\min(r,s)+1}^{\max(r,s)} |r-s|l^2} & \text{for } r \neq s \end{cases} \\ &= \begin{cases} 2\zeta(3) - H_r^{(3)} & \text{for } r=s \\ \frac{\psi_1(1 + \min(r,s)) - \psi_1(1 + \max(r,s))}{|r-s|} & \text{for } r \neq s, \end{cases} \end{aligned}$$

where $H_r^{(n)}$ is a generalized HARMONIC NUMBER and $\psi_k(x)$ is a POLYGAMMA FUNCTION (Hata 2000).

Sums related to $\zeta(3)$ are

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (k!)^2}{(2k)! k^3} \quad (7)$$

(used by Apéry), and

$$\lambda(3) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} = \frac{7}{8}\zeta(3) \quad (8)$$

$$\sum_{k=0}^{\infty} \frac{1}{(3k+1)^3} = \frac{2\pi^3}{81\sqrt{3}} + \frac{13}{27}\zeta(3) \quad (9)$$

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)^3} = \frac{\pi^3}{64} + \frac{7}{16}\zeta(3) \quad (10)$$

$$\sum_{k=0}^{\infty} \frac{1}{(6k+1)^3} = \frac{\pi^3}{36\sqrt{3}} + \frac{91}{216}\zeta(3), \quad (11)$$

where $\lambda(z)$ is the DIRICHLET LAMBDA FUNCTION. The above equations are special cases of a general result due to Ramanujan (Berndt 1985). Apéry's proof relied on showing that the sum

$$a(n) \equiv \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2, \quad (12)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT, satisfies the RECURRENCE RELATION

$$\begin{aligned} (n+1)^3 a(n+1) - (34n^3 + 51n^2 + 27n + 5)a(n) \\ + n^3 a(n-1) = 0 \end{aligned} \quad (13)$$

(van der Poorten 1979, Zeilberger 1991). The characteristic polynomial $x^2 - 34x + 1$ has roots $(1 + \pm\sqrt{2})^4$, so

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = (1 + \sqrt{2})^4 \quad (14)$$

is irrational and a_n cannot satisfy a two-term recurrence (Jin and Dickinson 2000).

Apéry's constant is also given by

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{S_{n,2}}{n!n}, \quad (15)$$

where $S_{n,m}$ is a STIRLING NUMBER OF THE FIRST KIND. This can be rewritten as

$$\zeta(3) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{n^2}, \quad (16)$$

where H_n is the n th HARMONIC NUMBER (Castellanos 1988).

INTEGRALS for $\zeta(3)$ include

$$\zeta(3) = \frac{1}{2} \int_0^{\infty} \frac{t^2}{e^t - 1} dt \quad (17)$$

$$= \frac{8}{7} \left[\frac{1}{4}\pi^2 \ln 2 + 2 \int_0^{\pi/4} x \ln(\sin x) dx \right]. \quad (18)$$

Gosper (1990) gave

$$\zeta(3) = \frac{1}{4} \sum_{k=1}^{\infty} \frac{30k - 11}{(2k-1)k^3 \binom{2k}{k}}. \quad (19)$$

A CONTINUED FRACTION involving Apéry's constant is

$$\frac{6}{\zeta(3)} = 5 - \frac{1^6}{117 -} \frac{2^6}{535 -} \dots \frac{n^6}{34n^3 + 51n^2 + 27n + 5 -} \dots \quad (20)$$

(Apéry 1979, Le Lionnais 1983). Amdeberhan (1996) used WILF-ZEILBERGER PAIRS (F, G) with

$$F(n, k) = \frac{(-1)^k k!^2 (sn - k - 1)!}{(sn + k + 1)!(k + 1)!}, \quad (21)$$

$s = 1$ to obtain

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\binom{2n}{n} n^3}, \quad (22)$$

For $s = 2$,

$$\zeta(3) = \frac{1}{4} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{56n^2 - 32n + 5}{(2n-1)^2} \frac{1}{\binom{3n}{n} \binom{2n}{n} n^3} \quad (23)$$

and for $s = 3$,

$$\begin{aligned} \zeta(3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{72 \binom{4n}{n} \binom{3n}{n}} \\ \times \frac{6120n + 5265n^4 + 13761n^2 + 13878n^3 + 1040}{(4n+1)(4n+3)(n+1)(3n+1)^2(3n+2)^2} \end{aligned} \quad (24)$$

(Amdeberhan 1996). The corresponding $G(n, k)$ for

$s = 1$ and 2 are

$$G(n, k) = \frac{2(-1)^k k!^2 (n-k)!}{(n+k+1)!(n+1)^2} \quad (25)$$

and

$$G(n, k) = \frac{(-1)^k k!^2 (2n-k)!(3+4n)(4n^2+6n+k+3)}{2(2n+k+2)!(n+1)^2(2n+1)^2}. \quad (26)$$

Gosper (1996) expressed $\zeta(3)$ as the MATRIX PRODUCT

$$\lim_{N \rightarrow \infty} \prod_{n=1}^N M_n = \begin{bmatrix} 0 & \zeta(3) \\ 0 & 1 \end{bmatrix}, \quad (27)$$

where

$$M_n = \begin{bmatrix} \frac{(n+1)^4}{4006(n+\frac{5}{2})^2(n+\frac{3}{4})^2} & \frac{24570n^4 + 64101n^3 + 62152n^2 + 26427n + 4154}{31104(n+\frac{1}{3})(n+\frac{1}{2})(n+\frac{2}{3})} \\ 0 & 1 \end{bmatrix} \quad (28)$$

which gives 12 bits per term. The first few terms are

$$M_1 = \begin{bmatrix} 1 & 2077 \\ 10600 & 1728 \\ 0 & 1 \end{bmatrix} \quad (29)$$

$$M_2 = \begin{bmatrix} 1 & 7501 \\ 9801 & 4320 \\ 0 & 1 \end{bmatrix} \quad (30)$$

$$M_3 = \begin{bmatrix} 9 & 50501 \\ 67600 & 20160 \\ 0 & 1 \end{bmatrix}, \quad (31)$$

which gives

$$\zeta(3) \approx \frac{423203577229}{352066176000} = 1.20205690315732 \dots \quad (32)$$

Given three INTEGERS chosen at random, the probability that no common factor will divide them all is

$$[\zeta(3)]^{-1} \approx 1.20206^{-1} \approx 0.831907. \quad (33)$$

B. Haible and T. Papanikolaou computed $\zeta(3)$ to 1,000,000 DIGITS using a WILF-ZEILBERGER PAIR identity with

$$F(n, k) = (-1)^k \frac{n!^6 (2n-k-1)! k!^3}{2(n+k+1)!^2 (2n)!^3}, \quad (34)$$

$s = 1$, and $t = 1$, giving the rapidly converging

$$\zeta(3) = \sum_{n=0}^{\infty} (-1)^n \frac{n!^{10} (205n^2 + 250n + 77)}{64(2n+1)!^5} \quad (35)$$

(Amdeberhan and Zeilberger 1997). The record as of Dec. 1998 was 128 million digits, computed by S. Wedeniwski.

See also RIEMANN ZETA FUNCTION, TRILOGARITHM, WILF-ZEILBERGER PAIR

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Aphylactic Projection

A term sometimes used to describe a MAP PROJECTION which is neither EQUAL-AREA nor CONFORMAL (Lee 1944; Snyder 1987, p. 4).

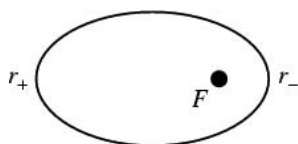
See also CONFORMAL MAPPING, EQUAL-AREA PROJECTION, MAP PROJECTION

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Snyder, J. P. *Map Projections--A Working Manual*. U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, 1987.

Apoapsis



The greatest radial distance of an ELLIPSE as measured from a FOCUS. Taking $v = \pi$ in the equation of an ELLIPSE

$$r = \frac{a(1 - e^2)}{1 + e \cos v}$$

gives the apoapsis distance

$$r_+ = a(1 + e).$$

Apoapsis for an orbit around the Earth is called apogee, and apoapsis for an orbit around the Sun is called aphelion.

See also ECCENTRICITY, ELLIPSE, FOCUS, PERIAPSIS

Apocalypse Number

A number having 666 DIGITS (where 666 is the BEAST NUMBER) is called an apocalypse number. The FIBONACCI NUMBER F_{3184} is an apocalypse number.

See also APOCALYPTIC NUMBER, BEAST NUMBER, LEVIATHAN NUMBER

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Apocalyptic Number

A number OF THE FORM 2^n which contains the digits 666 (the BEAST NUMBER) is called an APOCALYPTIC

NUMBER. 2^{157} is an apocalyptic number. The first few such powers are 157, 192, 218, 220, ... (Sloane's A007356).

See also APOCALYPSE NUMBER, BEAST NUMBER, LEVIATHAN NUMBER

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Apodization

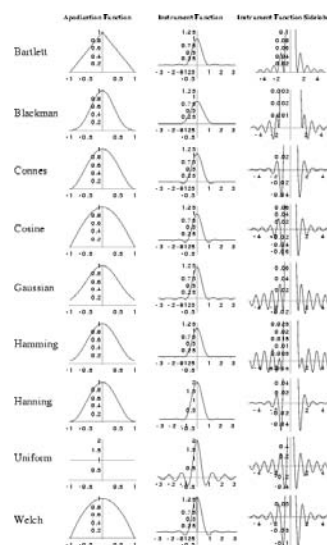
The application of an APODIZATION FUNCTION.

Apodization Function

A function (also called a TAPERING FUNCTION) used to bring an interferogram smoothly down to zero at the edges of the sampled region. This suppresses side-lobes which would otherwise be produced, but at the expense of widening the lines and therefore decreasing the resolution.

The following are apodization functions for symmetrical (2-sided) interferograms, together with the INSTRUMENT FUNCTIONS (or APPARATUS FUNCTIONS) they produce and a blowup of the INSTRUMENT FUNCTION sidelobes. The INSTRUMENT FUNCTION $I(k)$ corresponding to a given apodization function $A(x)$ can be computed by taking the finite FOURIER COSINE TRANSFORM,

$$I(k) = \int_{-\alpha}^{\alpha} \cos(2\pi kx) A(x) dx. \quad (1)$$



Type	Apodization Function	INSTRUMENT FUNCTION
BARTLETT	$1 - \frac{ x }{a}$	$a \operatorname{sinc}^2(\pi ka)$
BLACKMAN	$B_A(x)$	$B_I(k)$
CONNES	$\left(1 - \frac{x^2}{a^2}\right)^2$	$8a\sqrt{2\pi} \frac{J_{5/2}(2\pi ka)}{(2\pi ka)^{5/2}}$
COSINE	$\cos\left(\frac{\pi x}{2a}\right)$	$\frac{4a \cos(2\pi ak)}{\pi(1 - 16a^2k^2)}$
GAUSSIAN	$e^{-x^2/(2a^2)}$	$2\int_0^x \cos(2\pi kx) e^{-x^2/(2a^2)} dx$
HAMMING	$Hm_A(x)$	$Hm_I(k)$
HANNING	$Hn_A(x)$	$Hn_I(k)$
UNIFORM	1	$2a \operatorname{sinc}(2\pi ka)$
WELCH	$1 - \frac{x^2}{a^2}$	$W_I(k)$

where

$$B_A(x) = 0.42 + 0.5\cos\left(\frac{\pi x}{a}\right) + 0.08\cos\left(\frac{2\pi x}{a}\right) \quad (2)$$

$$B_I(k) = \frac{a(0.84 - 0.36a^2k^2 - 2.17 \times 10^{-19}a^4k^4)\operatorname{sinc}(2\pi ak)}{(1 - a^2k^2)(1 - 4a^2k^3)} \quad (3)$$

$$Hm_A(x) = 0.54 + 0.46\cos\left(\frac{\pi x}{a}\right) \quad (4)$$

$$Hm_I(k) = \frac{a(1.08 - 0.64a^2k^2)\operatorname{sinc}(2\pi ak)}{1 - 4a^2k^2} \quad (5)$$

$$Hn_A(x) = \cos^2\left(\frac{\pi x}{2a}\right) \quad (6)$$

$$= \frac{1}{2} \left[1 + \cos\left(\frac{\pi x}{a}\right) \right] \quad (7)$$

$$Hn_I(k) = \frac{a \operatorname{sinc}(2\pi ak)}{1 - 4a^2k^2} \quad (8)$$

$$= a[\operatorname{sinc}(2\pi ka) + \frac{1}{2} \operatorname{sinc}(2\pi ka - \pi) + \frac{1}{2} \operatorname{sinc}(2\pi ka + \pi)] \quad (9)$$

$$W_I(k) = a2\sqrt{2\pi} \frac{J_{3/2}(2\pi ka)}{(2\pi ka)^{3/2}} \quad (10)$$

$$= a \frac{\sin(2\pi ka) - 2\pi ak \cos(2\pi ak)}{2a^3k^3\pi^3}. \quad (11)$$

Type	Instrument Function FWHM	IF Peak	Peak(−)Sidelobe Peak	Peak(+)Sidelobe Peak
Bartlett	1.77179	1	0.00000000	0.0471904
Blackman	2.29880	0.84	−0.00106724	0.00124325
Connes	1.90416	$\frac{16}{15}$	−0.0411049	0.0128926
Cosine	1.63941	$\frac{4}{\pi}$	−0.0708048	0.0292720
Gaussian	—	1	—	—
Hamming	1.81522	1.08	−0.00689132	0.00734934
Hanning	2.00000	1	−0.0267076	0.00843441
Uniform	1.20671	2	−0.217234	0.128375
Welch	1.59044	$\frac{4}{3}$	−0.0861713	0.356044

A general symmetric apodization function $A(x)$ can be written as a FOURIER SERIES

$$A(x) = a_0 + 2 \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{b}\right). \quad (12)$$

where the COEFFICIENTS satisfy

$$a_0 + 2 \sum_{n=1}^{\infty} a_n = 1. \quad (13)$$

The corresponding apparatus function is

$$I(t) \equiv \int_{-b}^b A(x) e^{-2\pi i k x} dx = 2b \{ a_0 \operatorname{sinc}(2\pi kb) + \sum_{n=1}^{\infty} [\operatorname{sinc}(2\pi kb + n\pi) + \operatorname{sinc}(2\pi kb - n\pi)] \}. \quad (14)$$

To obtain an APODIZATION FUNCTION with zero at $ka = 3/4$, use

$$a_0 \operatorname{sinc}\left(\frac{3}{2}\pi\right) + a_1 [\operatorname{sinc}\left(\frac{5}{2}\pi\right) + \operatorname{sinc}\left(\frac{1}{2}\pi\right)] = 0. \quad (15)$$

Plugging in (14),

$$\begin{aligned} -(1 - 2a_1) \frac{2}{3\pi} + a_1 \left(\frac{2}{5\pi} + \frac{2}{\pi} \right) \\ = -\frac{1}{3}(1 - 2a_1) + a_1 \left(\frac{1}{5} + 1 \right) = 0 \end{aligned} \quad (16)$$

$$a_1 \left(\frac{6}{5} + \frac{2}{3} \right) = \frac{1}{3} \quad (17)$$

$$a_1 = \frac{\frac{1}{3}}{\frac{6}{5} + \frac{2}{3}} = \frac{5}{6 \cdot 3 + 2 \cdot 5} = \frac{5}{28} \quad (18)$$

$$a_0 = 1 - 2a_1 = \frac{28 - 2 \cdot 5}{28} = \frac{18}{28} = \frac{9}{14}. \quad (19)$$

The HAMMING FUNCTION is close to the requirement that the APPARATUS FUNCTION goes to 0 at $ka = 5/4$, giving

$$a_0 = \frac{25}{46} \approx 0.5435 \quad (20)$$

$$a_1 = \frac{21}{92} \approx 0.2283. \quad (21)$$

The BLACKMAN FUNCTION is chosen so that the APPARATUS FUNCTION goes to 0 at $ka = 5/4$ and $ka = 9/4$, giving

$$a_0 = \frac{3969}{9304} \approx 0.42659 \quad (22)$$

$$a_1 = \frac{1155}{4652} \approx 0.24828 \quad (23)$$

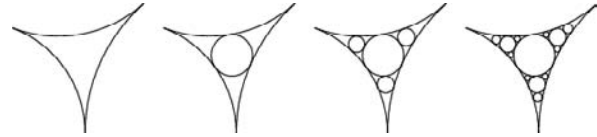
$$a_2 = \frac{715}{18608} \approx 0.38424, \quad (24)$$

See also BARTLETT FUNCTION, BLACKMAN FUNCTION, CONNES FUNCTION, COSINE APODIZATION FUNCTION, FULL WIDTH AT HALF MAXIMUM, GAUSSIAN FUNCTION, HAMMING FUNCTION, HANN FUNCTION, HANNING FUNCTION, MERTZ APODIZATION FUNCTION, PARZEN APODIZATION FUNCTION, UNIFORM APODIZATION FUNCTION, WELCH APODIZATION FUNCTION

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Apollonian Gasket



Consider three mutually tangent circles, and draw their inner SODDY CIRCLES. Then draw the inner SODDY CIRCLES of this circle with each pair of the original three, and continue iteratively. The points which are never inside a circle form a set of measure 0 having fractal dimension approximately 1.3058 (Mandelbrot 1983, p. 172).

See also BOWL OF INTEGERS, FORD CIRCLE, SODDY CIRCLES

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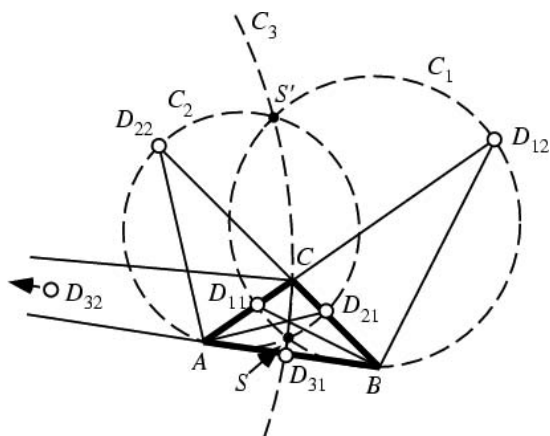
Apollonius Circles

There are two completely different definitions of the so-called Apollonius circles:

1. The set of all points whose distances from two fixed points are in a constant ratio $1 : \mu$ (Durell 1928, Ogilvy 1990).
2. The eight CIRCLES (two of which are nondegenerate) which solve APOLLONIUS' PROBLEM for three CIRCLES.

Given one side of a TRIANGLE and the ratio of the lengths of the other two sides, the LOCUS of the third VERTEX is the Apollonius circle (of the first type) whose CENTER is on the extension of the given side. For a given TRIANGLE, there are three circles of Apollonius.

Denote the three Apollonius circles (of the first type) of a TRIANGLE by k_1 , k_2 , and k_3 , and their centers L_1 , L_2 , and L_3 . The center L_1 is the intersection of the side A_2A_3 with the tangent to the CIRCUMCIRCLE at A_1 . L_1 is also the pole of the SYMMEDIAN POINT K with respect to CIRCUMCIRCLE. The centers L_1 , L_2 , and L_3 are COLLINEAR on the POLAR of K with regard to its CIRCUMCIRCLE, called the LEMOINE LINE. The circle of Apollonius k_1 is also the locus of a point whose PEDAL TRIANGLE is ISOSCELES such that $\overline{P_1P_2} = \overline{P_1P_3}$.



Let U and V be points on the side line BC of a TRIANGLE $\triangle ABC$ met by the interior and exterior ANGLE BISECTORS of ANGLES A . The CIRCLE with DIAMETER UV is called the A -Apollonian circle. Similarly, construct the B - and C -Apollonian circles. The Apollonian circles pass through the VERTICES A , B , and C , and through the two ISODYNAMIC POINTS S and S' . The VERTICES of the D -TRIANGLE lie on the respective Apollonius circles.

See also APOLLONIUS' PROBLEM, APOLLONIUS PURSUIT PROBLEM, CASEY'S THEOREM, HART'S THEOREM, HEXLET, ISODYNAMIC POINTS, SODDY CIRCLES, TANGENT CIRCLES, TANGENT SPHERES

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Apollonius Point

Consider the EXCIRCLES Γ_A , Γ_B , and Γ_C of a TRIANGLE, and the CIRCLE Γ internally TANGENT to all three. Denote the contact point of Γ and Γ_A by A' , etc. Then the LINES AA' , BB' , and CC' CONCUR in this point. It has TRIANGLE CENTER FUNCTION

$$\alpha = \sin^2 A \cos^2 \left[\frac{1}{2}(B - C) \right].$$

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Apollonius Pursuit Problem

Given a ship with a known constant direction and speed v , what course should be taken by a chase ship in pursuit (traveling at speed V) in order to intercept the other ship in as short a time as possible? The problem can be solved by finding all points which can be simultaneously reached by both ships, which is an APOLLONIUS CIRCLE with $\mu = v/V$. If the CIRCLE cuts the path of the pursued ship, the intersection is the point towards which the pursuit ship should steer. If the CIRCLE does not cut the path, then it cannot be caught.

See also APOLLONIUS CIRCLES, APOLLONIUS' PROBLEM, PURSUIT CURVE

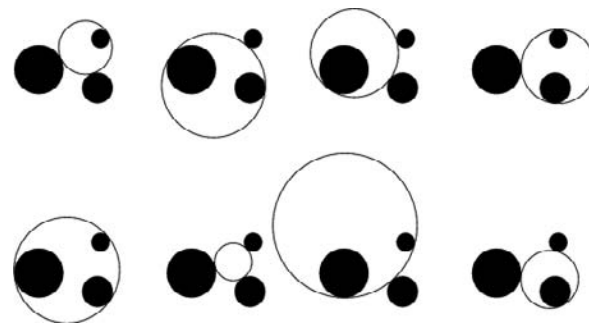
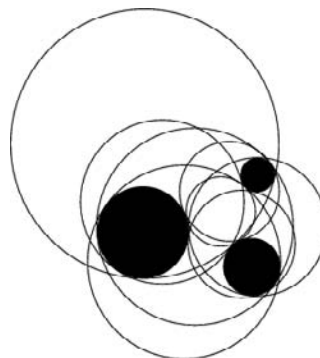
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Apollonius Spheres

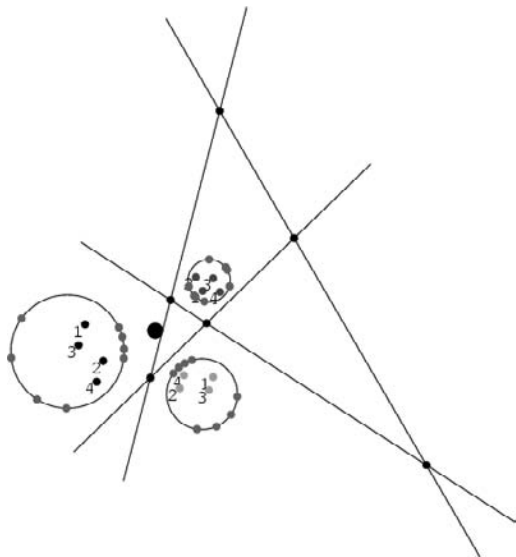
TANGENT SPHERES

Apollonius' Problem



Given three objects, each of which may be a POINT, LINE, or CIRCLE, draw a CIRCLE that is TANGENT to

each. There are a total of ten cases. The two easiest involve three points or three LINES, and the hardest involves three CIRCLES. Euclid solved the two easiest cases in his *Elements*, and the others (with the exception of the three CIRCLE problem), appeared in the *Tangencies* of Apollonius which was, however, lost. The general problem is, in principle, solvable by STRAIGHTEDGE and COMPASS alone.



The three-CIRCLE problem was solved by Viète (Boyer 1968), and the solutions are called APOLLONIUS CIRCLES. There are eight total solutions. The simplest solution is obtained by solving the three simultaneous quadratic equations

$$(x - x_1)^2 + (y - y_1)^2 - (r \pm r_1)^2 = 0 \quad (1)$$

$$(x - x_2)^2 + (y - y_2)^2 - (r \pm r_2)^2 = 0 \quad (2)$$

$$(x - x_3)^2 + (y - y_3)^2 - (r \pm r_3)^2 = 0 \quad (3)$$

in the three unknowns x, y, r for the eight triplets of signs (Courant and Robbins 1996). Expanding the equations gives

$$(x^2 + y^2 - r^2) - 2xx_i - 2yy_i \mp 2rr_i + (x_i^2 + y_i^2 - r_i^2) = 0 \quad (4)$$

for $i = 1, 2, 3$. Since the first term is the same for each equation, taking (2) - (1) and (3) - (1) gives

$$ax + by + cr = d \quad (5)$$

$$a'x + b'y + c'r = d', \quad (6)$$

where

$$a = 2(x_1 - x_2) \quad (7)$$

$$b = 2(y_1 - y_2) \quad (8)$$

$$c = \pm 2(r_1 - r_2) \quad (9)$$

$$d = (x_1^2 + y_1^2 - r_1^2) - (x_2^2 + y_2^2 - r_2^2) \quad (10)$$

and similarly for a', b', c' and d' (where the 2 subscripts are replaced by 3s). Solving these two simultaneous linear equations gives

$$x = \frac{b'd - bd' - b'cr + bc'r}{ab' - ba'} \quad (11)$$

$$y = \frac{-a'd + ad' + a'cr - ac'r}{ab' - a'b}, \quad (12)$$

which can then be plugged back into the QUADRATIC EQUATION (1) and solved using the QUADRATIC FORMULA.

Perhaps the most elegant solution is due to Gergonne. It proceeds by locating the six HOMOTHETIC CENTERS (three internal and three external) of the three given CIRCLES. These lie three by three on four lines (illustrated above). Determine the POLES of one of these with respect to each of the three CIRCLES and connect the POLES with the RADICAL CENTER of the CIRCLES. If the connectors meet, then the three pairs of intersections are the points of tangency of two of the eight circles (Petersen 1879, Johnson 1929, Dörrie 1965). To determine *which* two of the eight Apollonius circles are produced by the three pairs, simply take the two which intersect the original three CIRCLES only in a single point of tangency. The procedure, when repeated, gives the other three pairs of CIRCLES.

If the three CIRCLES are mutually tangent, then the eight solutions collapse to two, known as the SODDY CIRCLES.

Larmor (1891) and Lachlan (1893, pp. 244-51) consider the problem of four circles having a common tangent circle.

See also APOLLONIUS PURSUIT PROBLEM, BEND (CURVATURE), CASEY'S THEOREM, CIRCULAR TRIANGLE, DESCARTES CIRCLE THEOREM, FOUR COINS PROBLEM, HART CIRCLE, HART'S THEOREM, SODDY CIRCLES

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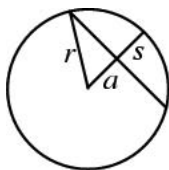
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Apollonius' Theorem

STEWART'S THEOREM

Apothem



Given a CIRCLE, the PERPENDICULAR distance a from the MIDPOINT of a CHORD to the CIRCLE's center is called the apothem. It is also equal to the RADIUS r minus the SAGITTA s ,

$$a = r - s.$$

See also CHORD, RADIUS, SAGITTA, SECTOR, SEGMENT

Apparatus Function

INSTRUMENT FUNCTION

Appell Cross Sequence

A sequence

$$s_n^{(\lambda)}(x) = [h(t)]^\lambda s_n(x),$$

where $s_n(x)$ is a SHEFFER SEQUENCE, $h(t)$ is invertible, and λ ranges over the real numbers is called a STEFFENSEN SEQUENCE. If $s_n(x)$ is an associated SHEFFER SEQUENCE, then $s_n^{(\lambda)}$ is called a CROSS SEQUENCE. If $s_n(x) = x^n$, then

$$s_n^\lambda(x) = [h(t)]^\lambda x^n$$

is called an Appell cross sequence.

Examples include the BERNOULLI POLYNOMIAL, EULER POLYNOMIAL, and HERMITE POLYNOMIAL.

See also APPELL SEQUENCE, CROSS SEQUENCE, SHEFFER SEQUENCE, STEFFENSEN SEQUENCE

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Appell Hypergeometric Function

A formal extension of the HYPERGEOMETRIC FUNCTION to two variables, resulting in four kinds of functions (Appell 1925; Whittaker and Watson 1990, Ex. 22, p. 300),

$$F_1(x; \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{m!n! (\gamma)_{m+n}} x^m y^n \quad (1)$$

$$F_2(x; \beta, \beta'; \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{m!n! (\gamma)_m (\gamma')_n} x^m y^n \quad (2)$$

$$F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{m!n! (\gamma)_{m+n}} x^m y^n \quad (3)$$

$$F_4(x; \beta; \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{m!n! (\gamma)_m (\gamma')_n} x^m y^n. \quad (4)$$

Appell defined the functions in 1880, and Picard showed in 1881 that they may all be expressed by INTEGRALS OF THE FORM

$$\int_0^1 u^x(1-u)^\beta(1-xu)^\gamma(1-yu)^\delta du \quad (5)$$

(Bailey 1934, pp. 76–9). The Appell functions are special cases of the KAMPÉ DE FÉRIET FUNCTION, and are the first four in the set of HORN FUNCTIONS.

In particular, the general integral

$$\int (a + b \sin x + c \cos x)^v dx$$

$$= CF_1 \left(n+1; \frac{1}{2}, \frac{1}{2}, n+2; \frac{a + c \cos x + b \sin x}{a - b \sqrt{1 + \frac{c^2}{b^2}}}, \right.$$

$$\left. \frac{a + c \cos x + b \sin x}{a + b \sqrt{1 + \frac{c^2}{b^2}}} \right), \quad (6)$$

where

$$C = \sec[x + \tan^{-1}(\frac{c}{b})](a + c \cos x + b \sin x)^{n+1}$$

$$\times \left[b(n+1) \sqrt{1 + \frac{c^2}{b^2}} \right]^{-1}$$

$$\times \frac{b(\sqrt{1 + \frac{c^2}{b^2}} - \sin x) - c \cos x}{b \sqrt{1 + \frac{c^2}{b^2}} + a}$$

$$\times \frac{b(\sqrt{1 + \frac{c^2}{b^2}} + \sin x) + c \cos x}{b \sqrt{1 + \frac{c^2}{b^2}} - a}, \quad (7)$$

has a closed form in terms of F_1 .

$F_1(x; \beta, \beta'; \gamma; x, y)$ reduces to the HYPERGEOMETRIC FUNCTION in the cases

$$F_1(x; \beta; \beta'; \gamma; 0, y) = {}_2F_1(\alpha, \beta'; \gamma; y) \quad (8)$$

$$F_1(x; \beta; \beta'; \gamma; x, 0) = {}_2F_1(\alpha, \beta; \gamma; x) \quad (9)$$

The F_1 function is built into *Mathematica* 4.0 as AppellF1[a, b1, b2, c, x, y].

See also ELLIPTIC INTEGRAL, HORN FUNCTION, HYPERGEOMETRIC FUNCTION, KAMPÉ DE FÉRIET FUNCTION, LAURICELLA FUNCTIONS

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Appell Polynomial

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Appell Sequence

An Appell sequence is a SHEFFER SEQUENCE for $(g(t), t)$. Roman (1984, pp. 86–06) summarizes properties of Appell sequences and gives a number of specific examples.

The sequence $s_n(x)$ is Appell for $g(t)$ IFF

$$\frac{1}{g^{(t)}} e^{y(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (1)$$

for all y in the field C of characteristic 0, and IFF

$$s_n(x) = \frac{x^n}{g(t)} \quad (2)$$

(Roman 1984, p. 27). The Appell identity states that the sequence $s_n(x)$ is an Appell sequence IFF

$$s_n(x+y) = \sum_{k=0}^n \binom{n}{k} s_k(y) x^{n-k} \quad (3)$$

(Roman 1984, p. 27).

The BERNOULLI POLYNOMIALS, EULER POLYNOMIALS, and HERMITE POLYNOMIALS are Appell sequences (in fact, more specifically, they are APPELL CROSS SEQUENCES).

See also APPELL CROSS SEQUENCE, SHEFFER SEQUENCE, UMBRAL CALCULUS

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Appell Transformation

A HOMOGRAPHIC transformation

$$x_1 = \frac{ax + by + c}{a''x + b''y + c''}$$

$$y_1 = \frac{a'x + b'y + c'}{a''x + b''y + c''}$$

with t_1 substituted for t according to

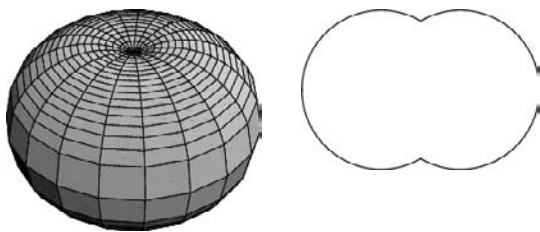
$$k dt_1 = \frac{dt}{(a''x + b''y + c'')^2}.$$

References

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AppellF1

APPELL HYPERGEOMETRIC FUNCTION

Apple

A SURFACE OF REVOLUTION defined by Kepler. It consists of more than half of a circular ARC rotated about an axis passing through the endpoints of the ARC. The equations of the upper and lower boundaries in the x - z PLANE are

$$z_{\pm} = \pm \sqrt{R^2 - (x - r)^2}$$

for $R > r$ and $x \in [-(r+R), r+R]$. It is the outside surface of a SPINDLE TORUS.

See also BUBBLE, LEMON, OBLATE SPHEROID, SPHERE-

SPHERE INTERSECTION, SPINDLE TORUS

Approximate Zero

An initial point that provides safe convergence of NEWTON'S METHOD (Smale 1981; Petkovic *et al.* 1997, p. 1).

See also ALPHA-TEST, NEWTON'S METHOD, POINT ESTIMATION THEORY

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- Petkovic, M. S.; Herceg, D. D.; and Ilic, S. M. *Point Estimation Theory and Its Applications*. Novi Sad, Yugoslavia: Institute of Mathematics, 1997.
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Approximately Equal

If two quantities A and B are approximately equal, this is written $A \approx B$.

See also DEFINED, EQUAL

Approximately Equal To

APPROXIMATELY EQUAL

Approximation Theory

The mathematical study of how given quantities can be approximated by other (usually simpler) ones under appropriate conditions. Approximation theory also studies the size and properties of the ERROR introduced by approximation. Approximations are often obtained by POWER SERIES expansions in which the higher order terms are dropped.

See also LAGRANGE REMAINDER

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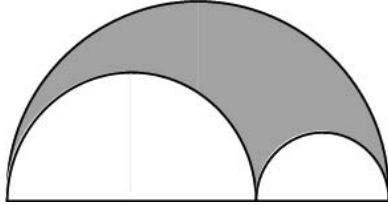
Arakelov Theory

A formal mathematical theory which introduces "components at infinity" by defining a new type of divisor class group of INTEGERS of a NUMBER FIELD.

The divisor class group is called an "arithmetic surface."

See also ARITHMETIC GEOMETRY

Arbelos



The term "arbelos" means SHOEMAKER'S KNIFE in Greek, and this term is applied to the shaded AREA in the above figure which resembles the blade of a knife used by ancient cobblers (Gardner 1979). Archimedes himself is believed to have been the first mathematician to study the mathematical properties of this figure. The position of the central notch is arbitrary and can be located anywhere along the DIAMETER.

The arbelos satisfies a number of unexpected identities (Gardner 1979, Schoch).

1. Call the diameters of the left and right SEMICIRCLES $r < 1$ and $1 - r$, respectively, so the diameter of the enclosing SEMICIRCLE is 1. Then the arc length along the bottom of the arbelos is

$$L = \pi r + \pi(1 - r) = \pi_1$$

so the arc length along the enclosing semicircle is the same as the arc length along the two smaller semicircles.

2. Draw the PERPENDICULAR BD from the tangent of the two SEMICIRCLES to the edge of the large CIRCLE. Then the AREA of the arbelos is the same as the AREA of the CIRCLE with DIAMETER BD . Let $AC = 1$ and $r = AB$, then simultaneously solve the equations

$$r^2 + h^2 = x^2 \quad (1)$$

$$(1 - r)^2 + h^2 = y^2 \quad (2)$$

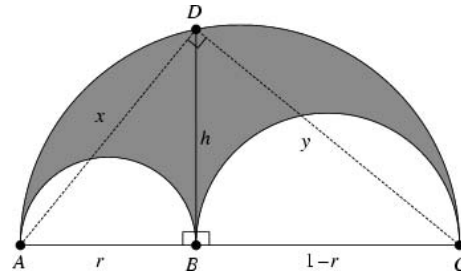
$$x^2 + y^2 = 1^2 \quad (3)$$

for the sides

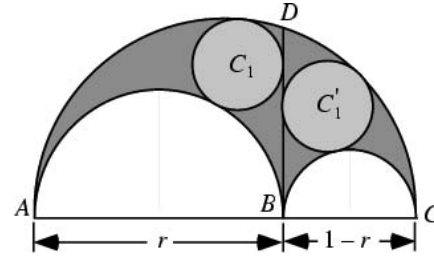
$$x = AD = \sqrt{r} \quad (4)$$

$$y = CD = \sqrt{1 - r} \quad (5)$$

$$h = BD = \sqrt{r(1 - r)}. \quad (6)$$

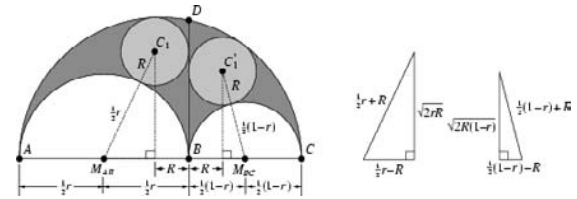


3. The CIRCLES C_1 and C'_1 inscribed on each half of BD on the arbelos (called ARCHIMEDES' CIRCLES) each have DIAMETER $(AB)(BC)/(AC)$.



If $AC = 1$ and $AB = r$, then the radius of the Archimedes' circles is

$$R = \frac{1}{2}r(1 - r). \quad (7)$$



The positions of the circles can be found using the triangles shown above. The lengths of the horizontal legs and hypotenuses are known as indicated, so the vertical legs can be found using the PYTHAGOREAN THEOREM. This then gives the centers of the circles as

$$x_1 = r - R = \frac{1}{2}r(1 + r) \quad (8)$$

$$y_1 = \sqrt{2rR} = r\sqrt{1 - r} \quad (9)$$

and

$$x'_1 = r + R = \frac{1}{2}r(3 - r) \quad (10)$$

$$y'_1 = \sqrt{2R(1 - r)} = (1 - r)\sqrt{r}. \quad (11)$$

4. Let A' be the point at which the CIRCLE centered at A and of RADIUS $r = AB$ intersects the enclosing SEMICIRCLE, and let C' be the point at which the CIRCLE centered at C of RADIUS $1 - r = BC$ intersects the enclosing SEMICIRCLE. Then the smallest CIRCLE C_2 passing through A' and tangent to BD is equal to the smallest CIRCLE C'_2 passing through C'

and tangent to BD (Schoch). Moreover, the radii R of these circles are the same as ARCHIMEDES' CIRCLES. Solving

$$(x - \frac{1}{2})^2 + y^2 = (\frac{1}{2})^2 \quad (12)$$

$$x^2 + y^2 = r^2 \quad (13)$$

gives $(x, y) = (r^2, r\sqrt{1-r^2})$, so the center of C_2 is

$$x_2 = r^2 + \frac{1}{2}r(1-r) = \frac{1}{2}r(r+1) \quad (14)$$

$$y_2 = r\sqrt{1-r^2}. \quad (15)$$

Similarly, solving

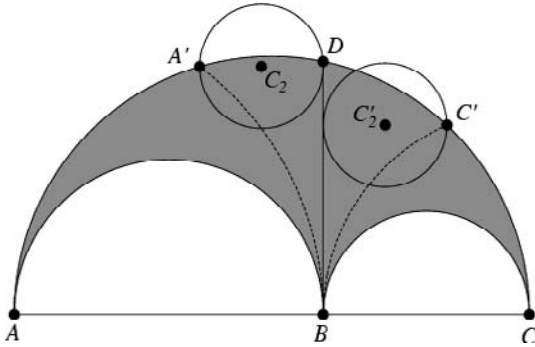
$$(x - \frac{1}{2})^2 + y^2 = (\frac{1}{2})^2 \quad (16)$$

$$(x - 1)^2 + y^2 = (1-r)^2 \quad (17)$$

gives $(x, y) = (r(2-r), (1-r)\sqrt{r(2-r)})$, so the center of C'_2 is

$$x'_2 = r(2-r) - \frac{1}{2}r(1-r) = \frac{1}{2}r(r-3) \quad (18)$$

$$y'_2 = (1-r)\sqrt{r(2-r)}. \quad (19)$$

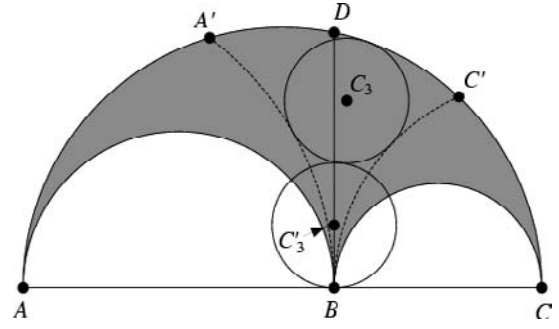


5. The APOLLONIUS CIRCLE C_3 of the circles with arcs BA' , BC' , and $AA'DC'C$ is located at a position

$$x = \frac{1}{2}r(1 + 3r - 2r^2) \quad (20)$$

$$y = r(1-r)\sqrt{(2-r)(1+r)} \quad (21)$$

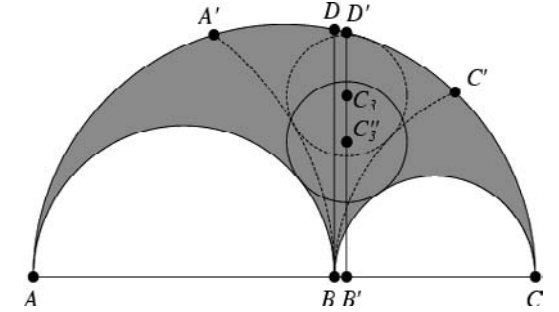
and has radius R equal to that of ARCHIMEDES' CIRCLES (Schoch), as does the smallest circle C'_3 passing through B and tangent to C_3 .



Furthermore, letting $B'D'$ be the line parallel to BD through the center of CIRCLE C_3 , the CIRCLE C''_3 with center on $B'D'$ and tangent to the small semicircles of the arbelos also has radius R (Schoch). The position of the center of C''_3 is given by

$$x''_3 = x = \frac{1}{2}r(1 + 3r - 2r^2) \quad (22)$$

$$\begin{aligned} y''_3 &= \sqrt{(\frac{1}{2}r + R) - (x - \frac{1}{2}r)^2} \\ &= r(1-r)\sqrt{1+r-r^2}. \end{aligned} \quad (23)$$



The vertical h' position of D' is

$$\begin{aligned} h' &= \sqrt{\frac{1}{4} - \frac{1}{4}(2r^3 - 3r^2 - r + 1)^2} \\ &= \frac{1}{2}\sqrt{r(1-r)(2r^2 - 3r - 1)(2r^2 - r - 2)}. \end{aligned} \quad (24)$$

6. Let P be the MIDPOINT of AB , and let Q be the MIDPOINT of BC . Then draw the SEMICIRCLE having PQ as a DIAMETER with center M . This CIRCLE has RADIUS

$$R_{PQ} = \frac{1}{2}\{1 - \frac{1}{2}[r + (1-r)]\} = \frac{1}{4}. \quad (25)$$

The smallest circle C_4 through D' touching arc PQ then has radius R (Schoch). Using similar triangles, the center of this circle is at

$$x_4 = \frac{r(2r^4 - 5r^3 + 3r + 1)}{1 + 4r - 4r^2} \quad (26)$$

$$y_4 = \frac{2r^2 - 2r - 1}{2(4r^2 - 4r - 1)}$$

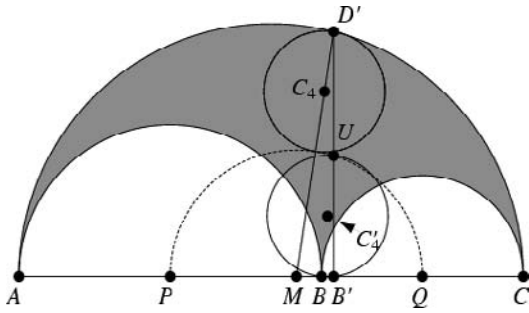
$$\times \sqrt{r(1-r)(2r^2-3r-1)(2r^2-r-2)}. \quad (27)$$

Similarly, let U be the point of intersection of $B'D'$ and the SEMICIRCLE PQ , then the CIRCLE through B , B' , and U also has RADIUS R (Schoch). The center of this CIRCLE is at

$$x'_4 = \frac{1}{4}r(3+3r-2r^2) \quad (28)$$

$$y'_4 = \frac{1}{4}r(1-r)$$

$$\times \sqrt{(2r+1)(3-2r)}. \quad (29)$$



Consider the circle X of RADIUS r_X which is tangent to the two interior semicircles. Its position and radius are obtained by solving the simultaneous equations

$$h^2 + z^2 = (\frac{1}{2}r + r_X)^2 \quad (30)$$

$$h^2 + (\frac{1}{2} - z)^2 = [\frac{1}{2}(1-r) + r_X]^2 \quad (31)$$

$$(\frac{1}{2}r + r_X)^2 + [\frac{1}{2}(1-r) + r_X]^2 = (\frac{1}{4})^2. \quad (32)$$

giving

$$z = \frac{1}{4} + \frac{1}{4}(2r-1)\sqrt{1+4r-4r^2} \quad (33)$$

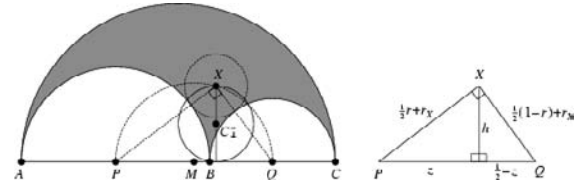
$$h = r(1-r) \quad (34)$$

$$r_X = \frac{1}{4}(\sqrt{1+4r-4r^2} - 1). \quad (35)$$

Letting C''_4 be the smallest CIRCLE through X and tangent to ABC , the radius of C''_4 is therefore $h/2 = r(1-r)/2 = R$ (Schoch), and its center is located at

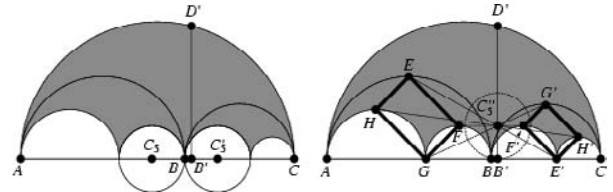
$$x''_4 = \frac{1}{4} + \frac{1}{2}r + \frac{1}{4}(2r-1)\sqrt{1+4r-4r^2} \quad (36)$$

$$y''_4 = \frac{1}{2}r(1-r). \quad (37)$$



7. Within each small semicircle of an arbelos, construct arbeloses similar to the original. Then the circles C_5 and C'_5 are congruent and have radius R (Schoch). Moreover, connect the mid-points of the arcs and their cusp points to form the RECTANGLES $\square EFGH$ and $\square E'F'G'H'$. Then these rectangles are similar with respect to the point C''_5 (Schoch). This point lies on the line $B'D'$, and the circle with center C''_5 and radius C''_5B' also has radius R , so C''_5 has coordinates $(\frac{1}{2}r(1+3r-2r^2), \frac{1}{2}r(1-r))$. The following tables summarized the positions of the rectangle vertices.

X	Coordinates	X'	Coordinates
E	$(\frac{1}{2}r, \frac{1}{2}r)$	E'	$(r(2-r), 0)$
F	$(\frac{1}{2}r(1+r), \frac{1}{2}r(1-r))$	F'	$(\frac{1}{2}r(3-r), \frac{1}{2}r(1-r))$
G	$(r^2, 0)$	G'	$(\frac{1}{2}(1+r), \frac{1}{2}(1-r))$
H	$(\frac{1}{2}r^2, \frac{1}{2}r^2)$	H'	$(\frac{1}{2}(1+2r-r^2), \frac{1}{2}(1-r)^2)$



8. Let MM' be the PERPENDICULAR BISECTOR of AC , let B be the cusp of the arbelos and D lie above it, let E and G' be the tops of the large and small semicircles, respectively. Let EG' intersect the lines MM' and BD in points I and J , respectively. Then the smallest circle C_6 passing through I and tangent to arc AC at M' , the smallest circle C'_6 through J and tangent to the outside semicircle at P_C , and the circle C''_6 with diameter JB are all equal to the Archimedean circles (Schoch). The circle C''_6 is called the BANKOFF CIRCLE, and is also the CIRCUMCIRCLE of the point B and tangent points P_A and P_C of the first Pappus circle. The centers of the circles C_6 , C'_6 , and C''_6 are given by

$$x_6 = \frac{1}{2}$$

$$y_6 = \frac{1}{2}(1-r+r^2) \quad (38)$$

$$x'_6 = \frac{r(1-r+2r^2)}{2(1-2r+2r^2)} \quad (39)$$

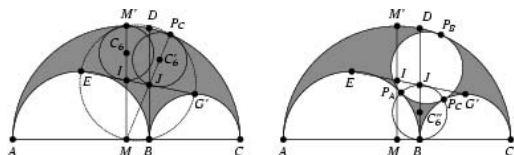
$$y'_6 = \frac{r(1-r)(1-r+r^2)}{1-2r+2r^2} \quad (40)$$

$$x''_6 = r \quad (41)$$

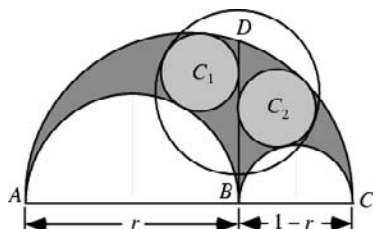
$$y''_6 = \frac{1}{2}r(1-r). \quad (42)$$

Rather amazingly, the points E, M, B, G', P_C, D , and M' are CONCYCLIC (Schoch) in a circle with center $((1+2r)/4, 1/4)$ and radius

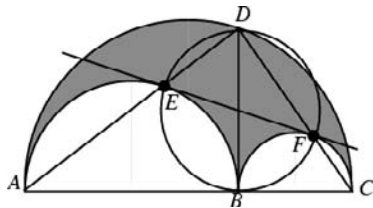
$$R_{EMBG'P_CDM'} = \frac{1}{4}\sqrt{2(1-2r+2r^2)}. \quad (43)$$



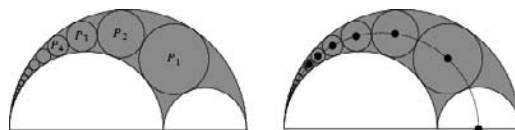
9. The smallest CIRCUMCIRCLE of the Archimedean circles has an area equal to that of the arbelos.



10. The line tangent to the semicircles AB and BC contains the point E and F which lie on the lines AD and CD , respectively. Furthermore, BD and EF bisect each other, and the points B, D, E , and F are CONCYCLIC.



11. Construct a chain of TANGENT CIRCLES starting with the CIRCLE TANGENT to the two small ones and large one (a so-called PAPPUS CHAIN). The centers of the CIRCLES lie on an ELLIPSE, and the DIAMETER of the n th CIRCLE C_n is $((1/n))$ th PERPENDICULAR distance to the base of the SEMICIRCLE. This result is most easily proven using INVERSION, but was known to Pappus, who referred to it as an ancient theorem (Hood 1961, Cadwell 1966, Gardner 1979, Bankoff 1981).



12. If B divides AC in the GOLDEN RATIO ϕ , then the circles in the chain satisfy a number of other special properties (Bankoff 1955).

See also ARCHIMEDES' CIRCLES, BANKOFF CIRCLE, COXETER'S LOXODROMIC SEQUENCE OF TANGENT CIRCLES, GOLDEN RATIO, INVERSION, PAPPUS CHAIN, STEINER CHAIN

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Arborescence

A DIRECTED GRAPH is called an arborescence if, from a given node x known as the ROOT NODE, there is

exactly one elementary path from x to every other node y .

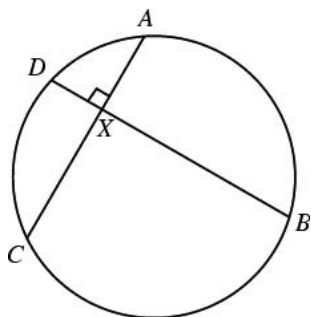
See also ARBORICITY, DIRECTED GRAPH, ROOT NODE

Arboricity

Given a GRAPH G , the arboricity is the MINIMUM number of line-disjoint acyclic SUBGRAPHS whose UNION is G .

See also ANARBORICITY

Arc



In general, any smooth curve joining two points. In particular, any portion (other than the entire curve) of a CIRCLE or ELLIPSE. As Archimedes proved, for CHORDS AC and BD which are PERPENDICULAR to each other,

$$\text{arc } AB + \text{arc } CD = \text{arc } BC + \text{arc } DA$$

(Wells 1991).

The prefix "arc" is also used to denote the INVERSE FUNCTIONS of TRIGONOMETRIC FUNCTIONS and HYPERBOLIC FUNCTIONS. Finally, any path through a graph which passes through no vertex twice is called an arc (Gardner 1984, p. 96).

See also APPLE, ARC LENGTH, CHORD, CIRCLE-CIRCLE INTERSECTION, CIRCULAR TRIANGLE, FIVE DISKS PROBLEM, FLOWER OF LIFE, LEMON, LENS, PIECEWISE CIRCULAR CURVE, REULEAUX POLYGON, REULEAUX TRIANGLE, SALINON, SEED OF LIFE, TRIANGLE ARCS, VENN DIAGRAM, YIN-YANG

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Arc Length

Arc length is defined as the length along a curve,

$$s \equiv \int_a^b |dl|. \quad (1)$$

Defining the line element $ds^2 \equiv |dl|^2$, parameterizing the curve in terms of a parameter t , and noting that

ds/dt is simply the magnitude of the VELOCITY with which the end of the RADIUS VECTOR $\mathbf{r}$ moves gives

$$s = \int_a^b ds = \int_a^b \frac{ds}{dt} dt = \int_a^b |\mathbf{r}'(t)| dt. \quad (2)$$

In POLAR COORDINATES,

$$d\ell = \hat{\mathbf{r}} dr + r \hat{\theta} d\theta = \left(\frac{dr}{d\theta} \hat{\mathbf{r}} + r \hat{\theta} \right) d\theta, \quad (3)$$

so

$$ds = |d\ell| = \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta \quad (4)$$

$$s = \int |d\ell| = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta. \quad (5)$$

In CARTESIAN COORDINATES,

$$d\ell = dy\hat{\mathbf{x}} + dx\hat{\mathbf{y}} \quad (6)$$

$$ds = |d\ell| = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dy}{dx} \right)^2 + 1} dx. \quad (7)$$

Therefore, if the curve is written

$$\mathbf{r}(x) = x\hat{\mathbf{x}} + f(x)\hat{\mathbf{y}}, \quad (8)$$

then

$$s = \int_a^b \sqrt{1 + f'^2(x)} dx. \quad (9)$$

If the curve is instead written

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}, \quad (10)$$

then

$$s = \int_a^b \sqrt{x'^2(t) + y'^2(t)} dt. \quad (11)$$

Or, in three dimensions,

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}, \quad (12)$$

so

$$s = \int_a^b \sqrt{x'^2(t) + y'^2(t) + z'^2(t)} dt. \quad (13)$$

See also CURVATURE, GEODESIC, NORMAL VECTOR, RADIUS OF CURVATURE, RADIUS OF TORSION, SPEED, SURFACE AREA, TANGENTIAL ANGLE, TANGENT VECTOR, TORSION (DIFFERENTIAL GEOMETRY), VELOCITY

Arc Minute

A unit of ANGULAR measure equal to 60 ARC SECONDS, or 1/60 of a DEGREE. The arc minute is denoted ' (not to be confused with the symbol for feet).

See also ARC SECOND, DEGREE

Arc Second

A unit of ANGULAR measure equal to 1/60 of an ARC MINUTE, or 1/3600 of a DEGREE. The arc second is denoted (not to be confused with the symbol for inches).

See also ARC MINUTE, DEGREE

Arccos

INVERSE COSINE

ArcCos

INVERSE COSINE

Arccosecant

INVERSE COSECANT

ArcCosh

INVERSE HYPERBOLIC COSINE

Arccosine

INVERSE COSINE

ArcCot

INVERSE COTANGENT

Arccot

INVERSE COTANGENT

Arccotangent

INVERSE COTANGENT

Arccoth

INVERSE HYPERBOLIC COTANGENT

ArcCoth

INVERSE HYPERBOLIC COTANGENT

ArcCsc

INVERSE COSECANT

Arccsc

INVERSE COSECANT

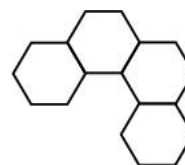
Arccsch

INVERSE HYPERBOLIC COSECANT

ArcCsch

INVERSE HYPERBOLIC COSECANT

Arch



A 4-POLYHEX (Gardner 1978, p. 147).

The term is also used by Gradshteyn and Ryzhik (2000, p. xxx) to denote

$$\text{Arch } z = i \cos^{-1} z,$$

where $\cos^{-1} z$ is the INVERSE COSINE.

See also ARCTH, ARSH, ARTH, INVERSE COSINE

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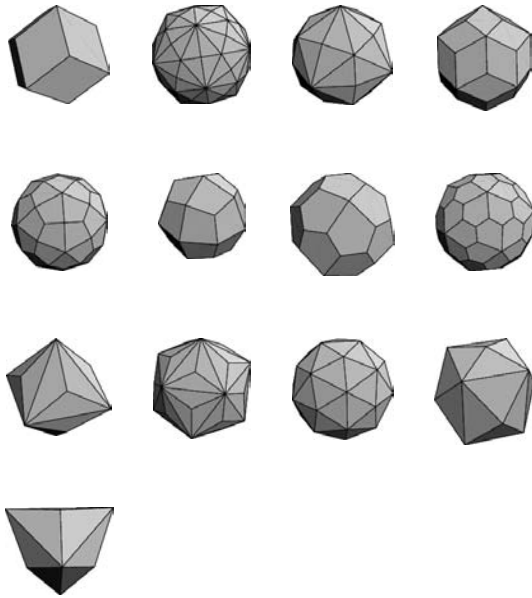
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Archimedean Dual

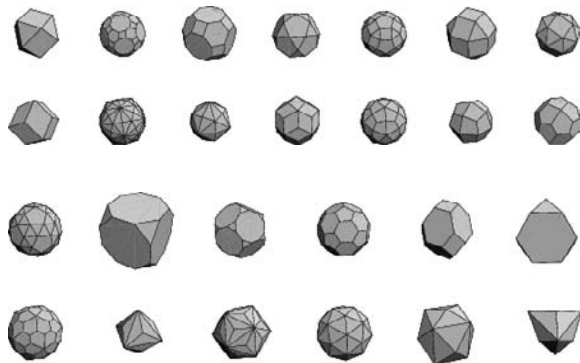
The DUALS of the ARCHIMEDEAN SOLIDS, sometimes called the CATALAN SOLIDS, are given in the following table. Hume (1986) gives exact solutions for the side lengths, angles, and DIHEDRAL ANGLES of the Archimedean duals.

<i>n</i>	ARCHIMEDEAN SOLID	DUAL
1	CUBOCTAHEDRON	RHOMBIC DODECAHEDRON
2	GREAT RHOMBICOSIDODECAHEDRON	DISDYAKIS TRIACONTAHE- DRON
3	GREAT RHOMBICUBOCTAHE- DRON	DISDYAKIS DODECAHEDRON
4	ICOSIDODECAHEDRON	RHOMBIC TRIACONTAHEDRON
5	SMALL RHOMBICOSIDODECA- HEDRON	DELTOIDAL HEXECONTAHE- DRON
6	SMALL RHOMBICUBOCTAHE- DRON	DELTOIDAL ICOSITETRAHE- DRON
7	SNUB CUBE (laevo)	PENTAGONAL ICOSITETRAHE- DRON (dextro)
8	SNUB DODECAHEDRON (lae- vo)	PENTAGONAL HEXECONTAHE- DRON (dextro)
9	TRUNCATED CUBE	SMALL TRIAKIS OCTAHEDRON
10	TRUNCATED DODECAHEDRON	TRIAKIS ICOSAHEDRON
11	TRUNCATED ICOSAHEDRON	PENTAKIS DODECAHEDRON
12	TRUNCATED OCTAHEDRON	TETRAKIS HEXAHEDRON
13	TRUNCATED TETRAHEDRON	TRIAKIS TETRAHEDRON

Here are the Archimedean DUALS (Pearce 1978, Holden 1991) displayed in the order listed above (left to right, then continuing to the next row).



Here are the Archimedean solids paired with their DUALS.



See also ARCHIMEDEAN SOLID, CATALAN SOLID

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Archimedean Solid

The Archimedean solids are convex POLYHEDRA which have a similar arrangement of nonintersecting reg-

ular plane CONVEX POLYGONS of two or more different types arranged in the same way about each VERTEX with all sides the same length (Cromwell 1997, pp. 91–2). The Archimedean solids are distinguished from the regular PRISMS and ANTIPRISMS by having very high symmetry, thus excluding solids belonging to a DIHEDRAL GROUP of symmetries (e.g., prisms and antiprisms with unit side lengths) and the ELONGATED SQUARE GYROBICUPOLA (because that surface's symmetry-breaking twist allows vertices "near the equator" and those "in the polar regions" to be distinguished; Cromwell 1997, p. 92). The Archimedean solids are sometimes also referred to as the SEMIREGULAR POLYHEDRA.

Nine of the Archimedean solids can be obtained by TRUNCATION of a PLATONIC SOLID, and two further can be obtained by a second truncation. The remaining two solids, the SNUB CUBE and SNUB DODECAHEDRON, are obtained by moving the faces of a CUBE and DODECAHEDRON outward while giving each face a twist. The resulting spaces are then filled with ribbons of EQUILATERAL TRIANGLES (Wells 1991).

Pugh (1976, p. 25) points out the Archimedean solids are all capable of being circumscribed by a regular TETRAHEDRON so that four of their faces lie on the faces of that TETRAHEDRON. A method of constructing the Archimedean solids using a method known as "expansion" has been enumerated by Stott (Stott 1910; Ball and Coxeter 1987, pp. 139–40).

Let the cyclic sequence $S = (p_1, p_2, \dots, p_q)$ represent the degrees of the faces surrounding a vertex (i.e., S is a list of the number of sides of all polygons surrounding any vertex). Then the definition of an Archimedean solid requires that the sequence must be the same for each vertex to within ROTATION and REFLECTION. Walsh (1972) demonstrates that S represents the degrees of the faces surrounding each vertex of a semiregular convex polyhedron or TESSELLATION of the plane IFF

1. $q \geq 3$ and every member of S is at least 3,
2. $\sum_{i=1}^q \frac{1}{p_i} \geq \frac{1}{2}q - 1$, with equality in the case of a plane TESSELLATION, and
3. for every ODD NUMBER $p \in S$, S contains a subsequence (b, p, b) .

Condition (1) simply says that the figure consists of two or more polygons, each having at least three sides. Condition (2) requires that the sum of interior angles at a vertex must be equal to a full rotation for the figure to lie in the plane, and less than a full rotation for a solid figure to be convex.

The usual way of enumerating the semiregular polyhedra is to eliminate solutions of conditions (1) and (2) using several classes of arguments and then prove that the solutions left are, in fact, semiregular (Kepler 1864, pp. 116–26; Catalan 1865, pp. 25–2; Coxeter 1940, p. 394; Coxeter *et al.* 1954; Lines 1965,

pp. 202–03; Walsh 1972). The following table gives all possible regular and semiregular polyhedra and tessellations. In the table, ‘P’ denotes PLATONIC SOLID, ‘M’ denotes a PRISM or ANTIPRISM, ‘A’ denotes an Archimedean solid, and ‘T’ a plane tessellation.

S	Figure	Solid	SCHLÄFLI SYMBOL
(3, 3, 3)	P	TETRAHEDRON	{3, 3}
(3, 4, 4)	M	Triangular PRISM	$t\{2, 3\}$
(3, 6, 6)	A	TRUNCATED TETRAHEDRON	$t\{3, 3\}$
(3, 8, 8)	A	TRUNCATED CUBE	$t\{4, 3\}$
(3, 10, 10)	A	TRUNCATED DODECAHEDRON	$t\{5, 3\}$
(3, 12, 12)	T	(Plane TESSELLATION)	$t\{6, 3\}$
(4, 4, n)	M	n -gonal PRISM	$t\{2, n\}$
(4, 4, 4)	P	CUBE	{4, 3}
(4, 6, 6)	A	TRUNCATED OCTAHEDRON	$t\{3, 4\}$
(4, 6, 8)	A	GREAT RHOMBICUBOCTAHEDRON	$t\{3\}$
(4, 6, 10)	A	GREAT RHOMBICOSIDODECAHEDRON	$t\{5\}$
(4, 6, 12)	T	(Plane TESSELLATION)	$t\{6\}$
(4, 8, 8)	T	(Plane TESSELLATION)	$t\{4, 4\}$
(5, 5, 5)	P	DODECAHEDRON	{5, 3}
(5, 6, 6)	A	TRUNCATED ICOSAHEDRON	$t\{3, 5\}$
(6, 6, 6)	T	(Plane TESSELLATION)	{6, 3}
(3, 3, 3, n)	M	n -gonal ANTIPRISM	$s\{2\}$
(3, 3, 3, 3)	P	OCTAHEDRON	{3, 4}
(3, 4, 3, 4)	A	CUBOCTAHEDRON	$\{3\}$
(3, 5, 3, 5)	A	ICOSIDODECAHEDRON	$\{5\}$
(3, 6, 3, 6)	T	(Plane TESSELLATION)	$\{6\}$
(3, 4, 4, 4)	A	SMALL RHOMBICUBOCTAHEDRON	$r\{3\}$
(3, 4, 5, 4)	A	SMALL RHOMBICOSIDODECAHEDRON	$r\{5\}$
(3, 4, 6, 4)	T	(Plane TESSELLATION)	$r\{6\}$
(4, 4, 4, 4)	T	(Plane TESSELLATION)	{4, 4}
(3, 3, 3, 3, 3)	P	ICOSAHEDRON	{3, 5}
(3, 3, 3, 3, 4)	A	SNUB CUBE	$s\{4\}$
(3, 3, 3, 3, 5)	A	SNUB DODECAHEDRON	$s\{5\}$
(3, 3, 3, 3, 6)	T	(Plane TESSELLATION)	$s\{6\}$
(3, 3, 3, 4, 4)	T	(Plane TESSELLATION)	—
(3, 3, 4, 3, 4)	T	(Plane TESSELLATION)	$s\{4\}$
(3, 3, 3, 3, 3)	T	(Plane TESSELLATION)	{3, 6}

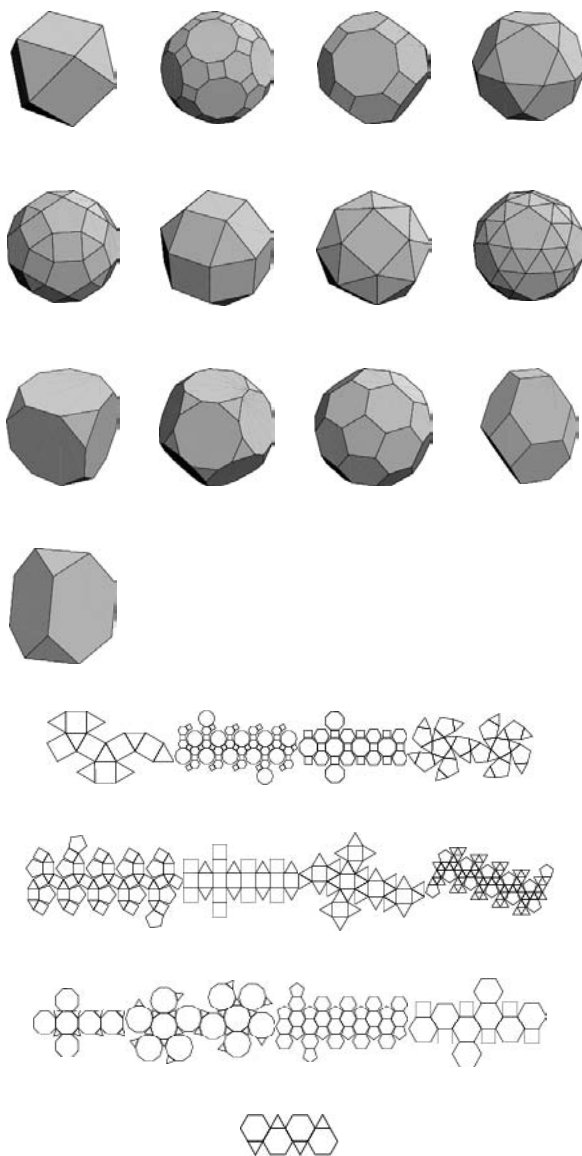
As shown in the above table, there are exactly 13 Archimedean solids (Walsh 1972, Ball and Coxeter 1987). They are called the CUBOCTAHEDRON, GREAT

RHOMBICOSIDODECAHEDRON, GREAT RHOMBICUBOCTAHEDRON, ICOSIDODECAHEDRON, SMALL RHOMBICOSIDODECAHEDRON, SMALL RHOMBICUBOCTAHEDRON, SNUB CUBE, SNUB DODECAHEDRON, TRUNCATED CUBE, TRUNCATED DODECAHEDRON, TRUNCATED ICOSAHEDRON (soccer ball), TRUNCATED OCTAHEDRON, and TRUNCATED TETRAHEDRON. The Archimedean solids satisfy

$$(2\pi - \sigma)V = 4\pi,$$

where σ is the sum of face-angles at a vertex and V is the number of vertices (Steinitz and Rademacher 1934, Ball and Coxeter 1987).

Here are the Archimedean solids shown in alphabetical order (left to right, then continuing to the next row).



The following table lists the symbols for the Archimedean solids (Wenninger 1989, p. 9).

n	Solid	SCHLÄFLI SYMBOL	WYTHOFF SYMBOL	C&R Symbol
1	CUBOCTAHEDRON	$\{3\}$	2 2 34 3 4	$(3.4)^2$
2	GREAT RHOMBICOSIDODECAHEDRON	$t\{3\}$	2 3 5 2 34	
3	GREAT RHOMBICUBOCTAHEDRON	$t\{3\}$	2 3 4 2 34	
4	ICOSIDODECAHEDRON	$\{3\}$	2 2 34 3 5	$(3.5)^2$
5	SMALL RHOMBICOSIDODECAHEDRON	$t\{3\}$	3 5 2 34 2	3.4.5.4
6	SMALL RHOMBICUBOCTAHEDRON	$r\{3\}$	3 4 2 34 2	3.4^3
7	SNUB CUBE	$s\{3\}$	2 34 2 3 4	$3^4.4$
8	SNUB DODECAHEDRON	$s\{3\}$	2 34 2 3 5	$3^4.5$
9	TRUNCATED CUBE	$t\{4, 3\}$	2 3 2 34 4	3.8^2
10	TRUNCATED DODECAHEDRON	$t\{5, 3\}$	2 3 2 34 5	3.10^2
11	TRUNCATED ICOSAHEDRON	$t\{3, 5\}$	2 5 2 34 3	5.6^2
12	TRUNCATED OCTAHEDRON	$t\{3, 4\}$	2 4 2 34 3	4.6^2
13	TRUNCATED TETRAHEDRON	$t\{3, 3\}$	2 3 2 34 3	3.6^2

The following table gives the number of vertices v , edges e , and faces f , together with the number of n -gonal faces f_n for the Archimedean solids.

n	Solid	v	e	f	f_3	f_4	f_5	f_6	f_8	f_{10}
1	CUBOCTAHEDRON	12	24	14	8	6				
2	GREAT RHOMBICOSIDODECAHEDRON	120	180	62	30			20		12
3	GREAT RHOMBICUBOCTAHEDRON	48	72	26	12			8	6	
4	ICOSIDODECAHEDRON	30	60	32	20		12			
5	SMALL RHOMBICOSIDODECAHEDRON	60	120	62	20	30	12			
6	SMALL RHOMBICUBOCTAHEDRON	24	48	26	8	18				
7	SNUB CUBE	24	60	38	32	6				
8	SNUB DODECAHEDRON	60	150	92	80		12			
9	TRUNCATED CUBE	24	36	14	8					6
10	TRUNCATED DODECAHEDRON	60	90	32	20					12
11	TRUNCATED ICOSAHEDRON	60	90	32			12	20		
12	TRUNCATED OCTAHEDRON	24	36	14		6		8		
13	TRUNCATED TETRAHEDRON	12	18	8	4			4		

Let r be the INRADIUS of the dual polyhedron (corresponding to the INSHERE, which touches the faces of the dual solid), ρ be the MIDRADIUS of both the polyhedron and its dual (corresponding to the MIDSPHERE, which touches the edges of both the polyhedron and its duals), and R the CIRCUMRADIUS

(corresponding to the CIRCUMSPHERE of the solid which touches the vertices of the solid). Since the CIRCUMSPHERE and INSHERE are dual to each other, they obey the relationship

$$R_r = \rho^2 \quad (1)$$

(Cundy and Rollett 1989, Table II following p. 144). The following tables give the analytic and numerical values of r , ρ , and R for the Archimedean solids with EDGES of unit length (Coxeter *et al.* 1954; Cundy and Rollett 1989, Table II following p. 144). Hume (1986) gives approximate expressions for the DIHEDRAL ANGLES of the Archimedean solid (and exact expressions for their duals).

n	Solid	r	ρ	R
1	CUBOCTAHEDRON	$\frac{2}{4}$	$\frac{1}{2}\sqrt{3}$	1
2	GREAT RHOMBICOSIDODECAHEDRON	$\frac{1}{24}(105 + 6\sqrt{5}) \times \sqrt{31 + 12\sqrt{5}}$	$\frac{1}{2}\sqrt{30 + 12\sqrt{5}}$	$\frac{1}{2}\sqrt{31 + 12\sqrt{5}}$
3	GREAT RHOMBICUBOCTAHEDRON	$\frac{3}{92}(14 + \sqrt{2}) \times \sqrt{13 + 6\sqrt{2}}$	$\frac{1}{2}\sqrt{12 + 6\sqrt{2}}$	$\frac{1}{2}\sqrt{13 + 6\sqrt{2}}$
4	ICOSIDODECAHEDRON	$\frac{1}{8}(5 + 3\sqrt{5})$	$\frac{1}{2}\sqrt{5 + 2\sqrt{5}}$	$\frac{1}{2}(1 + \sqrt{5})$
5	SMALL RHOMBICOSIDODECAHEDRON	$\frac{1}{41}(15 + 2\sqrt{5}) \times \sqrt{11 + 4\sqrt{5}}$	$\frac{1}{2}\sqrt{10 + 4\sqrt{5}}$	$\frac{1}{2}\sqrt{11 + 4\sqrt{5}}$
6	SMALL RHOMBICUBOCTAHEDRON	$\frac{1}{11}(6 + \sqrt{2}) \times \sqrt{5 + 2\sqrt{2}}$	$\frac{1}{2}\sqrt{4 + 2\sqrt{2}}$	$\frac{1}{2}\sqrt{5 + 2\sqrt{2}}$
7	SNUB CUBE	*	*	*
8	SNUB DODECAHEDRON	*	*	*
9	TRUNCATED CUBE	$\frac{1}{17}(5 + 2\sqrt{2}) \times \sqrt{7 + 4\sqrt{2}}$	$\frac{1}{2}(2 + \sqrt{2})$	$\frac{1}{2}\sqrt{7 + 4\sqrt{2}}$
10	TRUNCATED DODECAHEDRON	$\frac{5}{48}(17\sqrt{2} + 3\sqrt{10}) \times \sqrt{37 + 15\sqrt{5}}$	$\frac{1}{4}(5 + 3\sqrt{5})$	$\frac{1}{4}\sqrt{74 + 30\sqrt{5}}$
11	TRUNCATED ICOSAHEDRON	$\frac{9}{872}(21 + \sqrt{5}) \times \sqrt{58 + 18\sqrt{5}}$	$\frac{3}{4}(1 + \sqrt{5})$	$\frac{1}{4}\sqrt{58 + 18\sqrt{5}}$
12	TRUNCATED OCTAHEDRON	$\frac{9}{20}\sqrt{10}$	$\frac{3}{2}$	$\frac{1}{2}\sqrt{10}$
13	TRUNCATED TETRAHEDRON	$\frac{9}{44}\sqrt{22}$	$\frac{3}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{22}$

*The complicated analytic expressions for the CIRCUMRADI of these solids are given in the entries for the SNUB CUBE and SNUB DODECAHEDRON.

n	Solid	r	ρ	R
1	CUBOCTAHEDRON	0.75	0.86603	1
2	GREAT RHOMBICOSIDODECAHEDRON	3.73665	3.76938	3.80239
3	GREAT RHOMBICUBOCTAHEDRON	2.20974	2.26303	2.31761
4	ICOSIDODECAHEDRON	1.46353	1.53884	1.61803
5	SMALL RHOMBICOSIDODECAHEDRON	2.12099	2.17625	2.23295
6	SMALL RHOMBICUBOCTAHEDRON	1.22026	1.30656	1.39897
7	SNUB CUBE	1.15763	1.24719	1.34371
8	SNUB DODECAHEDRON	2.03969	2.09688	2.15583

9	TRUNCATED CUBE	1.63828	1.70711	1.77882
10	TRUNCATED DODECAHEDRON	2.88526	2.92705	2.96945
11	TRUNCATED ICOSAHEDRON	2.37713	2.42705	2.47802
12	TRUNCATED OCTAHEDRON	1.42302	1.5	1.58114
13	TRUNCATED TETRAHEDRON	0.95940	1.06066	1.17260

The Archimedean solids and their DUALS are all CANONICAL POLYHEDRA. Since the Archimedean solids of convex, the CONVEX HULL of each Archimedean solid is the solid itself.

See also ARCHIMEDEAN SOLID STELLATION, CATALAN SOLID, DELTAHEDRON, ISOHEDRON, JOHNSON SOLID, KEPLER-POINSOT SOLID, PLATONIC SOLID, QUASIREGULAR POLYHEDRON, SEMIREGULAR POLYHEDRON, UNIFORM POLYHEDRON

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Archimedean Solid Stellation

A large class of POLYHEDRA which includes the DODECADODECAHEDRON and GREAT ICOSIDODECAHEDRON. No complete enumeration (even with restrictive uniqueness conditions) has been worked out. There are at least four stellations of the CUBOCTAHEDRON (Wenninger 1989), although the exact number depends on what type of cells formed by plane intersections are allowed.

There are also many stellations of the Archimedean solid duals. The RHOMBIC DODECAHEDRON has three stellations (Wells 1991, pp. 216–17).

See also ARCHIMEDEAN SOLID, CATALAN SOLID

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Archimedean Spiral

A SPIRAL with POLAR equation

$$r = a\theta^{1/n}, \quad (1)$$

where r is the radial distance, θ is the polar angle, and n is a constant which determines how tightly the spiral is "wrapped." The CURVATURE of an Archimedean spiral is given by

$$\kappa = \frac{n\theta^{1-1/n}(1 + n + n^2\theta^2)}{a(1 + n^2\theta^2)^{3/2}}, \quad (2)$$

and the ARC LENGTH by

$$s = a\theta^{1/n} {}_2F_1((2n)^{-1}, -\frac{1}{2}; 1 + (2n)^{-1}; -n^2\theta^2), \quad (3)$$

where ${}_2F_1(a, b; c; x)$ is a HYPERGEOMETRIC FUNCTION. Various special cases are given in the following table.

Name	n
LITUUS	-2
HYPERBOLIC SPIRAL	-1
ARCHIMEDES' SPIRAL	1
FERMAT'S SPIRAL	2

If a fly crawls radially outward along a uniformly spinning disk, the curve it traces with respect to a reference frame in which the disk is at rest is an Archimedean spiral (Steinhaus 1999, p. 137). Furthermore, a heart-shaped frame composed of two arcs of an Archimedean spiral which is fixed to a rotating disk converts uniform rotational motion to uniform back-and-forth motion (Steinhaus 1999, pp. 136–37).

See also ARCHIMEDES' SPIRAL, DAISY, FERMAT'S SPIRAL, HYPERBOLIC SPIRAL, LITUUS, SPIRAL

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Archimedean Spiral Inverse Curve

The INVERSE CURVE of the ARCHIMEDEAN SPIRAL

$$r = a\theta^{1/n}$$

with INVERSION CENTER at the origin and inversion RADIUS k is the ARCHIMEDEAN SPIRAL

$$r = ka\theta^{1/n}.$$

Archimedean Tessellation

TESSELLATION

Archimedean Valuation

A VALUATION for which $|x| \leq 1$ IMPLIES $|1 + x| \leq C$ for the constant $C = 1$ (independent of x). Such a VALUATION does not satisfy the strong TRIANGLE INEQUALITY

$$|x + y| \leq \max(|x|, |y|).$$

Archimedes Algorithm

Successive application of ARCHIMEDES' RECURRENCE FORMULA gives the Archimedes algorithm, which can be used to provide successive approximations to π (PI). The algorithm is also called the BORCHARDT-PFAFF ALGORITHM. Archimedes obtained the first rigorous approximation of π by CIRCUMSCRIBING and INSCRIBING $n = G \cdot 2^k$ -gons on a CIRCLE. From ARCHIMEDES' RECURRENCE FORMULA, the CIRCUMFERENCES a and b of the circumscribed and inscribed POLYGONS are

$$a(n) = 2n \tan\left(\frac{\pi}{n}\right) \quad (1)$$

$$b(n) = 2n \sin\left(\frac{\pi}{n}\right), \quad (2)$$

where

$$b(n) < C = 2\pi r = 2\pi \cdot 1 = 2\pi < a(n). \quad (3)$$

For a HEXAGON, $n = 6$ and

$$a_0 \equiv a(6) = 4\sqrt{3} \quad (4)$$

$$b_0 \equiv b(6) = 6, \quad (5)$$

where $a_k \equiv a(6 \cdot 2^k)$. The first iteration of ARCHIMEDES' RECURRENCE FORMULA then gives

$$a_1 = \frac{2 \cdot 6 \cdot 4\sqrt{3}}{6 + 4\sqrt{3}} = \frac{24\sqrt{3}}{3 + 2\sqrt{3}} = 24(2 - \sqrt{3}) \quad (6)$$

$$b_1 = \sqrt{24(2 - \sqrt{3})} \cdot 6 = 12\sqrt{2 - \sqrt{3}}$$

$$= 6(\sqrt{6} - \sqrt{2}). \quad (7)$$

Additional iterations do not have simple closed forms, but the numerical approximations for $k = 0, 1, 2, 3, 4$ (corresponding to 6-, 12-, 24-, 48-, and 96-gons) are

$$3.00000 < \pi < 3.46410 \quad (8)$$

$$3.10583 < \pi < 3.21539 \quad (9)$$

$$3.13263 < \pi < 3.15966 \quad (10)$$

$$3.13935 < \pi < 3.14609 \quad (11)$$

$$3.14103 < \pi < 3.14271. \quad (12)$$

By taking $k = 4$ (a 96-gon) and using strict inequalities to convert irrational bounds to rational bounds at each step, Archimedes obtained the slightly looser result

$$\frac{223}{71} = 3.14084 \dots < \pi < \frac{22}{7} = 3.14285 \dots \quad (13)$$

See also P1

References

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Archimedes' Axiom

An AXIOM actually attributed to Eudoxus (Boyer and Merzbach 1991, pp. 89–90) which states that

$$\frac{a}{b} = \frac{c}{d}$$

IFF the appropriate one of following conditions is satisfied for INTEGERS m and n :

1. If $ma < nb$, then $mc < nd$.
2. If $ma = nb$, then $mc = nd$.
3. If $ma > nb$, then $mc > nd$.

Also known as the continuity axiom or Archimedes' lemma, this axiom survives in the writings of Eudoxus (Boyer and Merzbach 1991). It states that, given two magnitudes having a ratio, one can find a multiple of either which will exceed the other. This principle was the basis for the EXHAUSTION METHOD which Archimedes invented to solve problems of AREA and VOLUME.

Formally, Archimedes' axiom states that if AB and CD are two line segments, then there exist a finite number of points $A_1, A_2, \dots, A_n$ on $A \cup B$ such that

$$CD \equiv AA_1 \equiv AA_2 \equiv \dots \equiv A_{n-1}A_n,$$

and B is between A and A_n (Itô 1986, p. 611). A geometry in which Archimedes' lemma does not hold is called a NON-ARCHIMEDEAN GEOMETRY.

See also CONTINUITY AXIOMS, FRACTION, INEQUALITY, NON-ARCHIMEDEAN GEOMETRY

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Archimedes' Cattle Problem

Also called the BOVINUM PROBLEMA. It is stated as follows: "The sun god had a herd of cattle consisting of bulls and cows, one part of which was white, a second black, a third spotted, and a fourth brown. Among the bulls, the number of white ones was one half plus one third the number of the black greater than the brown; the number of the black, one quarter plus one fifth the number of the spotted greater than the brown; the number of the spotted, one sixth and one seventh the number of the white greater than the brown. Among the cows, the number of white ones was one third plus one quarter of the total black cattle; the number of the black, one quarter plus one fifth the total of the spotted cattle; the number of spotted, one fifth plus one sixth the total of the brown cattle; the number of the brown, one sixth plus one seventh the total of the white cattle. What was the composition of the herd?"

Solution consists of solving the simultaneous DIOPHANTINE EQUATIONS in INTEGERS W, X, Y, Z (the number of white, black, spotted, and brown bulls) and w, x, y, z (the number of white, black, spotted, and brown cows),

$$W = \frac{5}{6}X + Z \quad (1)$$

$$X = \frac{9}{20}Y + Z \quad (2)$$

$$Y = \frac{13}{42}W + Z \quad (3)$$

$$w = \frac{7}{12}(X + x) \quad (4)$$

$$x = \frac{9}{20}(Y + y) \quad (5)$$

$$y = \frac{11}{30}(Z + z) \quad (6)$$

$$z = \frac{13}{42}(W + w). \quad (7)$$

The smallest solution in INTEGERS is

$$W = 10,366,482 \quad (8)$$

$$X = 7,460,514 \quad (9)$$

$$Y = 7,358,060 \quad (10)$$

$$Z = 4,149,387 \quad (11)$$

$$w = 7,206,360 \quad (12)$$

$$x = 4,893,246 \quad (13)$$

$$y = 3,515,820 \quad (14)$$

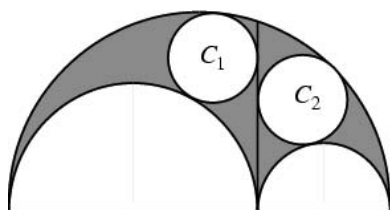
$$z = 5,439,213. \quad (15)$$

A more complicated version of the problem requires that $W+X$ be a SQUARE NUMBER and $Y+Z$ a TRIANGULAR NUMBER. The solution to this PROBLEM are numbers with 206544 or 206545 digits.

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Archimedes' Circles



Draw the PERPENDICULAR LINE from the intersection of the two small SEMICIRCLES in the ARBELOS. The two CIRCLES C_1 and C_2 TANGENT to this line, the large

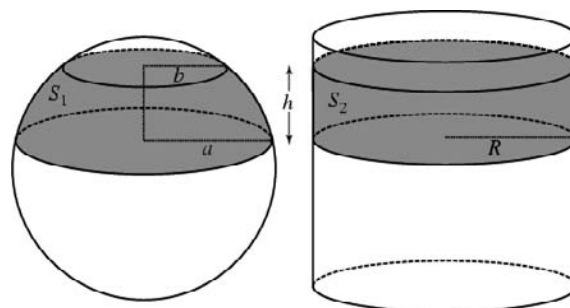
SEMICIRCLE, and each of the two SEMICIRCLES are then congruent and known as Archimedes' circles.

See also ARBELOS, BANKOFF CIRCLE, SEMICIRCLE

Archimedes' Constant

PI

Archimedes' Hat-Box Theorem



Enclose a SPHERE in a CYLINDER and cut out a SPHERICAL SEGMENT by slicing twice PERPENDICULARLY to the CYLINDER'S axis. Then the lateral SURFACE AREA of the SPHERICAL SEGMENT S_1 is equal to the lateral SURFACE AREA cut out of the CYLINDER S_2 by the same slicing planes, i.e.,

$$S \equiv S_1 = S_2 = 2\pi Rh,$$

where R is the RADIUS of the CYLINDER (and tangent SPHERE) and h is the height of the cylindrical (and spherical) segment.

See also ARCHIMEDES' PROBLEM, CYLINDER, SPHERE, SPHERICAL SEGMENT

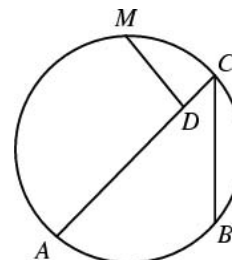
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Archimedes' Lemma

ARCHIMEDES' AXIOM

Archimedes' Midpoint Theorem



Let M be the MIDPOINT of the ARC AMB . Pick C at random and pick D such that $MD \perp AC$ (where $\perp$

denotes PERPENDICULAR). Then

$$AD = DC + BC.$$

See also MIDPOINT

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Archimedes' Postulate

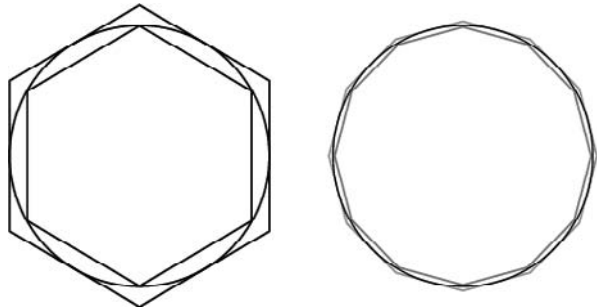
ARCHIMEDES' LEMMA

Archimedes' Problem

Cut a SPHERE by a PLANE in such a way that the VOLUMES of the SPHERICAL SEGMENTS have a given RATIO.

See also ARCHIMEDES' HAT-BOX THEOREM, SPHERICAL SEGMENT

Archimedes' Recurrence Formula



Let a_n and b_n be the PERIMETERS of the CIRCUMSCRIBED and INSCRIBED n -gon and a_{2n} and b_{2n} the PERIMETERS of the CIRCUMSCRIBED and INSCRIBED $2n$ -gon. Then

$$a_{2n} = \frac{2a_n b_n}{a_n + b_n} \quad (1)$$

$$b_{2n} = \sqrt{a_{2n} b_n}. \quad (2)$$

The first follows from the fact that side lengths of the POLYGONS on a CIRCLE of RADIUS $r = 1$ are

$$s_R = 2 \tan\left(\frac{\pi}{n}\right) \quad (3)$$

$$s_r = 2 \sin\left(\frac{\pi}{n}\right), \quad (4)$$

so

$$a_n = 2n \tan\left(\frac{\pi}{n}\right) \quad (5)$$

$$b_n = 2n \sin\left(\frac{\pi}{n}\right). \quad (6)$$

But

$$\begin{aligned} \frac{2a_n b_n}{a_n + b_n} &= \frac{2 \cdot 2n \tan\left(\frac{\pi}{n}\right) \cdot 2n \sin\left(\frac{\pi}{n}\right)}{2n \tan\left(\frac{\pi}{n}\right) + 2n \sin\left(\frac{\pi}{n}\right)} \\ &= 4n \frac{\tan\left(\frac{\pi}{n}\right) \sin\left(\frac{\pi}{n}\right)}{\tan\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)}. \end{aligned} \quad (7)$$

Using the identity

$$\tan\left(\frac{1}{2}x\right) = \frac{\tan x \sin x}{\tan x + \sin x} \quad (8)$$

then gives

$$\frac{2a_n b_n}{a_n + b_n} = 4n \tan\left(\frac{\pi}{2n}\right) = a_{2n}. \quad (9)$$

The second follows from

$$\sqrt{a_{2n} b_n} = \sqrt{4n \tan\left(\frac{\pi}{2n}\right) \cdot 2n \sin\left(\frac{\pi}{2n}\right)} \quad (10)$$

Using the identity

$$\sin x = 2 \sin\left(\frac{1}{2}x\right) \cos\left(\frac{1}{2}x\right) \quad (11)$$

gives

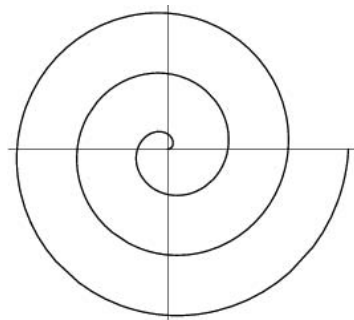
$$\begin{aligned} \sqrt{a_{2n} b_n} &= 2n \sqrt{2 \tan\left(\frac{x}{2n}\right) \cdot 2 \sin\left(\frac{\pi}{2n}\right) \cos\left(\frac{\pi}{2n}\right)} \\ &= 4n \sqrt{\sin^2\left(\frac{\pi}{2n}\right)} = 4n \sin\left(\frac{\pi}{2n}\right) = b_{2n}. \end{aligned} \quad (12)$$

Successive application gives the ARCHIMEDES ALGORITHM, which can be used to provide successive approximations to PI (π).

See also ARCHIMEDES ALGORITHM, PI

References

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Archimedes' Spiral

An ARCHIMEDEAN SPIRAL with POLAR equation

$$r = a\theta.$$

This spiral was studied by Conon, and later by Archimedes in *On Spirals* about 225 BC. Archimedes was able to work out the lengths of various tangents to the spiral.

Archimedes' spiral can be used for COMPASS and STRAIGHTEDGE division of an ANGLE into n parts (including ANGLE TRISECTION) and can also be used for CIRCLE SQUARING. In addition, the curve can be used as a cam to convert uniform circular motion into uniform linear motion (Steinhaus 1983, p. 137; Brown). The cam consists of one arch of the spiral above the x -AXIS together with its reflection in the x -AXIS. Rotating this with uniform angular velocity about its center will result in uniform linear motion of the point where it crosses the y -AXIS.

See also ARCHIMEDEAN SPIRAL

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Archimedes' Spiral Inverse

Taking the ORIGIN as the INVERSION CENTER, ARCHIMEDES' SPIRAL $r = a\theta$ inverts to the HYPERBOLIC SPIRAL $r = a/\theta$.

ArcSec

INVERSE SECANT

Arcsec

INVERSE SECANT

Arcsecant

INVERSE SECANT

ArcSech

INVERSE HYPERBOLIC SECANT

Arcsech

INVERSE HYPERBOLIC SECANT

ArcSin

INVERSE SINE

Arcsin

INVERSE SINE

Arcsine

INVERSE SINE

Arcsinh

INVERSE HYPERBOLIC SINE

ArcSinh

INVERSE HYPERBOLIC SINE

Arctan

INVERSE TANGENT

ArcTan

INVERSE TANGENT

Arctangent

INVERSE TANGENT

Arctangent Integral

INVERSE TANGENT INTEGRAL

Arctanh

INVERSE HYPERBOLIC TANGENT

ArcTanh

INVERSE HYPERBOLIC TANGENT

Arcth

$$\text{Arcth } z = \frac{1}{i} \cot^{-1}(-iz),$$

where $\cot^{-1} z$ is the INVERSE COTANGENT.

See also ARCH, ARSH, ARTH, INVERSE COTANGENT

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Arcwise-Connected

See also CONNECTED SET, LOCALLY PATHWISE-CONNECTED, PATH-CONNECTED, PATHWISE-CONNECTED

Arcwise-Connected Set

See also CONNECTED SET, PATH-CONNECTED SET

Area

The AREA of a SURFACE is the amount of material needed to "cover" it completely. The AREA of a TRIANGLE is given by

$$A_{\Delta} = \frac{1}{2}lh, \quad (1)$$

where l is the base length and h is the height, or by HERON'S FORMULA

$$A_{\Delta} = \sqrt{s(s-a)(s-b)(s-c)}, \quad (2)$$

where the side lengths are a , b , and c and s the SEMIPERIMETER. The AREA of a RECTANGLE is given by

$$A_{\text{rectangle}} = ab, \quad (3)$$

where the sides are length a and b . This gives the special case of

$$A_{\text{square}} = a^2 \quad (4)$$

for the SQUARE. The AREA of a REGULAR POLYGON with n sides and side length s is given by

$$A_{n\text{-gon}} = \frac{1}{4}ns^2 \cot\left(\frac{\pi}{n}\right). \quad (5)$$

CALCULUS and, in particular, the INTEGRAL, are powerful tools for computing the AREA between a curve $f(x)$ and the x -AXIS over an INTERVAL $[a, b]$, giving

$$A = \int_a^b f(x) dx. \quad (6)$$

The AREA of a POLAR curve with equation $r = r(\theta)$ is

$$A = \frac{1}{2} \int r^2 d\theta. \quad (7)$$

Written in CARTESIAN COORDINATES, this becomes

$$A = \frac{1}{2} \int \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) dt \quad (8)$$

$$= \frac{1}{2} \int (x dy - y dx). \quad (9)$$

For the AREA of special surfaces or regions, see the entry for that region. The generalization of AREA to 3-

D is called VOLUME, and to higher DIMENSIONS is called CONTENT.

See also ARC LENGTH, AREA ELEMENT, CONTENT, SURFACE AREA, VOLUME

References

Gray, A. "The Intuitive Idea of Area on a Surface." §15.3 in *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2nd ed. Boca Raton, FL: CRC Press, pp. 351–53, 1997.

Area Element

The area element for a SURFACE with RIEMANNIAN METRIC

$$ds^2 = E du^2 + 2F du dv + G dv^2$$

is

$$dA = \sqrt{EG - F^2} du \wedge dv,$$

where $du \wedge dv$ is the WEDGE PRODUCT.

See also AREA, LINE ELEMENT, RIEMANNIAN METRIC, VOLUME ELEMENT

References

Gray, A. "The Intuitive Idea of Area on a Surface." §15.3 in *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2nd ed. Boca Raton, FL: CRC Press, pp. 351–53, 1997.

Area Integral

A double integral over three coordinates giving the AREA within some region R ,

$$A = \iint_R dx dy.$$

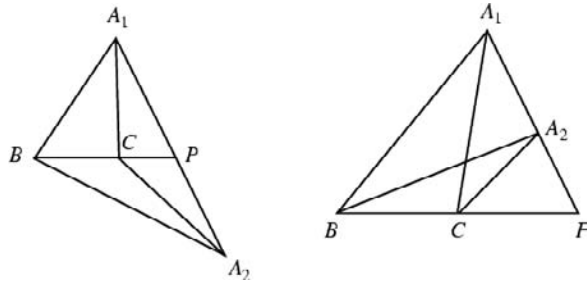
If a plane curve is given by $y = f(x)$, then the area between the curve and the x -AXIS from $x = a$ to $x = b$ is given by

$$A = \int_a^b f(x) dx.$$

See also INTEGRAL, LINE INTEGRAL, LUSIN AREA INTEGRAL, MULTIPLE INTEGRAL, SURFACE INTEGRAL, VOLUME INTEGRAL

Area Principle

There are at least two results known as "the area principle."



The geometric area principle states that

$$\frac{|A_1P|}{|A_2P|} = \frac{|A_1BC|}{|A_2BC|}. \quad (1)$$

This can also be written in the form

$$\left[\frac{|A_1P|}{|A_2P|} \right] = \left[\frac{|A_1BC|}{|A_2BC|} \right], \quad (2)$$

where

$$\left[\frac{AB}{CD} \right] \quad (3)$$

is the ratio of the lengths $[A, B]$ and $[C, D]$ for $AB \parallel CD$ with a PLUS or MINUS SIGN depending on if these segments have the same or opposite directions, and

$$\left[\frac{ABC}{DEF} \right] \quad (4)$$

is the RATIO of signed AREAS of the TRIANGLES. Grünbaum and Shepard (1995) show that CEVA'S THEOREM, HOEHN'S THEOREM, and MENELAUS' THEOREM are the consequences of this result.

The area principle of complex analysis states that if f is a SCHLICHT FUNCTION and if

$$h(z) = \frac{1}{f(z)} = \frac{1}{z} + \sum_{j=0}^{\infty} b_j z^j, \quad (5)$$

then

$$\sum_{j=1}^{\infty} j |b_j|^2 \leq 1 \quad (6)$$

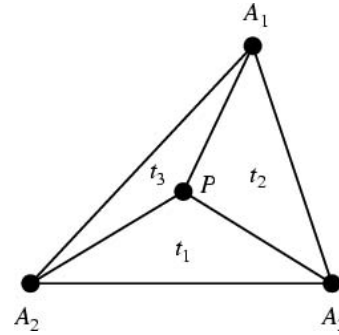
(Krantz 1999, p. 150).

See also CEVA'S THEOREM, HOEHN'S THEOREM, MENELAUS' THEOREM, SCHLICHT FUNCTION, SELF-TRANSVERSALITY THEOREM

References

- Grünbaum, B. and Shepard, G. C. "Ceva, Menelaus, and the Area Principle." *Math. Mag.* **68**, 254–68, 1995.
Krantz, S. G. "Schlicht Functions." §12.1.1 in *Handbook of Complex Analysis*. Boston, MA: Birkhäuser, p. 149, 1999.

Areal Coordinates



BARYCENTRIC COORDINATES (t_1, t_2, t_3) normalized so that they become the AREAS of the TRIANGLES PA_1A_2 , PA_1A_3 , and PA_2A_3 , where P is the point whose coordinates have been specified, *normalized by the area of the original triangle* $\Delta A_1A_2A_3$. This is equivalent to application of the normalization relation

$$t_1 + t_2 + t_3 = 1$$

(Coxeter 1969, p. 218).

See also BARYCENTRIC COORDINATES, TRILINEAR COORDINATES

References

- Coxeter, H. S. M. *Introduction to Geometry*, 2nd ed. New York: Wiley, p. 218, 1969.

Area-Preserving Map

A MAP F from $\mathbb{R}^n$ to $\mathbb{R}^n$ is AREA-preserving if

$$m(F(A)) = m(A)$$

for every subregion A of $\mathbb{R}^n$, where $m(A)$ is the n -D MEASURE of A . A linear transformation is AREA-preserving if its corresponding DETERMINANT is equal to 1.

See also CONFORMAL MAP, SYMPLECTIC MAP

Arf Invariant

A LINK invariant which always has the value 0 or 1. A KNOT has ARF INVARIANT 0 if the KNOT is "pass equivalent" to the UNKNOT and 1 if it is pass equivalent to the TREFOIL KNOT. If K_+ , K_- , and L are projections which are identical outside the region of the crossing diagram, and K_+ and K_- are KNOTS while l is a 2-component LINK with a nonintersecting crossing diagram where the two left and right strands belong to the different LINKS, then

$$a(K_+) = a(K_-) + l(L_1, L_2), \quad (1)$$

where l is the LINKING NUMBER of L_1 and L_2 . The Arf invariant can be determined from the ALEXANDER POLYNOMIAL or JONES POLYNOMIAL for a KNOT. For Δ_K the ALEXANDER POLYNOMIAL of K , the Arf invariant is given by

$$\Delta_K(-1) \equiv \begin{cases} 1 \pmod{8} & \text{if } \text{Arf}(K) = 0 \\ 5 \pmod{8} & \text{if } \text{Arf}(K) = 1 \end{cases} \quad (2)$$

(Jones 1985). For the JONES POLYNOMIAL W_K of a KNOT K ,

$$\text{Arf}(K) = W_K(i) \quad (3)$$

(Jones 1985), where i is the IMAGINARY NUMBER.

References

- Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. New York: W. H. Freeman, pp. 223–31, 1994.
- Jones, V. "A Polynomial Invariant for Knots via von Neumann Algebras." *Bull. Amer. Math. Soc.* **12**, 103–11, 1985.
- Weisstein, E. W. "Knots." MATHEMATICA NOTEBOOK KNOTS.M.

Arg

ARGUMENT (COMPLEX NUMBER)

Argand Diagram

A plot of COMPLEX NUMBERS as points

$$z = x + iy$$

using the x -AXIS as the REAL AXIS and y -AXIS as the IMAGINARY AXIS. An Argand diagram is also called the COMPLEX PLANE or ARGAND PLANE. The Argand plane was described by C. Wessel prior to Argand.

See also COMPLEX PLANE, IMAGINARY NUMBER, REAL NUMBER

References

- Argand, R. *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques*. Paris: Albert Blanchard, 1971. Reprint of the 2nd ed., published by G. J. Hoel in 1874. First edition published Paris, 1806.

Argand Plane

ARGAND DIAGRAM

Argoh's Conjecture

Let B_k be the k th BERNOULLI NUMBER. Then does

$$nB_{n-1} \equiv -1 \pmod{n}$$

IFF n is PRIME? For example, for $n = 1, 2, \dots, nB_{n-1} \pmod{n}$ is 0, -1, -1, 0, -1, 0, -1, 0, -3, 0, -1, ... (Sloane's A046094). There are no counterexamples less than $n = 5,600$. Any counterexample to Argoh's conjecture would be a contradiction to GIUGA'S CONJECTURE, and vice versa.

See also BERNOULLI NUMBER, GIUGA'S CONJECTURE

References

- Borwein, D.; Borwein, J. M.; Borwein, P. B.; and Girgensohn, R. "Giuga's Conjecture on Primality." *Amer. Math. Monthly* **103**, 40–0, 1996.

Argument (Elliptic Integral)

Sloane, N. J. A. Sequences A046094 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.

Argument (Complex Number)

A COMPLEX NUMBER z may be REPRESENTED AS

$$z \equiv x + iy = |z|e^{i\theta}, \quad (1)$$

where $|z|$ is called the MODULUS of z , and θ is called the argument (or PHASE) and is given by

$$\arg(x + iy) \equiv \tan^{-1}\left(\frac{y}{x}\right). \quad (2)$$

Here, θ , sometimes also denoted ϕ , corresponds to the counterclockwise ANGLE from the POSITIVE REAL AXIS, i.e., the value of θ such that $x = \cos \theta$ and $y = \sin \theta$. The special kind of INVERSE TANGENT used here takes into account the quadrant in which z lies and is returned by the FORTRAN command ATAN2 (X, Y) and the Mathematica command ArcTan[x, y], and is often restricted to the range $-\pi < \theta \leq \pi$. In the degenerate case when $x = 0$,

$$\phi = \begin{cases} -\frac{1}{2}\pi & \text{if } y < 0 \\ \text{undefined} & \text{if } y = 0 \\ \frac{1}{2}\pi & \text{if } y > 0. \end{cases} \quad (3)$$

From the definition of the argument,

$$\begin{aligned} \arg(zw) &= \arg(|z|e^{i\theta_z}|w|e^{i\theta_w}) = \arg(e^{i\theta_z}e^{i\theta_w}) \\ &= \arg[e^{i(\theta_z+\theta_w)}] = \arg(z) + \arg(w). \end{aligned} \quad (4)$$

Extending this procedure gives

$$\arg(z^n) = n \arg(z). \quad (5)$$

The argument of a COMPLEX NUMBER is sometimes called the PHASE.

See also AFFIX, COMPLEX NUMBER, DE MOIVRE'S IDENTITY, EULER FORMULA, IMAGINARY PART, INVERSE TANGENT, MODULUS (COMPLEX NUMBER), PHASE, PHASOR, REAL PART

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- Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover, p. 16, 1972.
- Krantz, S. G. "The Argument of a Complex Number." §1.2.6 in *Handbook of Complex Analysis*. Boston, MA: Birkhäuser, p. 11, 1999.
- Silverman, R. A. *Introductory Complex Analysis*. New York: Dover, 1984.

Argument (Elliptic Integral)

Given an AMPLITUDE ϕ in an ELLIPTIC INTEGRAL, the argument u is defined by the relation

$$\phi \equiv am u.$$

See also AMPLITUDE, ELLIPTIC INTEGRAL

Argument (Function)

An argument of a FUNCTION $f(x_1, \dots, x_n)$ is one of the n parameters on which the function's value depends. For example, the SINE $\sin x$ is a one-argument function, the BINOMIAL COEFFICIENT $\binom{n}{m}$ is a two-argument function, and the HYPERGEOMETRIC FUNCTION ${}_2F_1(a, b; c; z)$ is a four-argument function.

Argument Addition Relation

A mathematical relationship relating $f(x+y)$ to $f(x)$ and $f(y)$.

See also ARGUMENT MULTIPLICATION RELATION, RECURRENCE RELATION, REFLECTION RELATION, TRANSLATION RELATION

Argument Multiplication Relation

A mathematical relationship relating $f(nx)$ to $f(x)$ for INTEGER n .

See also ARGUMENT ADDITION RELATION, RECURRENCE RELATION, REFLECTION RELATION, TRANSLATION RELATION

Argument Principle

If $f(z)$ is MEROMORPHIC in a region R enclosed by a CONTOUR γ , let N be the number of COMPLEX ROOTS of $f(z)$ in γ , and P be the number of POLES in γ , then

$$N - P = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z) dz}{f(z)}$$

Defining $w \equiv f(z)$ and $\sigma \equiv f(\gamma)$ gives

$$N - P = \frac{1}{2\pi i} \int_{\sigma} \frac{dw}{w}.$$

See also CAUCHY INTEGRAL FORMULA, CAUCHY INTEGRAL THEOREM, HURWITZ'S ROOT THEOREM, MEROMORPHIC FUNCTION, POLE, ROOT, ROUCHÉ'S THEOREM, VARIATION OF ARGUMENT

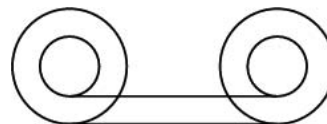
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- Krantz, S. G. "The Argument Principle." Ch. 5 in *Handbook of Complex Analysis*. Boston, MA: Birkhäuser, pp. 69–8, 1999.

Argument Variation

VARIATION OF ARGUMENT

Aristotle's Wheel Paradox



A PARADOX mentioned in the Greek work *Mechanica*, dubiously attributed to Aristotle. Consider the above diagram depicting a wheel consisting of two concentric CIRCLES of different DIAMETERS (a wheel within a wheel). there is a 1:1 correspondence of points on the large CIRCLE with points on the small CIRCLE, so the wheel should travel the same distance regardless of whether it is rolled from left to right on the top straight line or on the bottom one. this seems to imply that the two CIRCUMFERENCES of different sized CIRCLES are equal, which is impossible.

The fallacy lies in the assumption that a 1:1 correspondence of points means that two curves must have the same length. In fact, the CARDINALITIES of points in a LINE SEGMENT of any length (or even an INFINITE LINE, a PLANE, a 3-D SPACE, or an infinite dimensional EUCLIDEAN SPACE) are all the same: $\aleph_1$ (ALEPH-1), so the points of any of these can be put in a ONE-TO-ONE correspondence with those of any other.

See also ZENO'S PARADOXES

References

- Ballew, D. "The Wheel of Aristotle." *Math. Teacher* **65**, 507–09, 1972.
- Costabel, P. "The Wheel of Aristotle and French Consideration of Galileo's Arguments." *Math. Teacher* **61**, 527–34, 1968.
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- vos Savant, M. *The World's Most Famous Math Problem*. New York: St. Martin's Press, pp. 48–0, 1993.

Arithmetic

The branch of mathematics dealing with INTEGERS or, more generally, numerical computation. Arithmetical operations include ADDITION, CONGRUENCE calculation, DIVISION, FACTORIZATION, MULTIPLICATION, POWER computation, ROOT EXTRACTION, and SUBTRACTION. Arithmetic was part of the QUADRIVIVUM taught in medieval universities.

The FUNDAMENTAL THEOREM OF ARITHMETIC, also called the UNIQUE FACTORIZATION THEOREM, states that any POSITIVE INTEGER can be represented in exactly one way as a PRODUCT of PRIMES.

The LÖWENHEIM-SKOLEM THEOREM, which is a fundamental result in MODEL THEORY, establishes the existence of "nonstandard" models of arithmetic.

See also ALGEBRA, CALCULUS, FLOATING-POINT ARITHMETIC, FUNDAMENTAL THEOREM OF ARITHMETIC, GROUP THEORY, HIGHER ARITHMETIC, LINEAR ALGEBRA, LÖWENHEIM-SKOLEM THEOREM, MODEL THEORY, NUMBER THEORY, TRIGONOMETRY

References

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 Weisstein, E. W. "Books about Arithmetic." <http://www.treasure-troves.com/books/Arithmetic.html>.

Arithmetic Function

A function $\psi(n)$ such that

$$\psi(n+m) = \psi(\psi(n) + \psi(m))$$

and

$$\psi(n, m) = \psi(\psi(n)\psi(m)).$$

See also ARITHMETICAL FUNCTION

References

- Atanassov, K. *Bull. Number Th.* **9**, 18, 1985.
 Trott, M. "Numerical Computations." §1.2.1 in *The Mathematica Guidebook, Vol. 1: Programming in Mathematica*. New York: Springer-Verlag, 2000.

Arithmetic Geometry

A vaguely defined branch of mathematics dealing with VARIETIES, the MORDELL CONJECTURE, ARAKELOV THEORY, and ELLIPTIC CURVES.

References

- Cornell, G. and Silverman, J. H. (Eds.). *Arithmetic Geometry*. New York: Springer-Verlag, 1986.
 Lorenzini, D. *An Invitation to Arithmetic Geometry*. Providence, RI: Amer. Math. Soc., 1996.

Arithmetic Mean

For a CONTINUOUS DISTRIBUTION FUNCTION, the arithmetic mean of the population, denoted μ , $\tilde{x}$, $\langle x \rangle$, or $A(x)$, is given by

$$\mu = \langle f(x) \rangle \equiv \int_{-\infty}^{\infty} P(x)f(x) dx, \quad (1)$$

where $\langle x \rangle$ is the EXPECTATION VALUE. For a DISCRETE DISTRIBUTION,

$$\mu = \langle f(x) \rangle \equiv \frac{\sum_{n=0}^N P(x_n)f(x_n)}{\sum_{n=0}^N P(x_n)} = \sum_{n=0}^N P(x_n)f(x_n). \quad (2)$$

The population mean satisfies

$$\langle f(x) + g(x) \rangle = \langle f(x) \rangle + \langle g(x) \rangle \quad (3)$$

Arithmetic Mean

$$\langle cf(x) \rangle = c\langle f(x) \rangle, \quad (4)$$

and

$$\langle f(x)g(y) \rangle = \langle f(x) \rangle \langle g(y) \rangle \quad (5)$$

if x and y are INDEPENDENT STATISTICS. The "sample mean," which is the mean estimated from a statistical sample, is an UNBIASED ESTIMATOR for the population mean.

For small samples, the mean is more efficient than the MEDIAN and approximately $\pi/2$ less (Kenney and Keeping 1962, p. 211). A general expression which often holds approximately is

$$\text{mean} - \text{mode} \approx 3(\text{mean} - \text{median}). \quad (6)$$

Given a set of samples $\{x_i\}$, the arithmetic mean is

$$A(x) \equiv \tilde{x} \equiv \mu \equiv \langle x \rangle = \frac{1}{N} \sum_{i=1}^N x_i. \quad (7)$$

Hoehn and Niven (1985) show that

$$\begin{aligned} A(a_1 + c, a_2 + c, \dots, a_n + c) \\ = c + A(a_1, a_2, \dots, a_n) \end{aligned} \quad (8)$$

for any POSITIVE constant c . For positive arguments, the arithmetic mean satisfies

$$A \geq G \geq H, \quad (9)$$

where G is the GEOMETRIC MEAN and H is the HARMONIC MEAN (Hardy *et al.* 1952; Mitrinovic 1970; Beckenbach and Bellman 1983; Bullen *et al.* 1988; Mitrinovic *et al.* 1993; Alzer 1996). This can be shown as follows. For $a, b > 0$,

$$\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right)^2 \geq 0 \quad (10)$$

$$\frac{1}{a} - \frac{2}{\sqrt{ab}} + \frac{1}{b} \geq 0 \quad (11)$$

$$\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}} \quad (12)$$

$$\sqrt{ab} \geq \frac{2}{\frac{1}{a} + \frac{1}{b}} \quad (13)$$

$$G \geq H, \quad (14)$$

with equality IFF $b = a$. To show the second part of the inequality,

$$(\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b \geq 0 \quad (15)$$

$$\frac{a+b}{2} \geq \sqrt{ab} \quad (16)$$

$$A \geq G, \quad (17)$$

with equality IFF $a = b$. Combining (14) and (17) then gives (9).

Given n independent random GAUSSIAN DISTRIBUTED variates x_i , each with population mean $\mu_i = \mu$ and VARIANCE $\sigma_i^2 = \sigma^2$,

$$\tilde{x} \equiv \frac{1}{N} \sum_{i=1}^N x_i \quad (18)$$

$$\begin{aligned} \langle x \rangle &= \frac{1}{N} \left\langle \sum_{i=1}^N x_i \right\rangle = \frac{1}{N} \sum_{i=1}^N \langle x_i \rangle \\ &= \frac{1}{N} \sum_{i=1}^N \mu = \frac{1}{N} (N\mu) = \mu, \end{aligned} \quad (19)$$

so the sample mean is an UNBIASED ESTIMATOR of population mean. However, the distribution of $\tilde{x}$ depends on the sample size. For large samples, $\tilde{x}$ is approximately NORMAL. For small samples, STUDENT'S T -DISTRIBUTION should be used.

The VARIANCE of the sample mean is independent of the distribution.

$$\begin{aligned} \text{var}(\tilde{x}) &= \text{var} \left(\frac{1}{N} \sum_{i=1}^N x_i \right) = \frac{1}{N^2} \text{var} \left(\sum_{i=1}^N x_i \right) \\ &= \frac{1}{N^2} \sum_{i=1}^N \text{var}(x_i) = \left(\frac{1}{N^2} \right) \sum_{i=1}^N \sigma^2 = \frac{\sigma^2}{N}. \end{aligned} \quad (20)$$

From K -STATISTIC for a GAUSSIAN DISTRIBUTION, the UNBIASED ESTIMATOR for the VARIANCE is given by

$$\sigma^2 = \frac{N}{N-1} s^2, \quad (21)$$

where

$$s \equiv \frac{1}{N} \sum_{i=1}^N (x_i - \tilde{x})^2, \quad (22)$$

so

$$\text{var}(\tilde{x}) = \frac{s^2}{N-1}. \quad (23)$$

The SQUARE ROOT of this,

$$\sigma_x = \frac{s}{\sqrt{N-1}}, \quad (24)$$

is called the STANDARD ERROR.

$$\text{var}(\tilde{x}) \equiv \langle \tilde{x}^2 \rangle - \langle \tilde{x} \rangle^2, \quad (25)$$

so

$$\langle \tilde{x}^2 \rangle = \text{var}(\tilde{x}) + \langle \tilde{x} \rangle^2 = \frac{\sigma^2}{N} + \mu^2. \quad (26)$$

See also ARITHMETIC-GEOMETRIC MEAN, ARITHMETIC-HARMONIC MEAN, CARLEMAN'S INEQUALITY, CUMU-

LANT, GENERALIZED MEAN, GEOMETRIC MEAN, HARMONIC MEAN, HARMONIC-GEOMETRIC MEAN, KURTOSIS, MEAN, MEAN DEVIATION, MEDIAN (STATISTICS), MODE, MOMENT, QUADRATIC MEAN, ROOT-MEAN-SQUARE, SAMPLE VARIANCE, SKEWNESS, STANDARD DEVIATION, TRIMEAN, VARIANCE

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Arithmetic Progression

ARITHMETIC SEQUENCE

Arithmetic Sequence

A SEQUENCE of n numbers $\{d_0 + kd\}_{k=0}^{n-1}$ such that the differences between successive terms is a constant d .

See also ARITHMETIC SERIES, BAUDET'S CONJECTURE, NONARITHMETIC PROGRESSION SEQUENCE, SEQUENCE, SZEMERÉD'S THEOREM

Arithmetic Series

An arithmetic series is the SUM of a SEQUENCE $\{a_k\}$, $k = 1, 2, \dots$, in which each term is computed from the previous one by adding (or subtracting) a constant d . Therefore, for $k > 1$,

$$a_k = a_{k-1} + d = a_{k-2} + 2d = \dots = a_1 + d(k-1). \quad (1)$$

The sum of the sequence of the first n terms is then given by

$$\begin{aligned} S_n &\equiv \sum_{k=1}^n a_k = \sum_{k=1}^n [a_1 + (k-1)d] = na_1 + d \sum_{k=1}^n (k-1) \\ &= na_1 + d \sum_{k=2}^n (k-1) \end{aligned}$$

$$= na_1 + d \sum_{k=1}^{n-1} k \quad (2)$$

Using the SUM identity

$$\sum_{k=1}^n k = \frac{1}{2}n(n+1) \quad (3)$$

then gives

$$S_n = na_1 + \frac{1}{2}dn(n-1) = \frac{1}{2}n[2a_1 + d(n-1)]. \quad (4)$$

Note, however, that

$$a_1 + an = a_1 + [a_1 + d(n-1)] = 2a_1 + d(n-1), \quad (5)$$

so

$$S_n = \frac{1}{2}n(a_1 + a_n), \quad (6)$$

or n times the AVERAGE of the first and last terms! This is the trick Gauss used as a schoolboy to solve the problem of summing the INTEGERS from 1 to 100 given as busy-work by his teacher. While his classmates toiled away doing the ADDITION longhand, Gauss wrote a single number, the correct answer

$$\frac{1}{2}(100)(1+100) = 50 \cdot 101 = 5050 \quad (7)$$

on his slate (Burton 1989, pp. 80–1; Hoffman 1998, p. 207). When the answers were examined, Gauss's proved to be the only correct one.

See also ARITHMETIC SEQUENCE, GEOMETRIC SERIES, HARMONIC SERIES, PRIME ARITHMETIC PROGRESSION

References

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Arithmetical Function

INTEGER FUNCTION

Arithmetic-Geometric Mean

The arithmetic-geometric mean (often abbreviated AGM) $M(a, b)$ of two numbers a and b is defined by starting with $a_0 \equiv a$ and $b_0 \equiv b$, then iterating

$$a_{n+1} = \frac{1}{2}(a_n + b_n) \quad (1)$$

Arithmetic-Geometric Mean

$$b_{n+1} = \sqrt{a_n b_n} \quad (2)$$

until $a_n = b_n$. a_n and b_n converge towards each other since

$$\begin{aligned} a_{n+1} - b_{n+1} &= \frac{1}{2}(a_n + b_n) - \sqrt{a_n b_n} \\ &= \frac{a_n - 2\sqrt{a_n b_n} + b_n}{2}. \end{aligned} \quad (3)$$

But $\sqrt{b_n} < \sqrt{a_n}$, so

$$2b_n < 2\sqrt{a_n b_n}. \quad (4)$$

Now, add $a_n - b_n - 2\sqrt{a_n b_n}$ to each side

$$a_n + b_n - 2\sqrt{a_n b_n} < a_n - b_n, \quad (5)$$

so

$$a_{n+1} - b_{n+1} < \frac{1}{2}(a_n - b_n). \quad (6)$$

The AGM is very useful in computing the values of complete ELLIPTIC INTEGRALS and can also be used for finding the INVERSE TANGENT. In terms of the complete ELLIPTIC INTEGRAL OF THE FIRST KIND $K(k)$,

$$M(a, b) = \frac{(a+b)\pi}{4K\left(\frac{a-b}{a+b}\right)}. \quad (7)$$

The special value $1/M(1, \sqrt{2})$ is called GAUSS'S CONSTANT.

The AGM has the properties

$$\lambda M(a, b) = M(\lambda a, \lambda b) \quad (8)$$

$$M(a, b) = M\left(\frac{1}{2}(a+b), \sqrt{ab}\right) \quad (9)$$

$$M(1, \sqrt{1-x^2}) = M(1+x, 1-x) \quad (10)$$

$$M(1, b) = \frac{1+b}{2} M\left(1, \frac{2\sqrt{b}}{1+b}\right). \quad (11)$$

The Legendre form is given by

$$M(1, x) = \prod_{n=0}^{\infty} \frac{1}{2}(1+k_n), \quad (12)$$

where $k_0 \equiv x$ and

$$k_{n+1} \equiv \frac{2\sqrt{k_n}}{1+k_n}. \quad (13)$$

Solutions to the differential equation

$$(x^3 - x) \frac{d^2 y}{dx^2} + (3x^2 - 1) \frac{dy}{dx} + xy = 0 \quad (14)$$

are given by $[M(1+x, 1-x)]^{-1}$ and $[M(1, x)]^{-1}$.

A generalization of the ARITHMETIC-GEOMETRIC MEAN is

$$I_p(a, b) = \int_0^\infty \frac{x^{p-2} dx}{(x^p + a^p)^{1/p} (x^p + b^p)^{(p-1)/p}} \quad (15)$$

which is related to solutions of the differential equation

$$x(1-x^p)Y'' + [1-(p+1)x^p]Y' - (p-1)x^{p-1}Y = 0. \quad (16)$$

When $p = 2$ or $p = 3$, there is a modular transformation for the solutions of (16) that are bounded as $x \rightarrow 0$. Letting $J_p(x)$ be one of these solutions, the transformation takes the form

$$J_p(\lambda) = \mu J_p(x), \quad (17)$$

where

$$\lambda = \frac{1-u}{1+(p-1)u} \quad (18)$$

$$\mu = \frac{1+(p-1)u}{p} \quad (19)$$

and

$$x^p + u^p = 1. \quad (20)$$

The case $p = 2$ gives the ARITHMETIC-GEOMETRIC MEAN, and $p = 3$ gives a cubic relative discussed by Borwein and Borwein (1990, 1991) and Borwein (1996) in which, for $a, b > 0$ and $I(a, b)$ defined by

$$I(a, b) = \int_0^\infty \frac{t dt}{[(a^3 + t^3)(b^3 + t^3)^2]^{1/3}}, \quad (21)$$

$$I(a, b) = I\left(\frac{a+2b}{3}, \left[\frac{b}{3}(a^2+ab+b^2)\right]\right) \quad (22)$$

For iteration with $a_0 = a$ and $b_0 = b$ and

$$a_{n+1} = \frac{a_n + 2b_n}{3} \quad (23)$$

$$b_{n+1} = \frac{b_n}{3}(a_n^2 + a_n b_n + b_n^2), \quad (24)$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \frac{I(1, 1)}{I(a, b)}. \quad (25)$$

Modular transformations are known when $p = 4$ and $p = 6$, but they do not give identities for $p = 6$ (Borwein 1996).

See also ARITHMETIC-HARMONIC MEAN

References

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Arithmetic-Harmonic Mean

Let

$$a_{n+1} = \frac{1}{2}(a_n + b_n) \quad (1)$$

$$b_{n+1} = \frac{2a_n b_n}{a_n + b_n}. \quad (2)$$

Then

$$A(a_0, b_0) = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n \sqrt{a_0 b_0}, \quad (3)$$

which is just the GEOMETRIC MEAN.

Arithmetic-Logarithmic-Geometric Mean Inequality

$$\frac{a+b}{2} > \frac{b-a}{\ln b - \ln a} > \sqrt{ab}.$$

See also NAPIER'S INEQUALITY

References

- Nelson, R. B. "Proof without Words: The Arithmetic-Logarithmic-Geometric Mean Inequality." *Math. Mag.* **68**, 305, 1995.

Armstrong Number

The n -digit numbers equal to sum of n th powers of their digits (a finite sequence), also called plus perfect numbers. They first few are given by 1, 2, 3, 4, 5, 6, 7, 8, 9, 153, 370, 371, 407, 1634, 8208, 9474, 54748, ... (Sloane's A005188).

See also HARSHAD NUMBER, NARCISSISTIC NUMBER

References

- Sloane, N. J. A. Sequences A005188/M0488 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.

Arnold Diffusion

The nonconservation of ADIABATIC INVARIANTS which arises in systems with three or more DEGREES OF FREEDOM.

References

- Lichtenberg, A. and Lieberman, M. *Regular and Stochastic Motion*, 2nd ed. New York: Springer-Verlag, 1994.

Rasband, S. N. "Arnold Diffusion." §8.6 in *Chaotic Dynamics of Nonlinear Systems*. New York: Wiley, pp. 179–81, 1990.

Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An Introduction*. New York: Wiley, p. 74, 1989.

Arnold Tongue

Consider the CIRCLE MAP. If K is NONZERO, then the motion is periodic in some FINITE region surrounding each rational Ω . This execution of periodic motion in response to an irrational forcing is known as MODE LOCKING. If a plot is made of K versus Ω with the regions of periodic MODE-LOCKED parameter space plotted around rational Ω values (the WINDING NUMBERS), then the regions are seen to widen upward from 0 at $K = 0$ to some FINITE width at $K = 1$. The region surrounding each RATIONAL NUMBER is known as an ARNOLD TONGUE.

At $K = 0$, the Arnold tongues are an isolated set of MEASURE zero. At $K = 1$, they form a general CANTOR SET of dimension $d = 0.8700 \pm 3.7 \times 10^{-4}$ (Rasband 1990, p. 131). In general, an Arnold tongue is defined as a resonance zone emanating out from RATIONAL NUMBERS in a two-dimensional parameter space of variables.

See also CIRCLE MAP, DEVIL'S STAIRCASE

References

Rasband, S. N. *Chaotic Dynamics of Nonlinear Systems*. New York: Wiley, pp. 130–31, 1990.

Arnold's Cat Map

The best known example of an ANOSOV DIFFEOMORPHISM. It is given by the TRANSFORMATION

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad (1)$$

where x_{n+1} and y_{n+1} are computed mod 1. The Arnold cat mapping is non-Hamiltonian, nonanalytic, and mixing. However, it is AREA-PRESERVING since the DETERMINANT is 1. The LYAPUNOV CHARACTERISTIC EXPONENTS are given by

$$\begin{vmatrix} 1-\sigma & 1 \\ 1 & 2-\sigma \end{vmatrix} = \sigma^2 - 3\sigma + 1 = 0, \quad (2)$$

so

$$\sigma_{\pm} = \frac{1}{2}(3 \pm \sqrt{5}). \quad (3)$$

The EIGENVECTORS are found by plugging $\sigma_{\pm}$ into the MATRIX EQUATION

$$\begin{bmatrix} 1-\sigma_{\pm} & 1 \\ 1 & 2-\sigma_{\pm} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (4)$$

For σ_{+} , the solution is

$$y = \frac{1}{2}(1 + \sqrt{5})x \equiv \phi x, \quad (5)$$

where ϕ is the GOLDEN RATIO, so the unstable

(normalized) EIGENVECTOR is

$$\xi_{+} = \frac{1}{10}\sqrt{50 - 10\sqrt{5}} \begin{bmatrix} 1 \\ \frac{1}{2}(1 + \sqrt{5}) \end{bmatrix}. \quad (6)$$

Similarly, for σ_{-} , the solution is

$$y = -\frac{1}{2}(\sqrt{5} - 1)x \equiv \phi^{-1}x, \quad (7)$$

so the stable (normalized) EIGENVECTOR is

$$\xi_{-} = \frac{1}{10}\sqrt{50 + 10\sqrt{5}} \begin{bmatrix} 1 \\ \frac{1}{2}(1 - \sqrt{5}) \end{bmatrix}. \quad (8)$$

See also ANOSOV MAP

Aronhold Process

The process used to generate an expression for a covariant in the first degree of any one of the equivalent sets of COEFFICIENTS for a curve.

See also CLEBSCH-ARONHOLD NOTATION, JOACHIMSTHAL'S EQUATION

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves*. New York: Dover, p. 74, 1959.

Aronson's Sequence

The sequence whose definition is: "t is the first, fourth, eleventh, ... letter of this sentence." The first few values are 1, 4, 11, 16, 24, 29, 33, 35, 39, ... (Sloane's A005224).

References

Hofstadter, D. R. *Metamagical Themas: Questing of Mind and Pattern*. New York: BasicBooks, p. 44, 1985.
Sloane, N. J. A. Sequences A005224/M3406 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.

Arrangement

In general, an arrangement of objects is simply a grouping of them. The number of "arrangements" of n items is given either by a COMBINATION (order is ignored) or PERMUTATION (order is significant).

The division of SPACE into cells by a collection of HYPERPLANES (Agarwal and Sharir 2000) is also called an arrangement.

See also COMBINATION, CONFIGURATION, CUTTING, HYPERPLANE, ORDERING, PERMUTATION

References

Agarwal, P. K. and Sharir, M. "Arrangements and Their Applications." Ch. 2 in *Handbook of Computational Geometry* (Ed. J.-R. Sack and J. Urrutia). Amsterdam, Netherlands: North-Holland, pp. 49–19, 2000.

Arrangement Number

PERMUTATION

Array

An array is a "list of lists" with the length of each level of list the same. The size (sometimes called the "shape") of a d -dimensional array is then indicated as $\underbrace{m \times n \times x \cdots \times p}_d$. The most common type of array

encountered is the 2-D $m \times n$ rectangular array having m columns and n rows. If $m = n$, a square array results. Sometimes, the order of the elements in an array is significant (as in a MATRIX), whereas at other times, arrays which are equivalent modulo reflections (and rotations, in the case of a square array) are considered identical (as in a MAGIC SQUARE or PRIME ARRAY).

In order to exhaustively list the number of distinct arrays of a given shape with each element being one of k possible choices, the naive algorithm of running through each case and checking to see whether it's equivalent to an earlier one is already just about as efficient as can be. The running time must be at least the number of answers, and this is so close to $k^{mn \cdots p}$ that the difference isn't significant.

However, finding the *number* of possible arrays of a given shape is much easier, and an exact formula can be obtained using the POLYA ENUMERATION THEOREM. For the simple case of an $m \times n$ array, even this proves unnecessary since there are only a few possible symmetry types, allowing the possibilities to be counted explicitly. For example, consider the case of m and n EVEN and distinct, so only reflections need be included. To take a specific case, let $m = 6$ and $n = 4$ so the array looks like

$$\begin{array}{cccccc} a & b & c & \vdots & d & e & f \\ g & h & i & \vdots & j & k & l \\ \cdots & \cdots & \cdots & + & \cdots & \cdots & \cdots \\ m & n & o & \vdots & p & q & r \\ s & t & u & \vdots & v & w & x \end{array}$$

where each $a, b, \dots, x$ can take a value from 1 to k . The total number of possible arrangements is k^{24} (k^{mn} in general). The number of arrangements which are equivalent to their left-right mirror images is k^{12} (in general, $k^{mn/2}$), as is the number equal to their up-down mirror images, or their rotations through 180° . There are also k^6 arrangements (in general, $k^{mn/4}$) with full symmetry.

In general, it is therefore true that

$$\begin{cases} k^{mn/4} & \text{with full symmetry} \\ k^{mn/2} - k^{mn/4} & \text{with only left-right reflection} \\ k^{mn/2} - k^{mn/4} & \text{with only up-down reflection} \\ k^{mn/2} - k^{mn/4} & \text{with only } 180^\circ \text{ rotation,} \end{cases}$$

so there are

$$k^{mn} - 3k^{mn/2} + 2k^{mn/4}$$

arrangements with no symmetry. Now dividing by the number of images of each type, the result, for $m \neq n$ with m, n EVEN, is

$$\begin{aligned} N(m, n, k) &= \frac{1}{4} k^{mn} + \left(\frac{1}{2}\right)(3)(k^{mn/2} - k^{mn/4}) \\ &\quad + \frac{1}{4}(k^{mn} - 3k^{mn/2} + 2k^{mn/4}) \\ &= \frac{1}{4} k^{mn} + \frac{3}{4} k^{mn/2} + \frac{1}{2} k^{mn/4}. \end{aligned}$$

The number is therefore of order $\mathcal{O}(k^{mn/4})$, with "correction" terms of much smaller order.

See also ANTIMAGIC SQUARE, EULER SQUARE, KIRKMAN'S SCHOOLGIRL PROBLEM, LATIN RECTANGLE, LATIN SQUARE, MAGIC SQUARE, MATRIX, MRS. PERKINS' QUILT, MULTIPLICATION TABLE, ORTHOGONAL ARRAY, PERFECT SQUARE, PRIME ARRAY, QUOTIENT-DIFFERENCE TABLE, ROOM SQUARE, STOLARSKY ARRAY, TRUTH TABLE, WYTHOFF ARRAY

Arrow Notation

A NOTATION invented by Knuth (1976) to represent LARGE NUMBERS in which evaluation proceeds from the right (Conway and Guy 1996, p. 60).

$$\begin{array}{cc} m \uparrow n & \overbrace{m \cdot m \cdots m}^n \\ m \uparrow \uparrow n & \overbrace{m \uparrow m \uparrow \cdots \uparrow m}^n \\ m \uparrow \uparrow \uparrow n & \underbrace{\overbrace{m \uparrow \uparrow m \uparrow \uparrow \cdots \uparrow \uparrow m}^n}_n \end{array}$$

For example,

$$m \uparrow n = m^n \quad (1)$$

$$\begin{aligned} m \uparrow \uparrow n &= \underbrace{m \uparrow \cdots \uparrow m}_n = \underbrace{m^{m \cdots m}}_n \\ m \uparrow \uparrow 2 &= \underbrace{m \uparrow m}_2 = m \uparrow m = m^m \end{aligned} \quad (2)$$

$$\begin{aligned} m \uparrow \uparrow 3 &= \underbrace{m \uparrow m \uparrow m}_3 = m \uparrow (m \uparrow m) \\ &= m \uparrow m^m = m^{m^m} \end{aligned} \quad (3)$$

$$\begin{aligned} m \uparrow \uparrow \uparrow 2 &= \underbrace{m \uparrow m}_2 = m \uparrow m = \underbrace{m^{m \cdots m}}_m \\ m \uparrow \uparrow \uparrow 3 &= \underbrace{m \uparrow m m \uparrow m}_3 = m \uparrow \uparrow m = \underbrace{m^{m \cdots m}}_m \end{aligned} \quad (4)$$

$$= \underbrace{m \uparrow \cdots \uparrow m}_m = \underbrace{m^{m \cdot m}}_m \quad (5)$$

$m \uparrow \uparrow m$ is sometimes called a POWER TOWER. The values $\underbrace{n \uparrow \cdots \uparrow n}$ are called ACKERMANN NUMBERS.

See also ACKERMANN NUMBER, CHAINED ARROW NOTATION, DOWN ARROW NOTATION, LARGE NUMBER, POWER TOWER, STEINHAUS-MOSER NOTATION

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Arrow's Paradox

Perfect democratic VOTING is, not just in practice but in principle, impossible.

See also SOCIAL CHOICE THEORY, VOTING

References

- Erickson, G. W. and Fossa, J. A. *Dictionary of Paradox*. Lanham, MD: University Press of America, pp. 13–5, 1998.
- Gardner, M. *Time Travel and Other Mathematical Bewilderments*. New York: W. H. Freeman, p. 56, 1988.

Arrowhead Curve

SIERPINSKI ARROWHEAD CURVE

Arsh

$$\text{Arsh } z = \frac{1}{i} \sin^{-1}(iz),$$

where $\sin^{-1} z$ is the INVERSE SINE.

See also ARCH, ARCTH, ARTH, INVERSE SINE

References

- Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products*, 6th ed. San Diego, CA: Academic Press, p. xxx, 2000.

Art Gallery Theorem

Also called Chvátal's art gallery theorem. If the walls of an art gallery are made up of n straight LINE SEGMENTS, then the entire gallery can always be supervised by $\lfloor n/3 \rfloor$ watchmen placed in corners, where $\lfloor x \rfloor$ is the FLOOR FUNCTION. This theorem was proved by Chvátal (1975). It was conjectured that an art gallery with n walls and h HOLES requires

$\lfloor (n+h)/3 \rfloor$ watchmen, which has now been proven by Bjorling-Sachs and Souvaine (1991, 1995) and Hoffman *et al.* (1991).

See also ILLUMINATION PROBLEM, TRIANGULATION, VORONOI DIAGRAM

References

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Arth

$$\text{Arth } z = \frac{1}{i} \tan^{-1}(iz).$$

where $\tan^{-1} z$ is the INVERSE TANGENT.

See also ARCH, ARSH, ARCTH, INVERSE TANGENT

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products*, 6th ed. San Diego, CA: Academic Press, p. xxx, 2000.

Articulation Vertex

An articulation of a CONNECTED GRAPH is a node whose removal will disconnect the graph (Chartrand 1985). In general, an articulation vertex is node of a GRAPH whose removal increases the number of components (Harary 1994, p. 26). Articulation vertices are also called cut-vertices or "cutpoints" (Harary 1994, p. 26).

A GRAPH with no articulation vertices is called a BICONNECTED GRAPH.

See also BICONNECTED GRAPH, BLOCK, BRIDGE, CUT SET, NONSEPARABLE GRAPH, VERTEX (GRAPH)

References

Chartrand, G. "Cut-Vertices and Bridges." §2.4 in *Introductory Graph Theory*. New York: Dover, pp. 45–9, 1985.
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Artin Braid Group

BRAID GROUP

Artin L-Function

An Artin L -function over the RATIONALS $\mathbb{Q}$ encodes in a GENERATING FUNCTION information about how an irreducible MONIC POLYNOMIAL over $\mathbb{Z}$ factors when reduced modulo each PRIME. For the POLYNOMIAL $x^2 + 1$, the Artin L -function is

$$L(s, \mathbb{Q}(i)/\mathbb{Q}, \text{sgn}) = \prod_{p \text{ odd prime}} \frac{1}{1 - \left(\frac{-1}{p}\right) p^{-s}},$$

where $(-1/p)$ is a LEGENDRE SYMBOL, which is equivalent to the EULER L -FUNCTION. The definition over arbitrary POLYNOMIALS generalizes the above expression.

See also LANGLANDS RECIPROCITY

References

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Artin Reciprocity

ARTIN'S RECIPROCITY THEOREM

Artin's Conjecture

There are at least two statements which go by the name of Artin's conjecture. The first is the RIEMANN HYPOTHESIS.

The second states that every INTEGER not equal to -1 or a SQUARE NUMBER is a primitive root modulo p for infinitely many p and proposes a density for the set of such p which are always rational multiples of a constant known as ARTIN'S CONSTANT. There is an analogous theorem for functions instead of numbers which has been proved by Billharz (Shanks 1993, p. 147).

See also ARTIN'S CONSTANT, RIEMANN HYPOTHESIS

References

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Ram Murty, M. "Artin's Conjecture for Primitive Roots." *Math. Intell.* **10**, 59–7, 1988.
Shanks, D. *Solved and Unsolved Problems in Number Theory*, 4th ed. New York: Chelsea, pp. 31, 80–3, and 147, 1993.

Artin's Constant

If $n \neq -1$ and n is not a PERFECT SQUARE, then Artin conjectured that the SET $S(n)$ of all PRIMES for which n is a PRIMITIVE ROOT is infinite. Under the assumption of the EXTENDED RIEMANN HYPOTHESIS, Artin's conjecture was solved by Hooley (1967).

If, in addition, n is not an r th POWER for any $r > 1$ then let n' be the SQUAREFREE PART of n and suppose that $n' \not\equiv 1 \pmod{4}$. Then Artin conjectured that the density of $S(n)$ relative to the PRIMES is given by C_{Artin} , where

$$C_{\text{Artin}} = \prod_{k=1}^{\infty} \left[1 - \frac{1}{p_k(p_k - 1)} \right] = 0.3739558136 \dots, \quad (1)$$

and p_k is the k th PRIME, independently of the choice of n .

C_{Artin} is connected with the PRIME ZETA FUNCTION $P(n)$ by

$$\ln C_{\text{Artin}} = - \sum_{n=2}^{\infty} \frac{(u_n - 1)P(n)}{n}, \quad (2)$$

where

$$u_n = u_{n-1} + u_{n-2} \quad (3)$$

with $u_1 = 1$, $u_2 = 3$ (Ribenoim 1998, Gourdon and Sebah). Wrench (1961) gave 45 digits of C_{Artin} , and Gourdon and Sebah give 60.

If $n' \equiv 1 \pmod{4}$ and n is still restricted not to be an r th power, then the density is not C_{Artin} itself, but a rational multiple thereof. The explicit formula for computing the density in this case is conjectured to be

$$C'_{\text{Artin}} = \left[1 - \mu(n') \prod_{\substack{\text{prime } q \\ q | n'}} \frac{1}{q^2 - q - 1} \right] C_{\text{Artin}} \quad (4)$$

(Finch, Matthews 1976), where $\mu(n)$ is the MÖBIUS

FUNCTION. Special cases can be written down explicitly for $n' = p$ a PRIME,

$$C'_{\text{Artin}} = \left(1 + \frac{1}{p^2 - p - 1}\right) C_{\text{Artin}} \quad (5)$$

or $n' = pq$, where p, q are both PRIMES with $u, v \equiv 1 \pmod{4}$,

$$C'_{\text{Artin}} = \left(1 + \frac{1}{p^2 - p - 1} \frac{1}{q^2 - q - 1}\right) C_{\text{Artin}}, \quad (6)$$

If n is a perfect cube (which is not a perfect square), a perfect fifth power (which is not a perfect square or perfect cube), etc., other formulas apply (Hooley 1967, Western and Miller 1968).

The significance of Artin's constant is more easily seen by describing it as the fraction of PRIMES p for which $1/p$ has a maximal DECIMAL EXPANSION, i.e., p is a FULL REPTEND PRIME, (Conway and Guy 1996).

See also ARTIN'S CONJECTURE, DECIMAL EXPANSION, FULL REPTEND PRIME, PRIMITIVE ROOT, STEPHENS' CONSTANT

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- Western, A. E. and Miller, J. C. P. *Tables of Indices and Primitive Roots*. Cambridge, England: Cambridge University Press, pp. xxxvii-xlii, 1968.

Wrench, J. W. "Evaluation of Artin's Constant and the Twin Prime Constant." *Math. Comput.* **15**, 396–98, 1961.

Artin's Reciprocity Theorem

A general RECIPROCITY THEOREM for all orders which covered all other known reciprocity theorems when proved by E. Artin in 1927. If R is a NUMBER FIELD and R' a finite integral extension, then there is a SURJECTION from the group of fractional IDEALS prime to the discriminant, given by the Artin symbol. For some cycle c , the kernel of this SURJECTION contains each PRINCIPAL fractional IDEAL generated by an element congruent to 1 mod c .

See also LANGLANDS PROGRAM

Artinian Group

A GROUP in which any decreasing CHAIN of distinct SUBGROUPS terminates after a FINITE number.

Artinian Ring

A noncommutative SEMISIMPLE RING satisfying the "descending chain condition."

See also GORENSTEIN RING, SEMISIMPLE RING

References

- Artin, E. "Zur Theorie der hyperkomplexer Zahlen." *Hamb. Abh.* **5**, 251–60, 1928.
- Artin, E. "Zur Arithmetik hyperkomplexer Zahlen." *Hamb. Abh.* **5**, 261–89, 1928.

Artistic Sequence

A SERIES is called artistic if every three consecutive terms have a common three-way ratio

$$P[a_i, a_{i+1}, a_{i+2}] = \frac{(a_i + a_{i+1} + a_{i+2})a_{i+1}}{a_i a_{i+2}}.$$

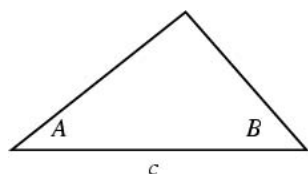
A SERIES is also artistic IFF its BIAS is a constant. A GEOMETRIC SERIES with RATIO $r > 0$ is an artistic series with

$$P = \frac{1}{r} + 1 + r \geq 3.$$

See also BIAS (SERIES), GEOMETRIC SERIES, MELODIC SEQUENCE

References

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ASA Theorem

Specifying two adjacent ANGLES A and B and the side between them c uniquely determines a TRIANGLE with AREA

$$K = \frac{c^2}{2(\cot A + \cot B)} \quad (1)$$

The angle C is given in terms of A and B by

$$C = \pi - A - B, \quad (2)$$

and the sides a and b can be determined by using the LAW OF SINES

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad (3)$$

to obtain

$$a = \frac{\sin A}{\sin(\pi - A - B)} c \quad (4)$$

$$b = \frac{\sin B}{\sin(\pi - A - B)} c. \quad (5)$$

See also AAA THEOREM, AAS THEOREM, ASS THEOREM, SAS THEOREM, SSS THEOREM, TRIANGLE

Aschbacher's Component Theorem

Suppose that $E(G)$ (the commuting product of all components of G) is SIMPLE and G contains a semisimple INVOLUTION. Then there is some semisimple INVOLUTION x such that $C_G(x)$ has a NORMAL SUBGROUP K which is either QUASISIMPLE or ISOMORPHIC to $O + (4, q)'$ and such that $Q = C_G(K)$ is TIGHTLY EMBEDDED.

See also INVOLUTION (GROUP), ISOMORPHIC GROUPS, NORMAL SUBGROUP, QUASISIMPLE GROUP, SIMPLE GROUP, TIGHTLY EMBEDDED

A-Sequence

N.B. A detailed online essay by S. Finch was the starting point for this entry.

An INFINITE SEQUENCE of POSITIVE INTEGERS a^i 'S satisfying

$$1 \leq a_1 < a_2 < a_3 < \dots \quad (1)$$

is an A-sequence if no a_k is the SUM of two or more distinct earlier terms (Guy 1994). Such sequences are sometimes also known as sum-free sets.

Erdos (1962) proved

$$S(A) \equiv \sup_{\text{all A sequences}} \sum_{k=1}^{\infty} \frac{1}{a_k} < 103. \quad (2)$$

Any A-sequence satisfies the CHI INEQUALITY (Levine and O'Sullivan 1977), which gives $S(A) < 3.9998$. Abbott (1987) and Zhang (1992) have given a bound from below, so the best result to date is

$$2.0649 < S(A) < 3.9998. \quad (3)$$

Levine and O'Sullivan (1977) conjectured that the sum of RECIPROCALs of an A-sequence satisfies

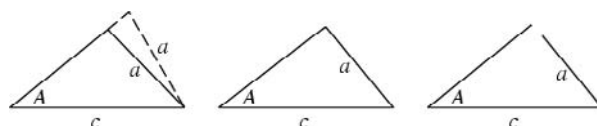
$$S(A) \leq \sum_{k=1}^{\infty} \frac{1}{\chi_k} = 3.01\dots, \quad (4)$$

where χ_i are given by the LEVINE-O'SULLIVAN GREEDY ALGORITHM.

See also B2-SEQUENCE, MIAN-CHOWLA SEQUENCE, SUM-FREE SET

References

- Abbott, H. L. "On Sum-Free Sequences." *Acta Arith.* **48**, 93–6, 1987.
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 Levine, E. and O'Sullivan, J. "An Upper Estimate for the Reciprocal Sum of a Sum-Free Sequence." *Acta Arith.* **34**, 9–4, 1977.
 Zhang, Z. X. "A Sum-Free Sequence with Larger Reciprocal Sum." Unpublished manuscript, 1992.

ASS Theorem

Specifying two adjacent side lengths a and c of a TRIANGLE (with $a < c$) and one ACUTE ANGLE A opposite a does not, in general, uniquely determine a triangle. If $\sin A < a/c$, there are two possible TRIANGLES satisfying the given conditions. If $\sin A = a/c$, there is one possible TRIANGLE. If $\sin A > a/c$, there are no possible TRIANGLES. Remember: don't try to prove congruence with the ASS theorem or you will make an ASS out of yourself.

See also AAA THEOREM, AAS THEOREM, SAS THEOREM, SSS THEOREM, TRIANGLE

Associate

Let p be an ODD PRIME, a a positive number such that $p \nmid a$ (i.e., p does not DIVIDE a), and let x be one of the numbers $1, 2, 3, \dots, p-1$. Then there is a unique x' ,

called the associate of x , such that

$$xx' \equiv a \pmod{p}$$

with $0 < x' < p$ (Hardy and Wright 1979, p. 67). If $x' = x$, then a is called a QUADRATIC RESIDUE of p .

See also QUADRATIC RESIDUE

References

Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers*, 5th ed. Oxford, England: Clarendon Press, p. 67, 1979.

Associated Fiber Bundle

Given a GROUP ACTION $G \times F \rightarrow F$ and a PRINCIPAL BUNDLE $\pi : A \rightarrow M$, the associated fiber bundle on M is

$$\tilde{\pi} : A \times F/G \rightarrow M. \quad (1)$$

In particular, it is the QUOTIENT SPACE $A \times F/G$ where $(a, x) \sim (ga, g^{-1}x)$.

For example, the torus $\mathbb{T} = \{e^{is}, e^{it}\}$ has a $\mathbb{S}^1$ action given by

$$\phi(e^{i\theta})(e^{is}, e^{it}) = (e^{i(s+\theta)}, e^{i(t+\theta)}) \quad (2)$$

and the frame bundle on the sphere,

$$\pi : SO(3) \rightarrow \mathbb{S}^2, \quad (3)$$

is a principal $\mathbb{S}^1$ bundle. The associated fiber bundle is a fiber bundle on the sphere, with fiber the torus. It is an example of a four-dimensional MANIFOLD.

See also BUNDLE, FIBER BUNDLE, GROUP ACTION, PRINCIPAL BUNDLE, QUOTIENT SPACE

Associated Laguerre Polynomial

LAGUERRE POLYNOMIAL

Associated Legendre Polynomial

LEGENDRE POLYNOMIAL

Associated Principal Bundle

See also BUNDLE

Associated Sequence

A SHEFFER SEQUENCE for $(1, f(t))$ is called the associated sequence for $f(t)$, and a sequence $s_n(x)$ of polynomials satisfying the orthogonality conditions

$$\langle [f(t)]^k | s_n(x) \rangle = n! \delta_{nk},$$

where δ_{nk} is the DELTA FUNCTION, is said to be associated to $f(t)$.

See also SHEFFER SEQUENCE

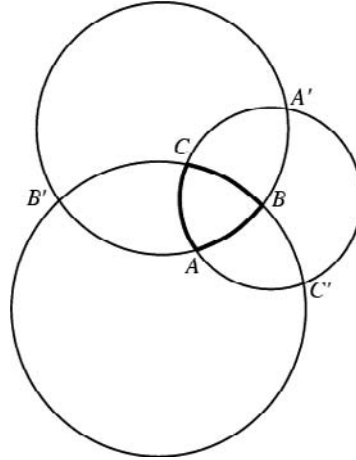
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Associated Stirling Number of the First Kind

STIRLING NUMBER OF THE FIRST KIND

Associated Triangles



The three CIRCULAR TRIANGLES $A'B'C'$, $AB'C'$, $A'BC'$, and $A'B'C$ obtained by extending the arcs of a CIRCULAR TRIANGLE ABC into complete circles.

See also CIRCULAR TRIANGLE

References

Lachlan, R. *An Elementary Treatise on Modern Pure Geometry*. London: Macmillan, pp. 251–52, 1893.

Associated Vector Bundle

Given a PRINCIPAL BUNDLE $\pi : A \rightarrow M$, with fiber a LIE GROUP G and BASE MANIFOLD M , and a REPRESENTATION of G , say $\phi : G \times V \rightarrow V$, then the associated vector bundle is

$$\tilde{\pi} : A \times V/G \rightarrow M. \quad (1)$$

In particular, it is the QUOTIENT SPACE $A \times V/G$ where $(a, v) \sim (ga, g^{-1}v)$.

This construction has many uses. For instance, any REPRESENTATION of the ORTHOGONAL GROUP gives rise to a BUNDLE of TENSORS on a RIEMANNIAN MANIFOLD as the vector bundle associated to the FRAME BUNDLE.

For example, $\pi : SO(3) \rightarrow \mathbb{S}^2$ is the frame bundle on $\mathbb{S}^2$ where

$$\pi \left(\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right) = w_1, \quad (2)$$

writing the special orthogonal matrix with rows w_i . It is a $SO(2)$ bundle with the action defined by

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} A, \quad (3)$$

which preserves the map π .

The TANGENT BUNDLE is the associated vector bundle with the standard REPRESENTATION of $SO(2)$ on $V = \mathbb{R}^2$ given by pairs (v, A) , with $v = (a, b) \in \mathbb{R}^2$ and $A \in SO(3)$. Two pairs (v_1, A_1) and (v_2, A_2) represent the same tangent vector IFF there is a $g \in SO(2)$ such that $v_2 = gv_1$ and $A_1 = g \cdot A_2$.

See also ASSOCIATED FIBER BUNDLE, FRAME BUNDLE, GROUP ACTION, LIE GROUP, PRINCIPAL BUNDLE, REPRESENTATION, QUOTIENT SPACE

Associative

Three elements x, y and z of a set S are said to be associative under a binary operation $*$ if they satisfy

$$x*(y*z) = (x*y)*z.$$

Real numbers are associative under addition

$$x + (y + z) = (x + y) + z$$

and multiplication

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

See also ASSOCIATIVE ALGEBRA, COMMUTATIVE, DISTRIBUTIVE, TRANSITIVE

Associative Algebra

In simple terms, let x, y , and z be members of an ALGEBRA. Then the ALGEBRA is said to be associative if

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z, \quad (1)$$

where $\cdot$ denotes MULTIPLICATION. More formally, let A denote an $\mathbb{R}$ -algebra, so that A is a VECTOR SPACE over $\mathbb{R}$ and

$$A \times A \rightarrow A \quad (2)$$

$$(x, y) \rightarrow x \cdot y. \quad (3)$$

Then A is said to be m -associative if there exists an m -dimensional SUBSPACE S of A such that

$$(y \cdot x) \cdot z = y \cdot (x \cdot z) \quad (4)$$

for all $y, z \in A$ and $x \in S$. Here, VECTOR MULTIPLICATION $x \cdot y$ is assumed to be BILINEAR. An n -dimensional n -associative ALGEBRA is simply said to be "associative."

See also ASSOCIATIVE

References

Finch, S. "Zero Structures in Real Algebras." <http://www.mathsoft.com/asolve/zerodiv/zerodiv.html>.

Associative Magic Square

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

An $n \times n$ MAGIC SQUARE for which every pair of numbers symmetrically opposite the center sum to $n^2 + 1$. The LO SHU is associative but not PANMAGIC. Order four squares can be PANMAGIC or associative, but not both. Order five squares are the smallest which can be both associative and PANMAGIC, and 16 distinct associative PANMAGIC SQUARES exist, one of which is illustrated above (Gardner 1988).

See also MAGIC SQUARE, PANMAGIC SQUARE

References

Gardner, M. "Magic Squares and Cubes." Ch. 17 in *Time Travel and Other Mathematical Bewilderments*. New York: W. H. Freeman, pp. 213–25, 1988.

Associator

For an ALGEBRA A , the associator is the trilinear map $A \times A \times A \rightarrow A$ given by

$$(x, y, z) = (xy)z - x(yz).$$

The associator is identically zero IFF A is associative.

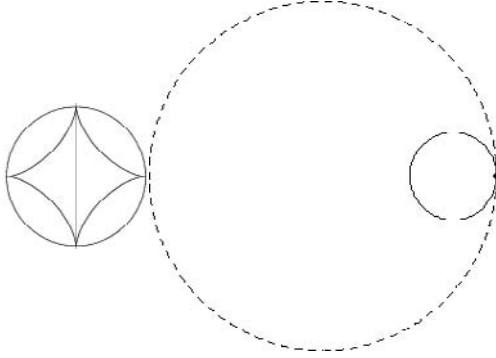
See also ALTERNATIVE ALGEBRA, COMMUTATOR, POWER ASSOCIATIVE ALGEBRA

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Asterisk

STAR

Astroid

A 4-cusped HYPOCYCLOID which is sometimes also called a TETRACUSPID, CUBOCYCLOID, or PARACYCLE. The PARAMETRIC EQUATIONS of the astroid can be obtained by plugging in $n \equiv a/b = 4$ or $4/3$ into the equations for a general HYPOCYCLOID, giving

$$x = 3b \cos \phi + b \cos(3\phi) = 4b \cos^3 \phi = a \cos^3 \phi \quad (1)$$

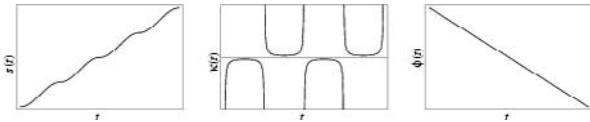
$$y = 3b \sin \phi - b \sin(3\phi) = 4b \sin^3 \phi = a \sin^3 \phi. \quad (2)$$

In CARTESIAN COORDINATES,

$$x^{2/3} + y^{2/3} = a^{2/3}. \quad (3)$$

In PEDAL COORDINATES with the PEDAL POINT at the center, the equation is

$$r^2 + 3p^2 = a^2 \quad (4)$$



The ARC LENGTH, CURVATURE, and TANGENTIAL ANGLE are

$$s(t) = \frac{3}{2} \int_0^t |\sin(2t')| dt' = \frac{3}{2} \sin^2 t \quad (5)$$

$$\kappa(t) = -\frac{2}{3} \csc(2t) \quad (6)$$

$$\phi(t) = -t. \quad (7)$$

As usual, care must be taken in the evaluation of $s(t)$ for $t > \pi/2$. Since (5) comes from an integral involving the ABSOLUTE VALUE of a function, it must be monotonic increasing. Each QUADRANT can be treated correctly by defining

$$n = \left\lfloor \frac{2t}{\pi} \right\rfloor + 1, \quad (8)$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION, giving the formula

$$s(t) = (-1)^{+1[n \pmod{2}]} \frac{3}{2} \sin^2 t + 3 \lfloor \frac{1}{2} n \rfloor. \quad (9)$$

The overall ARC LENGTH of the astroid can be

computed from the general HYPOCYCLOID formula

$$s_n = \frac{Sa(n-1)}{n} \quad (10)$$

with $n = 4$,

$$s_4 = 6a. \quad (11)$$

The AREA is given by

$$A_n = \frac{(n-1)(n-2)}{n^2} \pi a^2 \quad (12)$$

with $n = 4$,

$$A_4 = \frac{3}{8} \pi a^2. \quad (13)$$

The EVOLUTE of an ELLIPSE is a stretched HYPOCYCLOID. The gradient of the TANGENT T from the point with parameter p is $-\tan p$. The equation of this TANGENT T is

$$x \sin p + y \cos p = \frac{1}{2} a \sin(2p) \quad (14)$$

(MacTutor Archive). Let T cut the x -AXIS and the y -AXIS at X and Y , respectively. Then the length XY is a constant and is equal to a .



The astroid can also be formed as the ENVELOPE produced when a LINE SEGMENT is moved with each end on one of a pair of PERPENDICULAR AXES (e.g., it is the curve enveloped by a ladder sliding against a wall or a garage door with the top corner moving along a vertical track; left figure above). The astroid is therefore a GLISSETTE. To see this, note that for a ladder of length L , the points of contact with the wall and floor are $(x_0, 0)$ and $(0, \sqrt{L^2 - x_0^2})$, respectively. The equation of the LINE made by the ladder with its foot at $(x_0, 0)$ is therefore

$$y - 0 = \frac{\sqrt{L^2 - x_0^2}}{-x_0} (x - x_0) \quad (15)$$

which can be written

$$U(x, y, x_0) = y + \frac{\sqrt{L^2 - x_0^2}}{x_0} (x - x_0). \quad (16)$$

The equation of the ENVELOPE is given by the simultaneous solution of

$$\begin{cases} U(x, y, x_0) = y + \frac{\sqrt{L^2 - x_0^2}}{x_0}(x - x_0) = 0 \\ \frac{\partial U}{\partial x_0} = \frac{x_0^2 - L^2 x}{x_0^2 \sqrt{L^2 - x_0^2}} = 0, \end{cases} \quad (17)$$

which is

$$x = \frac{x_0^3}{L^2} \quad (18)$$

$$y = \frac{(L^2 - x_0^2)^{3/2}}{L^2} \quad (19)$$

Noting that

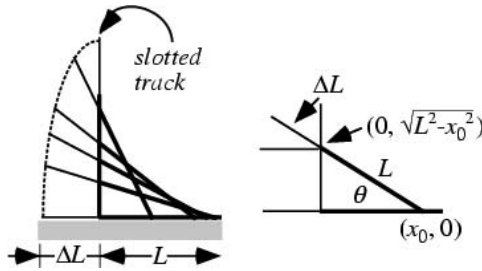
$$x^{2/3} = \frac{x_0^2}{L^{4/3}} \quad (20)$$

$$y^{2/3} = \frac{L^2 - x_0^2}{L^{4/3}} \quad (21)$$

allows this to be written implicitly as

$$x^{2/3} + y^{2/3} = L^{2/3}, \quad (22)$$

the equation of the astroid, as promised.



The related problem obtained by having the "garage door" of length L with an "extension" of length ΔL move up and down a slotted track also gives a surprising answer. In this case, the position of the "extended" end for the foot of the door at horizontal position x_0 and ANGLE θ is given by

$$x = -\Delta L \cos \theta \quad (23)$$

$$y = \sqrt{L^2 - x_0^2} + \Delta L \sin \theta. \quad (24)$$

Using

$$x_0 = L \cos \theta \quad (25)$$

then gives

$$x = -\frac{\Delta L}{L} x_0 \quad (26)$$

$$y = \sqrt{L^2 - x_0^2} \left(1 + \frac{\Delta L}{L} \right) \quad (27)$$

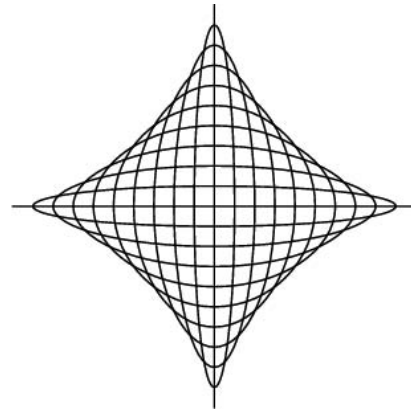
Solving (26) for x_0 , plugging into (27) and squaring then gives

$$y^2 = L^2 - \frac{L^2 x^2}{(\Delta L)^2} \left(1 + \frac{\Delta L}{L} \right)^2. \quad (28)$$

Rearranging produces the equation

$$\frac{x^2}{(\Delta L)^2} + \frac{y^2}{(L + \Delta L)^2} = 1, \quad (29)$$

the equation of a (QUADRANT of an) ELLIPSE with SEMIMAJOR and SEMIMINOR AXES of lengths δl and $l + \delta l$.



the astroid is also the ENVELOPE of the family of ELLIPSES

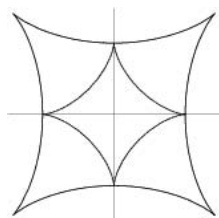
$$\frac{x^2}{c^2} + \frac{y^2}{(1-c)^2} - 1 = 0, \quad (30)$$

illustrated above (Wells 1991).

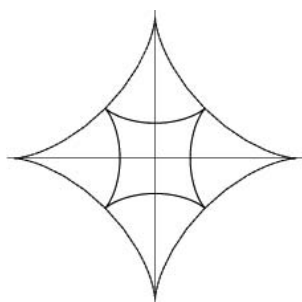
See also DELTOID, ELLIPSE ENVELOPE, LAMÉ CURVE, NEPHROID, RANUNCULOID

References

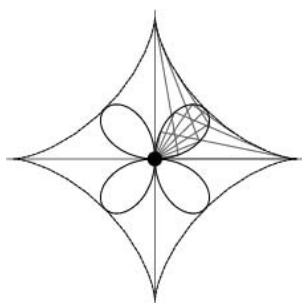
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Astroid Evolute

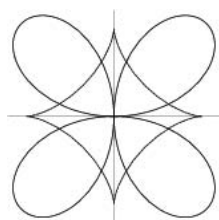
A HYPOCYCLOID EVOLUTE for $n = 4$ is another ASTROID scaled by a factor $n/(n-2) = 4/2 = 2$ and rotated $1/(2 \cdot 4) = 1/8$ of a turn.

Astroid Involute

A HYPOCYCLOID INVOLUTE for $n = 4$ is another ASTROID scaled by a factor $(n-2)/n = 2/4 = 1/2$ and rotated $1/(2 \cdot 4) = 1/8$ of a turn.

Astroid Pedal Curve

The PEDAL CURVE of an ASTROID with PEDAL POINT at the center is a QUADRIFOLIUM.

Astroid Radial Curve**The QUADRIFOLIUM**

$$x = x_0 + 3a \cos t - 3a \cos(3t)$$

$$y = y_0 + 3a \sin t + 3 \sin(3t).$$

Astroidal Ellipsoid

The surface which is the inverse of the ELLIPSOID in the sense that it "goes in" where the ELLIPSOID "goes out." It is given by the PARAMETRIC EQUATIONS

$$x = (a \cos u \cos v)^3$$

$$y = (b \sin u \cos v)^3$$

$$z = (c \sin v)^3$$

for $u \in [-\pi/2, \pi/2]$ and $v \in [-\pi, \pi]$. The special case $a = b = c = 1$ corresponds to the HYPERBOLIC OCTAHEDRON.

See also ELLIPSOID, HYPERBOLIC OCTAHEDRON

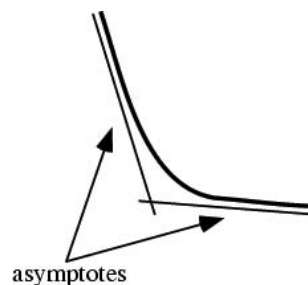
References

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Asymptosy

ASYMPTOTIC behavior. A useful yet endangered word, found rarely outside the captivity of the *Oxford English Dictionary*.

See also ASYMPTOTE, ASYMPTOTIC

Asymptote

A curve approaching a given curve arbitrarily closely, as illustrated in the above diagram.

See also ASYMPTOSY, ASYMPTOTIC, ASYMPTOTIC CURVE

References

Giblin, P. J. "What is an Asymptote?" *Math. Gaz.* **56**, 274–84, 1972.

Asymptotic

Approaching a value or curve arbitrarily closely (i.e., as some sort of LIMIT is taken). A CURVE A which is asymptotic to given CURVE C is called the ASYMPTOTE

of *C. Hardy and Wright* (1979, p. 7) use the symbol $\asymp$ to denote that one quantity is asymptotic to another. If $f \asymp \phi$, then Hardy and Wright say that f and ϕ are of the same ORDER OF MAGNITUDE.

See also ASYMPTOSY, ASYMPTOTE, ASYMPTOTIC CURVE, ASYMPTOTIC DIRECTION, ASYMPTOTIC NOTATION, ASYMPTOTIC SERIES, LANDAU SYMBOL, LIMIT, ORDER OF MAGNITUDE

References

Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers*, 5th ed. Oxford, England: Clarendon Press, 1979.

Asymptotic Curve

Given a REGULAR SURFACE M , an asymptotic curve is formally defined as a curve $\mathbf{x}(t)$ on M such that the NORMAL CURVATURE is 0 in the direction $\mathbf{x}'(t)$ for all t in the domain of $\mathbf{x}$. The differential equation for the parametric representation of an asymptotic curve is

$$eu'^2 + 2fu'v' + gv'^2 = 0, \quad (1)$$

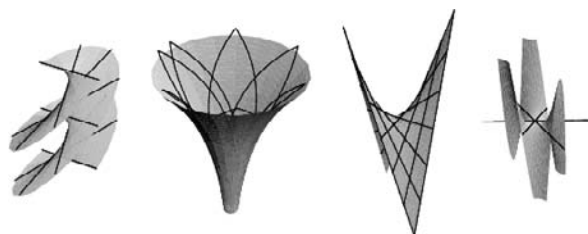
where e , f , and g are coefficients of the SECOND FUNDAMENTAL FORM. The differential equation for asymptotic curves on a MONGE PATCH $(u, v, h(u, v))$ is

$$h_{uu}u'^2 + 2h_{uv}u'v' + h_{vv}v'^2 = 0, \quad (2)$$

and on a polar patch $(r \cos \theta, r \sin \theta, h(r))$ is

$$h''(r)r'^2 + h'(r)r\theta'^2 = 0. \quad (3)$$

The images below show asymptotic curves for the ELLIPTIC HELICOID, FUNNEL, HYPERBOLIC PARABOLOID, and MONKEY SADDLE.



See also RULED SURFACE

References

Gray, A. "Asymptotic Curves," "Examples of Asymptotic Curves," and "Using Mathematica to Find Asymptotic Curves." §18.1, 18.2, and 18.3 in *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2nd ed. Boca Raton, FL: CRC Press, pp. 417–29, 1997.

Asymptotic Direction

An asymptotic direction at a point $\mathbf{p}$ of a REGULAR SURFACE $M \in \mathbb{R}^3$ is a direction in which the NORMAL CURVATURE of M vanishes.

1. There are no asymptotic directions at an ELLIPTIC POINT.
2. There are exactly two asymptotic directions at a HYPERBOLIC POINT.
3. There is exactly one asymptotic direction at a PARABOLIC POINT.
4. Every direction is asymptotic at a PLANAR POINT.

See also ASYMPTOTIC CURVE

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2nd ed. Boca Raton, FL: CRC Press, pp. 364 and 418, 1997.

Asymptotic Equipartition Property

This entry contributed by ERIK G. MILLER

A theorem from INFORMATION THEORY that is a simple consequence of the WEAK LAW OF LARGE NUMBERS. It states that if a set of values $X_1, X_2, \dots, X_n$ is drawn independently from a random variable X distributed according to $P(x)$ then the joint probability $P(X_1, \dots, X_n)$ satisfies

$$-\frac{1}{n} \ln P(X_1, X_2, \dots, X_n) \rightarrow H(X),$$

where $H(X)$ is the ENTROPY of the random variable X .

See also ENTROPY

References

Cover, T. M. and Thomas, J. A. *Elements of Information Theory*. New York: Wiley, 1991.

Asymptotic Expansion

ASYMPTOTIC SERIES

Asymptotic Notation

Let n be an integer variable which tends to infinity and let x be a continuous variable tending to some limit. Also, let $\phi(n)$ or $\phi(x)$ be a positive function and $f(n)$ or $f(x)$ any function. Then Hardy and Wright (1979) define

1. $f = \mathcal{O}(\phi)$ to mean that $|f| < A\phi$ for some constant A and all values of n and x ,
2. $f = o(\phi)$ to mean that $f/\phi \rightarrow 0$,
3. $f \sim \phi$ to mean that $f/\phi \rightarrow 1$,
4. $f \asymp \phi$ to mean the same as $f = o(\phi)$,
5. $f \prec \phi$ to mean $f/\phi \rightarrow \infty$, and
6. $f \succ \phi$ to mean $A_1\phi < f < A_2\phi$ for some positive constants A_1 and A_2 .

$f = o(\phi)$ implies and is stronger than $f = \mathcal{O}(\phi)$.

The term LANDAU SYMBOL is sometimes used to indicate the notation $o(\phi)$, and in general, $\mathcal{O}(x)$ and $o(x)$ are read as "is of order x ."

See also LANDAU SYMBOL

References

- Hardy, G. H. and Wright, E. M. "Some Notations." §1.6 in *An Introduction to the Theory of Numbers*, 5th ed. Oxford, England: Clarendon Press, pp. 7–, 1979.
- Jeffreys, H. and Jeffreys, B. S. "Increasing and Decreasing Functions." §1.065 in *Methods of Mathematical Physics*, 3rd ed. Cambridge, England: Cambridge University Press, p. 22, 1988.

Asymptotic Series

An asymptotic series is a SERIES EXPANSION of a FUNCTION in a variable x which may converge or diverge (Erdélyi 1987, p. 1), but whose partial sums can be made an arbitrarily good approximation to a given function for large enough x . To form an asymptotic series $R(x)$ of

$$f(x) \sim R(x), \quad (1)$$

take

$$x^n R_n(x) = x^n [f(x) - S_n(x)], \quad (2)$$

where

$$S_n(x) \equiv a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}. \quad (3)$$

The asymptotic series is defined to have the properties

$$\lim_{x \rightarrow \infty} x^n R_n(x) = 0 \quad \text{for fixed } n \quad (4)$$

$$\lim_{x \rightarrow \infty} x^n R_n(x) = \infty \quad \text{for fixed } x \quad (5)$$

Therefore,

$$f(x) \approx \sum_{n=0}^{\infty} a_n x^{-n} \quad (6)$$

in the limit $x \rightarrow \infty$. If a function has an asymptotic expansion, the expansion is unique. The symbol $\sim$ is also used to mean directly SIMILAR.

See also HYPERASYMPTOTIC SERIES, SUPERASYMPTOTIC SERIES

References

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Morse, P. M. and Feshbach, H. "Asymptotic Series; Method of Steepest Descent." §4.6 in *Methods of Theoretical Physics, Part I*. New York: McGraw-Hill, pp. 434–43, 1953.

Olver, F. W. J. *Asymptotics and Special Functions*. New York: Academic Press, 1974.

Wasow, W. R. *Asymptotic Expansions for Ordinary Differential Equations*. New York: Dover, 1987.

Weisstein, E. W. "Books about Asymptotic Series." <http://www.treasure-troves.com/books/AsymptoticSeries.html>.

Atiyah-Singer Index Theorem

A theorem which states that the analytic and topological "indices" are equal for any elliptic differential operator on an n -D COMPACT DIFFERENTIABLE C^∞ boundaryless MANIFOLD.

See also COMPACT MANIFOLD, DIFFERENTIABLE MANIFOLD

References

- Atiyah, M. F. and Singer, I. M. "The Index of Elliptic Operators on Compact Manifolds." *Bull. Amer. Math. Soc.* **69**, 322–33, 1963.
- Atiyah, M. F. and Singer, I. M. "The Index of Elliptic Operators I, II, III." *Ann. Math.* **87**, 484–04, 1968.
- Petkovsek, M.; Wilf, H. S.; and Zeilberger, D. *A=B*. Wellesley, MA: A. K. Peters, p. 4, 1996.

Atkin-Goldwasser-Kilian-Morain Certificate

A recursive PRIMALITY CERTIFICATE for a PRIME p . The certificate consists of a list of

1. A point on an ELLIPTIC CURVE C

$$y^2 = x^3 + g_2 x + g_3 \pmod{p}$$

for some numbers g_2 and g_3 .

2. A PRIME q with $q > (p^{1/4} + 1)^2$ such that for some other number k and $m = kq$ with $k \neq 1$, $mC(x, y, g_2, g_3, p)$ is the identity on the curve, but $kC(x, y, g_2, g_3, p)$ is not the identity. This guarantees PRIMALITY of p by a theorem of Goldwasser and Kilian (1986).

3. Each q has its recursive certificate following it. So if the smallest q is known to be PRIME, all the numbers are certified PRIME up the chain.

A PRATT CERTIFICATE is quicker to generate for small numbers. The *Mathematica* task ProvablePrimeQ[n] in the *Mathematica* add-on package NumberTheory`PrimeQ` (which can be loaded with the command <<NumberTheory`) therefore generates an Atkin-Goldwasser-Kilian-Morain certificate only for numbers above a certain limit (10^{10} by default), and a PRATT CERTIFICATE for smaller numbers.

See also ELLIPTIC CURVE PRIMALITY PROVING, ELLIPTIC PSEUDOPRIME, PRATT CERTIFICATE, PRIMALITY CERTIFICATE, WITNESS

References

- Atkin, A. O. L. and Morain, F. "Elliptic Curves and Primality Proving." *Math. Comput.* **61**, 29–8, 1993.
- Bressoud, D. M. *Factorization and Prime Testing*. New York: Springer-Verlag, 1989.
- Goldwasser, S. and Kilian, J. "Almost All Primes Can Be Quickly Certified." *Proc. 18th STOC*. pp. 316–29, 1986.
- Morain, F. "Implementation of the Atkin-Goldwasser-Kilian Primality Testing Algorithm." Rapport de Recherche 911, INRIA, Octobre 1988.
- Schoof, R. "Elliptic Curves over Finite Fields and the Computation of Square Roots mod p ." *Math. Comput.* **44**, 483–94, 1985.
- Wunderlich, M. C. "A Performance Analysis of a Simple Prime-Testing Algorithm." *Math. Comput.* **40**, 709–14, 1983.

Atlas

An atlas is a collection of consistent COORDINATE CHARTS on a MANIFOLD, where "consistent" most commonly means that the TRANSITION FUNCTIONS of the charts are SMOOTH. As the name suggests, an atlas corresponds to a collection of maps, each of which shows a piece of a MANIFOLD and looks like flat two-dimensional Euclidean space. To use an atlas, one needs to know how the maps overlap. To be useful, the maps must not be too different on these overlapping areas.

The overlapping maps from one chart to another are called transition functions. They represent the transition from one chart's point of view to that of another. Let the open unit ball in $\mathbb{R}^n$ be denoted B_1 . Then if $\phi : U \rightarrow B_1$ and $\psi : V \rightarrow B_1$ are two coordinate charts, the composition $\phi \circ \psi^{-1}$ is a function defined on $\psi(U \cap V)$. That is, it is a function from an open subset of B_1 to B_1 , and given such a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, there are conditions for it to be smooth or have k smooth derivatives (i.e., it is a C^k FUNCTION). Furthermore, when $\mathbb{R}^{2n}$ is isomorphic to $\mathbb{C}^n$ (in the even DIMENSIONAL case), a function can be HOLOMORPHIC.

A smooth atlas has transition functions that are C^∞ smooth (i.e., infinitely differentiable). The consequence is that a smooth function on one chart is smooth in any other chart (by the CHAIN RULE for higher derivatives). Similarly, one could have an atlas in class C^k , where the transition functions are in class C^k .

In the even-dimensional case, one may ask whether the transition functions are HOLOMORPHIC. In this case, one has a holomorphic atlas, and by the chain rule, it makes sense to ask if a function on the manifold is holomorphic.

It is possible for two atlases to be compatible, meaning the union is also an atlas. By ZORN'S LEMMA, there always exists a maximal atlas, where a maximal atlas is an atlas not contained in any other atlas. However, in typical applications, it is not necessary to use a

maximal atlas and any sufficiently refined atlas will do.

See also COORDINATE CHART, HOLOMORPHIC FUNCTION, MANIFOLD, SMOOTH FUNCTION, TRANSITION FUNCTION, ZORN'S LEMMA

Atom

ATOMIC STATEMENT, URELEMENT

Atomic Statement

In LOGIC, a statement which cannot be broken down into smaller statements.

Attraction Basin

BASIN OF ATTRACTION

Attractor

An attractor is a SET of states (points in the PHASE SPACE), invariant under the dynamics, towards which neighboring states in a given BASIN OF ATTRACTION asymptotically approach in the course of dynamic evolution. An attractor is defined as the smallest unit which cannot be itself decomposed into two or more attractors with distinct BASINS OF ATTRACTION. This restriction is necessary since a DYNAMICAL SYSTEM may have multiple attractors, each with its own BASIN OF ATTRACTION.

Conservative systems do not have attractors, since the motion is periodic. For dissipative DYNAMICAL SYSTEMS, however, volumes shrink exponentially so attractors have 0 volume in n -D phase space.

A stable FIXED POINT surrounded by a dissipative region is an attractor known as a SINK. Regular attractors (corresponding to 0 LYAPUNOV CHARACTERISTIC EXPONENTS) act as LIMIT CYCLES, in which trajectories circle around a limiting trajectory which they asymptotically approach, but never reach. STRANGE ATTRACTORS are bounded regions of PHASE SPACE (corresponding to POSITIVE LYAPUNOV CHARACTERISTIC EXPONENTS) having zero MEASURE in the embedding PHASE SPACE and a FRACTAL DIMENSION. Trajectories within a STRANGE ATTRACTOR appear to skip around randomly.

See also BARNSLEY'S FERN, BASIN OF ATTRACTION, CHAOS GAME, FRACTAL DIMENSION, LIMIT CYCLE, LYAPUNOV CHARACTERISTIC EXPONENT, MEASURE, SINK (MAP), STRANGE ATTRACTOR

Aubel's Theorem

VON AUBEL'S THEOREM

Auction

A type of sale in which members of a group of buyers offer ever increasing amounts. The bidder making the

last bid (for which no higher bid is subsequently made within a specified time limit: "going once, going twice, sold") must then purchase the item in question at this price. Variants of simple bidding are also possible, as in a VICKREY AUCTION.

See also VICKREY AUCTION

Augend

The first of several ADDENDS, or "the one to which the others are added," is sometimes called the augend. Therefore, while a , b , and c are ADDENDS in $a + b + c$, a is the augend.

See also ADDEND, ADDITION

Augmented Amicable Pair

A PAIR of numbers m and n such that

$$\sigma(m) = \sigma(n) = m + n - 1,$$

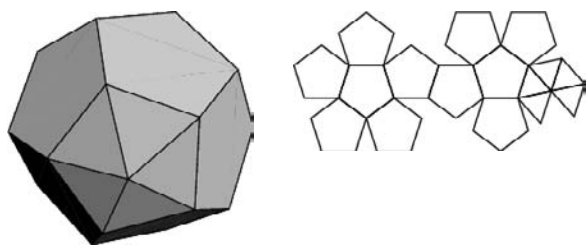
where $\sigma(m)$ is the DIVISOR FUNCTION. Beck and Najjar (1977) found 11 augmented amicable pairs.

See also AMICABLE PAIR, DIVISOR FUNCTION, QUASIAMICABLE PAIR

References

- Beck, W. E. and Najjar, R. M. "More Reduced Amicable Pairs." *Fib. Quart.* **15**, 331–32, 1977.
 Guy, R. K. *Unsolved Problems in Number Theory*, 2nd ed. New York: Springer-Verlag, p. 59, 1994.

Augmented Dodecahedron



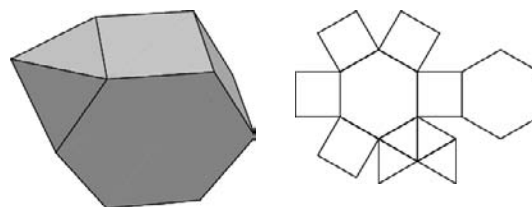
JOHNSON SOLID J_{58} .

References

- Weisstein, E. W. "Johnson Solids." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.M.
 Weisstein, E. W. "Johnson Solid Netlib Database." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.DAT.

Augmented Sphenocorona

Augmented Hexagonal Prism

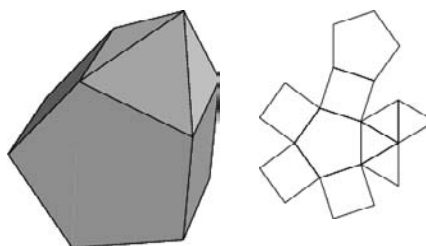


JOHNSON SOLID J_{54} .

References

- Weisstein, E. W. "Johnson Solids." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.M.
 Weisstein, E. W. "Johnson Solid Netlib Database." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.DAT.

Augmented Pentagonal Prism



JOHNSON SOLID J_{52} .

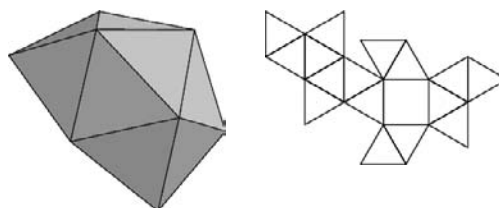
References

- Weisstein, E. W. "Johnson Solids." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.M.
 Weisstein, E. W. "Johnson Solid Netlib Database." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.DAT.

Augmented Polyhedron

A UNIFORM POLYHEDRON with one or more other solids adjoined.

Augmented Sphenocorona

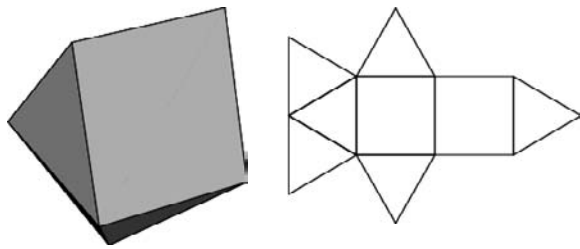


JOHNSON SOLID J_{87} .

References

Weisstein, E. W. "Johnson Solids." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.M.
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Augmented Triangular Prism

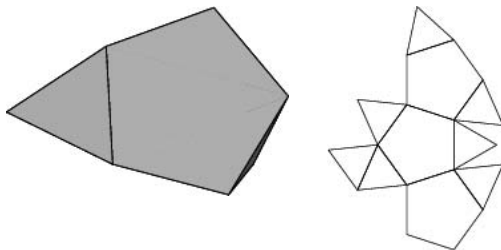


JOHNSON SOLID J_{49} .

References

Weisstein, E. W. "Johnson Solids." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.M.
Weisstein, E. W. "Johnson Solid Netlib Database." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.DAT.

Augmented Tridiminished Icosahedron

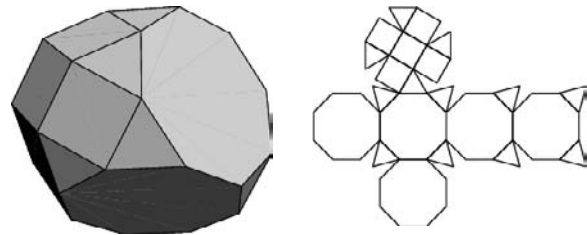


JOHNSON SOLID J_{64} .

References

Weisstein, E. W. "Johnson Solids." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.M.
Weisstein, E. W. "Johnson Solid Netlib Database." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.DAT.

Augmented Truncated Cube

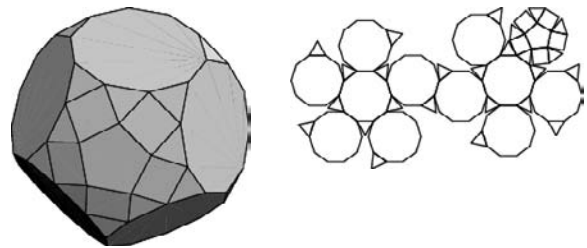


JOHNSON SOLID J_{66} .

References

Weisstein, E. W. "Johnson Solids." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.M.
Weisstein, E. W. "Johnson Solid Netlib Database." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.DAT.

Augmented Truncated Dodecahedron

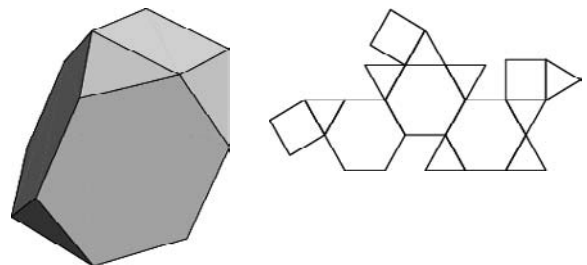


JOHNSON SOLID J_{68} .

References

Weisstein, E. W. "Johnson Solids." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.M.
Weisstein, E. W. "Johnson Solid Netlib Database." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.DAT.

Augmented Truncated Tetrahedron



JOHNSON SOLID J_{65} .

References

- Weisstein, E. W. "Johnson Solids." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.M.
 Weisstein, E. W. "Johnson Solid Netlib Database." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.DAT.

Augmenting Path

A path constructed by repeatedly finding a path of positive capacity from a source to a sink and then adding it to the flow (Skiena 1990, p. 237).

See also *BERGE'S THEOREM*

References

- Ford, L. R. and Fulkerson, D. R. *Flows in Networks*. Princeton, NJ: Princeton University Press, 1962.
 Skiena, S. *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*. Reading, MA: Addison-Wesley, 1990.

Aureum Theorema

Gauss's name for the QUADRATIC RECIPROCITY THEOREM.

Aurifeuillean Factorization

A factorization OF THE FORM

$$2^{4n+2} + 1 = (2^{2n+1} - 2^{n+1} + 1)(2^{2n+1} + 2^{n+1} + 1). \quad (1)$$

The factorization for $n = 14$ was discovered by Aurifeuille, and the general form was subsequently discovered by Lucas. The large factors are sometimes written as L and M as follows

$$2^{4k-2} + 1 = (2^{2k-1} - 2^k + 1)(2^{2k-1} + 2^k + 1) \quad (2)$$

$$3^{6k-3} + 1 = (3^{2k-1} + 1)(3^{2k-1} - 3^k + 1) \times (3^{2k-1} + 3^k + 1), \quad (3)$$

which can be written

$$2^{2h} + 1 = L_{2h} M_{2h} \quad (4)$$

$$3^{3h} + 1 = (3^h + 1) L_{3h} M_{3h} \quad (5)$$

$$5^{5k} - 1 = (5^h + 1) L_{5h} M_{5h}, \quad (6)$$

where $h = 2k - 1$ and

$$L_{2h}, M_{2h} = 2^h + 1 \mp 2^k \quad (7)$$

$$L_{3h}, M_{3h} = 3^h + 1 \mp 3^k \quad (8)$$

$$L_{5h}, M_{5h} = 5^{2h} + 3 \cdot 5^h + 1 \mp 5^k(5^k + 1). \quad (9)$$

See also *GAUSS'S CYCLOTOMIC FORMULA*

References

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 Wagstaff, S. S. Jr. "Aurifeuillean Factorizations and the Period of the Bell Numbers Modulo a Prime." *Math. Comput.* **65**, 383–91, 1996.

Ausdehnungslehre

EXTERIOR ALGEBRA

Aut

"Aut" is the term applied in PROPOSITIONAL CALCULUS to the XOR connective. "Aut" is Latin form for "either/or (but not both)," e.g., "Aut Caesar aut nihil" (Cesare Borgia; 1476–507).

The symbol Aut is also commonly used for the completely different purpose of denoting an AUTOMORPHISM.

See also *AUTOMORPHISM*, *XOR*

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Authalic Latitude

An AUXILIARY LATITUDE which gives a SPHERE equal SURFACE AREA relative to an ELLIPSOID. The authalic latitude is defined by

$$\beta = \sin^{-1} \left(\frac{q}{q_p} \right), \quad (1)$$

where

$$q = (1 - e^2) \left[\frac{\sin \phi}{1 - e^2 \sin^2 \phi} - \frac{1}{2e} \ln \left(\frac{1 - e \sin \phi}{1 + e \sin \phi} \right) \right] \quad (2)$$

and q_p is q evaluated at the north pole ($\phi = 90^\circ$). Let R_q be the RADIUS of the SPHERE having the same SURFACE AREA as the ELLIPSOID, then

$$R_q = a \sqrt{\frac{q_p}{2}}. \quad (3)$$

The series for β is

$$\begin{aligned} \beta = \phi - & \left(\frac{1}{3} e^2 + \frac{31}{180} e^4 + \frac{59}{560} e^6 + \dots \right) \sin(\phi) \\ & + \left(\frac{17}{360} e^4 + \frac{61}{1260} e^6 + \dots \right) \sin(4\phi) \\ & - \left(\frac{383}{45360} e^6 + \dots \right) \sin(6\phi) + \dots \end{aligned} \quad (4)$$

The inverse FORMULA is found from

$$\Delta\phi = \frac{(1 - e^2 \sin^2 \phi)^2}{2 \cos \phi} \times \left[\frac{q}{1 - e^2} - \frac{\sin \phi}{1 - e^2 \sin^2 \phi} + \frac{1}{2e} \ln \left(\frac{1 - e \sin \phi}{1 + e \sin \phi} \right) \right], \quad (5)$$

where

$$q = q_p \sin \beta \quad (6)$$

and $\phi_0 = \sin^{-1}(q/2)$. This can be written in series form as

$$\begin{aligned} \phi = & \beta + \left(\frac{1}{3}e^2 + \frac{31}{180}e^4 + \frac{517}{5040}e^6 + \dots\right) \sin(2\beta) \\ & + \left(\frac{23}{360}e^4 + \frac{251}{3780}e^6 + \dots\right) \sin(4\beta) \\ & + \left(\frac{761}{45360}e^6 + \dots\right) \sin(6\beta) + \dots \end{aligned} \quad (7)$$

See also LATITUDE

References

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Snyder, J. P. *Map Projections—A Working Manual*. U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, p. 16, 1987.

Authalic Projection

Lee (1944) defines an authalic MAP PROJECTION to be one in which at any point the scales in two orthogonal directions are inversely proportional.

See also EQUAL-AREA PROJECTION

References

- Lee, L. P. "The Nomenclature and Classification of Map Projections." *Empire Survey Review* 7, 190–00, 1944.

Autocorrelation

The autocorrelation function $R_f(t)$ of a real function $f(t)$ is defined by

$$R_f(t) \equiv \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\tau) f(T + \tau) d\tau \quad (1)$$

(Papoulis 1962, p. 241). For a complex function, the autocorrelation $\rho_f(t)$ is defined by

$$\rho_f(t) \equiv f \star f = \bar{f}(-t) * f(t) = \int_{-\infty}^{\infty} f(t + \tau) \bar{f}(\tau) d\tau. \quad (2)$$

where $*$ denotes CONVOLUTION, $\star$ denotes CROSS-CORRELATION, and $\bar{f}$ is the COMPLEX CONJUGATE (Papoulis 1962, pp. 241–42). The autocorrelation discards phase information, returning only the power, and is therefore an irreversible operation.

There is also a somewhat surprising and extremely important relationship between the autocorrelation and the FOURIER TRANSFORM known as the WIENER-KHINTCHINE THEOREM. Let $\mathcal{F}[f(x)] = F(k)$, and $\bar{F}$ denote the COMPLEX CONJUGATE of F , then the FOURIER TRANSFORM of the ABSOLUTE SQUARE of $F(k)$ is given by

$$\mathcal{F}[|F(k)|^2] = \int_{-\infty}^{\infty} \bar{f}(\tau) f(\tau + x) d\tau. \quad (3)$$

The autocorrelation is a HERMITIAN OPERATOR since $\rho_f(-t) = \bar{\rho}_f(t)$.

$f \star f$ is MAXIMUM at the ORIGIN; in other words,

$$\int_{-\infty}^{\infty} f(u) f(u + x) du \leq \int_{-\infty}^{\infty} f^2(u) du. \quad (4)$$

To see this, let ϵ be a REAL NUMBER. Then

$$\int_{-\infty}^{\infty} [f(u) + \epsilon f(u + x)]^2 du > 0 \quad (5)$$

$$\begin{aligned} \int_{-\infty}^{\infty} f^2(u) du + 2\epsilon \int_{-\infty}^{\infty} f(u) f(u + x) du \\ + \epsilon^2 \int_{-\infty}^{\infty} f^2(u + x) du > 0 \end{aligned} \quad (6)$$

$$\begin{aligned} \int_{-\infty}^{\infty} f^2(u) du + 2\epsilon \int_{-\infty}^{\infty} f(u) f(u + x) du \\ + \epsilon^2 \int_{-\infty}^{\infty} f^2(u + x) du > 0. \end{aligned} \quad (7)$$

Define

$$a \equiv \int_{-\infty}^{\infty} f^2(u) du \quad (8)$$

$$b \equiv 2 \int_{-\infty}^{\infty} f(u) f(u + x) du. \quad (9)$$

Then plugging into above, we have $a\epsilon^2 + b\epsilon + c > 0$. This QUADRATIC EQUATION does not have any REAL ROOT, so $b^2 - 4ac \leq 0$, i.e., $b/2 \leq a$. It follows that

$$\int_{-\infty}^{\infty} f(u) f(u + x) du \leq \int_{-\infty}^{\infty} f^2(u) du, \quad (10)$$

with the equality at $x = 0$. This proves that $f \star f$ is MAXIMUM at the ORIGIN.

See also AVERAGE POWER, CONVOLUTION, CROSS-CORRELATION, QUANTIZATION EFFICIENCY, WIENER-KHINTCHINE THEOREM

References

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FFT." §13.2 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing*, 2nd ed. Cambridge, England: Cambridge University Press, pp. 538–39, 1992.

Autogonal Projection

CONFORMAL PROJECTION

Automata Theory

The mathematical study of abstract computing machines (especially TURING MACHINES) and the analysis of algorithms used by such machines.

See also CELLULAR AUTOMATON, TURING MACHINE

References

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 Simon, M. *Automata Theory*. Singapore: World Scientific, 1999.
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Automatic Set

A k -automatic set is a set of integers whose base- k representations form a regular language, i.e., a language accepted by a finite automaton or state machine. If bases a and b are incompatible (do not have a common power) and if an a -automatic set S_a and b -automatic set S_b are both of density 0 over the integers, then it is believed that $S_a \cap S_b$ is finite. However, this problem has not been settled.

Some automatic sets, such as the 2-automatic consisting of numbers whose BINARY representations contain at most two 1s: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 17, 18, ... (Sloane's A048645) have a simple arithmetic expression. However, this is not the case for general k -automatic sets.

See also TURING MACHINE

References

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Automaton

AUTOMATIC SET, CELLULAR AUTOMATON, TURING MACHINE

Automorphic Form

See also AUTOMORPHIC FUNCTION, LANGLANDS PROGRAM

Automorphic Function

An automorphic function $f(z)$ of a COMPLEX variable z is one which is analytic (except for POLES) in a domain D and which is invariant under a DENUMERABLY INFINITE group of LINEAR FRACTIONAL TRANSFORMATIONS (also known as MÖBIUS TRANSFORMATIONS)

$$z' = \frac{az + b}{cz + d}.$$

Automorphic functions are generalizations of TRIGONOMETRIC FUNCTIONS and ELLIPTIC FUNCTIONS.

See also AUTOMORPHIC FORM, MODULAR FUNCTION, MÖBIUS TRANSFORMATIONS, ZETA FUCHSIAN

References

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 Siegel, C. L. *Topics in Complex Function Theory, Vol. 2: Automorphic Functions and Abelian Integrals*. New York: Wiley, 1988.

Automorphic Number

A number k such that nk^2 has its last digits equal to k is called n -automorphic. For example, $1 \cdot 5^2 \equiv 25$ (Wells 1986, pp. 58–9) and $1 \cdot 6^2 \equiv 36$ (Wells 1986, p. 68) are 1-automorphic and $2 \cdot 8^2 \equiv 128$ and $2 \cdot 88^2 = 15488$ are 2-automorphic. de Guerre and Fairbairn (1968) give a history of automorphic numbers.

The first few 1-automorphic numbers are 1, 5, 6, 25, 76, 376, 625, 9376, 90625, ... (Sloane's A003226, Wells 1986, p. 130). There are two 1-automorphic numbers with a given number of digits, one ending in 5 and one in 6 (except that the 1-digit automorphic numbers include 1), and each of these contains the previous number with a digit prepended. Using this fact, it is possible to construct automorphic numbers having more than 25,000 digits (Madachy 1979). The first few 1-automorphic numbers ending with 5 are 5, 25, 625, 0625, 90625, ... (Sloane's A007185), and the first few ending with 6 are 6, 76, 376, 9376, 09376, ... (Sloane's A016090). The 1-automorphic numbers $a(n)$ ending in 5 are IDEMPOTENT (mod 10^n) since

$$[a(n)]^2 \equiv a(n) \pmod{10^n}$$

(Sloane and Plouffe 1995).

The following table gives the 10-digit n -automorphic numbers.

n	n -Automorphic Numbers	Sloane
1	0000000001, 8212890625, 1787109376	–, A007185, A016090

2	0893554688	A030984
3	6666666667, 7262369792, 9404296875	–, A030985, A030986
4	0446777344	A030987
5	3642578125	A030988
6	3631184896	A030989
7	7142857143, 4548984375, 1683872768	A030990, A030991, A030992
8	0223388672	A030993
9	5754123264, 3134765625, 8888888889	A030994, A030995, –

The infinite 1-automorphic number ending in 5 is given by ...56259918212890625 (Sloane's A018247), while the infinite 1-automorphic number ending in 6 is given by ...740081787109376 (Sloane's A018248).

See also IDEMPOTENT, NARCISSISTIC NUMBER, NUMBER PYRAMID, TRIMORPHIC NUMBER

References

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- Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers*. Middlesex, England: Penguin Books, pp. 59 and 171, 178, 191–92, 1986.

Automorphism

AN ISOMORPHISM of a system of objects onto itself. The term derives from the Greek prefix *avto* (*auto*) "self" and *μορφωσις* (*morphosis*) "to form" or "to shape."

The automorphisms of a GRAPH always describe a GROUP (Skiena 1990, p. 19).

An automorphism of a region of the COMPLEX PLANE is a conformal SELF-MAP (Krantz 1999, p. 81).

See also ANOSOV AUTOMORPHISM, GRAPH AUTOMORPHISM

References

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- Skiena, S. *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*. Reading, MA: Addison-Wesley, 1990.

Automorphism Group

The GROUP of functions from an object G to itself which preserve the structure of the object, denoted $Aut(G)$. The automorphism group of a GROUP preserves the MULTIPLICATION table, the automorphism group of a GRAPH the INCIDENCE MATRICES, and that of a FIELD the ADDITION and MULTIPLICATION tables.

Autonomous

A differential equation or system of ORDINARY DIFFERENTIAL EQUATIONS is said to be autonomous if it does not explicitly contain the independent variable (usually denoted t). A second-order autonomous differential equation is OF THE FORM $F(y, y', y'') = 0$, where $y' \equiv dy/dt \equiv v$. By the CHAIN RULE, y'' can be expressed as

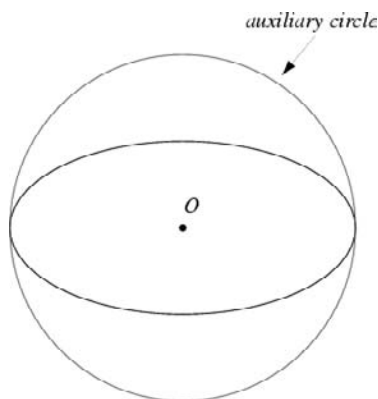
$$y'' = v' = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = \frac{dv}{dy} v.$$

For an autonomous ODE, the solution is independent of the time at which the initial conditions are applied. This means that all particles pass through a given point in phase space. A nonautonomous system of n first-order ODEs can be written as an autonomous system of $n + 1$ ODEs by letting $t \equiv x_{n+1}$ and increasing the dimension of the system by 1 by adding the equation

$$\frac{dx_{n+1}}{dt} = 1.$$

Autoregressive Model

MAXIMUM ENTROPY METHOD

Auxiliary Circle

The CIRCUMCIRCLE of an ELLIPSE, i.e., the CIRCLE whose CENTER concurs with that of the ELLIPSE and whose RADIUS is equal to the ELLIPSE'S SEMIMAJOR AXIS.

See also CIRCLE, ECCENTRIC ANGLE, ELLIPSE

References

Montenbruck, O. and Pfleger, T. *Astronomy on the Personal Computer*, 4th ed. Berlin: Springer-Verlag, p. 62, 2000.

Auxiliary Latitude

AUTHALIC LATITUDE, CONFORMAL LATITUDE, GEOCENTRIC LATITUDE, ISOMETRIC LATITUDE, LATITUDE, PARAMETRIC LATITUDE, RECTIFYING LATITUDE, REDUCED LATITUDE

Auxiliary Triangle

MEDIAL TRIANGLE

Average

MEAN

Average Absolute Deviation

$$\alpha = \frac{1}{N} \sum_{i=1}^N |x_i - \mu| = \langle |x_i - \mu| \rangle.$$

See also ABSOLUTE DEVIATION, DEVIATION, STANDARD DEVIATION, VARIANCE

Average Function

If f is CONTINUOUS on a CLOSED INTERVAL $[a, b]$, then there is at least one number x^* in $[a, b]$ such that

$$\int_a^b f(x) dx = f(x^*)(b - a).$$

The average value of the FUNCTION (f^-) on this interval is then given by $f(x^*)$.

See also MEAN-VALUE THEOREM

Average Power

The average power of a complex signal $f(t)$ as a function of time t is defined as

$$\langle f^2(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(t)|^2 dt,$$

where $|z|$ is the MODULUS (Papoulis 1962, p. 240).

See also AUTOCORRELATION

References

Papoulis, A. *The Fourier Integral and Its Applications*. New York: McGraw-Hill, 1962.

Average Seek Time

POINT-POINT DISTANCE-1-D

Avoided Pattern

A pattern $\tau = (\tau_1, \dots, \tau_n)$ is said to avoid $\alpha = (\alpha_1, \dots, \alpha_k)$ if α is not CONTAINED in τ . In other words, τ avoids α IFF no K -SUBSET of τ is ORDER ISOMORPHIC to α .

See also CONTAINED PATTERN, ORDER ISOMORPHIC, PERMUTATION PATTERN, WILF CLASS, WILF EQUIVALENT

References

Mansour, T. Permutations Avoiding a Pattern from S_k and at Least Two Patterns from S_3 . 31 Jul 2000. <http://xxx.lanl.gov/abs/math.CO/0007194/>.

Axial Vector

PSEUDOVECTOR

Axiom

A PROPOSITION regarded as self-evidently TRUE without PROOF. The word "axiom" is a slightly archaic synonym for POSTULATE. Compare CONJECTURE or HYPOTHESIS, both of which connote apparently TRUE but not self-evident statements.

See also ARCHIMEDES' AXIOM, AXIOM OF CHOICE, AXIOMATIC SYSTEM, CANTOR-DEDEKIND AXIOM, CONGRUENCE AXIOMS, CONJECTURE, CONTINUITY AXIOMS, COUNTABLE ADDITIVITY PROBABILITY AXIOM, DEDEKIND'S AXIOM, DIMENSION AXIOM, EILENBERG-STEENROD AXIOMS, EUCLID'S AXIOMS, EXCISION AXIOM, FANO'S AXIOM, FIELD AXIOMS, HAUSDORFF AXIOMS, HILBERT'S AXIOMS, HOMOTOPY AXIOM, INACCESSIBLE CARDINALS AXIOM, INCIDENCE AXIOMS, INDEPENDENCE AXIOM, INDUCTION AXIOM, LAW, LEMMA, LONG EXACT SEQUENCE OF A PAIR AXIOM, ORDERING AXIOMS, PARALLEL AXIOM, PASCH'S AXIOM, PEANO'S AXIOMS, PLAYFAIR'S AXIOM, PORISM, POSTULATE, PROBABILITY AXIOMS, PROCLUS' AXIOM, RULE, T2-SEPARATION AXIOM, THEOREM, ZERMELO'S AXIOM OF CHOICE, ZERMELO-FRAENKEL AXIOMS

Axiom A Diffeomorphism

Let $\phi : M \rightarrow M$ be a C^1 DIFFEOMORPHISM on a compact RIEMANNIAN MANIFOLD M . Then ϕ satisfies Axiom A if the NONWANDERING set $\Omega(\phi)$ of ϕ is hyperbolic and the PERIODIC POINTS of ϕ are DENSE in $\omega(\phi)$. although it was conjectured that the first of these conditions implies the second, they were shown to be independent in or around 1977. examples include the ANOSOV DIFFEOMORPHISMS and SMALE HORSESHOE MAP.

In some cases, Axiom A can be replaced by the condition that the DIFFEOMORPHISM is a hyperbolic diffeomorphism on a hyperbolic set (Bowen 1975, Parry and Pollicott 1990).

See also ANOSOV DIFFEOMORPHISM, AXIOM A FLOW, DIFFEOMORPHISM, DYNAMICAL SYSTEM, RIEMANNIAN MANIFOLD, SMALE HORSESHOE MAP

References

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Axiom A Flow

A FLOW defined analogously to the AXIOM A DIFFEOMORPHISM, except that instead of splitting the TANGENT BUNDLE into two invariant sub-BUNDLES, they are split into three (one exponentially contracting, one expanding, and one which is 1-dimensional and tangential to the flow direction).

See also DYNAMICAL SYSTEM

Axiom of Choice

An important and fundamental axiom in SET THEORY sometimes called ZERMELO'S AXIOM OF CHOICE. It was formulated by Zermelo in 1904 and states that, given any SET of mutually exclusive nonempty SETS, there exists at least one SET that contains exactly one element in common with each of the nonempty SETS. The axiom of choice is related to the first of HILBERT'S PROBLEMS.

In ZERMELO-FRAENKEL SET THEORY (in the form omitting the axiom of choice), the ZORN'S LEMMA, TRICHOTOMY LAW, and the WELL ORDERING PRINCIPLE are equivalent to the axiom of choice (Mendelson 1997, p. 275). In contexts sensitive to the axiom of choice, the notation "ZF" is often used to denote Zermelo-Fraenkel *without* the axiom of choice, while "ZFC" is used if the axiom of choice is included.

In 1940, Gödel proved that the axiom of choice is CONSISTENT with the axioms of VON NEUMANN-BERNAYS-GÖDEL SET THEORY (a conservative extension of ZERMELO-FRAENKEL SET THEORY). However, in 1963, Cohen (1963) unexpectedly demonstrated that the axiom of choice is also *independent* of ZERMELO-FRAENKEL SET THEORY (Mendelson 1997; Boyer and Merzbacher 1991, pp. 610–11).

See also HILBERT'S PROBLEMS, SET THEORY, VON NEUMANN-BERNAYS-GÖDEL SET THEORY, WELL ORDERED SET, WELL ORDERING PRINCIPLE, ZERMELO-FRAENKEL AXIOMS, ZERMELO-FRAENKEL SET THEORY, ZORN'S LEMMA

References

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 Moore, G. H. *Zermelo's Axiom of Choice: Its Origin, Development, and Influence*. New York: Springer-Verlag, 1982.

Axiom of Comprehension

AXIOM OF SEPARATION

Axiom of Extensionality

The axiom of ZERMELO-FRAENKEL SET THEORY which asserts that sets formed by the same elements are equal,

$$\forall x(x \in a \equiv x \in b) \Rightarrow a = b.$$

Using the notation $a \subset b$ (a is a SUBSET of b) for $x \in a(x \in b)$, the axiom can be rewritten

$$a \subset b \wedge b \subset a \Rightarrow a = b.$$

See also ZERMELO-FRAENKEL SET THEORY

References

- Itô, K. (Ed.). "Zermelo-Fraenkel Set Theory." §33B in *Encyclopedic Dictionary of Mathematics*, 2nd ed., Vol. I. Cambridge, MA: MIT Press, pp. 146–48, 1986.

Axiom of Foundation

One of the ZERMELO-FRAENKEL AXIOMS, also known the axiom of regularity (Rubin 1967, Suppes 1972). In the formal language of SET THEORY, it states that

$$x \neq 0 \Rightarrow \exists y(y \in x \wedge y \cap x = \emptyset),$$

where $\Rightarrow$ means IMPLIES, $\exists$ means EXISTS, $\wedge$ means AND, $\cap$ denotes INTERSECTION, and $\emptyset$ is the EMPTY

SET (Mendelson 1997, p. 288). More descriptively, "every nonempty set is disjoint from one of its elements."

The axiom of foundation can also be stated as "A set contains no infinitely descending (membership) sequence," or "A set contains a (membership) minimal element," i.e., there is an element of the set that shares no member with the set (Ciesielski 1997, p. 37; Moore 1982, p. 269; Rubin 1967, p. 81; Suppes 1972, p. 53).

Mendelson (1958) proved that the equivalence of these two statements necessarily relies on the AXIOM OF CHOICE. The dual expression is called ϵ -induction, and is equivalent to the axiom itself (Itô 1986, p. 147).

See also AXIOM OF CHOICE, ZERMELO-FRAENKEL AXIOMS

References

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- Zermelo, E. "Über Grenzzahlen und Mengenbereiche." *Fund. Math.* **16**, 29–7, 1930.

Axiom of Infinity

The axiom of ZERMELO-FRAENKEL SET THEORY which asserts the existence of a set containing all the natural numbers,

$$\exists r(\emptyset \in x \vee \forall y \in x(y' \in x)).$$

Here, following von Neumann, $0 = \phi$, $1 = 0' = \{0\}$, $2 = 1' = \{0, 1\}$, $3 = 2' = \{0, 1, 2\}$, ...

See also ZERMELO-FRAENKEL SET THEORY

References

- Itô, K. (Ed.). "Zermelo-Fraenkel Set Theory." §33B in *Encyclopedic Dictionary of Mathematics, 2nd ed., Vol. 1*. Cambridge, MA: MIT Press, pp. 146–48, 1986.

Axiom of Regularity

AXIOM OF FOUNDATION

Axiom of Replacement

One of the ZERMELO-FRAENKEL AXIOMS which asserts the existence for any set a of a set x such that, for any y of a , if there exists a z satisfying $A(y, z)$, then such z exists in x . This axiom was introduced by Fraenkel.

See also ZERMELO-FRAENKEL AXIOMS

References

- Itô, K. (Ed.). "Zermelo-Fraenkel Set Theory." §33B in *Encyclopedic Dictionary of Mathematics, 2nd ed., Vol. 1*. Cambridge, MA: MIT Press, pp. 146–48, 1986.

Axiom of Separation

The axiom of ZERMELO-FRAENKEL SET THEORY which asserts the existence for any set a and a formula $A(y)$ of a set x consisting of all elements of a satisfying $A(y)$,

$$\exists x \forall y(y \in x \equiv y \in a \wedge A(y)).$$

This axiom is also called the axiom of comprehension or axiom of subsets, and was introduced by Zermelo.

See also ZERMELO-FRAENKEL SET THEORY

References

- Itô, K. (Ed.). "Zermelo-Fraenkel Set Theory." §33B in *Encyclopedic Dictionary of Mathematics, 2nd ed., Vol. 1*. Cambridge, MA: MIT Press, pp. 146–48, 1986.

Axiom of the Empty Set

One of the ZERMELO-FRAENKEL AXIOMS which asserts the existence of the EMPTY SET ϕ . The axiom may be stated symbolically as

$$\exists x \forall y(!y \in x).$$

See also ZERMELO-FRAENKEL AXIOMS

References

- Itô, K. (Ed.). "Zermelo-Fraenkel Set Theory." §33B in *Encyclopedic Dictionary of Mathematics, 2nd ed., Vol. 1*. Cambridge, MA: MIT Press, pp. 146–48, 1986.

Axiom of the Power Set

One of the ZERMELO-FRAENKEL AXIOMS which asserts the existence for any set a of the POWER SET x consisting of all the SUBSETS of a . The axiom may be stated symbolically as

$$\forall x \exists y(y \in x \equiv \forall z(z \in y \Rightarrow z \in a)).$$

See also POWER SET, ZERMELO-FRAENKEL AXIOMS

References

Itô, K. (Ed.). "Zermelo-Fraenkel Set Theory." §33B in *Encyclopedic Dictionary of Mathematics, 2nd ed., Vol. 1.* Cambridge, MA: MIT Press, pp. 146–48, 1986.

Axiom of the Sum Set

The axiom of ZERMELO-FRAENKEL SET THEORY which asserts the existence for any set a of the sum (union) x of all sets that are elements of a . The axiom may be stated symbolically as

$$\exists x \forall y (y \in x \equiv \exists z \in a (y \in z)).$$

See also ZERMELO-FRAENKEL SET THEORY

References

Itô, K. (Ed.). "Zermelo-Fraenkel Set Theory." §33B in *Encyclopedic Dictionary of Mathematics, 2nd ed., Vol. 1.* Cambridge, MA: MIT Press, pp. 146–48, 1986.

Axiom of the Unordered Pair

The axiom of ZERMELO-FRAENKEL SET THEORY which asserts the existence for any sets a and b of a set x having a and b as its only elements. x is called the unordered pair of a and b , denoted $\{a, b\}$. The axiom may be stated symbolically as

$$\exists x \forall y (y \in x \equiv y = a \vee y = b).$$

See also ZERMELO-FRAENKEL SET THEORY

References

Itô, K. (Ed.). "Zermelo-Fraenkel Set Theory." §33B in *Encyclopedic Dictionary of Mathematics, 2nd ed., Vol. 1.* Cambridge, MA: MIT Press, pp. 146–48, 1986.

Axiomatic Set Theory

A version of SET THEORY in which axioms are taken as uninterpreted rather than as formalizations of pre-existing truths.

See also AXIOMATIC SYSTEM, COMPLETE AXIOMATIC THEORY, NAIVE SET THEORY, SET THEORY

References

Curry, H. B. *Foundations of Mathematical Logic.* New York: Dover, pp. 22–3, 1977.

Axiomatic System

A logical system which possesses an explicitly stated SET of AXIOMS from which THEOREMS can be derived.

See also AXIOMATIC SET THEORY, COMPLETE AXIOMATIC THEORY, CONSISTENCY, MODEL THEORY, THEOREM

Axioms of Subsets

This entry contributed by NICOLAS BRAY

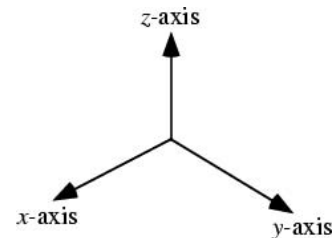
For any set theoretic formula $f(x, t_1, t_2, \dots, t_n)$,

$$(\forall t_1)(\forall t_2) \dots (\forall t_n)(\forall A)(\exists B)(\forall x).$$

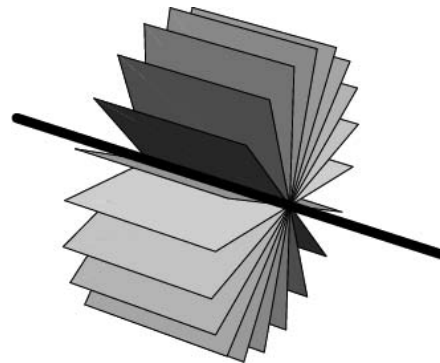
$$(x \in B \Leftrightarrow x \in A \wedge f(x, t_1, \dots, t_n))$$

In other words, for any formula and set A there is a SUBSET of A consisting exactly of those elements which satisfy the formula.

Axis



A LINE with respect to which a curve or figure is drawn, measured, rotated, etc.



The term is also used to refer to a LINE through a SHEAF OF PLANES (Woods 1961; Altshiller-Court 1979, p. 12).

See also ABCISSA, BROCARD AXIS, HOMOLOGY AXIS, LEMOINE AXIS, LINE, MAJOR AXIS, MEDIAL AXIS, MINOR AXIS, ORDINATE, ORTHIC AXIS, PERSPECTIVE AXIS, RADICAL AXIS, REAL AXIS, SEMIMAJOR AXIS, SEMIMINOR AXIS, SHEAF OF PLANES, SIMILARITY AXIS, X-AXIS, Y-AXIS, Z-AXIS

References

Altshiller-Court, N. *Modern Pure Solid Geometry.* New York: Chelsea, 1979.
Woods, F. S. *Higher Geometry: An Introduction to Advanced Methods in Analytic Geometry.* New York: Dover, p. 8, 1961.

Ax-Kochen Isomorphism Theorem

Let P be the SET of PRIMES, and let $\mathbb{Q}_p$ and $\mathbb{Z}_p(t)$ be the FIELDS of p -ADIC NUMBERS and formal POWER SERIES over $\mathbb{Z}_p = (0, 1, \dots, p-1)$. Further, suppose that D is a "nonprincipal maximal filter" on P . Then $\prod_{p \in P} \mathbb{Q}_p/D$ and $\prod_{p \in P} \mathbb{Z}_p(t)/D$ are ISOMORPHIC.

See also HYPERREAL NUMBER, NONSTANDARD ANALYSIS

Axonometry

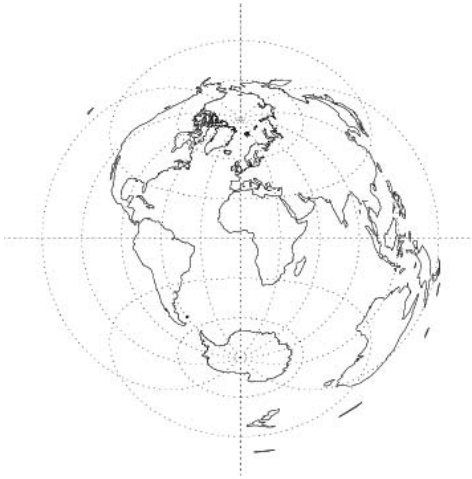
A METHOD for mapping 3-D figures onto the PLANE.

See also CROSS SECTION, MAP PROJECTION, POHLKE'S THEOREM, PROJECTION, STEREOLOGY

References

- Coxeter, H. S. M. *Regular Polytopes*, 3rd ed. New York: Dover, p. 313, 1973.
 Hazewinkel, M. (Managing Ed.). *Encyclopaedia of Mathematics: An Updated and Annotated Translation of the Soviet "Mathematical Encyclopaedia."* Dordrecht, Netherlands: Reidel, pp. 322–23, 1988.

Azimuthal Equidistant Projection



AN AZIMUTHAL PROJECTION which is neither EQUAL-AREA nor CONFORMAL. Let ϕ_1 and λ_0 be the LATITUDE and LONGITUDE of the center of the projection, then the transformation equations are given by

$$x = k' \cos \phi \sin(\lambda - \lambda_0) \quad (1)$$

$$y = k' [\cos \phi_1 \sin \phi - \sin \phi_1 \cos \phi \cos(\lambda - \lambda_0)]. \quad (2)$$

Here,

$$k' = \frac{c}{\sin c} \quad (3)$$

and

$$\cos c = \sin \phi_1 \sin \phi + \cos \phi_1 \cos \phi \cos(\lambda - \lambda_0), \quad (4)$$

where c is the angular distance from the center. The

inverse FORMULAS are

$$\phi = \sin^{-1} \left(\cos c \sin \phi_1 + \frac{y \sin c \cos \phi_1}{c} \right) \quad (5)$$

and

$$\lambda = \begin{cases} \lambda_0 + \tan^{-1} \left(\frac{x \sin c}{c \cos \phi_1 \cos c - y \sin \phi_1 \sin c} \right) & \text{for } \phi_1 \neq \pm 90^\circ \\ \lambda_0 + \tan^{-1} \left(-\frac{x}{y} \right) & \text{for } \phi_1 = 90^\circ \\ \lambda_0 + \tan^{-1} \left(\frac{x}{y} \right) & \text{for } \phi_1 = -90^\circ. \end{cases} \quad (6)$$

with the angular distance from the center given by

$$c = \sqrt{x^2 + y^2}. \quad (7)$$

See also AZIMUTHAL PROJECTION, EQUIDISTANT PROJECTION

References

- Snyder, J. P. *Map Projections--A Working Manual*. U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 191–02, 1987.

Azimuthal Projection

A MAP PROJECTION on which the azimuths of all points are shown correctly with respect to the center (Snyder 1987, p. 4). A plane tangent to one of the Earth's poles is the basis for polar azimuthal projection. The term "zenithal" is an older one for azimuthal projections (Hinks 1921, Lee 1944).

See also

AZIMUTHAL EQUIDISTANT PROJECTION, LAMBERT AZIMUTHAL EQUAL-AREA PROJECTION, ORTHOGRAPHIC PROJECTION, STEREOGRAPHIC PROJECTION

References

- Hinks, A. R. *Map Projections*, 2nd rev. ed. Cambridge, England: Cambridge University Press, 1921.
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B

B2-Sequence

N.B. A detailed online essay by S. Finch was the starting point for this entry.

Also called a SIDON SEQUENCE. An INFINITE SEQUENCE of POSITIVE INTEGERS

$$1 \leq b_1 < b_2 < b_3 < \dots \quad (1)$$

such that all pairwise sums

$$b_i + b_j \quad (2)$$

for $i \leq j$ are distinct (Guy 1994). An example is 1, 2, 4, 8, 13, 21, 31, 45, 66, 81, 97, 123, 148, 182, 204, 252, 290, 361, ... (Sloane's A005282).

Zhang (1993, 1994) showed that

$$S(B2) \equiv \sup_{\text{all } B2 \text{ sequences}} \sum_{k=1}^{\infty} \frac{1}{b_k} > 2.1597, \quad (3)$$

which has been increased to $S(B2) > 2.16086$ by R. Lewis using the sequence 1, 2, 4, 8, 13, 21, 31, 45, 66, 81, 97, 123, 148, 182, 204, 252, 291, 324, ... (Sloane's A046185). The definition can be extended to B_n -sequences (Guy 1994).

See also A-SEQUENCE, MIAN-CHOWLA SEQUENCE

References

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- Guy, R. K. "Packing Sums of Pairs," "Three-Subsets with Distinct Sums," and " B_2 -Sequences," and B_2 -Sequences Formed by the Greedy Algorithm." §C9, C11, E28, and E32 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 115–118, 121–123, 228–229, and 232–233, 1994.
- Mian, A. M. and Chowla, S. D. "On the B_2 -Sequences of Sidon." *Proc. Nat. Acad. Sci. India* **A14**, 3–4, 1944.
- Sloane, N. J. A. Sequences A005282/M1094 and A046185 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.
- Zhang, Z. X. "A B_2 -Sequence with Larger Reciprocal Sum." *Math. Comput.* **60**, 835–839, 1993.
- Zhang, Z. X. "Finding Finite B_2 -Sequences with Larger $m - a_m^{1/2}$." *Math. Comput.* **63**, 403–414, 1994.

Baby Monster Group

Also known as FISCHER'S BABY MONSTER GROUP. The SPORADIC FINITE GROUP B . It has ORDER

$$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47.$$

See also FINITE GROUP, MONSTER GROUP

References

- Wilson, R. A. "ATLAS of Finite Group Representation." <http://for.mat.bham.ac.uk/atlas/html/BM.html>.

BAC-CAB Identity

The VECTOR TRIPLE PRODUCT identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).$$

This identity can be generalized to n -D

$$\mathbf{a}_2 \times \dots \times \mathbf{a}_{n-1} \times (\mathbf{b}_1 \times \dots \times \mathbf{b}_{n-1}) \\ = (-1)^{n+1} \begin{vmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_{n-1} \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \dots & \mathbf{a}_2 \cdot \mathbf{b}_{n-1} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n-1} \cdot \mathbf{b}_1 & \dots & \mathbf{a}_{n-1} \cdot \mathbf{b}_{n-1} \end{vmatrix}.$$

See also LAGRANGE'S IDENTITY

BAC-CAB Rule

BAC-CAB IDENTITY

Bachelier Function

BROWN FUNCTION

Bachet Equation

The DIOPHANTINE EQUATION

$$x^2 + k = y^3.$$

which is also an ELLIPTIC CURVE. The general equation is still the focus of ongoing study.

Bachet's Conjecture

LAGRANGE'S FOUR-SQUARE THEOREM

Bachet's Theorem

LAGRANGE'S FOUR-SQUARE THEOREM

Backhouse's Constant

Let $P(x)$ be defined as the POWER SERIES whose n th term has a COEFFICIENT equal to the n th PRIME,

$$P(x) \equiv \sum_{k=0}^{\infty} p_k x^k = 1 + 2x + 3x^2 + 5x^3 + 7x^4 + 11x^5 + \dots,$$

and let $Q(x)$ be defined by

$$Q(x) = \frac{1}{P(x)} = \sum_{k=0}^{\infty} q_k x^k.$$

Then N. Backhouse conjectured that

$$\lim_{n \rightarrow \infty} \left| \frac{q_{n+1}}{q_n} \right| = 1.4560749485826896713995953511116 \dots$$

This list was subsequently shown to exist by P. Flajolet.

References

Finch, S. "Favorite Mathematical Constants." <http://www.mathsoft.com/asolve/constant/backhaus/backhaus.html>.

Bäcklund Transformation

A method for solving classes of nonlinear PARTIAL DIFFERENTIAL EQUATIONS.

See also INVERSE SCATTERING METHOD, SOLITON

References

- Anderson, R. L. and Ibragimov, N. H. *Lie-Bäcklund Transformation in Applications*. Philadelphia, PA: SIAM, 1979.
- Dodd, R. K.; Eilbeck, J. C.; and Morris, H. C. *Solitons and Nonlinear Equations*. London: Academic Press, 1984.
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- Lamb, G. L. Jr. *Elements of Soliton Theory*. New York: Wiley, 1980.
- Miura, R. M. (Ed.). *Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications*. New York: Springer-Verlag, 1974.
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- Zwillinger, D. "Bäcklund Transformations." §87 in *Handbook of Differential Equations*, 3rd ed. Boston, MA: Academic Press, pp. 321–24, 1997.

Backtracking

A method of solving combinatorial problems by means of an algorithm which is allowed to run forward until a dead end is reached, at which point previous steps are retraced and the algorithm is allowed to run forward again. Backtracking can greatly reduce the amount of work in an exhaustive search. Backtracking is implemented as Backtrack[s, partialQ, solutionQ] in the Mathematica add-on package DiscreteMath`Combinatorica` (which can be loaded with the command <<DiscreteMath`).

Backtracking also refers to a method of drawing FRACTALS by appropriate numbering of the corresponding tree diagram which does not require storage of intermediate results (Lauwerier 1991).

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- Wilf, H. "Backtrack: An $\iota(1)$ Expected Time Algorithm for the Graph Coloring Problem." *Info. Proc. Let.* **18**, 119–21, 1984.

Backus-Gilbert Method

A method which can be used to solve some classes of INTEGRAL EQUATIONS and is especially useful in implementing certain types of data inversion. It has been applied to invert seismic data to obtain density profiles in the Earth.

References

- Backus, G. and Gilbert, F. "The Resolving Power of Growth Earth Data." *Geophys. J. Roy. Astron. Soc.* **16**, 169–05, 1968.
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- Parker, R. L. "Understanding Inverse Theory." *Ann. Rev. Earth Planet. Sci.* **5**, 35–4, 1977.
- Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Backus-Gilbert Method." §18.6 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing*, 2nd ed. Cambridge, England: Cambridge University Press, pp. 806–09, 1992.

Backward Difference

The backward difference is a FINITE DIFFERENCE defined by

$$\nabla_p \equiv \nabla f_p \equiv f_p - f_{p-1}. \quad (1)$$

Higher order differences are obtained by repeated operations of the backward difference operator, so

$$\nabla_p^2 = \nabla(\nabla p) = \nabla(f_p - f_{p-1}) = \nabla f_p - \nabla f_{p-1} \quad (2)$$

$$\begin{aligned} &= (f_p - f_{p-1}) - (f_{p-1} - f_{p-2}) \\ &= f_p - 2f_{p-1} + f_{p-2} \end{aligned} \quad (3)$$

In general,

$$\nabla_p^k \equiv \nabla^k f_p \equiv \sum_{m=0}^k (-1)^m \binom{k}{m} f_{p-m}, \quad (4)$$

where $\binom{k}{m}$ is a BINOMIAL COEFFICIENT.

NEWTON'S BACKWARD DIFFERENCE FORMULA expresses f_p as the sum of the n th backward differences

$$\begin{aligned} f_p &= f_0 + p\nabla_0 + \frac{1}{2!} p(p+1)\nabla_0^2 + \frac{1}{3!} p(p+1)(p+2)\nabla_0^3 \\ &\quad + \dots, \end{aligned} \quad (5)$$

where ∇_0^n is the first n th difference computed from the difference table.

See also ADAMS' METHOD, DIFFERENCE EQUATION, DIVIDED DIFFERENCE, FINITE DIFFERENCE, FORWARD DIFFERENCE, NEWTON'S BACKWARD DIFFERENCE FORMULA, RECIPROCAL DIFFERENCE

References

- Beyer, W. H. *CRC Standard Mathematical Tables*, 28th ed. Boca Raton, FL: CRC Press, pp. 429 and 433, 1987.

Backward Stability

The property of certain algorithms that accurate answers are returned for well-conditioned problems, and the inaccuracy of the answers returned for ill-conditioned problems is proportional to the sensitivity.

Bader-Deuflhard Method

A generalization of the BULIRSCH-STOER ALGORITHM for solving ORDINARY DIFFERENTIAL EQUATIONS.

References

- Bader, G. and Deuflhard, P. "A Semi-Implicit Mid-Point Rule for Stiff Systems of Ordinary Differential Equations." *Numer. Math.* **41**, 373–98, 1983.
 Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing*, 2nd ed. Cambridge, England: Cambridge University Press, p. 730, 1992.

Baer Differential Equation

The Baer differential equation is given by

$$(x - a_1)(x - a_2)y'' + \frac{1}{2}[2x - (a_1 + a_2)]y' - (p^2x + q^2)y = 0,$$

while the Baer "wave equation" is

$$(x - a_1)(x - a_2)y'' + \frac{1}{2}[2x - (a_1 + a_2)]y' - (k^2x^2 - p^2x + q^2)y = 0$$

(Moon and Spencer 1961, pp. 156–57; Zwillinger 1997, p. 121).

References

- Moon, P. and Spencer, D. E. *Field Theory for Engineers*. New York: Van Nostrand, 1961.
 Zwillinger, D. *Handbook of Differential Equations*, 3rd ed. Boston, MA: Academic Press, p. 121, 1997.

Bagging

See also RESAMPLING STATISTICS

Baguenaudier



A PUZZLE involving disentangling a set of rings from a looped double rod, originally used by French peasants to lock chests (Steinhaus 1983). The word "baguenaudier" means "time-waster" in French, and the puzzle is also called the Chinese rings or Devil's needle puzzle. ("Bague" also means "ring," but this

appears to be an etymological coincidence. Interestingly, the bladder-senna tree is also known as "baguenaudier" in French.) Culin (1965) attributes the puzzle to Chinese general Hung Ming (A.D. 181–34), who gave it to his wife as a present to occupy her while he was away at the wars.

The solution of the baguenaudier is intimately related to the theory of GRAY CODES.

The minimum number of moves $a(n)$ needed for n rings is

$$a(n) = \lceil \frac{2}{3} (2^n - 1) \rceil = \begin{cases} \frac{1}{3} (2^{n+1} - 2) & n \text{ even} \\ \frac{1}{3} (2^{n+1} - 1) & n \text{ odd,} \end{cases} \quad (1)$$

where $\lceil x \rceil$ is the CEILING FUNCTION, giving 1, 2, 5, 10, 21, 42, 85, 170, 341, 682, ... (Sloane's A000975). The GENERATING FUNCTION for these numbers is

$$\frac{1}{(1 - 2x)(1 - x^2)} = 1 + 2x + 5x^2 + 10x^3 + 21x^4 + \dots \quad (2)$$

They are also given by the RECURRENCE RELATION

$$a(n) = a(n - 1) + 2a(n - 2) + 1 \quad (3)$$

with $a(1) = 1$ and $a(2) = 2$.

By simultaneously moving the two end rings, the number of moves for n rings can be reduced to

$$b(n) = \begin{cases} 2^{n-1} - 1 & n \text{ even} \\ 2^{n-1} & n \text{ odd,} \end{cases} \quad (4)$$

giving 1, 1, 4, 7, 16, 31, 64, 127, 256, 511, ... (Sloane's A051049).

Defining the complexity of a solution as the minimal number of times the ring passes through the arc from the last ring to the base of the puzzle, the minimal complexity of a solution is 2^{n-1} , as conjectured by Kauffman (1996) and proved by Przytycki and Sikora (2000).

See also GRAY CODE, HABIRO MOVE

References

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 University of Waterloo. "Wire and Ring Puzzles." <http://www.ahs.uwaterloo.ca/~museum/vexhibit/puzzles/wire/wire.html>.

Bailey's Lemma

If, for $n \geq 0$,

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r}(aq; q)_{n+r}}, \quad (1)$$

then

$$\beta'_n = \sum_{r=0}^n \frac{\alpha'_r}{(q; q)_{n-r}(aq; q)_{n+r}}, \quad (2)$$

where

$$\alpha'_r = \frac{(\rho_1; q)_r(\rho_2; q)_r(aq/\rho_1\rho_2)^r\alpha_r}{(aq/\rho_1; q)_r(aq/\rho_2; q)_r} \quad (3)$$

$$\beta'_n = \sum_{j \geq 0} \frac{(\rho_1; q)_j(\rho_2; q)_j(aq/\rho_1\rho_2; q)_{n-j}(aq/\rho_1\rho_2)^j\beta_j}{(q; q)_{n-j}(aq/\rho_1; q)_n(aq/\rho_2; q)_n}. \quad (4)$$

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Bailey's Method

LAMBERT'S METHOD

Bailey's Theorem

Let $\Gamma(z)$ be the GAMMA FUNCTION, then

$$\begin{aligned} & \left[\frac{\Gamma(m + \frac{1}{2})}{\Gamma(m)} \right]^2 \\ & \times \underbrace{\left[\frac{1}{m} + \left(\frac{1}{2} \right)^2 \frac{1}{m+1} + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \frac{1}{m+2} + \dots \right]}_n \\ & = \left[\frac{\Gamma(n + \frac{1}{2})}{\Gamma(n)} \right]^2 \\ & \times \underbrace{\left[\frac{1}{n} + \left(\frac{1}{2} \right)^2 \frac{1}{n+1} + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \frac{1}{n+2} + \dots \right]}_m. \end{aligned}$$

Writing the sums explicitly, Bailey's theorem states

$$\begin{aligned} & \left[\frac{\Gamma(m + \frac{1}{2})}{\Gamma(m)} \right]^2 \sum_{k=0}^{n-1} \frac{1}{m+k} \left[\frac{(2k-1)!!}{(2k)!!} \right]^2 \\ & \left[\frac{\Gamma(n + \frac{1}{2})}{\Gamma(n)} \right]^2 \sum_{k=0}^{m-1} \frac{1}{n+k} \left[\frac{(2k-1)!!}{(2k)!!} \right]^2. \end{aligned}$$

See also GAMMA FUNCTION

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Bailey's Transformation

The very general transformation

$$\begin{aligned} & {}_9F_8 \left[\begin{matrix} a, & 1 + \frac{1}{2}a, & b, & c, & d \\ & \frac{1}{2}a & 1+a-b, & 1+a-c, & 1+a-d. \end{matrix} \right. \\ & \quad \left. \begin{matrix} e, & f, & g, & -m; \\ 1+a-e, & 1+a-f, & 1+a-g, & 1+a+m \end{matrix} \right] \\ & = \frac{(1+a)_m(1+k-e)_m(1+k-f)_m(1+k-g)_m}{(1+k)_m(1+a-e)_m(1+a-f)_m(1+a-g)_m} \end{aligned}$$

$$\times_9 F_8 \left[\begin{matrix} k, & 1 + \frac{1}{2}k, & k + b - a, & k + c - a, & k + d - a, \\ & \frac{1}{2}k, & 1 + a - b, & a + a - c, & 1 + a - d, \\ & e, & f, & g, & -m; \\ & 1 + k - e, & 1 + k - f, & 1 + k - g, & 1 + k + m \end{matrix} \right],$$

where $k = 1 + 2a - b - c - d$, and the parameters are subject to the restriction

$$b + c + d + e + f + g - m = 2 + 3a$$

(Bailey 1935, p. 27).

Bhatnagar (1995, pp. 17–8) defines the Bailey transform as follows. Let $(a; q)_n$ be the q -POCHHAMMER SYMBOL, and let a be an indeterminate, and let the LOWER TRIANGULAR MATRICES $F = (F(n, k))$ and $F = (G(n, k))$ be defined as

$$F(n, k) = \frac{1}{(q; q)_{n-k}(aq; q)_{n+k}}$$

and

$$G(n, k) = \frac{(1 - aq^{2n})(a; q)_{n+k}}{(1 - a)(q; q)_{n-k}} (-1)^{n-k} q^{\binom{n-k}{2}}$$

Then F and G are MATRIX INVERSES.

See also DOUGALL-RAMANUJAN IDENTITY, GENERALIZED HYPERGEOMETRIC FUNCTION

References

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- Milne, S. C. and Lilly, G. M. "The A_ℓ and C_ℓ Bailey Transform and Lemma." *Bull. Amer. Math. Soc.* **26**, 258–63, 1992.

Bailey-Borwein-Plouffe Algorithm

The DIGIT-EXTRACTION ALGORITHM for calculating the digits of π given by the formula

$$\pi = \sum_{n=0}^{\infty} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right) \left(\frac{1}{16} \right)^n.$$

See also π , π FORMULAS

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- Adamchik, V. and Wagon, S. "A Simple Formula for π ." *Amer. Math. Monthly* **104**, 852–55, 1997.
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Finch, S. "Unsolved Mathematics Problems: The Miraculous Bailey-Borwein-Plouffe Pi Algorithm." <http://www.mathsoft.com/asolve/plouffe/plouffe.html>.

Baire Category Theorem

A nonempty complete METRIC SPACE cannot be REPRESENTED AS the UNION of a COUNTABLE family of NOWHERE DENSE SUBSETS.

See also COUNTABLE SET, METRIC SPACE, NOWHERE DENSE

Baire Function

References

- Feller, W. *An Introduction to Probability Theory and Its Applications, Vol. 2, 3rd ed.* New York: Wiley, pp. 104–06, 1971.

Baire Space

A TOPOLOGICAL SPACE X in which each SUBSET of X of the "first category" has an empty interior. A TOPOLOGICAL SPACE which is HOMEOMORPHIC to a complete METRIC SPACE is a Baire space.

Bairstow's Method

A procedure for finding the quadratic factors for the COMPLEX CONJUGATE ROOTS of a POLYNOMIAL $P(x)$ with REAL COEFFICIENTS.

$$\begin{aligned} [x - (a - ib)][x - (a + ib)] &= x^2 + 2ax + (a^2 + b^2) \\ &\equiv x^2 + Bx + C. \end{aligned} \quad (1)$$

Now write the original POLYNOMIAL as

$$P(x) = (x^2 + Bx + C)Q(x) + Rx + S \quad (2)$$

$$R(B + \delta B, C + \delta C) \approx R(B, C) + \frac{\partial R}{\partial B} \delta B + \frac{\partial R}{\partial C} \delta C \quad (3)$$

$$S(B + \delta B, C + \delta C) \approx S(B, C) + \frac{\partial S}{\partial B} \delta B + \frac{\partial S}{\partial C} \delta C \quad (4)$$

$$\frac{\partial P}{\partial C} = 0 = (x^2 + Bx + C) \frac{\partial Q}{\partial C} + Q(x) + \frac{\partial R}{\partial C} + \frac{\partial S}{\partial C} \quad (5)$$

$$-Q(x) = (x^2 + Bx + C) \frac{\partial Q}{\partial C} + \frac{\partial R}{\partial C} + \frac{\partial S}{\partial C} \quad (6)$$

$$\frac{\partial P}{\partial B} = 0 = (x^2 + Bx + C) \frac{\partial Q}{\partial B} + xQ(x) + \frac{\partial R}{\partial B} + \frac{\partial S}{\partial B} \quad (7)$$

$$-xQ(x) = (x^2 + Bx + C) \frac{\partial Q}{\partial B} + \frac{\partial R}{\partial B} + \frac{\partial S}{\partial B}. \quad (8)$$

Now use the 2-D NEWTON'S METHOD to find the simultaneous solutions.

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Baker's Dozen

The number 13.

See also 13, DOZEN

Baker's Map

The MAP

$$x_{n+1} = 2\mu x_n, \quad (1)$$

where x is computed modulo 1. A generalized Baker's map can be defined as

$$x_{n+1} = \begin{cases} \lambda_a x_n & y_n < \alpha \\ (1 - \lambda_b) + \lambda_b x_n & y_n > \alpha \end{cases} \quad (2)$$

$$y_{n+1} = \begin{cases} \frac{y_n}{\alpha} & y_n < \alpha \\ \frac{y_n - \alpha}{\beta} & y_n > \alpha, \end{cases} \quad (3)$$

where $\beta \equiv 1 - \alpha$, $\lambda_a + \lambda_b \leq 1$, and x and y are computed mod 1. The $q = 1$ Q-DIMENSION is

$$D_1 = 1 + \frac{\alpha \ln\left(\frac{1}{\alpha}\right) + \beta \ln\left(\frac{1}{\beta}\right)}{\alpha \ln\left(\frac{1}{\lambda_a}\right) + \beta \ln\left(\frac{1}{\lambda_b}\right)}. \quad (4)$$

If $\lambda_a = \lambda_b$, then the general Q-DIMENSION is

$$D_q = 1 + \frac{1}{q-1} \frac{\ln(\alpha^q + \beta^q)}{\ln \lambda_a}. \quad (5)$$

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Rasband, S. N. *Chaotic Dynamics of Nonlinear Systems*. New York: Wiley, p. 32, 1990.

Bakos' Compound

CUBE 4-COMPOUND

Balanced ANOVA

An ANOVA in which the number of REPLICATES (sets of identical observations) is restricted to be the same for each FACTOR LEVEL (treatment group).

See also ANOVA

Balanced Binomial Coefficient

An integer n is p -balanced for p a prime if, among all nonzero binomial coefficients $\binom{n}{k}$; for $k = 0, \dots, n \pmod{p}$, there are equal numbers of quadratic residues and nonresidues $\pmod{p}$. Let T_p be the set of integers n , $0 \leq n \leq p-1$, that are p -balanced. Among all the primes $< 1,000,000$, only those with $p = 2, 3$, and 11 have $T_p = \emptyset$.

p	T_p
2	$\emptyset$
3	$\emptyset$
5	{3}
7	{3}
11	$\emptyset$
13	{7, 11}
17	{3, 15}

See also BINOMIAL COEFFICIENT

References

Garfield, R. and Wilf, H. S. "The Distribution of the Binomial Coefficients Modulo p ." *J. Number Th.* **41**, 1, 1992.
Wilf, H. "On Crossing Numbers, and Some Unsolved Problems." In *Combinatorics, Geometry, and Probability: A Tribute to Paul Erdos. Papers from the Conference in Honor of Erdos' 80th Birthday Held at Trinity College, Cambridge, March 1993* (Ed. B. Bollobás and A. Thomason). Cambridge, England: Cambridge University Press, pp. 557–62, 1997.

Balanced Incomplete Block Design

BLOCK DESIGN

Ball

The n -ball, denoted $\mathbb{B}^n$, is the interior of a SPHERE $\mathbb{S}^{n-1}$, and sometimes also called the n -DISK. (Although physicists often use the term "SPHERE" to mean the solid ball, mathematicians definitely do not!) Let $\text{Vol}(\mathbb{B}^n)$ denote the volume of an n -D ball of RADIUS r . Then

$$\sum_{n=0}^{\infty} \text{Vol}(\mathbb{B}^n) = e^{\pi r^2} [1 + \text{erf}(r\sqrt{\pi})],$$

where $\text{erf}(x)$ is the ERF function.

See also ALEXANDER'S HORNED SPHERE, BALL LINE PICKING, BALL TRIANGLE PICKING, BANACH-TARSKI PARADOX, BING'S THEOREM, BISHOP'S INEQUALITY, BOUNDED SET, DISK, HYPERSPHERE, SPHERE, WILD POINT

References

Freden, E. Problem 10207. "Summing a Series of Volumes." *Amer. Math. Monthly* **100**, 882, 1993.

Ball Line Picking

Given an n -ball $\mathbb{B}^n$ of radius R , find the distribution of the lengths s of the lines determined by two points chosen at random within the ball. The probability distribution of lengths is given by

$$P_n(s) = n \frac{s^{n-1}}{R^n} I_x\left(\frac{1}{2}(n+1), \frac{1}{2}\right), \quad (1)$$

where

$$x \equiv 1 - \frac{s^2}{4R^2} \quad (2)$$

and

$$I_x(p, q) = \frac{B(x; p, q)}{B(p, q)} \quad (3)$$

is a REGULARIZED BETA FUNCTION, with $B(x; p, q)$ is an INCOMPLETE BETA FUNCTION and $B(p, q)$ is a BETA FUNCTION (Tu and Fischbach 2000). The first few are

$$P_1(s) = \frac{1}{R} - \frac{s}{2R} \quad (4)$$

$$P_2(s) = \frac{4s}{\pi R^2} \cos^{-1}\left(\frac{s}{2R}\right) - \frac{2s^2}{\pi R^3} \sqrt{1 - \frac{s^2}{4R^2}} \quad (5)$$

$$P_3(s) = \frac{3s^2}{R^3} - \frac{9s^3}{4R^4} + \frac{3s^5}{16R^6} \quad (6)$$

$$P_4(s) = \frac{8s^3}{\pi R^4} \cos^{-1}\left(\frac{s}{2R}\right) - \frac{8s^4}{3\pi R^5} \times \left(1 - \frac{s^2}{4R^2}\right)^{3/2} - \frac{4s^4}{\pi R^5} \sqrt{1 - \frac{s^2}{4R^2}}. \quad (7)$$

The average lengths are given by

$$\bar{s}_1 = \frac{2R}{3} \quad (8)$$

$$\bar{s}_2 = \frac{128R}{45\pi} \quad (9)$$

$$\bar{s}_3 = \frac{36R}{35} \quad (10)$$

$$\bar{s}_4 = \frac{16384R}{4725\pi}. \quad (11)$$

See also BALL POINT PICKING, SPHERE LINE PICKING

References

Kendall, M. G. and Moran, P. A. P. *Geometrical Probability*. New York: Hafner, 1963.

Santaló, L. A. *Integral Geometry and Geometric Probability*. Reading, MA: Addison-Wesley, 1976.

Tu, S.-J. and Fischbach, E. A New Geometric Probability Technique for an N -Dimensional Sphere and Its Applications 17 Apr 2000. <http://xxx.lanl.gov/abs/math-ph/0004021/>.

Ball Point Picking

See also BALL LINE PICKING, DISK POINT PICKING, NOISE SPHERE, SPHERE POINT PICKING

Ball Tetrahedron Picking

The mean volume of a TETRAHEDRON formed by four random points in a UNIT SPHERE is $\bar{V} = 12\pi/715$ (Hostinsky 1925; Solomon 1978, p. 124).

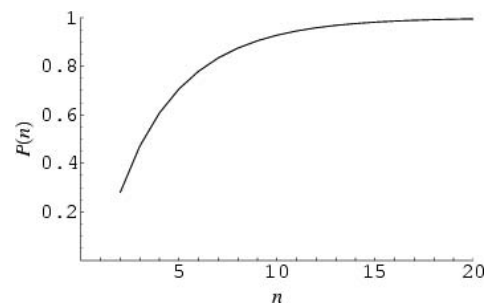
See also SPHERE TETRAHEDRON PICKING

References

Hostinsky, B. "Sur les probabilités géométriques." *Publ. Fac. Sci. Univ. Masaryk*, No. 50. Brno, Czechoslovakia, 1925.

Solomon, H. *Geometric Probability*. Philadelphia, PA: SIAM, 1978.

Ball Triangle Picking



The determination of the probability for obtaining an OBTUSE TRIANGLE by picking three points at random in the unit DISK was generalized by Hall (1982) to the n -dimensional BALL. Buchta (1986) subsequently gave closed form evaluations for Hall's integrals. Let P_n be the probability that that three points chosen independently and uniformly from the n -BALL

form an ACUTE TRIANGLE, then

$$\begin{aligned}
 P_{2m+1} = & -\frac{1}{2} - 2^{2m-1} \frac{\binom{2m}{m} \binom{4m}{2m}}{\binom{4m}{m} \binom{6m+1}{2m}} + m \binom{2m}{m}^2 2^{2m} \\
 & \times \sum_{k=0}^m \frac{\binom{2k}{k}}{\binom{2m+k}{m} \binom{4m+2k}{2m+k}} \\
 & \times \frac{3m+k+1}{(m+k)(3m+2k+1)} \quad (1) \\
 P_{2m+2} = & \frac{1}{4} - \frac{3}{2^{2m+4}} \frac{\binom{4m+4}{m+1}}{\binom{2m+2}{m+1}} + \frac{2^{4m}}{\binom{2m}{m} \pi^2} \\
 & \times \left[\frac{1}{(2m+1)^2 \binom{2m}{m}} \right. \\
 & \left. + \sum_{k=0}^m \frac{2^{2k}(3m+k-3)}{(2k+1) \binom{2k}{k} \binom{2m+k}{m} \binom{2m+k+2}{m}} \right], \quad (2)
 \end{aligned}$$

the first few being

$$P_2 = \frac{4}{\pi^2} - \frac{1}{8} \approx 0.280285 \quad (3)$$

$$P_3 = \frac{33}{70} \approx 0.471429 \quad (4)$$

$$P_4 = \frac{256}{45\pi^2} + \frac{1}{32} \approx 0.607655 \quad (5)$$

$$P_5 = \frac{1415}{2002} \approx 0.706793 \quad (6)$$

$$P_6 = \frac{2048}{315\pi^2} + \frac{31}{256} \approx 0.779842 \quad (7)$$

$$P_7 = \frac{231161}{277134} \approx 0.834113 \quad (8)$$

$$P_8 = \frac{4194304}{606375\pi^2} + \frac{89}{512} \approx 0.874668 \quad (9)$$

$$P_9 = \frac{9615369}{10623470} \approx 0.905106. \quad (10)$$

The case P_2 corresponds to DISK TRIANGLE PICKING case.

See also CUBE TRIANGLE PICKING, OBTUSE TRIANGLE, SPHERE POINT PICKING

References

- Buchta, C. "A Note on the Volume of a Random Polytope in a Tetrahedron." *Ill. J. Math.* **30**, 653–59, 1986.
 Hall, G. R. "Acute Triangles in the n -Ball." *J. Appl. Prob.* **19**, 712–15, 1982.

Ballantine

BORROMEAN RINGS

Ballieu's Theorem

Let the CHARACTERISTIC POLYNOMIAL of an $n \times n$ COMPLEX MATRIX A be written in the form

$$P(\lambda) = |\lambda I - A| = \lambda^n + b_1 \lambda^{n-1} + b_2 \lambda^{n-2} + \dots + b_{n-1} \lambda + b_n.$$

Then for any set $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ of POSITIVE numbers with $\mu_0 = 0$ and

$$M^- = \max_{0 \leq k \leq n-1} \frac{\mu_k + \mu_n |b_{n-k}|}{\mu_{k+1}},$$

all the EIGENVALUES λ_i (for $i = 1, \dots, n$) lie on the CLOSED DISK $|z| \leq M^-$ in the COMPLEX PLANE.

References

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Ballot Problem

Suppose A and B are candidates for office and there are $2n$ voters, n voting for A and n for B . In how many ways can the ballots be counted so that A is always ahead of or tied with B ? The solution is a CATALAN NUMBER C_n .

A related problem also called "the" ballot problem is to let A receive a votes and B b votes with $a > b$. This version of the ballot problem then asks for the probability that A stays ahead of B as the votes are counted (Vardi 1991). The solution is $(a-b)/(a+b)$, as first shown by M. Bertrand (Hilton and Pedersen 1991). Another elegant solution was provided by André (1887) using the so-called ANDRÉ'S REFLECTION METHOD.

The problem can also be generalized (Hilton and Pedersen 1991). Furthermore, the TAK FUNCTION is connected with the ballot problem (Vardi 1991).

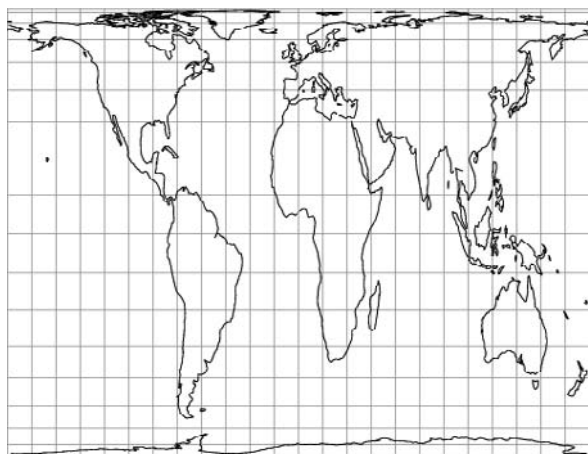
See also ANDRÉ'S REFLECTION METHOD, CATALAN NUMBER, STAIRCASE WALK, TAK FUNCTION

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- Vardi, I. *Computational Recreations in Mathematica*. Redwood City, CA: Addison-Wesley, pp. 185–87, 1991.

Balthasart Projection



A CYLINDRICAL EQUAL-AREA PROJECTION which uses a standard parallel of $\phi_s = 50^\circ$.

See also CYLINDRICAL EQUAL-AREA PROJECTION, BEHRMANN CYLINDRICAL EQUAL-AREA PROJECTION, GALL ORTHOGRAPHIC PROJECTION, LAMBERT AZIMUTHAL EQUAL-AREA PROJECTION, PETERS PROJECTION, TRISTAN EDWARDS PROJECTION

Banach Algebra

A Banach algebra is an ALGEBRA B over a FIELD F endowed with a NORM $\| \cdot \|$ such that B is a BANACH SPACE under the norm $\| \cdot \|$ and multiplication is continuous in the sense that if $x, y \in B$ then $\|xy\| \leq \|x\|\|y\|$. Continuity of multiplication is the most important property.

F is frequently taken to be the COMPLEX NUMBERS in order to assure that the SPECTRUM fully characterizes an OPERATOR (i.e., the spectral theorems for normal or compact normal operators do not, in general, hold in the SPECTRUM over the REAL NUMBERS).

If B has a unit, then $x \in B$ is invertible if and only if $\hat{x}(\phi) \neq 0$ for all ϕ , where $x \mapsto \hat{x}$ is the GELFAND TRANSFORM.

See also B*-ALGEBRA, BANACH SPACE, GELFAND TRANSFORM

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- Rudin, W. *Real and Complex Analysis*, 3rd ed. New York: McGraw-Hill, 1987.

Banach Fixed Point Theorem

Let f be a contraction mapping from a closed SUBSET F of a BANACH SPACE E into F . Then there exists a unique $z \in F$ such that $f(z) = z$.

See also FIXED POINT THEOREM

References

- Debnath, L. and Mikusinski, P. *Introduction to Hilbert Spaces with Applications*. San Diego, CA: Academic Press, 1990.

Banach Measure

An "AREA" which can be defined for every set—even those without a true geometric AREA—which is rigid and finitely additive.

Banach Space

A Banach space is a COMPLETE VECTOR SPACE B with a norm $\|v\|$. Its topology is determined by its norm, and the vector space operations of addition and scalar multiplication are required to be continuous. Two norms $\langle v \rangle_1$ and $\langle v \rangle_2$ are called equivalent if they give the same TOPOLOGY, which is equivalent to the existence of constants c and C such that

$$c\langle v \rangle_1 \leq \langle v \rangle_2 \leq C\langle v \rangle_1 \quad (1)$$

holds for all v . In the finite dimensional case, all norms are equivalent. An infinite dimensional space can have many different norms.

A basic example is n dimensional EUCLIDEAN SPACE with the Euclidean norm. Usually, the notion of Banach space is only used in the infinite dimensional setting, typically as a VECTOR SPACE of functions. For example, the set of continuous functions on the real line with the norm of a function f given by

$$\|f\| = \sup_{x \in \mathbb{R}} |f(x)| \quad (2)$$

is a Banach space, where $\sup$ denotes the SUPREMUM.

On the other hand, the set of continuous functions on the unit interval $[0, 1]$ with the norm of a function f given by

$$\|f\| = \int_0^1 |f(x)| dx \quad (3)$$

is not a Banach space because it is not complete. For instance, the CAUCHY SEQUENCE of functions

$$f_n \begin{cases} 1 & \text{for } x \leq 1/2 \\ \frac{1}{2}n + 1 - nx & \text{for } x \leq 1/2 + 1/n \\ 0 & \text{for } x > 1/2 + 1/n \end{cases} \quad (4)$$

does not converge to a continuous function.

HILBERT SPACES with their norm given by the inner product are examples of Banach spaces. While a HILBERT SPACE is always a Banach space, the converse need not hold. Therefore, it is possible for a Banach space not to have a norm given by an inner product. For instance, the supremum norm cannot be given by an INNER PRODUCT.

See also BESOV SPACE, COMPLETE SPACE, HILBERT SPACE, SCHAUDER FIXED POINT THEOREM, VECTOR SPACE

Banach-Hausdorff-Tarski Paradox

BANACH-TARSKI PARADOX

Banach-Steinhaus Theorem

UNIFORM BOUNDEDNESS PRINCIPLE

Banach-Tarski Paradox

First stated in 1924, the Banach-Tarski paradox states that it is possible to dissect a BALL into six pieces which can be reassembled by rigid motions to form two balls of the same size as the original. The number of pieces was subsequently reduced to five by R. M. Robinson in 1944, although the pieces are extremely complicated. (Actually, four pieces are sufficient as long as the single point at the center is neglected.) A generalization of this theorem is that any two bodies in $\mathbb{R}^3$ which do not extend to infinity and each containing a ball of arbitrary size can be dissected into each other (i.e., they are EQUIDECOMPOSABLE).

See also BALL, CIRCLE SQUARING, DISSECTION, EQUIDECOMPOSABLE

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Bandwidth

The bandwidth of a MATRIX $M = (m_{ij})$ is the maximum value of $|i - j|$ such that m_{ij} is nonzero.

The bandwidth of a GRAPH G is the minimum bandwidth among ADJACENCY MATRICES of GRAPHS isomorphic to G . Bounds for the bandwidth of a graph have been considered by (Harper 1964), and the bandwidth of the k -cube was determined by Harper (1966).

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Bang's Theorem

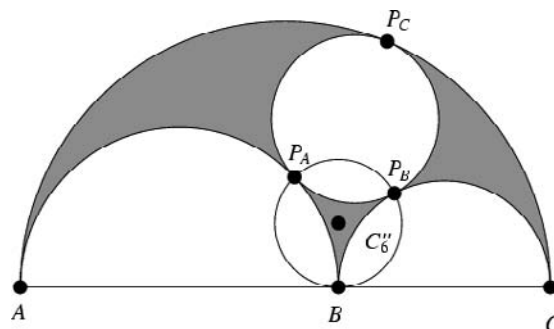
The lines drawn to the VERTICES of a face of a TETRAHEDRON from the point of contact of the FACE with the INSHERE form three ANGLES at the point of contact which are the same three ANGLES in each FACE.

See also TETRAHEDRON

References

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Bankoff Circle



The circle through the cusp of the ARBELOS and the tangent points of the first Pappus circle, which is congruent to the two ARCHIMEDES' CIRCLES. If $AB = r$

and $AC = 1$, then the radius of the Bankoff circle is

$$R = \frac{1}{2}r(1 - r).$$

See also ARCHIMEDES' CIRCLES, ARBELOS, PAPPUS CHAIN

References

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 Gardner, M. "Mathematical Games: The Diverse Pleasures of Circles that Are Tangent to One Another." *Sci. Amer.* **240**, 18–8, Jan. 1979.

Banzhaf Power Index

The number of ways in which a group of n with weights $\sum_{i=1}^n w_i = 1$ can change a losing coalition (one with $\sum w_i < 1/2$) to a winning one, or vice versa. It was proposed by the lawyer J. F. Banzhaf in 1965.

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Bar

A bar (also called an overbar) is a horizontal line written above a mathematical symbol to give it some special meaning. If the bar is placed over a single symbol, as in $\bar{x}$ (voiced "x-bar"), it is sometimes called a MACRON. If placed over multiple symbols (especially in the context of a RADICAL), it is known as a VINCULUM. Common uses of the bar symbol include the following.

1. The MEAN

$$\bar{x} \equiv \frac{1}{n} \sum_{i=1}^n x_i$$

of a set $\{x_i\}_{i=1}^n$.

2. The COMPLEX CONJUGATE

$$\bar{z} \equiv x - iy$$

for $z = x + iy$.

3. The COMPLEMENT $\bar{F}$ of a set F .
4. A SET stripped of any structure besides order, hence the ORDER TYPE of the set.

In conventional typography, "bar" refers to a vertical (instead a horizontal) bar, such as those used to denote ABSOLUTE VALUE ($|x|$) (Bringhurst 1997, p. 271).

See also DOUBLE BAR, HAT, MACRON, VINCULUM

References

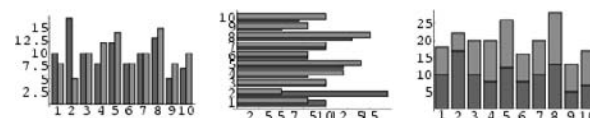
- Bringhurst, R. *The Elements of Typographic Style*, 2nd ed. Point Roberts, WA: Hartley and Marks, p. 271, 1997.

Bar (Edge)

The term in rigidity theory for the EDGES of a GRAPH.

See also CONFIGURATION, FRAMEWORK

Bar Chart



A bar graph is any plot of a set of data such that the number of data elements falling within one or more categories is indicated using a rectangle whose height or width is a function of the number of elements.

See also HISTOGRAM, PIE CHART

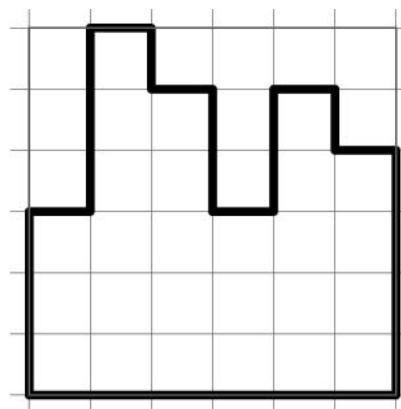
References

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Bar Graph

BAR CHART

Bar Graph Polygon



A column-convex SELF-AVOIDING POLYGON which contains the bottom edge of its minimal bounding rectangle. The anisotropic perimeter and area generating function

$$G(x, y, q) = \sum_{m \geq 1} \sum_{n \geq 1} \sum_{a \geq 0} C(m, n, a) x^m y^n q^a,$$

where $C(m, n, a)$ is the number of polygons with $2m$ horizontal bonds, $2n$ vertical bonds, and area a , has been computed exactly for the bar graph polygons (Bousquet-Mélou 1996, Bousquet-Mélou et al. 1999). The anisotropic area and perimeter generating function $G(x, y, q)$ and partial generating functions

$H_m(y, q)$, connected by

$$G(x, y, q) = \sum_{m \geq 1} H_m(y, q) x^m,$$

satisfy the self-reciprocity and inversion relations

$$H_m(1/y, 1/q) = \frac{(-1)^m}{yq^m} H_m(y, q)$$

and

$$G(x, y, q) - yG(-xq, 1/y, 1/q) = 0$$

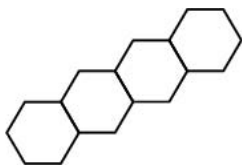
(Bousquet-Mélou *et al.* 1999).

See also LATTICE POLYGON, SELF-AVOIDING POLYGON

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Bar Polyhex



A POLYHEX consisting of HEXAGONS arranged along a line.

See also BAR POLYDIAMOND

References

- Gardner, M. *Mathematical Magic Show: More Puzzles, Games, Diversions, Illusions and Other Mathematical Sleight-of-Mind from Scientific American*. New York: Vintage, p. 147, 1978.

Bar Polyiamond



A POLYDIAMOND consisting of EQUILATERAL TRIANGLES arranged along a line.

See also BAR POLYHEX

References

- Golomb, S. W. *Polyominoes: Puzzles, Patterns, Problems, and Packings*, 2nd ed. Princeton, NJ: Princeton University Press, p. 92, 1994.

Barber Paradox

A man of Seville is shaved by the Barber of Seville IFF the man does not shave himself. Does the barber

shave himself? This PSEUDOPARADOX was proposed by Bertrand Russell.

See also PSEUDOPARADOX, RUSSELL'S PARADOX

References

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Barbier's Theorem

ALL CURVES OF CONSTANT WIDTH of width w have the same PERIMETER πw .

Bare Angle Center

The TRIANGLE CENTER with TRIANGLE CENTER FUNCTION

$$\alpha = A.$$

References

- Kimberling, C. "Major Centers of Triangles." *Amer. Math. Monthly* **104**, 431–38, 1997.

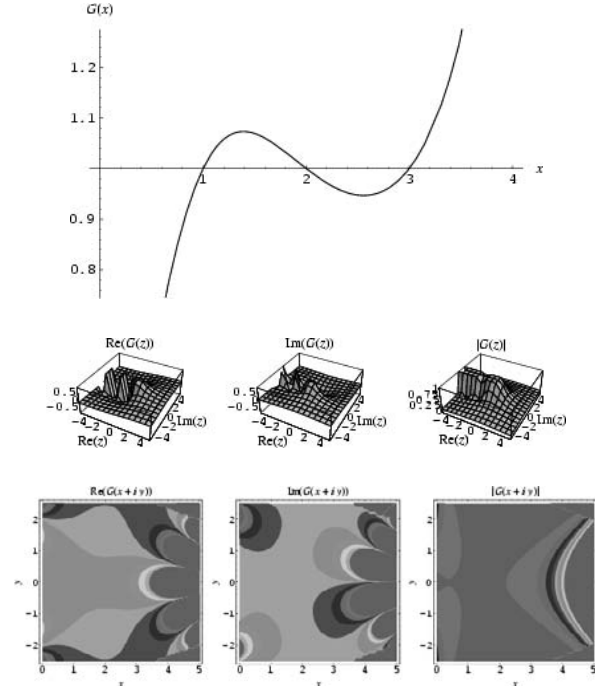
Barlow Packing

A face-centered cubic SPHERE PACKING obtained by placing layers of spheres one on top of another. Because there are two distinct ways to place each layer on top of the previous one, there are an infinite number of such packings as the number of layers is increased.

See also KEPLER CONJECTURE, SPHERE PACKING

References

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- Sloane, N. J. A. "Kepler's Conjecture Confirmed." *Nature* **395**, 435–36, 1998.

Barnes' *G*-Function

Barnes' *G*-function is defined by

$$G(z+1) \equiv (2\pi)^{z/2} e^{-[z(z+1)+\gamma z^2]/2} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right)^n e^{-z - z^2/(2n)} \right] \quad (1)$$

where γ is the EULER-MASCHERONI CONSTANT (Whittaker and Watson 1990, p. 264; Voros 1987). It is an ENTIRE FUNCTION analogous to $1/\Gamma(z)$, where $\Gamma(z)$ is the GAMMA FUNCTION, except that it has order 2 instead of 1.

This is an ANALYTIC CONTINUATION of the *G*-function defined in the construction of the GLAISHER-KINKELIN CONSTANT

$$G(n) \equiv \frac{[\Gamma(n)]^{n-1}}{K_n}, \quad (2)$$

where

$$K_n \equiv 0^0 1^1 2^2 3^3 \cdots (n-1)^{n-1}, \quad (3)$$

which has the special values

$$G(n) = \begin{cases} 0 & \text{if } n = 0, -1, -2, \dots \\ 1 & \text{if } n = 1 \\ 0!1!2! \cdots (n-2)! & \text{if } n = 2, 3, 4, \dots \end{cases} \quad (4)$$

for INTEGER n . This function is what Sloane and Plouffe (1995) call the SUPERFACTORIAL, and the first few values for $n = 1, 2, \dots$ are 1, 1, 1, 2, 12, 288, 34560, 24883200, 125411328000, 5056584744960000, ... (Sloane's A000178).

Barnes' *G*-function satisfies the functional equation

$$G(z+1) = \Gamma(z)G(z), \quad (5)$$

and has the TAYLOR SERIES

$$\ln G(1+z) = \frac{1}{2}[\ln(2\pi) - 1]z - (1+\gamma)\frac{z^2}{2} + \sum_{n=3}^{\infty} (-1)^{n-1} \zeta(n-1) \frac{z^n}{n} \quad (6)$$

in $|z| < 1$. It also gives an analytic solution to the finite product

$$\prod_{i=1}^n \Gamma(k+i) = \frac{G(n+k+1)}{G(k+1)}, \quad (7)$$

has the identities

$$\frac{[\Gamma(n)]^n}{G(n)} = K(n), \quad (8)$$

where $K(n)$ is the *K*-FUNCTION, and the equivalent reflection formulas

$$\frac{G'(z+1)}{G(z+1)} = \frac{1}{2} \ln(2\pi) - \frac{1}{2} - z + z \frac{\Gamma'(z)}{\Gamma(z)} \quad (9)$$

$$\ln \left[\frac{G(1-z)}{G(1+z)} \right] = \pi \int_0^z z \cot(\pi z) dz - z \ln(2\pi) \quad (10)$$

$$\frac{G(\frac{1}{2}+z)}{(\frac{1}{2}-z)} = \frac{(2\pi)^2}{\Gamma(\frac{1}{2}+z)} \sqrt{\frac{\pi}{\cos(\pi z)}} \exp \left[\pi \int_0^z \tan(\pi z) dz \right] \quad (11)$$

(Voros 1987; Whittaker and Watson 1990, p. 264). A Stirling-like ASYMPTOTIC SERIES as $z \rightarrow \infty$ is given by

$$\ln G(1+z) \sim z^2 \left(\frac{1}{2} \ln z - \frac{3}{4} \right) + \frac{1}{2} \ln(2\pi)z - \frac{1}{12} \ln z - \ln A + \mathcal{O}\left(\frac{1}{z}\right) \quad (12)$$

(Voros 1987).

$G(n)$ has the special values

$$G(\frac{1}{2}) = \pi^{-1/4} \exp \left[\frac{1}{24} \ln 2 + \frac{3}{2} \zeta'(-1) \right] \quad (13)$$

$$= A^{-3/2} \pi^{-1/4} e^{1/8} 2^{1/24} \quad (14)$$

$$G(\frac{3}{2}) = A^{-3/2} \pi^{1/4} e^{1/8} 2^{1/24}, \quad (15)$$

and so on, where $\zeta'(-1)$ is the derivative of the RIEMANN ZETA FUNCTION evaluated at -1 and the GLAISHER-KINKELIN CONSTANT A is defined by

$$A = \exp \left[\frac{1}{12} - \zeta'(-1) \right] = 1.28242712 \dots \quad (16)$$

(Voros 1987). *Mathematica* 4.0 implements the constant A as Glaisher. In general, for odd $n = 2k+1$,

$$G(\tfrac{1}{2}(2k+1)) = c_k \frac{A^{-3/2} \pi^{-(2k-3)/4} e^{1/8} 2^{1/24}}{2^{(k-1)(k-2)/2}}, \quad (17)$$

where

$$c_k = \prod_{i=1}^{k-2} \frac{2^i \Gamma(\tfrac{1}{2} + i)}{\sqrt{\pi}} \quad (18)$$

for $k > 1$, of which the first few terms are 1, 1, 1, 3, 45, 4725 4465125, ... (Sloane's A057863).

Barnes' G -function can arise in spectral functions in mathematical physics (Voros 1987).

Another G -FUNCTION is defined by Erdélyi *et al.* (1981, p. 20) as

$$G(z) \equiv \psi_0\left(\tfrac{1}{2} + hz\right) - \psi_0\left(\tfrac{1}{2}z\right), \quad (19)$$

where $\psi_0(z)$ is the DIGAMMA FUNCTION. An unrelated pair of functions are denoted g_n and G_n and are known as RAMANUJAN G - AND G -FUNCTIONS.

See also EULER-MASCHERONI CONSTANT, G -FUNCTION, GLAISHER-KINKELIN CONSTANT, K -FUNCTION, MEIJER'S G -FUNCTION, RAMANUJAN G - AND G -FUNCTIONS, SUPERFACTORIAL

References

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Barnes' Lemma

If a CONTOUR in the COMPLEX PLANE is curved such that it separates the increasing and decreasing sequences of POLES, then

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s) ds \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)},$$

where $\Gamma(z)$ is the GAMMA FUNCTION (Bailey 1935, p. 7).

Barnes' second lemma states that

$$\frac{2}{2\pi i} \frac{\Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\Gamma(\alpha_3+s)\Gamma(1-\beta_1-s)\Gamma(-s)}{\Gamma(\beta_2+s)} ds = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(1-\beta_1+\alpha_1)\Gamma(1-\beta_1+\alpha_2)\Gamma(1-\beta_1+\alpha_3)}{\Gamma(\beta_2-\alpha_1)\Gamma(\beta_2-\alpha_2)\Gamma(\beta_2-\alpha_3)}$$

provided that $\beta_1 + \beta_2 = \alpha_1 + \alpha_2 + \alpha_3 + 1$ (Bailey 1935, pp. 42–3).

References

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- Barnes, E. W. "A New Development in the Theory of the Hypergeometric Functions." *Proc. London Math. Soc.* **6**, 141–77, 1908.

Barnes-Wall Lattice

A lattice which can be constructed from the LEECH LATTICE A_{24} .

See also COXETER-TODD LATTICE, LATTICE POINT, LEECH LATTICE

References

- Barnes, E. S. and Wall, G. E. "Some Extreme Forms Defined in Terms of Abelian Groups." *J. Austral. Math. Soc.* **1**, 47–3, 1959.
- Conway, J. H. and Sloane, N. J. A. "The 16-Dimensional Barnes-Wall Lattice A_{16} ." §4.10 in *Sphere Packings, Lattices, and Groups*, 2nd ed. New York: Springer-Verlag, pp. 127–29, 1993.

Barnette's Conjecture

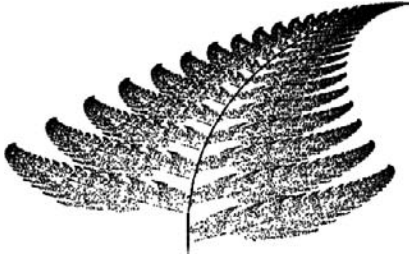
The conjecture that every 3-connected BIPARTITE CUBIC PLANAR GRAPH is HAMILTONIAN.

See also BIPARTITE GRAPH, CUBIC GRAPH, HAMILTONIAN GRAPH

References

- Barnette, D. Conjecture 5 in *Recent Progress in Combinatorics* (Ed. W. T. Tutte). New York: Academic Press, 1969.
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Barnsley's Fern



The ATTRACTOR of the ITERATED FUNCTION SYSTEM given by the set of "fern functions"

$$f_1(x, y) = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.00 \\ 1.60 \end{bmatrix} \quad (1)$$

$$f_2(x, y) = \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.00 \\ 0.44 \end{bmatrix} \quad (2)$$

$$f_3(x, y) = \begin{bmatrix} 0.20 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.00 \\ 1.60 \end{bmatrix} \quad (3)$$

$$f_4(x, y) = \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 0.16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (4)$$

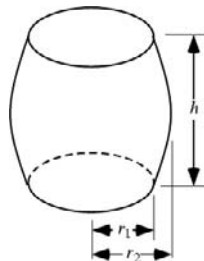
(Barnsley 1993, p. 86; Wagon 1991). These AFFINE TRANSFORMATIONS are contractions. The tip of the fern (which resembles the black spleenwort variety of fern) is the fixed point of f_1 , and the tips of the lowest two branches are the images of the main tip under f_2 and f_3 (Wagon 1991).

See also DYNAMICAL SYSTEM, FRACTAL, ITERATED FUNCTION SYSTEM

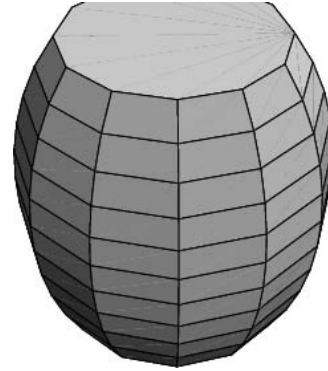
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 Gleick, J. *Chaos: Making a New Science*. New York: Penguin Books, p. 238, 1988.
 Wagon, S. "Biasing the Chaos Game: Barnsley's Fern." §5.3 in *Mathematica in Action*. New York: W. H. Freeman, pp. 156–63, 1991.
 Weisstein, E. W. "Fractals." MATHEMATICA NOTEBOOK FRACTAL.M.

Barrel



A SOLID OF REVOLUTION composed of parallel circular top and bottom with a common axis and a side formed by a smooth curve symmetrical about the midplane.



For sides consisting of an arc of an ELLIPSE, the equation of the side is given by

$$x(z) = r_2 \sqrt{1 - \frac{(z - \frac{1}{2}h)^2}{a^2}}, \quad (1)$$

with $x(0) = r_1$. Solving for a gives

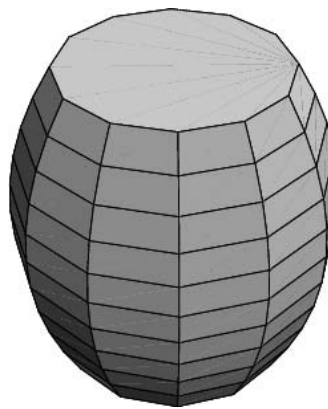
$$a = \frac{hr_2}{2\sqrt{r_2^2 - r_1^2}}, \quad (2)$$

so the sides have equation

$$x(z) = \sqrt{r_2^2 + \frac{(r_1 - r_2)(r_1 + r_2)(h - 2z)^2}{h^2}}. \quad (3)$$

Using the equation for a SOLID OF REVOLUTION then gives

$$V = \pi \int_0^h [x(z)]^2 dz = \frac{1}{3} \pi h (2r_2^2 + r_1^2). \quad (4)$$



For sides consisting of a PARABOLIC SEGMENT, the equation of the side is given by

$$x(z) = r_2 + a(z - \frac{1}{2}h)^2 \quad (5)$$

with $x(0) = r_1$. Solving for a gives

$$a = \frac{4(r_1 - r_2)}{h^2}, \quad (6)$$

so the sides have equation

$$x(z) = r_2 + \frac{(r_1 - r_2)(h - 2z)^2}{h^2}. \quad (7)$$

Using the equation for a SOLID OF REVOLUTION then gives

$$V = \pi \int_0^h [x(z)]^2 dz = \frac{1}{15} \pi h (3r_1^2 + 4r_1 r_2 + 8r_2^2). \quad (8)$$

See also CYLINDER

References

Harris, J. W. and Stocker, H. "Barrel." §4.10.4 in *Handbook of Mathematics and Computational Science*. New York: Springer-Verlag, p. 112, 1998.

Barrier

A number n is called a barrier of a number-theoretic function $f(m)$ if, for all $m < n$, $m + f(m) \leq n$. Neither the TOTIENT FUNCTION $\phi(n)$ nor the DIVISOR FUNCTION $\sigma(n)$ has a barrier.

Let $U \subseteq \mathbb{C}$ be an OPEN SET and $x_0 \in \partial U$, then a function $b : \bar{U} \rightarrow \mathbb{R}$ is called a barrier for U at a point x_0 if

1. b is continuous,
2. b is SUBHARMONIC on U ,
3. $b|_{\partial U} \leq 0$,
4. $\{z \in \partial U : b(z) = 0\} = \{x_0\}$

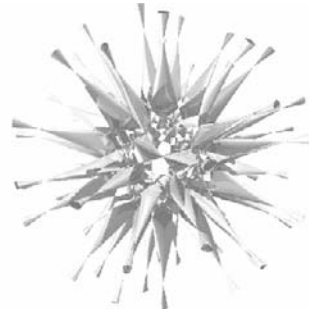
(Krantz 1999, pp. 100–01).

See also SUBHARMONIC FUNCTION

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Barth Decic



The Barth decic is a DECIC SURFACE in complex three-dimensional projective space having the maximum possible number of ORDINARY DOUBLE POINTS (345). It is given by the implicit equation

$$\begin{aligned} &8(x^2 - \phi^4 y^2)(y^2 - \phi^4 z^2)(z^2 - \phi^4 x^2) \\ &\quad \times (x^4 + y^4 + z^4 - 2x^2 y^2 - 2x^2 z^2 - 2y^2 z^2) + (3 + 5\phi) \\ &\quad \times (x^2 + y^2 + z^2 - w^2)^2 [x^2 + y^2 + z^2 - (2 - \phi)w^2]^2 w^2 \\ &= 0, \end{aligned}$$

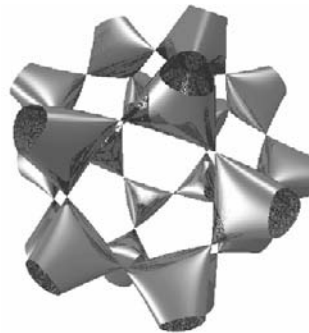
where ϕ is the GOLDEN MEAN and w is a parameter (Endraß, Nordstrand), taken as $w = 1$ in the above plot. The Barth decic is invariant under the ICOSAHEDRAL GROUP.

See also ALGEBRAIC SURFACE, BARTH SEXTIC, DECIC SURFACE, ORDINARY DOUBLE POINT

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Barth Sextic



The Barth-sextic is a SEXTIC SURFACE in complex three-dimensional projective space having the maximum possible number of ORDINARY DOUBLE POINTS (65). Of these, 20 nodes are at the vertices of a regular DODECAHEDRON of side length $2/\phi$, and 30 are at the midpoints of the edges of a concentric DODECAHEDRON of side length $2/\phi^2$, where ϕ is the GOLDEN RATIO. The surface was discovered by W. Barth in 1994, and is given by the implicit equation

$$4(\phi^2 x^2 - y^2)(\phi^2 y^2 - z^2)(\phi^2 z^2 - x^2) - (1 + 2\phi) \\ \times (x^2 + y^2 + z^2 - w^2)^2 w^2 = 0,$$

where ϕ is the GOLDEN MEAN, and w is a parameter (Endraß, Nordstrand), taken as $w = 1$ in the above plot.

The Barth sextic is invariant under the ICOSAHEDRAL GROUP. Under the map

$$(x, y, z, w) \rightarrow (x^2, y^2, z^2, w^2),$$

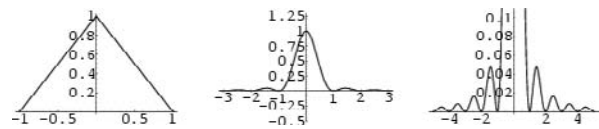
the surface is the eightfold cover of the CAYLEY CUBIC (Endraß).

See also ALGEBRAIC SURFACE, BARTH DECIC, CAYLEY CUBIC, ORDINARY DOUBLE POINT, SEXTIC SURFACE

References

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Bartlett Function



The APODIZATION FUNCTION

$$f(x) = 1 - \frac{|x|}{a} \quad (1)$$

which is a generalization of the one-argument TRIANGLE FUNCTION. Its FULL WIDTH AT HALF MAXIMUM is a .

It has INSTRUMENT FUNCTION

$$I(x) = \int_{-a}^a e^{-2\pi i k x} \left(1 - \frac{|x|}{a}\right) dx \\ = \int_{-a}^0 e^{-2\pi i k x} \left(1 + \frac{x}{a}\right) dx \\ + \int_0^a e^{-2\pi i k x} \left(1 - \frac{x}{a}\right) dx. \quad (2)$$

Letting $x' \equiv -x$ in the first part therefore gives

$$\int_{-a}^0 e^{-2\pi i k x} \left(1 + \frac{x}{a}\right) dx = \int_a^0 e^{-2\pi i k x'} \left(1 - \frac{x'}{a}\right) (-dx') \\ = \int_0^a e^{-2\pi i k x} \left(1 - \frac{x}{a}\right) dx. \quad (3)$$

Rewriting (2) using (3) gives

$$I(x) = (e^{2\pi i k x} + e^{-2\pi i k x}) \left(1 - \frac{x}{a}\right) dx \\ = 2 \int_0^a \cos(2\pi k x) \left(1 - \frac{x}{a}\right) dx. \quad (4)$$

Integrating the first part and using the integral

$$\int x \cos(bx) dx = \frac{1}{b^2} \cos(bx) + \frac{x}{b} \sin(bx) \quad (5)$$

for the second part gives

$$I(x) = 2 \left[\frac{\sin(2\pi k x)}{2\pi k} - \frac{1}{a} \left\{ \frac{1}{4\pi^2 k^2} \cos(2\pi k x) + \frac{x}{2\pi k} \sin(2\pi k x) \right\} \right]_0^a \\ = 2 \left\{ \left[\frac{\sin(2\pi k a)}{2\pi k} - 0 \right] - \frac{1}{a} \left[\frac{\cos(2\pi k a) - 1}{4\pi^2 k^2} + \frac{a \sin(2\pi k a)}{2\pi k} \right] \right\} \\ = \frac{1}{2\pi^2 a k^2} [\cos(2\pi k a) - 1] = a \frac{\sin^2(\pi k a)}{\pi^2 k^2 a^2} = a \operatorname{sinc}^2(\pi k a) \quad (6)$$

where $\operatorname{sinc} x$ is the SINC FUNCTION. The peak (in units of a) is 1. The function $I(x)$ is always positive, so there are no NEGATIVE sidelobes. The extrema are given by letting $\beta \equiv \pi k a$ and solving

$$\frac{d}{d\beta} \left(\frac{\sin \beta}{\beta} \right)^2 = 2 \frac{\sin \beta}{\beta} \frac{\sin \beta - \beta \cos \beta}{\beta^2} = 0 \quad (7)$$

$$\sin \beta (\sin \beta - \beta \cos \beta) = 0 \quad (8)$$

$$\sin \beta - \beta \cos \beta = 0 \quad (9)$$

$$\tan \beta = \beta. \quad (10)$$

Solving this numerically gives $\beta = 4.49341$ for the first maximum, and the peak POSITIVE sidelobe is 0.047190. The full width at half maximum is given by

setting $x \equiv \pi ka$ and solving

$$\operatorname{sinc}^2 x = \frac{1}{2} \quad (11)$$

for $x_{1/2}$, yielding

$$x_{1/2} = \pi k_{1/2} a = 1.39156. \quad (12)$$

Therefore, with $L \equiv 2a$,

$$\text{FWHM} = 2k_{1/2} = \frac{0.885895}{a} = \frac{1.77179}{L}. \quad (13)$$

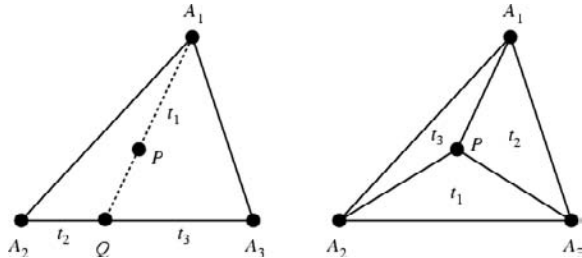
See also APODIZATION FUNCTION, PARZEN APODIZATION FUNCTION, TRIANGLE FUNCTION

References

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Barycentric Coordinates

Barycentric coordinates are triples of numbers (t_1, t_2, t_3) corresponding to masses placed at the vertices of a reference triangle $\Delta A_1 A_2 A_3$. These masses then determine a point P , which is the centroid of the three masses, and is identified with coordinates (t_1, t_2, t_3) . The vertices of the triangle are given by $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Barycentric coordinates were discovered by Möbius in 1827 (Coxeter 1969, p. 217; Fauvel *et al.* 1993).



To find the barycentric coordinates for an arbitrary point P , find t_2 and t_3 from the point Q at the intersection of the line A_1P with the side A_2A_3 , and then determine t_1 as the mass at A_n that will balance a mass $t_2 + t_3$ at Q , thus making P the centroid (left figure). Furthermore, the areas of the triangles $\Delta A_1 A_2 P$, $\Delta A_1 A_3 P$, and $\Delta A_2 A_3 P$ are proportional to the barycentric coordinates t_3 , t_2 , and t_1 of P (right figure; Coxeter 1969, p. 217).

Barycentric coordinates are homogeneous, so

$$(t_1, t_2, t_3) = (\mu t_1, \mu t_2, \mu t_3) \quad (1)$$

for $\mu \neq 0$. Barycentric coordinates normalized so that they become the actual areas of the subtriangles are called homogeneous barycentric coordinates, and

barycentric coordinates normalized so that

$$t_1 + t_2 + t_3 = 1, \quad (2)$$

so that the coordinates give the areas of the subtriangles *normalized by the area of the original triangle* are called AREAL COORDINATES (Coxeter 1969, p. 218). Barycentric and areal coordinates can provide particular elegant proofs of geometric theorems such as ROUTH'S THEOREM, CEVA'S THEOREM, and MENELAUS' THEOREM (Coxeter 1969, pp. 219–21).

The homogeneous barycentric coordinates corresponding to TRILINEAR COORDINATES $\alpha : \beta : \gamma$ are $(\alpha x, \beta y, \gamma z)$, and the TRILINEAR COORDINATES corresponding to homogeneous barycentric coordinates (t_1, t_2, t_3) are $t_1/a : t_2/b : t_3/c$. The homogeneous barycentric coordinates for some common triangle centers are summarized in the following table, where $s = (a + b + c)/2$ is the SEMIPERIMETER.

triangle center	homogeneous barycentric coordinates
CENTROID (TRIANGLE)	(1, 1, 1)
CIRCUMCENTER	$(a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^3(a^2 + b^2 - c^2))$
EXCENTERS	$(-a, b, c)$ $(a, -b, c)$ $(a, b, -c)$
GERGONNE POINT	$((s-b)(s-c), (s-c)(s-a), (s-a)(s-b))$
INCENTER	(a, b, c)
NAGEL POINT	$(s-a, s-b, s-c)$
ORTHOCENTER	$((a^2 + b^2 - c^2)(c^2 + a^2 - b^2), (b^2 + c^2 - a^2)(a^2 + b^2 - c^2), (c^2 + a^2 - b^2)(b^2 + c^2 - a^2))$
SYMMEDIAN POINT	(a^2, b^2, c^2)

In barycentric coordinates, a line has a linear homogeneous equation. In particular, the line joining points (r_1, r_2, r_3) and (s_1, s_2, s_3) has equation

$$\begin{vmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0 \quad (3)$$

(Loney 1962, pp. 39 and 57; Coxeter 1969, p. 219; Bottema 1982). If the vertices P_i of a triangle $\Delta P_1 P_2 P_3$ have barycentric coordinates (x_i, y_i, z_i) , then the area of the triangle is

$$\Delta P_1 P_2 P_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Delta ABC \quad (4)$$

(Bottema 1982, Yiu 2000).

See also AREAL COORDINATES, TRILINEAR COORDINATES

References

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Base (Logarithm)

The number used to define the number system in which a LOGARITHM is computed. In general, the logarithm of a number x in base b is written $\log_b x$. The symbol $\log x$ is an abbreviation regrettably used both for the COMMON LOGARITHM $\log_{10} x$ (by engineers and physicists and indicated on pocket calculators) and for the NATURAL LOGARITHM $\log_e x$ (by mathematicians). $\ln x$ denotes the NATURAL LOGARITHM $\log_e x$ (as used by engineers and physicists and indicated on pocket calculators), and $\lg x$ denotes $\log_2 x$. In this work, the notations $\log x = \log_{10} x$ and $\ln x = \log_e x$ are used.

To convert between logarithms in different bases, the formula

$$\log_b x = \frac{\ln x}{\ln b}$$

can be used.

See also COMMON LOGARITHM, E , LG, LN, LOGARITHM, NAPIERIAN LOGARITHM, NATURAL LOGARITHM, BASE (NUMBER)

Base (Neighborhood System)

A base for a neighborhood system of a point x is a collection N of OPEN SETS such that x belongs to every member of N , and any OPEN SET containing x also contains a member of N as a SUBSET.

Base (Number)

A REAL NUMBER x can be represented using any INTEGER number b as a base (sometimes also called a RADIX or SCALE). The choice of a base yields to a representation of numbers known as a NUMBER SYSTEM. In base b , the DIGITS 0, 1, ..., $b - 1$ are used (where, by convention, for bases larger than 10, the symbols A, B, C, ...are generally used as symbols representing the DECIMAL numbers 10, 11, 12, ...).

Base	Name
2	BINARY
3	TERNARY
4	QUATERNARY
5	Quinary
6	Senary
7	Septenary
8	OCTAL
9	Nonary
10	DECIMAL
11	Undenary
12	DUODECIMAL
16	HEXADECIMAL
20	VIGESIMAL
60	SEXAGESIMAL

Let the base b representation of a number x be written

$$(a_n a_{n-1} \dots a_0 . a_{-1} \dots)_b, \quad (1)$$

(e.g., 123.456_{10}), then the index of the leading DIGIT needed to represent the number is

$$n \equiv \lfloor \log_b x \rfloor, \quad (2)$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION. Now, recursively compute the successive DIGITS

$$a_i = \left\lfloor \frac{r_i}{b^i} \right\rfloor, \quad (3)$$

where $r_n \equiv x$ and

$$r_{i-1} = r_i - a_i b^i \quad (4)$$

for $i = n, n - 1, \dots, 1, 0, \dots$. This gives the base b representation of x . Note that if x is an INTEGER, then i need only run through 0, and that if x has a fractional part, then the expansion may or may not terminate. For example, the HEXADECIMAL representation of 0.1 (which terminates in DECIMAL notation) is the infinite expression $0.19999 \dots_h$.

Some number systems use a mixture of bases for counting. Examples include the Mayan calendar and the old British monetary system (in which ha'pennies, pennies, threepence, sixpence, shillings, half crowns, pounds, and guineas corresponded to units of 1/2, 1, 3, 6, 12, 30, 240, and 252, respectively).

Knuth (1998) has considered using TRANSCENDENTAL bases. This leads to some rather unfamiliar results, such as equating π to 1 in "base π ," $\pi = 10_\pi$.

See also BINARY, DECIMAL, DUODECIMAL, HEREDI-

TARY REPRESENTATION, HEXADECIMAL, OCTAL, QUATERNARY, SEXAGESIMAL, TERNARY, VIGESIMAL

References

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Base Curve

DIRECTRIX (RULED SURFACE)

Base Manifold

The base manifold in a BUNDLE is analogous to the domain for a set of functions. In fact, a bundle, by definition, comes with a map to the base manifold, often called π or projection.

For example, the base manifold to the TANGENT BUNDLE of a MANIFOLD M is the MANIFOLD M . A VECTOR FIELD is a function from the manifold to the TANGENT BUNDLE, with the restriction that every point gets mapped to a vector at that point. In general, a BUNDLE has SECTIONS, at least locally, which are maps from the base manifold to the BUNDLE.

See also BUNDLE, MANIFOLD, SECTION (BUNDLE), TANGENT BUNDLE, VECTOR BUNDLE

Base Space

The SPACE B of a FIBER BUNDLE given by the MAP $f: E \rightarrow B$, where E is the TOTAL SPACE of the FIBER BUNDLE.

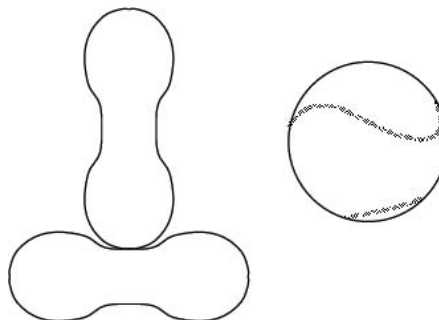
See also FIBER BUNDLE, TOTAL SPACE

Baseball

The numbers three and four appear prominently in the game of baseball. There are three strikes for an out, and three outs per inning, $3 \cdot 3 = 9$ innings in a game, giving $3^3 = 27$ outs per game (assuming no extra innings). In addition, there are 3×3 players per team. Four balls are needed for a walk. The number of bases can either be regarded as three (excluding HOME PLATE) or four (including it).

See also BASEBALL COVER, HOME PLATE

Baseball Cover



A pair of identical plane regions (mirror symmetric about two perpendicular lines through the center) which can be stitched together to form a baseball (or tennis ball). A baseball has a CIRCUMFERENCE of $9 \frac{1}{8}$ inches. The practical consideration of separating the regions far enough to allow the pitcher a good grip requires that the "neck" distance be about $1 \frac{3}{16}$ inches. The baseball cover was invented by Elias Drake as a boy in the 1840s. (Thompson's attribution of the current design to trial and error development by C. H. Jackson in the 1860s is apparently unsubstantiated, as discovered by George Bart.)

One way to produce a baseball cover is to draw the regions on a SPHERE, then cut them out. However, it is difficult to produce two identical regions in this manner. Thompson (1996) gives mathematical expressions giving baseball cover curves both in the plane and in 3-D. J. H. Conway has humorously proposed the following "baseball curve conjecture:" no two definitions of "the" baseball curve will give the same answer unless their equivalence was obvious from the start.

See also BASEBALL, HOME PLATE, TENNIS BALL THEOREM, YIN-YANG

References

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Basepoint

See also LOOP

Basic Polynomial Sequence

A POLYNOMIAL SEQUENCE $p_n(x)$ is called the basic polynomial sequence for a DELTA OPERATOR Q if

1. $p_0(x) = 1$,
2. $p_n(0) = 0$ for all $n > 0$,
3. $Qp_n(x) = np_{n-1}(x)$.

If $p_n(x)$ is a basic polynomial sequence for some DELTA OPERATOR Q , then it is a BINOMIAL-TYPE SEQUENCE of polynomials. Furthermore, if $p_n(x)$ is a BINOMIAL-TYPE SEQUENCE of polynomials, then it is a basic polynomial sequence for some DELTA OPERATOR.

See also BINOMIAL-TYPE SEQUENCE, DELTA OPERATOR, POLYNOMIAL SEQUENCE, UMBRAL OPERATOR

References

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Basin of Attraction

The set of points in the space of system variables such that initial conditions chosen in this set dynamically evolve to a particular ATTRACTOR.

See also WADA BASIN

Basis

The word basis can arise in several different contexts. Speaking in general terms, an object is "generated" by a basis in whatever manner is appropriate. For example, a VECTOR SPACE can have a BASIS which SPANS the vector space by finite LINEAR COMBINATIONS.

See also BASIS POINT, BASIS (TOPOLOGY), BASIS (VECTOR SPACE), HAMEL BASIS, HILBERT BASIS, ORTHONORMAL BASIS, VECTOR BASIS

Basis (Topology)

If X is a SET, a basis for a TOPOLOGY on X is a collection B of SUBSETS of X (called basis elements) satisfying the following properties.

1. For each $x \in X$, there is at least one basis element B containing x .
2. If x belongs to the intersection of two basis elements B_1 and B_2 , then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

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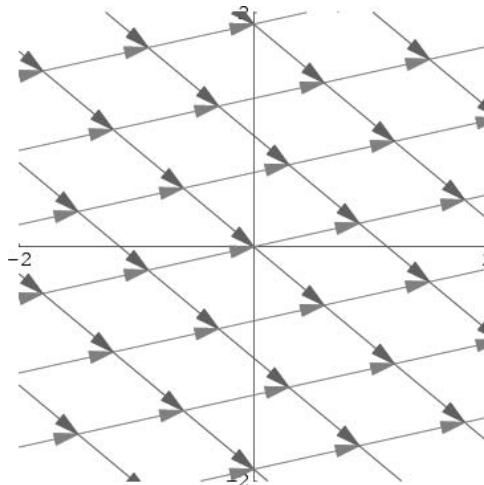
Basis (Vector Space)

A basis of a VECTOR SPACE V is defined as a subset $v_1, \dots, v_n$ of vectors in V that are LINEARLY INDEPENDENT and SPAN V . Consequently, if $(v_1, v_2, \dots, v_n)$ is a list of vectors in V , then these vectors form a basis if and only if every $v \in V$ can be

uniquely written as

$$v = a_1 b_1 + a_2 b_2 + \dots + a_n b_n,$$

where $a_1, \dots, a_n$ are elements of $\mathbb{R}$ or $\mathbb{C}$. A VECTOR SPACE V will have many different bases, but there are always the same number of basis vectors in each of them. The number of basis vectors in V is called the DIMENSION of V . Every spanning list in a vector space can be reduced to a basis of the vector space.



The simplest example of a basis is the standard basis in $\mathbb{R}^n$ consisting of the coordinate axes. For example, in $\mathbb{R}^2$, the standard basis consists of two VECTORS $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Any VECTOR $w = (a, b)$ can be written uniquely as the LINEAR COMBINATION $w = ae_1 + be_2$. Indeed, a vector is defined by its coordinates. The VECTORS $v_1 = (3, 2)$ and $v_2 = (2, 1)$ are also a basis for $\mathbb{R}^2$ because any VECTOR $w = (a, b)$ can be uniquely written as $w = (-a + 2b)v_1 + (2a - 3b)v_2$. The above figure shows $(0.6, -0.5)n + (0.9, .02)m$, which are linear combinations of the basis $\{(0.6, -0.5), (0.9, 0.2)\}$.

Here is a *Mathematica* function which will return the coefficients a_i given a basis v_i .

```
LinearCombination[v_List?MatrixQ, w_] :=  
  LinearSolve[Transpose[v], w]
```

For example, `LinearCombo[{{1, 2}, {0, 1}}, {-3, 4}]` yields `{3, -2}`, since $3(-1, 2) - 2(0, 1) = (-3, 4)$.

When a VECTOR SPACE is infinite dimensional, then a basis exists, as long as one assumes the AXIOM OF CHOICE. A subset of the basis which is linearly independent and whose span is DENSE is called a complete set, and is similar to a basis. When V is a HILBERT SPACE, a complete set is called a HILBERT BASIS.

See also BASIS, DIMENSION, HILBERT BASIS, LINEAR COMBINATION, ORTHONORMAL BASIS, SPAN (VECTOR SPACE), VECTOR SPACE

Basis Element

A collection B of subsets of a set X forming a topological BASIS.

See also BASIS (TOPOLOGY)

Basis Point

One basis point is defined to be 0.01 PERCENTAGE POINTS. Therefore, a change of 0.21% could also be expressed as a change by 21 "basis points."

See also PERCENTAGE POINT

Basis Theorem

HILBERT BASIS THEOREM

Basler Problem

The problem of analytically finding the value of $\zeta(2)$, where $\zeta(n)$ is the RIEMANN ZETA FUNCTION.

See also APÉRY'S CONSTANT, RIEMANN ZETA FUNCTION

References

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Basset Function

MODIFIED BESSEL FUNCTION OF THE SECOND KIND

Bat

CHEVRON

Batch

A set of values of similar meaning obtained in any manner.

References

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Bateman Equation**References**

Fairlie, D. B. and Leznov, A. N. The Complex Bateman Equation in a Space of Arbitrary Dimension. 16 Sep 1999. <http://xxx.lanl.gov/abs/solv-int/9909013/>.

Bateman Function

$$k_\nu(x) \equiv \frac{e^{-x}}{\Gamma(1 + \frac{1}{2}\nu)} U(-\frac{1}{2}\nu, 0, 2x)$$

for $x > 0$, where U is a CONFLUENT HYPERGEOMETRIC FUNCTION OF THE SECOND KIND.

See also CONFLUENT HYPERGEOMETRIC DIFFERENTIAL EQUATION, HYPERGEOMETRIC FUNCTION

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Batrachion

A class of CURVE defined at INTEGER values which hops from one value to another. Their name derives from the Greek word *βατραχίον* *batrachion*, which means "small frog." Many batrachions are FRACTAL. Examples include the BLANCMANGE FUNCTION, HOFSTADTER-CONWAY \$10,000 SEQUENCE, HOFSTADTER'S Q -SEQUENCE, and MALLOWS' SEQUENCE.

References

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Baudet's Conjecture

If $C_1, C_2, \dots, f > C_r$ are sets of positive integers and

$$\bigcup_{i=1}^r C_i = \mathbb{N},$$

where $\mathbb{N}$ is the set of positive integers, then some C_i contains arbitrarily long ARITHMETIC SEQUENCES. The conjecture was proved in 1928 by B. L. van der Waerden.

See also ARITHMETIC SEQUENCE, VAN DER WAERDEN'S THEOREM

References

van der Waerden, B. L. "How the Proof of Baudet's Conjecture Was Found." *Studies in Pure Mathematics (Presented to Richard Rado)*. London: Academic Press, pp. 251–60, 1971.

Bauer's Identical Congruence

Let $T(m)$ denote the set of the $\phi(m)$ numbers less than and RELATIVELY PRIME to m , where $\phi(n)$ is the TOTIENT FUNCTION. Define

$$f_m(x) = \prod_{t \in T(m)} (x - t). \quad (1)$$

Then a theorem of Lagrange states that

$$f_p(x) \equiv x^{\phi(p)} - 1 \pmod{p} \quad (2)$$

for p an ODD PRIME (Hardy and Wright 1979, p. 98).

This can be generalized as follows. Let p be an ODD PRIME DIVISOR of m and p^a the highest POWER which divides m , then

$$f_m(x) \equiv (x^{p-1} - 1)^{\phi(m)/(p-1)} \pmod{p^a} \quad (3)$$

and, in particular,

$$f_{p^a}(x) \equiv (x^{p-1} - 1)^{p^{a-1}} \pmod{p^a}. \quad (4)$$

Now, if $m > 2$ is EVEN and 2^a is the highest POWER of 2 that divides m , then

$$f_m(x) \equiv (x^2 - 1)^{\phi(m)/2} \pmod{2^a} \quad (5)$$

and, in particular,

$$f_{2^a}(x) \equiv (x^2 - 1)^{2^{a-2}} \pmod{2^a}. \quad (6)$$

See also CONGRUENCE, LEUDESORF THEOREM

References

- Bauer. *Nouvelles annales* **2**, 256–64, 1902.
 Hardy, G. H. and Wright, E. M. *J. London Math. Soc.* **9**, 38–1 and 240, 1934.
 Hardy, G. H. and Wright, E. M. "Bauer's Identical Congruence." §8.5 in *An Introduction to the Theory of Numbers*, 5th ed. Oxford, England: Clarendon Press, pp. 98–00, 1979.

Bauer's Theorem

Let $m \geq 3$ be an integer and let

$$f(x) = \sum_{k=0}^n a_k x^{n-k}$$

be an INTEGER POLYNOMIAL that has at least one real zero. Then $f(x)$ has infinitely many PRIME DIVISORS that are not congruent to 1 (mod m) (Nagell 1951, p. 168).

See also BAUER'S IDENTICAL CONGRUENCE, PRIME DIVISOR

References

- Nagell, T. "A Theorem of Bauer on the Prime Divisors of Certain Polynomials." §49 in *Introduction to Number Theory*. New York: Wiley, pp. 168–69, 1951.

Bauer-Muir Transformation

A transformation formula for CONTINUED FRACTIONS (Lorentzen and Waadeland 1992) which can, for example, be used to prove identities such as

$$1 + \frac{\frac{2+q}{1 + \frac{2+q^2}{1 + \frac{2+q^3}{1 + \dots}}}}{2 + q + \frac{q^2}{2 + q^2 + \frac{q^3}{2 + q^3 + \dots}}} = \frac{1}{q}$$

(Berndt *et al.*).

See also CONTINUED FRACTION

References

- Berndt, B. C.; Huang, S.-S.; Sohn, J.; and Son, S. H. "Some Theorems on the Rogers-Ramanujan Continued Fraction

in Ramanujan's Lost Notebook." To appears in *Trans. Amer. Math. Soc.*
 Lorentzen, L. and Waadeland, H. *Continued Fractions with Applications*. Amsterdam, Netherlands: North-Holland, p. 76, 1992.

Bauspiel

A construction for the RHOMBIC DODECAHEDRON.

References

- Coxeter, H. S. M. *Regular Polytopes*, 3rd ed. New York: Dover, pp. 26 and 50, 1973.

Baxter-Hickerson Function

In April 1999, Ed Pegg conjectured on sci.math that there were only finitely many ZEROFREE cubes, to which D. Hickerson responded with a counterexample. A few days later, Lew Baxter posted the slightly simpler example

$$f(n) = \frac{1}{3}(2 \cdot 10^{5n} - 10^{4n} + 2 \cdot 10^{3n} + 10^{2n} + 10^n + 1),$$

which produces numbers whose cubes lack zeros. The first few terms for $n = 0, 1, \dots$ are 2, 64037, 6634003367, 666334000333667, ... (Sloane's A052427). Primes occur for $n = 0, 1, 7, 133, \dots$ (Sloane's A051832) with no others ≤ 470 (Weisstein, Dec. 15, 1999), corresponding to 2, 64037, ... (Sloane's A051833).

See also NUMBER PATTERN, ZEROFREE

References

- Pegg, E. Jr. "Fun with Numbers." <http://www.mathpuzzle.com/numbers.html>.
 Sloane, N. J. A. Sequences A051832, A051833, and A052427 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.

Bayes' Formula

BAYES' THEOREM

Bayes' Theorem

Let A and B_j be SETS. CONDITIONAL PROBABILITY requires that

$$P(A \cap B_j) = P(A)P(B_j|A), \quad (1)$$

where $\cap$ denotes INTERSECTION ("and"), and also that

$$P(A \cap B_j) = P(B_j \cap A) = P(B_j)P(A|B_j). \quad (2)$$

Therefore,

$$P(B_j|A) = \frac{P(B_j)P(A|B_j)}{P(A)}. \quad (3)$$

Now, let

$$S \equiv \bigcup_{i=1}^N A_i, \quad (4)$$

so A_i is an event in S and $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$A = A \cap S = A \cap \left(\bigcup_{i=1}^N A_i \right) = \bigcup_{i=1}^N (A \cap A_i) \quad (5)$$

$$P(A) = P\left(\bigcup_{i=1}^N (A \cap A_i)\right) = \sum_{i=1}^N P(A \cap A_i). \quad (6)$$

But this can be written

$$P(A) = \sum_{i=1}^N P(A_i)P(A|A_i), \quad (7)$$

so

$$P(A_i|A) = \frac{P(A_i)P(A|A_i)}{\sum_{j=1}^N P(A_j)P(A|A_j)} \quad (8)$$

(Papoulis 1984, pp. 38–9).

See also CONDITIONAL PROBABILITY, INCLUSION-EXCLUSION PRINCIPLE, INDEPENDENT STATISTICS, TOTAL PROBABILITY THEOREM

References

- Papoulis, A. "Bayes' Theorem in Statistics" and "Bayes' Theorem in Statistics (Reexamined)." §3– and 4– in *Probability, Random Variables, and Stochastic Processes*, 2nd ed. New York: McGraw-Hill, pp. 38–9, 78–1, and 112–14, 1984.
- Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing*, 2nd ed. Cambridge, England: Cambridge University Press, p. 810, 1992.

Bayesian Analysis

A statistical procedure which endeavors to estimate parameters of an underlying distribution based on the observed distribution. Begin with a "PRIOR DISTRIBUTION" which may be based on anything, including an assessment of the relative likelihoods of parameters or the results of non-Bayesian observations. In practice, it is common to assume a UNIFORM DISTRIBUTION over the appropriate range of values for the PRIOR DISTRIBUTION.

Given the PRIOR DISTRIBUTION, collect data to obtain the observed distribution. Then calculate the LIKELIHOOD of the observed distribution as a function of parameter values, multiply this likelihood function by the PRIOR DISTRIBUTION, and normalize to obtain a unit probability over all possible values. This is called the POSTERIOR DISTRIBUTION. The MODE of the distribution is then the parameter estimate, and "probability intervals" (the Bayesian analog of CONFIDENCE INTERVALS) can be calculated using the standard procedure. Bayesian analysis is somewhat controversial because the validity of the result depends on how valid the PRIOR DISTRIBUTION is, and this cannot be assessed statistically.

See also MAXIMUM LIKELIHOOD, PRIOR DISTRIBUTION, UNIFORM DISTRIBUTION

References

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Bays' Shuffle

A shuffling algorithm used in a class of RANDOM NUMBER generators.

References

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- Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing*, 2nd ed. Cambridge, England: Cambridge University Press, pp. 270–71, 1992.

Beal's Conjecture

A generalization of FERMAT'S LAST THEOREM which states that if $a^x + b^y = c^z$, where a, b, c, x, y , and z are POSITIVE INTEGERS and $x, y, z > 2$, then a, b , and c have a common factor. The conjecture was announced in Mauldin (1997), and a cash prize of \$75,000 has been offered for its proof or a counterexample.

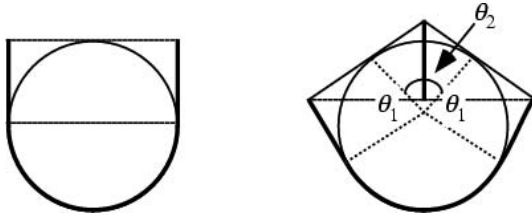
See also ABC CONJECTURE, FERMAT'S LAST THEOREM

References

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- Mauldin, R. D. "A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem." *Not. Amer. Math. Soc.* **44**, 1436–437, 1997.
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Beam Detector

N.B. A detailed online essay by S. Finch was the starting point for this entry.



A "beam detector" for a given curve C is defined as a curve (or set of curves) through which every LINE tangent to or intersecting C passes. The shortest 1-arc beam detector, illustrated in the upper left figure, has length $L_1 = \pi + 2$. The shortest known 2-arc beam detector, illustrated in the right figure, has angles

$$\theta_1 \approx 1.286 \text{ rad} \quad (1)$$

$$\theta_2 \approx 1.191 \text{ rad}, \quad (2)$$

given by solving the simultaneous equations

$$2 \cos \theta_1 - \sin(\tfrac{1}{2}\theta_2) = 0 \quad (3)$$

$$\tan(\tfrac{1}{2}\theta_1)\cos(\tfrac{1}{2}\theta_2) + \sin(\tfrac{1}{2}\theta_2)[\sec^2(\tfrac{1}{2}\theta_2) + 1] = 2. \quad (4)$$

The corresponding length is

$$\begin{aligned} L_2 &= 2\pi - 2\theta_1 - \theta_2 + 2 \tan\left(\tfrac{1}{2}\theta_1\right) + \sec\left(\tfrac{1}{2}\theta_2\right) - \cos\left(\tfrac{1}{2}\theta_2\right) \\ &\quad + \tan\left(\tfrac{1}{2}\theta_1\right) \sin\left(\tfrac{1}{2}\theta_2\right) \\ &= 4.8189264563 \dots \end{aligned} \quad (5)$$

A more complicated expression gives the shortest known 3-arc length $L_3 = 4.799891547 \dots$. Finch defines

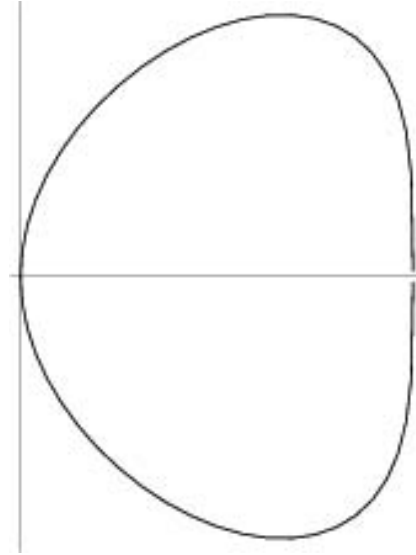
$$L = \inf_{n \geq 1} L_n \quad (6)$$

as the beam detection constant, or the TRENCH DIGGERS' CONSTANT. It is known that $L \geq \pi$.

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Beast Curve



The PLANE CURVE given by the Cartesian equation

$$x^4 + x^2y^2 + y^4 = x(x^2 + y^2).$$

References

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Beast Number

The occult "number of the beast" associated in the Bible with the Antichrist. It has figured in many numerological studies. It is mentioned in Revelation 13:18: "Here is wisdom. Let him that hath understanding count the number of the beast: for it is the number of a man; and his number is 666." The origin of this number is not entirely clear, although it may be as simple as the number containing the concatenation of one symbol of each type (exclude $M = 1000$) in ROMAN NUMERALS: $DCLXVI = 666$ (Wells 1986).

The first few numbers containing the beast number in their digits are 666, 1666, 2666, 3666, 4666, 5666, 6660, ... (Sloane's A051003).

The beast number has several interesting properties which numerologists may find particularly interesting (Keith 1982–3). In particular, the beast number is equal to the sum of the squares of the first 7 PRIMES

$$2^2 + 3^2 + 5^2 + 7^2 + 11^2 + 13^2 + 17^2 = 666, \quad (1)$$

satisfies the identity

$$\phi(666) = 6 \cdot 6 \cdot 6, \quad (2)$$

where ϕ is the TOTIENT FUNCTION, as well as the sum

$$\sum_{i=1}^{6 \cdot 6} i = 666 \quad (3)$$

which is the sum of numbers on a roulette wheel (Emanouilidis 1998). Emanouilidis (1998) also gives additional more obscure connections between 666 and the numbers on a roulette wheel. The number 666 is a sum and difference of the first three 6th POWERS,

$$666 = 1^6 - 2^6 + 3^6 \quad (4)$$

(Keith). Another curious identity is that there are exactly two ways to insert "+" signs into the sequence 123456789 to make the sum 666, and exactly one way for the sequence 987654321,

$$\begin{aligned} 666 &= 1 + 2 + 3 + 4 + 567 + 89 \\ &= 123 + 456 + 78 + 9 \end{aligned} \quad (5)$$

$$666 = 9 + 87 + 6 + 543 + 21 \quad (6)$$

(Keith). 666 is a REPDIGIT, and is also a TRIANGULAR NUMBER

$$T_{6 \cdot 6} = T_{36} = 666. \quad (7)$$

In fact, it is the largest REPDIGIT TRIANGULAR NUMBER (Bellew and Weger 1975–6). 666 is also a SMITH NUMBER. The first 144 DIGITS of $\pi - 3$, where π is PI, add to 666. In addition $144 = (6 + 6) \times (6 + 6)$ (Blatner 1997). Finally,

$$\sum_{i=0}^5 2048^i \equiv 691 \pmod{666}. \quad (8)$$

A number OF THE FORM 2^i which contains the digits of the beast number "666" is called an APOCALYPTIC NUMBER, and a number having 666 digits is called an APOCALYPSE NUMBER.

See also APOCALYPSE NUMBER, APOCALYPTIC NUMBER, BIMONSTER, MONSTER GROUP, ROMAN NUMERAL

References

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Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers*. Middlesex, England: Penguin Books, 1986.

Beatty Sequence

The Beatty sequence is a SPECTRUM SEQUENCE with an IRRATIONAL base. In other words, the Beatty sequence corresponding to an IRRATIONAL NUMBER θ is given by $[\theta]$, $[2\theta]$, $[3\theta]$, ..., where $[x]$ is the FLOOR FUNCTION. If α and β are POSITIVE IRRATIONAL NUMBERS such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

then the Beatty sequences $[\alpha]$, $[2\alpha]$, ... and $[\beta]$, $[2\beta]$, ... together contain all the POSITIVE INTEGERS without repetition.

The sequences for particular values of α and β are given in the following table (Sprague 1963; Wells 1986, pp. 35 and 40), where ϕ is the GOLDEN RATIO.

parameter	Sloane	sequence
$\alpha = \sqrt{2}$	A001951	1, 2, 4, 5, 7, 8, 9, 11, 12, ...
$\beta = 2 + \sqrt{2}$	A001952	3, 6, 10, 13, 17, 20, 23, 27, 30, ...
$\alpha = \sqrt{3}$	A022838	1, 3, 5, 6, 8, 10, 12, 13, 15, 17, ...
$\beta = \frac{1}{2}(3 + \sqrt{3})$	A054406	2, 4, 7, 9, 11, 14, 16, 18, 21, 23, 26, ...
$\alpha = e$	A022843	2, 5, 8, 10, 13, 16, 19, 21, 24, 27, 29, ...
$\beta = e/(e-1)$	A054385	1, 3, 4, 6, 7, 9, 11, 12, 14, 15, 17, 18, ...
$\alpha = \pi$	A022844	3, 6, 9, 12, 15, 18, 21, 25, 28, 31, 34, ...
$\beta = \pi/(\pi-1)$	A054386	1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, ...
$\alpha = \phi$	A000201	1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, ...
$\beta = \phi^2$	A001950	2, 5, 7, 10, 13, 15, 18, 20, 23, 26, 28, 31, 34, ...

See also FRACTIONAL PART, WYTHOFF ARRAY, WYTHOFF'S GAME

References

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Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers*. Middlesex, England: Penguin Books, p. 35, 1986.

Beauzamy and Dégot's Identity

For P , Q , R , and S POLYNOMIALS in n variables

$$[P \cdot Q, R \cdot S] = \sum_{i_1, \dots, i_n \geq 0} \frac{A}{i_1! \cdots i_n!},$$

where

$$A \equiv [R^{(i_1, \dots, i_n)}(D_1, \dots, D_n)Q(x_1, \dots, x_n) \\ \times P^{(i_1, \dots, i_n)}(D_1, \dots, D_n)S(x_1, \dots, x_n)],$$

$D_i = \partial/\partial x_i$ is the DIFFERENTIAL OPERATOR, $[X, Y]$ is the BOMBIERI INNER PRODUCT, and

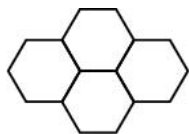
$$P^{(i_1, \dots, i_n)} = D_1^{i_1} \cdots D_n^{i_n} P.$$

See also REZNIK'S IDENTITY

Bed-of-Nails Function

SHAH FUNCTION

Bee



A 4-POLYHEX.

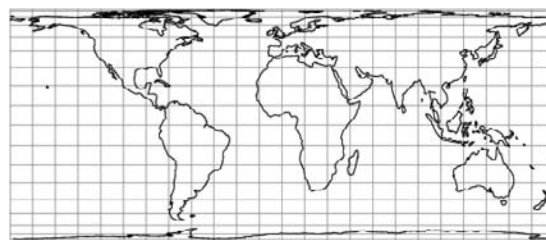
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Behrens-Fisher Test

FISHER-BEHRENS PROBLEM

Behrmann Cylindrical Equal-Area Projection



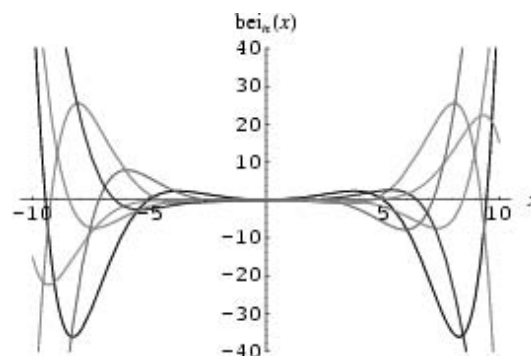
A CYLINDRICAL EQUAL-AREA PROJECTION which uses a standard parallel of $\phi_s = 30^\circ$.

See also BALTHASART PROJECTION, CYLINDRICAL EQUAL-AREA PROJECTION, EQUAL-AREA PROJECTION, GALL ORTHOGRAPHIC PROJECTION, LAMBERT AZIMUTHAL EQUAL-AREA PROJECTION, PETERS PROJECTION, TRISTAN EDWARDS PROJECTION

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Dana, P. H. "Map Projections." http://www.colorado.edu/geography/gcraft/notes/mapproj/mapproj_f.html.

Bei



The IMAGINARY PART of

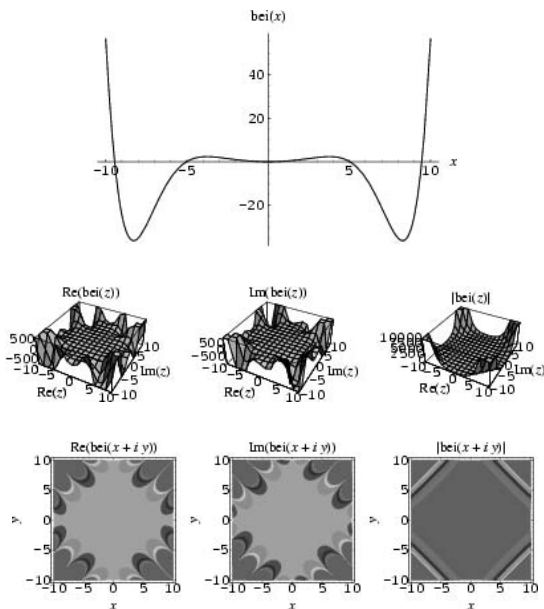
$$J_v(xe^{3\pi i/4}) = \text{ber}_v(x) + i \text{bei}_v(x). \quad (1)$$

The function $\text{bei}_v(x)$ has the series expansion

$$\text{bei}_v(x) = \left(\frac{1}{2}x\right)^v \sum_{k=0}^{\infty} \frac{\sin[(\frac{3}{4}v + \frac{1}{2}k)\pi]}{k! \Gamma(v + k + 1)} \left(\frac{1}{4}x^2\right)^k, \quad (2)$$

where $\Gamma(x)$ is the GAMMA FUNCTION (Abramowitz and

Stegun 1972, p. 379).



The special case $v=0$ gives

$$J_0(i\sqrt{x}) \equiv \text{ber}(x) + i \text{bei}(x), \quad (3)$$

where $J_0(x)$ is the zeroth order BESSEL FUNCTION OF THE FIRST KIND. The function $\text{bei}_0(x) \equiv \text{bei}(x)$ has the series expansion

$$\text{bei}(x) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}x)^{2+4n}}{[(2n+1)!]^2}. \quad (4)$$

See also BER, BESSEL FUNCTION, KEI, KELVIN FUNCTIONS, KER

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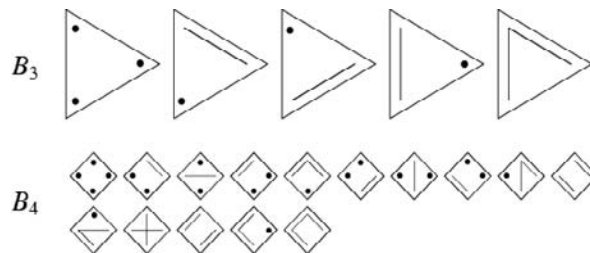
Bell Curve

GAUSSIAN DISTRIBUTION, NORMAL DISTRIBUTION

Bell Number

The number of ways a SET of n elements can be PARTITIONED into nonempty SUBSETS is called a BELL NUMBER and is denoted B_n . For example, there are five ways the numbers $\{1, 2, 3\}$ can be partitioned:

$\{\{1\}, \{2\}, \{3\}\}$, $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{1\}, \{2, 3\}\}$, and $\{\{1, 2, 3\}\}$, so $B_3 = 5$. $B_0 = 1$ and the first few Bell numbers for $n = 1, 2, \dots$ are 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, ... (Sloane's A000110).



Bell numbers are closely related to CATALAN NUMBERS. The diagram above shows the constructions giving $B_3 = 5$ and $B_4 = 15$, with line segments representing elements in the same SUBSET and dots representing subsets containing a single element (Dickau). The INTEGERS B_n can be defined by the sum

$$B_n = \sum_{k=1}^n S(n, k), \quad (1)$$

where $S(n, k)$ is a STIRLING NUMBER OF THE SECOND KIND, i.e., as the STIRLING TRANSFORM of the sequence 1, 1, 1, ...

The Bell number are given by the EXPONENTIAL GENERATING FUNCTION

$$e^{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n. \quad (2)$$

The Bell numbers can also be generated using the BELL TRIANGLE, using the RECURRENCE RELATION

$$B_{n+1} = \sum_{k=0}^n B_k \binom{n}{k}, \quad (3)$$

where $\binom{a}{b}$ is a BINOMIAL COEFFICIENT, or using the formula of Comtet (1974)

$$B_n = \left\lceil e^{-1} \sum_{m=1}^{2n} \frac{m^n}{m!} \right\rceil, \quad (4)$$

where $\lceil x \rceil$ denotes the CEILING FUNCTION.

The Bell number B_n is also equal to $\phi_n(1)$, where $\phi_n(x)$ is an EXPONENTIAL POLYNOMIAL. DOBINSKI'S FORMULA gives the n th Bell number

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}. \quad (5)$$

Lovász (1993) showed that this formula gives the asymptotic limit

$$B_n \sim n^{-1/2} [\lambda(n)]^{n+1/2} e^{\lambda(n)-n-1}, \quad (6)$$

where $\lambda(n)$ is defined implicitly by the equation

$$\lambda(n) \log[\lambda(n)] = n. \quad (7)$$

A variation of DOBINSKI'S FORMULA gives

$$B_n = \sum_{k=1}^n \frac{k^n}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \quad (8)$$

(Pitman 1997). de Bruijn (1958) gave the asymptotic formula

$$\frac{\ln B_n}{n} = \ln n - \ln \ln n - 1 + \frac{\ln \ln n}{\ln n} + \frac{1}{\ln n} + \frac{1}{2} \left(\frac{\ln \ln n}{\ln n} \right)^2 + \mathcal{O} \left[\frac{\ln \ln n}{(\ln n)^2} \right] \quad (9)$$

TOUCHARD'S CONGRUENCE states

$$B_{p+k} \equiv B_k + B_{k+1} \pmod{p}, \quad (10)$$

when p is PRIME. The only PRIME Bell numbers for $n \leq 1000$ are B_2 , B_3 , B_7 , B_{13} , B_{42} , and B_{55} . The Bell numbers also have the curious property that

$$\begin{vmatrix} B_0 & B_1 & B_2 & \cdots & B_n \\ B_1 & B_2 & B_3 & \cdots & B_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_n & B_{n+1} & B_{n+2} & \cdots & B_{2n} \end{vmatrix} = \prod_{i=1}^n i! \quad (11)$$

(Lenard 1986), where the product is simply a SUPER-FACTORIAL, the first few of which for $n = 0, 1, 2, \dots$ are 1, 1, 2, 12, 288, 34560, 24883200, ... (Sloane's A000178).

See also BELL TRIANGLE, DOBINSKI'S FORMULA, EXPONENTIAL POLYNOMIAL, STIRLING NUMBER OF THE SECOND KIND, TOUCHARD'S CONGRUENCE

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Bell Polynomial

The Bell polynomial are defined by

$$B_{n,k}(x_1, x_2, \dots) = \sum_{\substack{j_1+j_2+\dots=k \\ j_1+2j_2+\dots=n}} \frac{n!}{j_1!j_2!\dots} \left(\frac{x_1}{1!} \right)^{j_1} \left(\frac{x_2}{2!} \right)^{j_2} \dots$$

They have GENERATING FUNCTION

$$\sum_{k=0}^{\infty} \frac{b_k(x; x_1, x_2, \dots)}{k!} t^k = e^x \left(\sum_{k=1}^{\infty} \frac{x_k}{k!} t^k \right).$$

See also EXPONENTIAL POLYNOMIAL, IDEMPOTENT NUMBER, LAH NUMBER

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Bell Triangle

$$\begin{array}{ccccccc} 1 & 2 & 5 & 15 & 52 & 203 & 877 \dots \\ 1 & 3 & 10 & 37 & 151 & 674 & \ddots \\ 2 & 7 & 27 & 114 & 523 & \ddots & \\ 5 & 20 & 87 & 409 & \ddots & & \\ 15 & 67 & 322 & \ddots & & & \\ 52 & 255 & \ddots & & & & \\ 203 & \ddots & & & & & \end{array}$$

A triangle of numbers which allow the BELL NUMBERS to be computed using the RECURRENCE RELATION

$$B_{n+1} = \sum_{k=0}^n B_k \binom{n}{k}.$$

See also BELL NUMBER, CLARK'S TRIANGLE, LEIBNIZ HARMONIC TRIANGLE, LOSSNITSCH'S TRIANGLE, NUMBER TRIANGLE, PASCAL'S TRIANGLE, SEIDEL-ENTRINGER-ARNOLD TRIANGLE

Bellows Conjecture

The conjecture proposed by Dennis Sullivan that all FLEXIBLE POLYHEDRA keep a constant VOLUME as they

are flexed (Cromwell 1997). This conjecture was proven by Connelly *et al.* (1997).

See also FLEXIBLE POLYHEDRON

References

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Beltrami Differential Equation

For a MEASURABLE FUNCTION μ , the Beltrami differential equation is given by

$$f_{\bar{z}} = \mu f_z,$$

where $f_{\bar{z}}$ is a PARTIAL DERIVATIVE and $\bar{z}$ denotes the COMPLEX CONJUGATE of z .

See also QUASICONFORMAL MAP

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Beltrami Field

A VECTOR FIELD $\mathbf{u}$ satisfying the vector identity

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \mathbf{0}$$

where $\mathbf{A} \times \mathbf{B}$ is the CROSS PRODUCT and $\nabla \times \mathbf{A}$ is the CURL is said to be a Beltrami field.

See also DIVERGENCELESS FIELD, IRROTATIONAL FIELD, SOLENOIDAL FIELD

Beltrami Identity

An identity in CALCULUS OF VARIATIONS discovered in 1868 by Beltrami. The EULER-LAGRANGE DIFFERENTIAL EQUATION is

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_x} \right) = 0. \quad (1)$$

Now, examine the DERIVATIVE of f with respect to x

$$\frac{df}{dx} = \frac{\partial f}{\partial y} y_x + \frac{\partial f}{\partial y_x} y_{xx} + \frac{\partial f}{\partial x}. \quad (2)$$

Solving for the $\partial f / \partial y$ term gives

$$\frac{\partial f}{\partial y} y_x = \frac{df}{dx} - \frac{\partial f}{\partial y_x} y_{xx} - \frac{\partial f}{\partial x}. \quad (3)$$

Now, multiplying (1) by y_x gives

Bend (Curvature)

$$y_x \frac{\partial f}{\partial y} - y_x \frac{d}{dx} \left(\frac{\partial f}{\partial y_x} \right) = 0. \quad (4)$$

Substituting (3) into (4) then gives

$$\frac{df}{dx} - \frac{\partial f}{\partial y_x} y_{xx} - \frac{\partial f}{\partial x} - y_x \frac{d}{dx} \left(\frac{\partial f}{\partial y_x} \right) = 0 \quad (5)$$

$$-\frac{\partial f}{\partial x} + \frac{d}{dx} \left(f - y_x \frac{\partial f}{\partial y_x} \right) = 0. \quad (6)$$

This form is especially useful if $f_x = 0$, since in that case

$$\frac{d}{dx} \left(f - y_x \frac{\partial f}{\partial y_x} \right) = 0, \quad (7)$$

which immediately gives

$$f - y_x \frac{\partial f}{\partial y_x} = C, \quad (8)$$

where C is a constant of integration (Weinstock 1974, pp. 24–5; Arfken 1985, pp. 928–29; Fox 1988, pp. 8–).

The Beltrami identity greatly simplifies the solution for the minimal AREA SURFACE OF REVOLUTION about a given axis between two specified points. It also allows straightforward solution of the BRACHISTOCHROME PROBLEM.

See also BRACHISTOCHROME PROBLEM, CALCULUS OF VARIATIONS, EULER-LAGRANGE DIFFERENTIAL EQUATION, SURFACE OF REVOLUTION

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Fox, C. *An Introduction to the Calculus of Variations*. New York: Dover, 1988.
Weinstock, R. *Calculus of Variations, with Applications to Physics and Engineering*. New York: Dover, 1974.

Beltrami's Theorem

Let $f : M \rightarrow N$ be a GEODESIC MAPPING. If either M or N has constant curvature, then both surfaces have constant curvature (Ambartzumian 1982, p. 26; Kreyszig 1991).

See also GEODESIC MAPPING

References

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Kreyszig, E. §91 in *Differential Geometry*. New York: Dover, 1991.

Bend (Curvature)

The bend of a circle C mutually tangent to three other circles is defined as the signed CURVATURE of C . If the contacts are all external, the signs of the bends of all

four circles are taken as POSITIVE, whereas if one circle surrounds the other three, the sign of this circle is taken as NEGATIVE (Coxeter 1969). Bends can also be defined for spheres.

See also CURVATURE, DESCARTES CIRCLE THEOREM, SODDY CIRCLES

References

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Bend (Knot)

A KNOT used to join the ends of two ropes together to form a longer length.

References

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Benford's Law

A phenomenological law also called the first digit law, first digit phenomenon, or leading digit phenomenon. Benford's law states that in listings, tables of statistics, etc., the DIGIT 1 tends to occur with PROBABILITY ~30%, much greater than the expected 10% (i.e., one digit out of 10). Benford's law can be observed, for instance, by examining tables of LOGARITHMS and noting that the first pages are much more worn and smudged than later pages (Newcomb 1881). While Benford's law unquestionably applies to many situations in the real world, a satisfactory explanation has been given only recently through the work of Hill (1996).

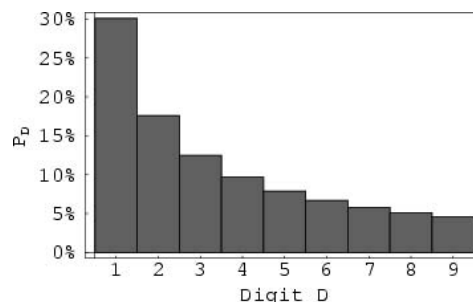
Benford's law applies to data that are *not* dimensionless, so the numerical values of the data depend on the units. If there exists a universal probability distribution $P(x)$ over such numbers, then it must be invariant under a change of scale, so

$$P(kx) = f(k)P(x). \quad (1)$$

If $\int P(x) dx = 1$, then $\int P(kx) dx = 1/k$, and normalization implies $f(k) = 1/k$. Differentiating with respect to k and setting $k = 1$ gives

$$xP'(x) = -P(x), \quad (2)$$

having solution $P(x) = 1/x$. Although this is not a proper probability distribution (since it diverges), both the laws of physics and human convention impose cutoffs. For example, if street addresses are distributed uniformly over the range of 1 to some maximum cutoff value, then they'll obey something close to Benford's law.



If many powers of 10 lie between the cutoffs, then the probability that the first (decimal) digit is D is given by the LOGARITHMIC DISTRIBUTION

$$P_D = \frac{\int_D^{D+1} P(x) dx}{\int_1^{10} P(x) dx} = \frac{\ln\left(\frac{D+1}{D}\right)}{\ln 10} = \frac{\ln(D+1) - \ln(D)}{\ln 10} \quad (3)$$

for $D = 1, \dots, 9$, illustrated above and tabulated below.

D	P_D	D	P_D
1	0.30103	6	0.0669468
2	0.176091	7	0.0579919
3	0.124939	8	0.0511525
4	0.09691	9	0.0457575
5	0.0791812		

However, Benford's law applies not only to scale-invariant data, but also to numbers chosen from a variety of different sources. Explaining this fact requires a more rigorous investigation of CENTRAL LIMIT-like theorems for the MANTISSAS of random variables under MULTIPLICATION. As the number of variables increases, the density function approaches that of a LOGARITHMIC DISTRIBUTION. Hill (1996) rigorously demonstrated that the "distribution of distributions" given by random samples taken from a variety of different distributions is, in fact, Benford's law (Matthews 1999).

One striking example of Benford's law is given by the 54 million real constants in Plouffe's "Inverse Symbolic Calculator" database, 30% of which begin with the DIGIT 1. Taking data from several disparate sources, the table below, shows the distribution of first digits as compiles by Benford (1938) in his original paper.

Col.	Title	First Digit									Samples
		1	2	3	4	5	6	7	8	9	
A	Rivers, Area	31.0	16.4	10.7	11.3	7.2	8.6	5.5	4.2	5.1	335
B	Population	33.9	20.4	14.2	8.1	7.2	6.2	4.1	3.7	2.2	3259
C	Constants	41.3	14.4	4.8	8.6	10.6	5.8	1.0	2.9	10.6	104
D	Newspapers	30.0	18.0	12.0	10.0	8.0	6.0	6.0	5.0	5.0	100
E	Specific Heat	24.0	18.4	16.2	14.6	10.6	4.1	3.2	4.8	4.1	1389
F	Pressure	29.6	18.3	12.8	9.8	8.3	6.4	5.7	4.4	4.7	703
G	H.P. Lost	30.0	18.4	11.9	10.8	8.1	7.0	5.1	5.1	3.6	690
H	Mol. Wgt.	26.7	25.2	15.4	10.8	6.7	5.1	4.1	2.8	3.2	1800
I	Drainage	27.1	23.9	13.8	12.6	8.2	5.0	5.0	2.5	1.9	159
J	Atomic Wgt.	47.2	18.7	5.5	4.4	6.6	4.4	3.3	4.4	5.5	91
K	n^{-1} , $\sqrt{n}$	25.7	20.3	9.7	6.8	6.6	6.8	7.2	8.0	8.9	5000
L	Design	26.8	14.8	14.3	7.5	8.3	8.4	7.0	7.3	5.6	560
M	Reader's Digest	33.4	18.5	12.4	7.5	7.1	6.5	5.5	4.9	4.2	308
N	Cost Data	32.4	18.8	10.1	10.1	9.8	5.5	4.7	5.5	3.1	741
O	X-Ray Volts	27.9	17.5	14.4	9.0	8.1	7.4	5.1	5.8	4.8	707
P	Am. League	32.7	17.6	12.6	9.8	7.4	6.4	4.9	5.6	3.0	1458
Q	Blackbody	31.0	17.3	14.1	8.7	6.6	7.0	5.2	4.7	5.4	1165
R	Addresses	28.9	19.2	12.6	8.8	8.5	6.4	5.6	5.0	5.0	342
S	$n^1, n^2 \dots n!$	25.3	16.0	12.0	10.0	8.5	8.8	6.8	7.1	5.5	900
T	Death Rate	27.0	18.6	15.7	9.4	6.7	6.5	7.2	4.8	4.1	418
	Average	30.6	18.5	12.4	9.4	8.0	6.4	5.1	4.9	4.7	1011
	Probable Error	± 0.8	± 0.4	± 0.4	± 0.3	± 0.2	± 0.2	± 0.2	± 0.3	± 0.3	

The following table gives the distribution of the first digit of the mantissa following Benford's Law using a number of different methods.

method	Sloane	sequence
Sainte-Lague	A055439	1, 2, 3, 1, 4, 5, 6, 1, 2, 7, 8, 9, ...
d'Hondt	A055440	1, 2, 1, 3, 1, 4, 2, 5, 1, 6, 3, 1, ...
largest remainder, Hare quotas	A055441	1, 2, 3, 4, 1, 5, 6, 7, 1, 2, 8, 1, ...
largest remainder, Droop quotas	A055442	1, 2, 3, 1, 4, 5, 6, 1, 2, 7, 8, 1, ...

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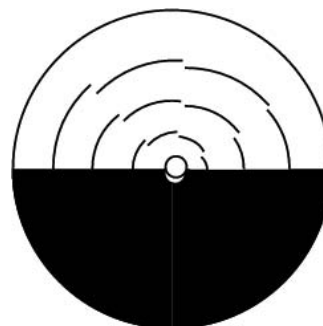
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Benham's Wheel



An optical ILLUSION consisting of a spinnable top marked in black with the pattern shown above. When

the wheel is spun (especially slowly), the black broken lines appear as green, blue, and red colored bands!

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Benjamin-Bona-Mahony Equation

THE PARTIAL DIFFERENTIAL EQUATION

$$u_t - u_{xxx} + uu_x = 0$$

(Arvin and Goldstein 1985; Zwillinger 1997, p. 130). A generalized version is given by

$$u_t - \nabla^2 u_t + \div(\phi(u)) = 0$$

(Goldstein and Wichnoski 1980; Zwillinger 1997, p. 132).

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Bennequin's Conjecture

A BRAID with M strands and R components with P positive crossings and N negative crossings satisfies

$$|P - N| \leq 2U + M - R \leq P + N,$$

where U is the UNKNOTTING NUMBER. While the second part of the INEQUALITY was already known to be true (Boileau and Weber, 1983, 1984) at the time the conjecture was proposed, the proof of the entire conjecture was completed using results of Kronheimer and Mrowka on MILNOR'S CONJECTURE (and, independently, using MENASCO'S THEOREM).

See also BRAID, MENASCO'S THEOREM, MILNOR'S CONJECTURE, UNKNOTTING NUMBER

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Benson's Formula

An equation for a LATTICE SUM with $n = 3$

$$\begin{aligned} -b_3(1) &= \sum_{i,j,k=-\infty}^{\infty} \frac{(-1)^{i+j+k+1}}{\sqrt{i^2 + j^2 + k^2}} \\ &= 12\pi \sum_{m,n=1,3,\dots}^{\infty} \operatorname{sech}^2\left(\frac{1}{2}\pi\sqrt{m^2 + n^2}\right). \end{aligned}$$

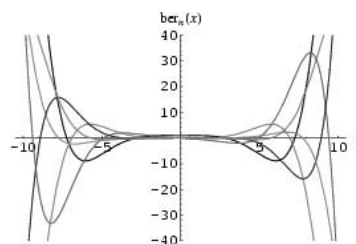
Here, the prime denotes that summation over $(0, 0, 0)$ is excluded. The sum is numerically equal to $-1.74756\dots$, a value known as "the" MADELUNG CONSTANT.

See also MADELUNG CONSTANTS

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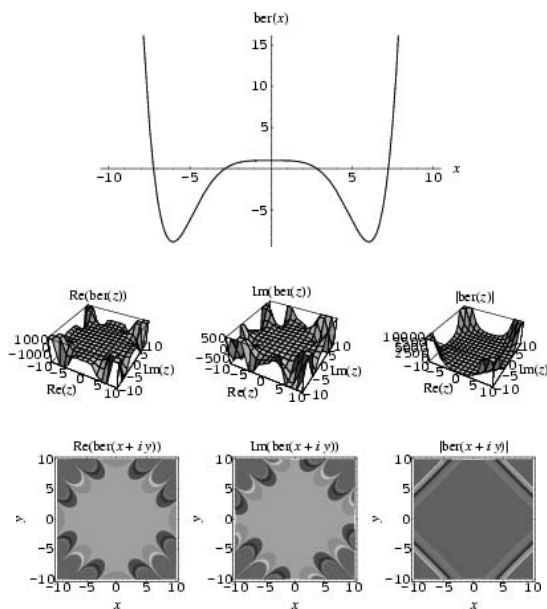
THE REAL PART OF

$$J_\nu(xe^{3\pi i/4}) = \operatorname{ber}_\nu(x) + i \operatorname{bei}_\nu(x). \quad (1)$$

The function $\operatorname{ber}_\nu(x)$ has the series expansion

$$\operatorname{ber}_\nu(x) = \left(\frac{1}{2}x\right)^\nu \sum_{k=0}^{\infty} \frac{\cos[(\frac{3}{4}\nu + \frac{1}{2}k)\pi]}{k! \Gamma(\nu + k + 1)} \left(\frac{1}{4}x^2\right)^k, \quad (2)$$

where $\Gamma(x)$ is the GAMMA FUNCTION (Abramowitz and Stegun 1972, p. 379).



The special case $v=0$ gives

$$J_0(i\sqrt{i}x) \equiv \text{ber}(x) + i \text{bei}(x), \quad (3)$$

where $J_0(x)$ is the zeroth order BESSEL FUNCTION OF THE FIRST KIND. The function $\text{ber}_0(x) \equiv \text{ber}(x)$ has the series expansion

$$\text{ber}(x) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}x)^{4n}}{[(2n)!]^2}. \quad (4)$$

See also BEI, BESSEL FUNCTION, KEI, KELVIN FUNCTIONS, KER

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Beraha Constants

The n th Beraha constant (or number) is given by

$$B(n) \equiv 2 + 2 \cos\left(\frac{2\pi}{n}\right).$$

They appear to be ROOTS of the CHROMATIC POLY-

NOMIALS of planar triangular GRAPHS. $B(5)$ is $\phi + 1$, where ϕ is the GOLDEN RATIO, $B(7)$ is the SILVER CONSTANT, and $B(10) = \phi + 2$. The following table summarizes the first few Beraha numbers.

n	$B(n)$	Approx.
1		4
2		0
3		1
4		2
5	$\frac{1}{2}(3 + \sqrt{5})$	2.618
6		3
7	$2 + 2 \cos(\frac{2}{7}\pi)$	3.247
8	$2 + \sqrt{2}$	3.414
9	$2 + 2 \cos(\frac{2}{9}\pi)$	3.532
10	$\frac{1}{2}(5 + \sqrt{5})$	3.618

See also CHROMATIC POLYNOMIAL, GOLDEN RATIO, SILVER CONSTANT

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Berezin Transform

The operator $\tilde{B}$ defined by

$$\tilde{B}f(x) = \int_D \frac{(1 - |z|^2)^2}{|1 - z\bar{w}|^4} f(w) dA(w)$$

for $z \in \mathbb{D}$, where $\mathbb{D}$ is the unit open disk and $\bar{w}$ is the COMPLEX CONJUGATE (Hedenmalm *et al.* 2000, p. 29).

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- Hedenmalm, H.; Korenblum, B.; and Zhu, K. "The Berezin Transform." Ch. 2 in *Theory of Bergman Spaces*. New York: Springer-Verlag, pp. 28–1, 2000.

Berge's Theorem

A MATCHING is maximal IFF it contains no AUGMENTING PATH.

See also MATCHING

References

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Berger-Kazdan Comparison Theorem

Let M be a compact n -D MANIFOLD with INJECTIVITY radius $\text{inj}(M)$. Then

$$\text{Vol}(M) \geq \frac{c_n \text{inj}(M)}{\pi},$$

with equality IFF M is ISOMETRIC to the standard round SPHERE $\mathbb{S}^n$ with RADIUS $\text{inj}(M)$, where $c_n(r)$ is the VOLUME of the standard n -HYPERSPHERE of RADIUS r .

See also BLASCHKE CONJECTURE, HYPERSPHERE, INJECTIVE, ISOMETRY

References

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Bergman Kernel

A Bergman kernel is a function of a COMPLEX VARIABLE with the "reproducing kernel" property defined for any DOMAIN in which there exist NONZERO ANALYTIC FUNCTIONS of class $L_2(d)$ with respect to the LEBESGUE MEASURE dv .

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Bergman Space

Let G be an open subset of the COMPLEX PLANE $\mathbb{C}$, and let $L_a^2(G)$ denote the collection of all ANALYTIC FUNCTIONS $f : G \rightarrow \mathbb{C}$ whose MODULUS is square integrable with respect to AREA measure. Then $L_a^2(G)$, sometimes also denoted $A^2(G)$, is called the Bergman space for G . Thus, the Bergman space consists of all the ANALYTIC FUNCTIONS in $L^2(G)$. The Bergman space can also be generalized to $L_a^p(G)$, where $0 < p < \infty$.

See also HARDY SPACE

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Berlekamp-Massey Algorithm

If a sequence takes only a small number of different values, then by regarding the values as the elements of a FINITE FIELD, the Berlekamp-Massey algorithm is an efficient procedure for finding the shortest linear recurrence from the field that will generate the sequence.

See also REED-SLOANE ALGORITHM

References

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Berlekamp-Zassenhaus Algorithm

An algorithm that can be used to find subsets S of a set for which the product of elements of S of a set of monic irreducible polynomials in $\mathbb{Z}_p$ for which the product of the elements of S has integer coefficients (van Hoeij 2000).

References

- van Hoeij, M. "Factoring Polynomials and the Knapsack Problem." Preprint. <http://www.math.fsu.edu/~aluffi/archive/paper124.ps.gz>.
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Bernays-Gödel Set Theory

VON NEUMANN-BERNAYS-GÖDEL SET THEORY

Bernoulli Differential Equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n. \quad (1)$$

Let $v \equiv y^{1-n}$ for $n \neq 1$, then

$$\frac{dv}{dx} = (1-n)y^{-n} \frac{dy}{dx}. \quad (2)$$

Rewriting (1) gives

$$y^{-n} \frac{dy}{dx} = q(x) - p(x)y^{1-n} = q(x) - vp(x). \quad (3)$$

Plugging (3) into (2),

$$\frac{dv}{dx} = (1-n)[q(x) - vp(x)]. \quad (4)$$

Now, this is a linear FIRST-ORDER ORDINARY DIFFERENTIAL EQUATION OF THE FORM

$$\frac{dv}{dx} + vP(x) = Q(x), \quad (5)$$

where $P(x) \equiv (1-n)p(x)$ and $Q(x) \equiv (1-n)q(x)$. It can therefore be solved analytically using an INTEGRATING FACTOR

$$v = \frac{\int e^{\int P(x) dx} Q(x) dx + C}{e^{\int P(x) dx}} = \frac{(1-n) \int e^{(1-n) \int p(x) dx} q(x) dx + C}{e^{(1-n) \int p(x) dx}}, \quad (6)$$

where C is a constant of integration. If $n = 1$, then equation (1) becomes

$$\frac{dy}{dx} = y(q-p) \quad (7)$$

$$\frac{dy}{y} = (q-p) dx \quad (8)$$

$$y = C_2 e^{\int [q(x) - p(x)] dx}. \quad (9)$$

The general solution is then, with C_1 and C_2 con-

stants,

$$y = \begin{cases} \left[\frac{(1-n) \int e^{(1-n) \int p(x) dx} q(x) dx + C_1}{e^{(1-n) \int p(x) dx}} \right]^{1/(1-n)} & \text{for } n \neq 1 \\ C_2 e^{\int [q(x) - p(x)] dx} & \text{for } n = 1. \end{cases} \quad (10)$$

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- Boyce, W. E. and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, 5th ed. New York: Wiley, p. 28, 1992.
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Bernoulli Distribution

A STATISTICAL DISTRIBUTION given by

$$P(n) = \begin{cases} q \equiv 1-p & \text{for } n = 0 \\ p & \text{for } n = 1 \end{cases} \quad (1)$$

$$= p^n (1-p)^{1-n} \quad \text{for } n = 0, 1. \quad (2)$$

The distribution of heads and tails in COIN TOSSING is a Bernoulli distribution with $p = q = 1/2$. The MOMENT-GENERATING FUNCTION of the Bernoulli distribution is

$$M(t) = \langle e^{tn} \rangle = \sum_{n=0}^1 e^{tn} p^n (1-p)^{1-n} = e^0 (1-p) + e^t p, \quad (3)$$

so

$$M(t) = (1-p) + pe^t \quad (4)$$

$$M'(t) = pe^t \quad (5)$$

$$M''(t) = pe^t \quad (6)$$

$$M^{(n)}(t) = pe^t, \quad (7)$$

and the MOMENTS about 0 are

$$\mu'_1 = \mu = M'(0) = p \quad (8)$$

$$\mu'_2 = M''(0) = p \quad (9)$$

$$\mu'_n = M^{(n)}(0) = p. \quad (10)$$

The MOMENTS about the MEAN are

$$\mu_2 = \mu'_2 - (\mu'_1)^2 = p - p^2 = p(1-p) \quad (11)$$

$$\begin{aligned}\mu_3 &= \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3 = p - 3p^2 + 2p^3 \\ &= p(1-p)(1-2p)\end{aligned}\quad (12)$$

$$\begin{aligned}\mu_4 &= \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4 \\ &= p - 4p^2 + 6p^3 - 3p^4 = p(1-p)(3p^2 - 3p + 1).\end{aligned}\quad (13)$$

The MEAN, VARIANCE, SKEWNESS, and KURTOSIS are then

$$\mu = p \quad (14)$$

$$\sigma^2 = \mu_2 = p(1-p) \quad (15)$$

$$\gamma_1 = \frac{\mu_3}{\sigma_3} = \frac{p(1-p)(1-2p)}{[p(1-p)]^{3/2}} = \frac{1-2p}{\sqrt{p(1-p)}} \quad (16)$$

$$\begin{aligned}\gamma_2 &= \frac{\mu_4}{\sigma_4} - 3 = \frac{p(1-2p)(2p^2 - 2p + 1)}{p^2(1-p)^2} - 3 \\ &= \frac{6p^2 - 6p + 1}{p(1-p)}.\end{aligned}\quad (17)$$

To find an estimator $\hat{p}$ for the mean of a Bernoulli population with actual mean p , let N trials be made and suppose n successes are obtained. Assume an estimator given by

$$\hat{p} \triangleq \frac{n}{N}, \quad (18)$$

so that the probability of obtaining the observed n successes in N trials is then

$$\binom{N}{n} p^n (1-p)^{N-n}. \quad (19)$$

The expectation value of the estimator $\hat{p}$ is therefore given by

$$\begin{aligned}\langle \hat{p} \rangle &\cong \sum_{n=0}^N p \binom{N}{n} p^n (1-p)^{N-n} \\ &= (1-p)^N \left(\frac{1}{1-p} \right)^N p = p,\end{aligned}\quad (20)$$

so $\langle \hat{p} \rangle$ is indeed an UNBIASED ESTIMATOR for the population mean p .

See also BERNOULLI TRIAL, BINOMIAL DISTRIBUTION, COIN TOSSING, RUN

References

Evans, M.; Hastings, N.; and Peacock, B. "Bernoulli Distribution." Ch. 4 in *Statistical Distributions*, 3rd ed. New York: Wiley, pp. 31–3, 2000.

Bernoulli Function

BERNOULLI POLYNOMIAL

Bernoulli Inequality

$$(1+x)^n > 1+nx, \quad (1)$$

where $x > -1 \neq 0$ is a REAL NUMBER and $n > 1$ an INTEGER. This inequality can be proven by taking a MACLAURIN SERIES of $(1+x)^n$,

$$(1+x)^n = 1+nx + \frac{1}{2}n(n-1)x^2 + \frac{1}{6}n(n-1)(n-2)x^3 + \dots \quad (2)$$

Since the series terminates after a finite number of terms for INTEGRAL n , the Bernoulli inequality for $x > 0$ is obtained by truncating after the first-order term.

When $-1 < x < 0$, slightly more finesse is needed. In this case, let $y = |x| = -x > 0$ so that $0 < y < 1$, and take

$$(1-y)^n = 1-ny + \frac{1}{2}n(n-1)y^2 - \frac{1}{6}n(n-1)(n-2)y^3 + \dots \quad (3)$$

Since each POWER of y multiplies by a number < 1 and since the ABSOLUTE VALUE of the COEFFICIENT of each subsequent term is smaller than the last, it follows that the sum of the third order and subsequent terms is a POSITIVE number. Therefore,

$$(1-y)^n > 1-ny, \quad (4)$$

or

$$(1+x)^n > 1+nx, \quad \text{for } -1 < x < 0, \quad (5)$$

completing the proof of the INEQUALITY over all ranges of parameters.

For $x > -1 \neq 0$, the following generalizations of Bernoulli inequality are valid for real exponents:

$$(1+x)^a > 1+ax \quad \text{if } a > 1 \text{ or } a < 0, \quad (6)$$

and

$$(1+x)^a < 1+ax \quad \text{if } 0 < a < 1 \quad (7)$$

(Mitrinovic 1970).

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Bernoulli Lemniscate

LEMNISCATE

Bernoulli Number

There are two definitions for the Bernoulli numbers. In modern usage, the Bernoulli numbers are written B_n , while the Bernoulli numbers encountered in older literature (where they are confusingly also denoted B_n) are distinguished by writing them as B_n^* . In each case, the Bernoulli numbers are a special case of the BERNOULLI POLYNOMIALS $B_n(x)$ or $B_n^*(x)$ with $B_n = B_n(0)$ and $B_n^* = B_n^*(0)$.

The older definition of the Bernoulli numbers, no longer in widespread use, defines B_n^* using the equations

$$\begin{aligned} \frac{x}{e^x - 1} + \frac{x}{2} - 1 &\equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n^* x^{2n}}{(2n)!} \\ &= \frac{B_1^* x^2}{2!} - \frac{B_2^* x^4}{4!} + \frac{B_3^* x^6}{6!} + \dots \end{aligned} \quad (1)$$

for $|x| < 2\pi$, or

$$\begin{aligned} 1 - \frac{x}{2} \cot\left(\frac{x}{2}\right) &\equiv \sum_{n=1}^{\infty} \frac{B_n^* x^{2n}}{(2n)!} \\ &= \frac{B_1^* x^2}{2!} - \frac{B_2^* x^4}{4!} + \frac{B_3^* x^6}{6!} + \dots \end{aligned} \quad (2)$$

for $|x| < \pi$ (Whittaker and Watson 1990, p. 125). Gradshteyn and Ryzhik (2000) denote these numbers B_n^* , while Bernoulli numbers defined by the newer (National Bureau of Standards) definition are denoted B_n . The B_n^* Bernoulli numbers may be calculated from the integral

$$B_n^* = 4n \int_0^{\infty} \frac{t^{2n-1} dt}{e^{2\pi t} - 1}, \quad (3)$$

and analytically from

$$B_n^* = \frac{2(2n)!}{(2\pi)^{2n}} \sum_{p=1}^{\infty} p^{-2n} = \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n) \quad (4)$$

for $n = 1, 2, \dots$, where $\zeta(z)$ is the RIEMANN ZETA FUNCTION.

The first few Bernoulli numbers b_n^* are

$$\begin{aligned} B_1^* &= \frac{1}{6} \\ B_2^* &= \frac{1}{30} \\ B_3^* &= \frac{1}{42} \\ B_4^* &= \frac{1}{30} \\ B_5^* &= \frac{5}{66} \\ B_6^* &= \frac{691}{2,730} \\ B_7^* &= \frac{7}{6} \\ B_8^* &= \frac{3,617}{510} \\ B_9^* &= \frac{43,867}{798} \\ B_{10}^* &= \frac{174,611}{330} \\ B_{11}^* &= \frac{854,513}{138}. \end{aligned}$$

Bernoulli numbers defined by the modern definition

are denoted B_n and sometimes called "even-index" Bernoulli numbers. These are the Bernoulli numbers returned, by example, by the *Mathematica* function `BernoulliB[n]`. The first few are

$$\begin{aligned} B_0 &= 1 \\ B_1 &= -\frac{1}{2} \\ B_2 &= \frac{1}{6} \\ B_4 &= -\frac{1}{30} \\ B_6 &= \frac{1}{42} \\ B_8 &= -\frac{1}{30} \\ B_{10} &= \frac{5}{66} \\ B_{12} &= -\frac{691}{2,730} \\ B_{14} &= \frac{7}{6} \\ B_{16} &= -\frac{3,617}{510} \\ B_{18} &= \frac{43,867}{798} \\ B_{20} &= -\frac{174,611}{330} \\ B_{22} &= \frac{854,513}{138} \end{aligned}$$

(Sloane's A000367 and A002445), with

$$B_{2n+1} = 0 \quad (5)$$

for $n = 1, 2, \dots$. The Bernoulli numbers B_n are a superset of the archaic ones B_n^* since

$$B_n \equiv \begin{cases} 1 & \text{for } n = 0 \\ -\frac{1}{2} & \text{for } n = 1 \\ (-1)^{(n/2)-1} B_{n/2}^* & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases} \quad (6)$$

The B_n can be defined by the identity

$$\frac{x}{e^x - 1} \equiv \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}. \quad (7)$$

These relationships can be derived using the generating function

$$F(x, t) = \sum_{n=0}^{\infty} \frac{B_n(x) t^n}{n!}, \quad (8)$$

which converges uniformly for $|t| < 2\pi$ and all x (Castellanos 1988). Taking the partial derivative gives

$$\frac{\partial F(x, t)}{\partial x} = \sum_{n=0}^{\infty} \frac{B_{n-1}(x) t^n}{(n-1)!} = t \sum_{n=0}^{\infty} \frac{B_n(x) t^n}{n!} = t F(x, t). \quad (9)$$

The solution to this differential equation can be found

using SEPARATION OF VARIABLES as

$$F(x, t) = T(t)e^{xt}, \quad (10)$$

so integrating gives

$$\int_0^1 F(x, t) dx = T(t) \int_0^1 e^{xt} dx = T(t) \frac{e^t - 1}{t}. \quad (11)$$

But integrating (11) explicitly gives

$$\begin{aligned} \int_0^1 F(x, t) dx &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^1 B_n(x) dx \\ &= 1 + \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^1 B_n(x) dx = 1, \end{aligned} \quad (12)$$

so

$$T(t) \frac{e^t - 1}{t} = 1. \quad (13)$$

Solving for $T(t)$ and plugging back into (10) then gives

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!} \quad (14)$$

(Castellanos 1988). Setting $x = 0$ and adding $t/2$ to both sides then gives

$$\frac{1}{2}t \coth\left(\frac{1}{2}t\right) = \sum_{n=0}^{\infty} \frac{B_{2n}t^{2n}}{(2n)!}. \quad (15)$$

Letting $t = 2ix$ then gives

$$x \cot x = \sum_{n=0}^{\infty} (-1)^n B_{2n} \frac{(2x)^{2n}}{(2n)!} \quad (16)$$

for $x \in [-\pi, \pi]$. The Bernoulli numbers may also be calculated from the integral

$$B_n = \frac{n!}{2\pi i} \int \frac{z}{e^z - 1} \frac{dz}{z^{n+1}}, \quad (17)$$

or from

$$B_n = \lim_{x \rightarrow 0} \frac{d^n}{dx^n} \frac{x}{e^x - 1}. \quad (18)$$

The Bernoulli numbers satisfy the identity

$$\binom{k+1}{1}B_k + \binom{k+1}{2}B_{k-1} + \cdots + \binom{k+1}{k}B_1 + B_0 = 0, \quad (19)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. They also satisfy the nice sum identity

$$\sum_{i=0}^n \frac{(1 - 2^{1-i})(1 - 2^{i-n+1})B_{n-i}B_i}{(n-i)!i!} = \frac{(1-n)B_n}{n!} \quad (20)$$

(Gosper).

An ASYMPTOTIC SERIES for the even Bernoulli numbers is

$$B_{2n} \sim (-1)^{n-1} 4\sqrt{\pi n} \left(\frac{n}{\pi e}\right)^{2n}. \quad (21)$$

Bernoulli numbers appear in expressions OF THE FORM $\sum_{k=1}^n k^p$, where $p = 1, 2, \dots$. Bernoulli numbers also appear in the series expansions of functions involving $\tan x$, $\cot x$, $\csc x$, $\ln|\sin x|$, $\ln|\cos x|$, $\ln|\tan x|$, $\tanh x$, $\coth x$, and $\operatorname{csch} x$. An analytic solution exists for EVEN orders,

$$B_{2n} = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{p=1}^{\infty} p^{-2n} = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \zeta(2n) \quad (22)$$

for $n = 1, 2, \dots$, where $\zeta(2n)$ is the RIEMANN ZETA FUNCTION. Another intimate connection with the RIEMANN ZETA FUNCTION is provided by the identity

$$B_n = (-1)^{n+1} n \zeta(1-n). \quad (23)$$

The DENOMINATOR of B_{2k} is given by the VON STAUDT-CLAUSEN THEOREM

$$\operatorname{denom}(B_{2k}) = \prod_{\substack{p \text{ prime} \\ (p-1)|2k}}^{2k+1} p, \quad (24)$$

which also implies that the DENOMINATOR of B_{2k} is SQUAREFREE (Hardy and Wright 1979). Another curious property is that the fraction part of B_n in DECIMAL has a DECIMAL PERIOD which divides n , and there is a single digit before that period (Conway 1996).

Bernoulli first used the Bernoulli numbers while computing $\sum_{k=1}^n k^p$. He used the property of the FIGURATE NUMBER TRIANGLE that

$$\sum_{i=0}^n a_{ij} = \frac{(n+1)a_{nj}}{j+1}, \quad (25)$$

along with a form for a_{nj} which he derived inductively to compute the sums up to $n = 10$ (Boyer 1968, p. 85). For $p \in \mathbb{Z} > 0$, the sum is given by

$$\sum_{k=1}^n k^p = \frac{(B+n+1)^{[p+1]} - B^{p+1}}{p+1}, \quad (26)$$

where the NOTATION $B^{[k]}$ means the quantity in question is raised to the appropriate POWER k , and all terms OF THE FORM B^m are replaced with the corresponding Bernoulli numbers B_m . Written explicitly in terms of a sum of POWERS,

$$\sum_{k=1}^n k^p = n^p + \sum_{k=0}^p \frac{B_k p!}{k!(p-k+1)!} n^{p-k+1}. \quad (27)$$

It is also true that the COEFFICIENTS of the terms in

such an expansion sum to 1 (which Bernoulli stated without proof). Ramanujan gave a number of curious infinite sum identities involving Bernoulli numbers (Berndt 1994).

G. J. Fee and S. Plouffe have computed $B_{200,000}$, which has $\sim 800,000$ DIGITS (Plouffe). Plouffe and collaborators have also calculated B_n for n up to 72,000.

See also ARGONH'S CONJECTURE, BERNOULLI FUNCTION, BERNOULLI NUMBER OF THE SECOND KIND, BERNOULLI POLYNOMIAL, DEBYE FUNCTIONS, EULER-MACLAURIN INTEGRATION FORMULAS, EULER NUMBER, FIGURATE NUMBER TRIANGLE, GENOCCHI NUMBER, MODIFIED BERNOULLI NUMBER, PASCAL'S TRIANGLE, RIEMANN ZETA FUNCTION, VON STAUDT-CLAUSEN THEOREM

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Bernoulli Number of the Second Kind

A number defined by $b_n = b_n(0)$, where $b_n(x)$ is a BERNOULLI POLYNOMIAL OF THE SECOND KIND (Roman 1974, p. 294), also called Cauchy numbers of the first kind. The first few for $n = 0, 1, 2, \dots$ are 1, $1/2$, $-1/6$, $1/4$, $-19/30$, $9/4$, $\dots$ (Sloane's A006232 and A006233). They are given by

$$b_n = \int_0^1 (x)_n dx,$$

where $(x)_n$ is a FALLING FACTORIAL, and have EXPONENTIAL GENERATING FUNCTION

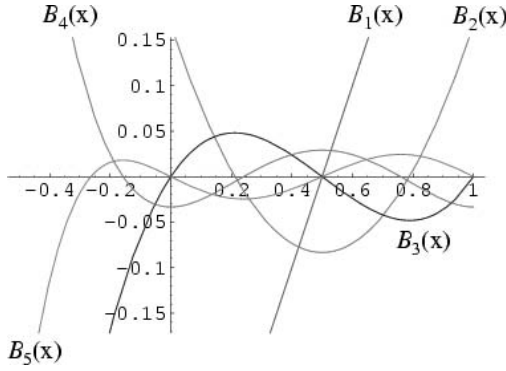
$$E(x) = \frac{x}{\ln(1+x)} = 1 + \frac{1!}{2} x - \frac{2!}{6} x^2 + \frac{3!}{4} x^3 + \dots$$

See also BERNOULLI NUMBER, BERNOULLI POLYNOMIAL OF THE SECOND KIND

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Bernoulli Polynomial



There are two definitions of Bernoulli polynomials in use. The n th Bernoulli polynomial is denoted here by $B_n(x)$ (Abramowitz and Stegun 1972), and the archaic form of the Bernoulli polynomial by $B_n^*(x)$ (or sometimes $\phi_n(x)$). When evaluated at zero, these definitions correspond to the BERNOLLI NUMBERS,

$$B_n \equiv B_n(0) \quad (1)$$

$$B_n^* \equiv B_n^*(0). \quad (2)$$

The Bernoulli polynomials are an APPELL SEQUENCE with

$$g(t) = \frac{e^t - 1}{t} \quad (3)$$

(Roman 1984, p. 31), giving the GENERATING FUNCTION

$$\frac{te^{tx}}{e^t - 1} \equiv \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (4)$$

(Abramowitz and Stegun 1972, p. 804), first obtained by Euler (1738). The first few Bernoulli polynomials are

$$B_0(x) = 1$$

$$B_1(x) = x - \frac{1}{2}$$

$$B_2(x) = x^2 - x + \frac{1}{6}$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$$

$$B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$$

$$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}.$$

Whittaker and Watson (1990, p. 126) define an older type of "Bernoulli polynomial" by writing

$$t \frac{e^{zt} - 1}{e^t - 1} = \sum_{n=1}^{\infty} \frac{\phi_n(z)t^n}{n!} \quad (5)$$

instead of (5). This gives the polynomials

$$\phi_n(x) = B_n(x) - B_n, \quad (6)$$

where B_n is a BERNOLLI NUMBER, the first few of which are

$$\phi_1(x) = x$$

$$\phi_2(x) = x^2 - x$$

$$\phi_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

$$\phi_4(x) = x^4 - 2x^3 + x^2$$

$$\phi_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x.$$

The Bernoulli polynomials also satisfy

$$B_n(1) = (-1)^n B_n(0) \quad (7)$$

and

$$B_n(1-x) = (-1)^n B_n(x) \quad (8)$$

(Lehmer 1988), as well as the relation

$$B_n(x+1) - B_n(x) = nx^{n-1} \quad (9)$$

(Whittaker and Watson 1990, p. 127).

Bernoulli (1713) defined the polynomials in terms of sums of the POWERS of consecutive integers,

$$\sum_{k=0}^{m-1} k^{n-1} = \frac{1}{n} [B_n(m) - B_n(0)]. \quad (10)$$

The Bernoulli polynomials satisfy the RECURRENCE RELATION

$$\frac{dB_n}{dx} = nB_{n-1}(x) \quad (11)$$

(Appell 1882), and obey the identity

$$B_n(x) = (B+x)^n, \quad (12)$$

where B^k is interpreted here as $B_k(x)$. Hurwitz gave the FOURIER SERIES

$$B_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{k=-\infty}^{\infty} k^{-n} e^{2\pi i k x}, \quad (13)$$

for $0 < x < 1$, where the prime in the summation indicates that the term $k = 0$ is omitted. Performing the sum gives

$$B_n(x) = -\frac{n!}{(2\pi i)^n} [(-1)^n \text{Li}_n(e^{-2\pi i x}) + \text{Li}_n(e^{2\pi i x})], \quad (14)$$

where $\text{Li}_n(x)$ is the POLYLOGARITHM function. Raabe (1851) found

$$\frac{1}{m} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) = m^{-n} B_n(mx). \quad (15)$$

A sum identity involving the Bernoulli polynomials is

$$\sum_{k=0}^m \binom{m}{k} B_k(\alpha) B_{m-k}(\beta) = -(m-1)B_m(\alpha+\beta) + m(\alpha+\beta-1)B_{m-1}(\alpha+\beta) \quad (16)$$

for m an INTEGER. A sum identity due to S. M. Ruiz is

$$\sum_{k=0}^n (-1)^{k+n} \binom{n}{k} B_n(k) = n!, \quad (17)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. The Bernoulli polynomials are also given by the formula

$$B_n(x) = B_n(0) + \sum_{k=1}^n \frac{n}{k} S(n-1, k-1)(x)_k, \quad (18)$$

where $S(n, m)$ is a STIRLING NUMBER OF THE SECOND KIND and $(x)_k$ is a FALLING FACTORIAL (Roman 1984, p. 94). A general identity is given by

$$(n)_m x^{n-m} = \sum_{k=m}^n \frac{(n)_k}{(k-m+1)!} B_{n-k}(x), \quad (19)$$

which simplifies to

$$nx^{n-1} = \sum_{k=1}^n \binom{n}{k} B_{n-k}(x) \quad (20)$$

(Roman 1984, p. 97). Gosper gave the identity

$$\sum_{j=0}^i \frac{[2(i-j)-1]3^{2f}(2^{2f+1}+1)B_{2(i-j)}B_{2j+1}(\frac{1}{3})}{[2(i-j)]!(2j+1)!} = \frac{2 \cdot 3^{2(i-1)}(2^{2i-1}+1)B_{2i-1}(\frac{1}{3}) - (i-\frac{1}{2})B_{2i}}{(2i)!}. \quad (21)$$

Roman (1984, p. 93) defines a generalization $B_n^{(x)}(x)$ of the Bernoulli numbers with an additional free parameter such that $B_n(x) = B_n^{(1)}(x)$.

See also BERNOULLI NUMBER, BERNOULLI POLYNOMIAL OF THE SECOND KIND, EULER-MACLAURIN INTEGRATION FORMULAS, EULER POLYNOMIAL

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Bernoulli Polynomial of the Second Kind

Polynomials $b_n(x)$ which form a SHEFFER SEQUENCE with

$$g(t) = \frac{t}{e^t - 1} \quad (1)$$

$$f(t) = e^t - 1, \quad (2)$$

giving GENERATING FUNCTION

$$\sum_{k=0}^{\infty} \frac{b_k(x)}{k!} t^k = \frac{t(t+1)^x}{\ln(1+t)}. \quad (3)$$

Roman (1984) defines BERNOULLI NUMBERS OF THE SECOND KIND as $b_n = b_n(0)$. They are related to the STIRLING NUMBERS OF THE FIRST KIND $s(n, m)$ by

$$b_n(x) = b_n(0) + \sum_{k=1}^n \frac{n}{k} s(n-1, k-1)x^k \quad (4)$$

(Roman 1984, p. 115), and obey the reflection formula

$$b_n(\frac{1}{2}n - 1 - x) = (-1)^n b_n(\frac{1}{2}n - 1 + x) \quad (5)$$

(Roman 1984, p. 119).

The first few Bernoulli polynomials of the second kind are

$$\begin{aligned} b_0(x) &= 1 \\ b_1(x) &= \frac{1}{2}(2x+1) \\ b_2(x) &= \frac{1}{6}(6x^2-1) \\ b_3(x) &= \frac{1}{4}(4x^3-6x^2+1) \\ b_4(x) &= \frac{1}{30}(30x^4-120x^3+120x^2-19). \end{aligned}$$

See also BERNOULLI NUMBER OF THE SECOND KIND, BERNOULLI POLYNOMIAL, SHEFFER SEQUENCE, STIRLING NUMBER OF THE FIRST KIND

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Bernoulli SchemeReferences

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Bernoulli Trial

An experiment in which s TRIALS are made of an event, with probability p of success in any given TRIAL.

See also BERNOULLI DISTRIBUTION, COIN TOSSING, RUN

References

Papoulis, A. "Bernoulli Trials." §3 – in *Probability, Random Variables, and Stochastic Processes*, 2nd ed. New York: McGraw-Hill, pp. 57–3, 1984.

Bernoulli's Method

In order to find a root of a polynomial equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \quad (1)$$

consider the difference equation

$$a_0y(t+n) + a_1y(t+n-1) + \cdots + a_ny(t),$$

which is known to have solution

$$y(t) = w_1x_1^t + w_2x_2^t + \cdots + w_nx_n^t + \cdots, \quad (2)$$

where $w_1, w_2, \dots$, are arbitrary functions of t with period 1, and $x_1, \dots, x_n$ are roots of (1). In order to find the absolutely greatest root (1), take any arbitrary values for $y(0), y(1), \dots, y(n-1)$. By repeated application of (2), calculate in succession the values $y(n), y(n+1), y(n+2), \dots$. Then the ratio of two successive members of this sequence tends in general to a limit, which is the absolutely greatest root of (1).

See also ROOT

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Bernoulli's Paradox

Suppose the HARMONIC SERIES converges to h :

$$\sum_{k=1}^{\infty} \frac{1}{k} = h.$$

Then rearranging the terms in the sum gives

$$h - 1 = h,$$

which is a contradiction.

See also HARMONIC SERIES

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Bernoulli's Theorem

WEAK LAW OF LARGE NUMBERS

BernoulliB

BERNOULLI NUMBER, BERNOULLI POLYNOMIAL

Bernstein Minimal Surface Theorem

If a MINIMAL SURFACE is given by the equation $z = f(x, y)$ and f has CONTINUOUS first and second PARTIAL DERIVATIVES for all REAL x and y , then f is a PLANE.

See also MINIMAL SURFACE

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Bernstein Polynomial

The POLYNOMIALS defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i},$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. The Bernstein polynomials of degree n form a basis for the POWER - POLYNOMIALS of degree n .

Another form of Bernstein polynomials is given by

$$B_n(f, x) = \sum_{j=0}^n \binom{n}{j} x^j (1-x)^{n-j} f\left(\frac{j}{n}\right)$$

(Gzyl and Palacios 1997, Mathé 1999).

See also BÉZIER CURVE

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Bernstein's Constant

N.B. A detailed online essay by S. Finch was the starting point for this entry.

Let $E_n(f)$ be the error of the best uniform approximation to a REAL function $f(x)$ on the INTERVAL $[-1, 1]$ by REAL POLYNOMIALS of degree at most n . If

$$\alpha(x) = |x|, \quad (1)$$

then Bernstein showed that

$$0.267 \dots < \lim_{n \rightarrow \infty} 2nE_{2n}(\alpha) < 0.286. \quad (2)$$

He conjectured that the lower limit (β) was $\beta = 1/(2 \times \sqrt{\pi})$. However, this was disproven by Varga and Carpenter (1987) and Varga (1990), who computed

$$\beta = 0.2801694990 \dots \quad (3)$$

For rational approximations $p(x)/q(x)$ for p and q of degree m and n , D. J. Newman (1964) proved

$$\frac{1}{2} e^{-9\sqrt{n}} \leq E_{n,n}(\alpha) \leq 3e^{-\sqrt{n}} \quad (4)$$

for $n \geq 4$. Gonchar (1967) and Bulanov (1975) improved the lower bound to

$$e^{-\pi\sqrt{n+1}} \leq E_{n,n}(\alpha) \leq 3e^{-\sqrt{n}}. \quad (5)$$

Vjacheslavo (1975) proved the existence of POSITIVE constants m and M such that

$$m \leq e^{\pi\sqrt{n}} E_{n,n}(\alpha) < M \quad (6)$$

(Petrushev 1987, pp. 105–06). Varga *et al.* (1993) conjectured and Stahl (1993) proved that

$$\lim_{n \rightarrow \infty} e^{\pi\sqrt{2n}} E_{2n, 2n}(\alpha) = 8. \quad (7)$$

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Bernstein's Inequality

Let P be a POLYNOMIAL of degree n with derivative P' . Then

$$\|P'\|_{\infty} \leq n \|P\|_{\infty},$$

where

$$\|P\|_{\infty} \equiv \max_{|z|=1} |P(z)|.$$

Bernstein's Polynomial Theorem

If $g(\theta)$ is a trigonometric POLYNOMIAL of degree m satisfying the condition $|g(\theta)| \leq 1$ where θ is arbitrary and real, then $g'(\theta) \leq m$.

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Bernstein-Bézier Curve

BÉZIER CURVE

Bernstein-Szego Polynomials

The POLYNOMIALS on the interval $[-1, 1]$ associated with the WEIGHT FUNCTIONS

$$w(x) = (1 - x^2)^{-1/2}$$

$$w(x) = (1 - x^2)^{1/2}$$

$$w(x) = \sqrt{\frac{1-x}{1+x}},$$

also called BERNSTEIN POLYNOMIALS.

References

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Berry Conjecture

The longstanding conjecture that the nonimaginary solutions E_n of

$$\zeta\left(\frac{1}{2} + iE_n\right) = 0,$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION, are the EIGENVALUES of an "appropriate" HERMITIAN OPERATOR H . Berry and Keating (1999) further conjecture that this operator is

$$H = xp = -i \left(x \frac{d}{dx} + \frac{1}{2} \right),$$

where x and p are the position and conjugate momentum operators, respectively.

See also RIEMANN HYPOTHESIS, RIEMANN ZETA FUNCTION

References

Berry, M. V. and Keating, J. P. " $H = xp$ and the Riemann Zeros." In *Supersymmetry and Trace Formulae: Chaos and Disorder* (Ed. I. V. Lerner, J. P. Keating, and D. E. Khmelnitskii). New York: Kluwer, pp. 355–67, 1999.

Berry Paradox

There are several versions of the Berry paradox, the original version of which was published by Bertrand Russell and attributed to Oxford University librarian Mr. G. Berry. In one form, the paradox notes that the number "one million, one hundred thousand, one hundred and twenty one" can be named by the description: "the first number not nameable in under ten words." However, this latter expression has only nine words, so the number *can* be named in under ten words, so there is an inconsistency in naming it in this manner!

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Berry-Esséen Theorem

If $F(x)$ is a probability distribution with zero mean and

$$\rho = \int_{-\infty}^{\infty} |x|^3 dF(x) < \infty, \quad (1)$$

where the above integral is a STIELTJES INTEGRAL, then for all x and n ,

$$|F_n(x) - \Phi(x) - \frac{1}{2}| < \frac{33}{4} \frac{\rho}{\sigma^3 \sqrt{n}}, \quad (2)$$

where $\Phi(x)$ is the NORMAL DISTRIBUTION FUNCTION, $\Phi(x) + 1/2 = \mathfrak{N}(x)$ in Feller's notation, and

$$F_n(x) = F^{n*}(x\sigma\sqrt{n}) \quad (3)$$

is the normalized n -fold CONVOLUTION of $F(x)$ (Wallace 1958, Feller 1971).

See also CENTRAL LIMIT THEOREM

References

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Bertelsen's Number

An erroneous value of $\pi(10^9)$, where $\pi(x)$ is the PRIME COUNTING FUNCTION. Bertelsen's value of 50,847,478 is 56 lower than the correct value of 50,847,534.

See also PRIME COUNTING FUNCTION

References

Brown, K. S. "Bertelsen's Number." <http://www.seanet.com/~ksbrown/kmath049.htm>.

Bertini's Theorem

The general curve of a system which is LINEARLY INDEPENDENT on a certain number of given irreducible curves will not have a singular point which is not fixed for all the curves of the system.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves*. New York: Dover, p. 115, 1959.

Bertrand Curves

Two curves which, at any point, have a common principal NORMAL VECTOR are called Bertrand curves. The product of the TORSIONS of Bertrand curves is a constant.

Bertrand's Paradox

BERTRAND'S PROBLEM

Bertrand's Postulate

If $n > 3$, there is always at least one PRIME between n and $2n - 2$. Equivalently, if $n > 1$, then there is always at least one PRIME between n and $2n$. The conjecture was first made by Bertrand in 1845 (Nagell 1951, p. 67). It was proved in 1850–1 by Chebyshev, and is therefore sometimes known as CHEBYSHEV'S THEOREM. An extension of this result is that if $n > k$, then there is a number containing a PRIME divisor $> k$ in the sequence $n, n + 1, \dots, n + k - 1$. (The case $n = k + 1$ then corresponds to Bertrand's postulate.) This was first proved by Sylvester, independently by Schur, and a simple proof was given by Erdos (Hoffman 1998, p. 37)

A related problem is to find the least value of θ so that there exists at least one PRIME between n and $n + \mathcal{O}(n^\theta)$ for sufficiently large n (Berndt 1994). The smallest known value is $\theta = 6/11 + \epsilon$ (Lou and Yao 1992).

See also CHOQUET THEORY, DE POLIGNAC'S CONJECTURE, PRIME NUMBER

References

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Bertrand's Problem

What is the PROBABILITY that a CHORD drawn at random on a CIRCLE of RADIUS r (i.e., CIRCLE LINE PICKING) has length $\geq r$ (or sometimes greater than or equal to the side length of an inscribed equilateral triangle; Solomon 1978, p. 2)? The answer depends on the interpretation of "two points drawn at random," or more specifically on the "natural" measure for the problem.

In the most commonly considered measure, the ANGLES θ_1 and θ_2 are picked at random on the CIRCUMFERENCE of the circle. Without loss of generality, this can be formulated as the probability that the chord length of a single point at random angle θ measured from the x -AXIS on the unit circle. Since the length as a function of θ (CIRCLE LINE PICKING) is given by

$$s(\theta) = 2 \left| \sin\left(\frac{1}{2}\theta\right) \right|, \quad (1)$$

solving for $s(\theta) = 1$ gives $\pi/3$, so the fraction of the top unit semicircle having chord length greater than 1 is

$$P = \frac{\pi - \frac{\pi}{3}}{\pi} = \frac{2}{3}. \quad (2)$$

However, if a point is instead placed at random on a RADIUS of the CIRCLE and a CHORD drawn PERPENDICULAR to it, then

$$P = \frac{\frac{\sqrt{3}}{2}r}{r} = \frac{\sqrt{3}}{2}. \quad (3)$$

The latter interpretation is more satisfactory in the sense that the result remains the same for a rotated CIRCLE, a slightly smaller CIRCLE INSCRIBED in the first, or for a CIRCLE of the same size but with its center slightly offset. Jaynes (1983) shows that the interpretation of "random" as a continuous UNIFORM DISTRIBUTION over the RADIUS is the only one possessing all these three invariances.

See also CHORD, CIRCLE LINE PICKING, GEOMETRIC PROBABILITY

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Bertrand's Test

A CONVERGENCE TEST also called DE MORGAN'S AND BERTRAND'S TEST. If the ratio of terms of a SERIES $\{a_n\}_{n=1}^\infty$ can be written in the form

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\rho_n}{n \ln n},$$

then the series converges if $\lim_{n \rightarrow \infty} \rho_n > 1$ and diverges if $\lim_{n \rightarrow \infty} \rho_n < 1$, where $\lim_{n \rightarrow \infty}$ is the LOWER LIMIT and $\lim_{n \rightarrow \infty}$ is the UPPER LIMIT.

See also KUMMER'S TEST

References

- Bromwich, T. J. I'a and MacRobert, T. M. *An Introduction to the Theory of Infinite Series, 3rd ed.* New York: Chelsea, p. 40, 1991.

Bertrand's Theorem

BERTRAND'S POSTULATE

Besov Space

A type of abstract SPACE which occurs in SPLINE and RATIONAL FUNCTION approximations. The Besov space $B_{p,q}^\alpha$ is a complete quasinormed space which is a BANACH SPACE when $1 \leq p, q \leq \infty$ (Petrushev and Popov 1987).

See also BANACH SPACE

References

- Bergh, J. and Löfström, J. *Interpolation Spaces*. New York: Springer-Verlag, 1976.
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 Triebel, H. *Interpolation Theory, Function Spaces, Differential Operators*. New York: Wiley, 1998.

Bessel Differential Equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - m^2)y = 0. \quad (1)$$

Equivalently, dividing through by x^2 ,

$$\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{m^2}{x^2}\right)y = 0, \quad (2)$$

The solutions to this equation define the BESSEL FUNCTIONS. The equation has a regular SINGULARITY at 0 and an irregular SINGULARITY at ∞ .

A transformed version of the Bessel differential equation given by Bowman (1958) is

$$x^2 \frac{d^2 y}{dx^2} + (2p+1)x \frac{dy}{dx} + (a^2 x^{2r} + \beta^2)y = 0. \quad (3)$$

The solution is

$$y = x^{-p} \left[C_1 J_{q/r} \left(\frac{\alpha}{r} x^r \right) + C_2 Y_{q/r} \left(\frac{\alpha}{r} x^r \right) \right], \quad (4)$$

where

$$q \equiv \sqrt{p^2 - \beta^2}, \quad (5)$$

$J_n(x)$ and $Y_n(x)$ are the BESSEL FUNCTIONS OF THE FIRST and SECOND KINDS, and C_1 and C_2 are constants. Another form is given by letting $y = x^\alpha J_n(\beta x^\gamma)$, $\eta = yx^{-\alpha}$, and $\xi = \beta x^\gamma$ (Bowman 1958, p. 117), then

$$\frac{d^2 y}{dx^2} - \frac{2\alpha-1}{x} \frac{dy}{dx} + \left(\beta^2 \gamma^2 x^{2\gamma-2} + \frac{\alpha^2 - n^2 \gamma^2}{x^2} \right) y = 0. \quad (6)$$

The solution is

$$y = \begin{cases} x^\alpha [A J_n(\beta x^\gamma) + B Y_n(\beta x^\gamma)] & \text{for integer } n \\ A J_n(\beta x^\gamma) + B J_{-n}(\beta x^\gamma) & \text{for noninteger } n. \end{cases} \quad (7)$$

See also AIRY FUNCTIONS, ANGER FUNCTION, BEI, BER, BESSEL FUNCTION, BOURGET'S HYPOTHESIS, CATALAN INTEGRALS, CYLINDRICAL FUNCTION, DINI EXPANSION, HANKEL FUNCTION, HANKEL'S INTEGRAL, HEMISPHERICAL FUNCTION, KAPTEYN SERIES, LIPSCHITZ'S INTEGRAL, LOMMEL DIFFERENTIAL EQUATION, LOMMEL FUNCTION, LOMMEL'S INTEGRALS, NEUMANN SERIES (BESSEL FUNCTION), PARSEVAL'S INTEGRAL, POISSON INTEGRAL, RAMANUJAN'S INTEGRAL, RICCATI DIFFERENTIAL EQUATION, SONINE'S INTEGRAL, STRUVE FUNCTION, WEBER FUNCTIONS, WEBER'S DISCONTINUOUS INTEGRALS

References

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Bessel Function

A function $Z_n(x)$ defined by the RECURRENCE RELATIONS

$$Z_{n+1} + Z_{n-1} - \frac{2n}{x} Z_n$$

and

$$Z_{n+1} - Z_{n-1} = -2 \frac{dZ_n}{dx}.$$

The Bessel functions are more frequently defined as solutions to the DIFFERENTIAL EQUATION

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0.$$

There are two classes of solution, called the BESSEL FUNCTION OF THE FIRST KIND $J_n(x)$ and BESSEL FUNCTION OF THE SECOND KIND $Y_n(x)$. (A BESSEL FUNCTION OF THE THIRD KIND is a special combination of the first and second kinds.) Several related functions are also defined by slightly modifying the defining equations.

See also BESSEL FUNCTION OF THE FIRST KIND, BESSEL FUNCTION OF THE SECOND KIND, BESSEL FUNCTION OF THE THIRD KIND, CYLINDER FUNCTION, HEMICYLINDRICAL FUNCTION, MODIFIED BESSEL FUNCTION OF THE FIRST KIND, MODIFIED BESSEL

FUNCTION OF THE SECOND KIND, SPHERICAL BESSEL
FUNCTION OF THE FIRST KIND, SPHERICAL BESSEL
FUNCTION OF THE SECOND KIND

References

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Bessel Function Fourier Expansion

Let $n \geq 1/2$ and $\alpha_1, \alpha_2, \dots$ be the POSITIVE ROOTS of $J_n(x) = 0$. An expansion of a function in the interval $(0, 1)$ in terms of BESSEL FUNCTIONS OF THE FIRST KIND

$$f(x) = \sum_{l=1}^{\infty} A_l J_n(x\alpha_l), \quad (1)$$

has COEFFICIENTS found as follows:

$$\int_0^1 xf(x)J_n(x\alpha_l) dx = \sum_{r=1}^{\infty} A_r \int_0^1 xJ_n(x\alpha_r)J_n(x\alpha_l) dx. \quad (2)$$

But ORTHOGONALITY of BESSEL FUNCTION ROOTS gives

$$\int_0^1 xJ_n(x\alpha_l)J_n(x\alpha_r) dx = \frac{1}{2}\delta_{l,r}J_{n+1}^2(\alpha_r) \quad (3)$$

(Bowman 1958, p. 108), so

$$\begin{aligned} \int_0^1 xf(x)J_n(x\alpha_l) dx &= \frac{1}{2} \sum_{r=1}^{\infty} A_r \delta_{l,r} J_{n+1}^2(\alpha_r) \\ &= \frac{1}{2} A_l J_{n+1}^2(\alpha_l), \end{aligned} \quad (4)$$

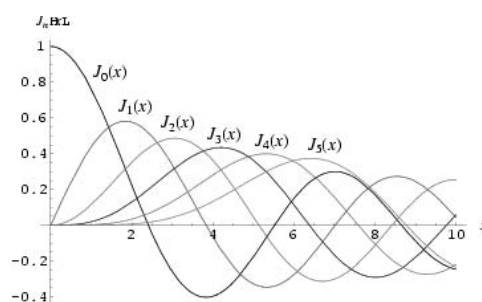
and the COEFFICIENTS are given by

$$A_l = \frac{2}{J_{n+1}^2(\alpha_l)} \int_0^1 xf(x)J_n(x\alpha_l) dx. \quad (5)$$

References

- Bowman, F. *Introduction to Bessel Functions*. New York: Dover, 1958.

Bessel Function of the First Kind



The Bessel functions of the first kind $J_n(x)$ are defined as the solutions to the BESSEL DIFFERENTIAL EQUATION

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - m^2)y = 0 \quad (1)$$

which are nonsingular at the origin. They are sometimes also called CYLINDER FUNCTIONS or CYLINDRICAL HARMONICS. The above plot shows $J_n(x)$ for $n = 1, 2, \dots, 5$.

To solve the differential equation, apply FROBENIUS METHOD using a series solution OF THE FORM

$$y = x^k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+k}. \quad (2)$$

Plugging into (1) yields

$$\begin{aligned} x^2 \sum_{n=0}^{\infty} (k+n)(k+n-1)a_n x^{k+n-2} \\ + x \sum_{n=0}^{\infty} (k+n)a_n x^{k+n-1} \\ + x^2 \sum_{n=0}^{\infty} a_n x^{k+n} - m^2 \sum_{n=0}^{\infty} a_n x^{n+k} = 0 \end{aligned} \quad (3)$$

$$\sum_{n=0}^{\infty} (k+n)(k+n-1)a_n x^{k+n} + \sum_{n=0}^{\infty} (k+n)a_n x^{k+n}$$

$$+ \sum_{n=2}^{\infty} a_{n-2} x^{k+n} - m^2 \sum_{n=0}^{\infty} a_n x^{n+k} = 0. \quad (4)$$

The INDICIAL EQUATION, obtained by setting $n = 0$, is

$$a_0[k(k-1) + k - m^2] = a_0(k^2 - m^2) = 0. \quad (5)$$

Since a_0 is defined as the first NONZERO term, $k^2 - m^2 = 0$, so $k = \pm m$. Now, if $k = m$,

$$\sum_{n=0}^{\infty} [(m+n)(m+n-1) + (m+n) - m^2] \times a_n x^{m+n} + \sum_{n=2}^{\infty} a_{n-2} x^{m+n} = 0 \quad (6)$$

$$\sum_{n=0}^{\infty} [(m+n)^2 - m^2] a_n x^{m+n} + \sum_{n=2}^{\infty} a_{n-2} x^{m+n} = 0 \quad (7)$$

$$\sum_{n=0}^{\infty} n(2m+n) a_n x^{m+n} + \sum_{n=2}^{\infty} a_{n-2} x^{m+n} = 0 \quad (8)$$

$$a_1(2m+1) + \sum_{n=2}^{\infty} [a_n n(2m+n) + a_{n-2}] x^{m+n} = 0. \quad (9)$$

First, look at the special case $m = -1/2$, then (9) becomes

$$\sum_{n=2}^{\infty} [a_n n(n-1) + a_{n-2}] x^{m+n} = 0, \quad (10)$$

so

$$a_n = -\frac{1}{n(n-1)} a_{n-2}. \quad (11)$$

Now let $n \equiv 2l$, where $l = 1, 2, \dots$

$$\begin{aligned} a_{2l} &= -\frac{1}{2l(2l-1)} a_{2l-2} \\ &= \frac{(-1)^l}{[2l(2l-1)[2(l-1)(2l-3)] \dots [2 \cdot 1 \cdot 1]} a_0 \\ &= \frac{(-1)^l}{2^l l! (2l-1)!!} a_0, \end{aligned} \quad (12)$$

which, using the identity $2^l l! (2l-1)!! = (2l)!$, gives

$$a_{2l} = \frac{(-1)^l}{(2l)!} a_0. \quad (13)$$

Similarly, letting $n \equiv 2l+1$,

$$\begin{aligned} a_{2l+1} &= -\frac{1}{(2l+1)(2l)} a_{2l-1} \\ &= \frac{(-1)^l}{[2l(2l+1)][2(l-1)(2l-1)] \dots [2 \cdot 1 \cdot 3][1]} a_1, \end{aligned} \quad (14)$$

which, using the identity $2^l l! (2l+1)!! = (2l+1)!$, gives

$$a_{2l+1} = \frac{(-1)^l}{2^l l! (2l+1)!!} a_1 = \frac{(-1)^l}{(2l+1)!} a_1. \quad (15)$$

Plugging back into (2) with $k = m = -1/2$ gives

$$\begin{aligned} y &= x^{-1/2} \sum_{n=0}^{\infty} a_n x^n \\ &= x^{-1/2} \left[\sum_{n=1,3,5,\dots}^{\infty} a_n x^n + \sum_{n=0,2,4,\dots}^{\infty} a_n x^n \right] \\ &= x^{-1/2} \left[\sum_{l=0}^{\infty} a_{2l} x^{2l} + \sum_{l=0}^{\infty} a_{2l+1} x^{2l+1} \right] \\ &= x^{-1/2} \left[a_0 \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} x^{2l} + a_1 \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!} x^{2l+1} \right] \\ &= x^{-1/2} (a_0 \cos x + a_1 \sin x). \end{aligned} \quad (16)$$

The BESSEL FUNCTIONS of order $\pm 1/2$ are therefore defined as

$$J_{-1/2}(x) \equiv \sqrt{\frac{2}{\pi x}} \cos x \quad (17)$$

$$J_{1/2}(x) \equiv \sqrt{\frac{2}{\pi x}} \sin x, \quad (18)$$

so the general solution for $m = \pm 1/2$ is

$$y = a'_0 J_{-1/2}(x) + a'_1 J_{1/2}(x). \quad (19)$$

Now, consider a general $m \neq -1/2$. Equation (9) requires

$$a_1(2m+1) = 0 \quad (20)$$

$$[a_n n(2m+n) + a_{n-2}] x^{m+n} = 0 \quad (21)$$

for $n = 2, 3, \dots$, so

$$a_1 = 0 \quad (22)$$

$$a_n = -\frac{1}{n(2m+n)} a_{n-2} \quad (23)$$

for $n = 2, 3, \dots$. Let $n \equiv 2l+1$, where $l = 1, 2, \dots$, then

$$\begin{aligned} a_{2l+1} &= -\frac{1}{(2l+1)[2(m+1)+1]} a_{2l-1} = \dots \\ &= \dots = f(n, m) a_1 = 0, \end{aligned} \quad (24)$$

where $f(n, m)$ is the function of l and m obtained by iterating the recursion relationship down to a_1 . Now let $n \equiv 2l$, where $l = 1, 2, \dots$, so

$$\begin{aligned} a_{2l} &= -\frac{1}{2l(2m+2l)} a_{2l-2} = -\frac{1}{4l(m+l)} a_{2l-2} \\ &= \frac{(-1)^l}{[4l(m+l)][4(l-1)(m+l-1)] \dots [4 \cdot (m+1)]} a_0. \end{aligned} \quad (25)$$

Plugging back into (9),

$$\begin{aligned}
 y &= \sum_{n=0}^{\infty} a_n x^{n+m} = \sum_{n=1, 3, 5, \dots}^{\infty} a_n x^{n+m} + \sum_{n=0, 2, 4, \dots}^{\infty} a_n x^{n+m} \\
 &= \sum_{l=0}^{\infty} a_{2l+1} x^{2l+m+1} + \sum_{l=0}^{\infty} a_{2l} x^{2l+m} \\
 &= a_0 \sum_{l=0}^{\infty} \frac{(-1)^l}{[4l(m+l)][4(l-1)(m+l-1)] \cdots [4(m+1)]} x^{2l+m} \\
 &= a_0 \sum_{l=0}^{\infty} \frac{[(-1)^l m(m-1) \cdots 1] x^{2l+m}}{[4l(m+l)][4(l-1)(m+l-1)] \cdots [4(m+1)m(m-1) \cdots 1]} \\
 &= a_0 \sum_{l=0}^{\infty} \frac{(-1)^l m!}{2^{2l} l! (m+l)!} x^{2l+m}, \quad (26)
 \end{aligned}$$

Now define

$$J_m(x) \equiv \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+m} l! (m+l)!} x^{2l+m}, \quad (27)$$

where the factorials can be generalized to GAMMA FUNCTIONS for nonintegral m . The above equation then becomes

$$y = a_0 2^m m! J_m(x) = a'_0 J_m(x). \quad (28)$$

Returning to equation (5) and examining the case $k = -m$,

$$a_1(1-2m) + \sum_{n=2}^{\infty} [a_n n(n-2m) + a_{n-2}] x^{n-m} = 0. \quad (29)$$

However, the sign of m is arbitrary, so the solutions must be the same for $+m$ and $-m$. We are therefore free to replace $-m$ with $-|m|$, so

$$a_1(1+2|m|) + \sum_{n=2}^{\infty} [a_n n(n+2|m|) + a_{n-2}] x^{|m|+n} = 0, \quad (30)$$

and we obtain the same solutions as before, but with m replaced by $|m|$.

$$J_m(x) = \begin{cases} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+|m|} l! (|m|+l)!} x^{2l+|m|} & \text{for } |m| \neq -\frac{1}{2} \\ \sqrt{\frac{2}{\pi x}} \cos x & \text{for } m = -\frac{1}{2} \\ \sqrt{\frac{2}{\pi x}} \sin x & \text{for } m = \frac{1}{2}. \end{cases} \quad (31)$$

We can relate J_m and J_{-m} (when m is an INTEGER) by writing

$$J_{-m}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l-m} l! (l-m)!} x^{2l-m}. \quad (32)$$

Now let $l \equiv l' + m$. Then

$$\begin{aligned}
 J_{-m}(x) &= \sum_{l'+m=0}^{\infty} \frac{(-1)^{l'+m}}{2^{2l'+m} (l'+m)! l!} x^{2l'+m} \\
 &= \sum_{l'=-m}^{-1} \frac{(-1)^{l'+m}}{2^{2l'+m} l'! (l'+m)!} x^{2l'+m} \\
 &\quad + \sum_{l'=0}^{\infty} \frac{(-1)^{l'+m}}{2^{2l'+m} l'! (l'+m)!} x^{2l'+m}. \quad (33)
 \end{aligned}$$

But $l'! = \infty$ for $l' = -m, \dots, -1$, so the DENOMINATOR is infinite and the terms on the right are zero. We therefore have

$$J_{-m}(x) = \sum_{l=0}^{\infty} \frac{(-1)^{l+m}}{2^{2l+m} l! (l+m)!} x^{2l+m} = (-1)^m J_m(x). \quad (34)$$

Note that the BESSEL DIFFERENTIAL EQUATION is second-order, so there must be two linearly independent solutions. We have found both only for $|m|=1/2$. For a general nonintegral order, the independent solutions are J_m and J_{-m} . When m is an INTEGER, the general (real) solution is OF THE FORM

$$Z_m \equiv C_1 J_m(x) + C_2 Y_m(x), \quad (35)$$

where J_m is a Bessel function of the first kind, Y_m (a.k.a. N_m) is the BESSEL FUNCTION OF THE SECOND KIND (a.k.a. NEUMANN FUNCTION or WEBER FUNCTION), and C_1 and C_2 are constants. Complex solutions are given by the HANKEL FUNCTIONS (a.k.a. BESSEL FUNCTIONS OF THE THIRD KIND).

The Bessel functions are ORTHOGONAL in $[0, 1]$ with respect to the weight factor x . Except when $2n$ is a NEGATIVE INTEGER,

$$J_m(z) = \frac{z^{-1/2}}{2^{2m+1/2} i^{m+1/2} \Gamma(m+1)} M_{0,m}(2iz), \quad (36)$$

where $\Gamma(x)$ is the GAMMA FUNCTION and $M_{0,m}$ is a WHITTAKER FUNCTION. In terms of a CONFLUENT HYPERGEOMETRIC FUNCTION OF THE FIRST KIND, the Bessel function is written

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)} {}_0F_1(\nu+1; -\frac{1}{4}z^2). \quad (37)$$

A derivative identity for expressing higher order Bessel functions in terms of $J_0(x)$ is

$$J_n(x) = i^n T_n \left(i \frac{d}{dx} \right) J_0(x), \quad (38)$$

where $T_n(x)$ is a CHEBYSHEV POLYNOMIAL OF THE FIRST KIND. Asymptotic forms for the Bessel functions are

$$J_m(x) \approx \frac{1}{\Gamma(m+1)} \left(\frac{x}{2} \right)^m \quad (39)$$

for $x \ll 1$ and

$$J_m(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \quad (40)$$

for $x \gg 1$.

A derivative identity is

$$\frac{d}{dx}[x^m J_m(x)] = x^m J_{m-1}(x). \quad (41)$$

An integral identity is

$$\int_0^u u' J_0(u') du' = u J_1(u). \quad (42)$$

Some sum identities are

$$1 = [J_0(x)]^2 + 2 \sum_{k=1}^{\infty} [J_k(x)]^2 \quad (43)$$

(Abramowitz and Stegun 1972, p. 363),

$$I = J_0(x) + 2 \sum_{k=1}^{\infty} J_{2k}(x) \quad (44)$$

(Abramowitz and Stegun 1972, p. 361),

$$0 = \sum_{k=0}^{2n} (-1)^k J_k(z) J_{2n-k}(z) + 2 \sum_{k=1}^{\infty} J_k(z) J_{2n+k}(z) \quad (45)$$

for $n \geq 1$ (Abramowitz and Stegun 1972, p. 361),

$$J_n(2z) = \sum_{k=0}^n J_k(z) J_{n-k}(z) + 2 \sum_{k=1}^{\infty} (-1)^k J_k(z) J_{n+k}(z) \quad (46)$$

(Abramowitz and Stegun 1972, p. 361), and the JACOBI-ANGER EXPANSION

$$e^{iz \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\theta}, \quad (47)$$

which can also be written

$$e^{iz \cos \theta} = J_0(z) + 2 \sum_{n=1}^{\infty} i^n J_n(z) \cos(n\theta). \quad (48)$$

The Bessel function addition theorem states

$$J_n(y+z) = \sum_{m=-\infty}^{\infty} J_m(y) J_{n-m}(z). \quad (49)$$

The first k roots $x_1, \dots, x_k$ of the Bessel function $J_n(x)$ can be found in *Mathematica* (Wolfram Research, Urbana, IL) using the command `BesselJZeros[n, k]` in the *Mathematica* add-on package `NumericalMath`BesselZeros`` (which can be loaded with the command `<<NumericalMath``). ROOTS of the FUNCTION $J_n(x)$ are given in the following table.

zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1336	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

The first k roots $x_1, \dots, x_k$ of the derivative of the Bessel function $J'_n(x)$ can be found in *Mathematica* using the command `BesselJPrimeZeros[n, k]` in the *Mathematica* add-on package `NumericalMath`BesselZeros`` (which can be loaded with the command `<<NumericalMath``). The first few such ROOTS are given in the following table.

zero	$J'_0(x)$	$J'_1(x)$	$J'_2(x)$	$J'_3(x)$	$J'_4(x)$	$J'_5(x)$
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

Various integrals can be expressed in terms of Bessel functions

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - n\theta) d\theta, \quad (50)$$

which is BESSEL'S FIRST INTEGRAL,

$$J_n(z) = \frac{i^{-n}}{\pi} \int_0^\pi e^{iz \cos \theta} \cos(n\theta) d\theta \quad (51)$$

$$J_n(z) = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{iz \cos \phi} e^{in\phi} d\phi \quad (52)$$

for $n = 1, 2, \dots$,

$$J_n(z) = \frac{2}{\pi} \frac{x^n}{(2rn-1)!!} \int_0^{\pi/2} \sin^{2n} u \cos(x \cos u) du \quad (53)$$

for $n = 1, 2, \dots$,

$$J_n(x) = \frac{1}{2\pi i} \int_\gamma e^{(x/2)(z-1/z)} z^{-n-1} dz \quad (54)$$

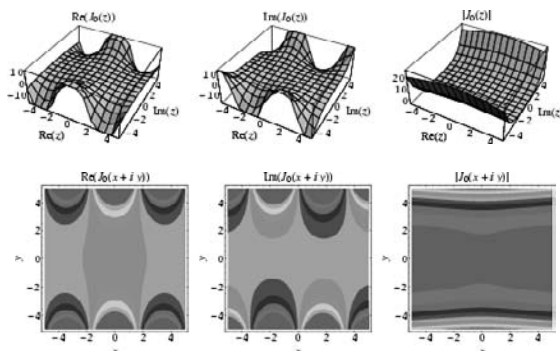
for $n > -1/2$. The Bessel functions are normalized so that

$$\int_0^\infty J_n(x) dx = 1 \quad (55)$$

for positive integral (and real) n . Integrals involving $J_1(x)$ include

$$\int_0^\infty \left[\frac{J_1(x)}{x} \right]^2 dx = \frac{4}{3\pi} \quad (56)$$

$$\int_0^\infty \left[\frac{J_1(x)}{x} \right]^2 x dx = \frac{1}{2}. \quad (57)$$



The special case of $n = 0$ gives $J_0(z)$ as the series

$$J_0(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2} \quad (58)$$

(Abramowitz and Stegun 1972, p. 360), or the integral

$$J_0(z) = \frac{1}{\pi} \int_0^\pi e^{iz \cos \theta} d\theta. \quad (59)$$

See also BESSEL FUNCTION OF THE SECOND KIND, DEBYE'S ASYMPTOTIC REPRESENTATION, DIXON-FERRAR FORMULA, HANSEN-BESSEL FORMULA, KAPTEYN SERIES, KNESER-SOMMERFELD FORMULA, MEHLER'S BESSEL FUNCTION FORMULA, NICHOLSON'S FORMULA, POISSON'S BESSEL FUNCTION FORMULA, RAYLEIGH FUNCTION, SCHLÄFLI'S FORMULA, SCHLÖMILCH'S SERIES, SOMMERFELD'S FORMULA, SONINE-SCHAFHEITLIN FORMULA, WATSON'S FORMULA, WATSON-NICHOLSON FORMULA, WEBER'S DISCONTINUOUS INTEGRALS, WEBER'S FORMULA, WEBER-SONINE FORMULA, WEYRICH'S FORMULA

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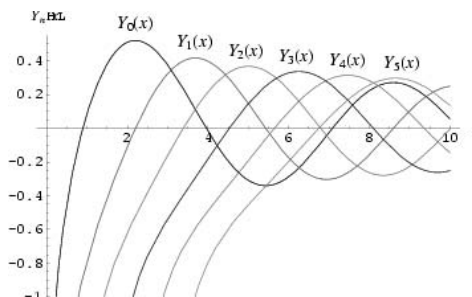
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Bessel Function of the Second Kind



A Bessel function of the second kind $Y_n(x)$ is a solution to the BESSEL DIFFERENTIAL EQUATION which is singular at the origin. Bessel functions of the second kind are also called NEUMANN FUNCTIONS or WEBER FUNCTIONS. The above plot shows $Y_n(x)$ for $n = 1, 2, \dots, 5$.

Let $v \equiv J_m(x)$ be the first solution and u be the other one (since the BESSEL DIFFERENTIAL EQUATION is second-order, there are two LINEARLY INDEPENDENT solutions). Then

$$xu'' + u' + xu = 0 \quad (1)$$

$$xv'' + v' + xv = 0. \quad (2)$$

Take $v \times (1)$ minus $u \times (2)$,

$$x(u''v - uv'') + u'v - uv' = 0 \quad (3)$$

$$\frac{d}{dx} [x(u'v - uv')] = 0, \quad (4)$$

so $x(u'v - uv') = B$, where B is a constant. Divide by xv^2 ,

$$\frac{u'v - uv'}{v^2} = \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{B}{xv^2} \quad (5)$$

$$\frac{u}{v} = A + B \int \frac{dx}{xv^2}. \quad (6)$$

Rearranging and using $v \equiv J_m(x)$ gives

$$\begin{aligned} u &= AJ_m(x) + BJ_m(x) \int \frac{dx}{xJ_m^2(x)} \\ &\equiv A'J_m(x) + B'Y_m(x), \end{aligned} \quad (7)$$

where Y_m is the so-called Bessel function of the second kind.

$Y_\nu(z)$ can be defined by

$$Y_\nu(z) = \frac{J_\nu(z) \cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \quad (8)$$

(Abramowitz and Stegun 1972, p. 358), where $J_\nu(z)$ is a BESSEL FUNCTION OF THE FIRST KIND and, for ν an integer n by the SERIES

$$Y_n(z) = -\frac{(\frac{1}{2}z)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} (\frac{1}{4}z^2)^k + \frac{2}{\pi} \ln(\frac{1}{2}z) J_n(z) - \frac{(\frac{1}{2}z)^n}{\pi} \sum_{k=0}^{\infty} [\psi_0(k+1) + \psi_0(n+k+1)] \frac{(-\frac{1}{4}z^2)^k}{k!(n+k)!}, \quad (9)$$

where $\psi_0(x)$ is the DIGAMMA FUNCTION (Abramowitz and Stegun 1972, p. 360).

The function has the integral representations

$$Y_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(z \sin \theta - \nu \theta) d\theta - \frac{1}{\pi} \int_0^\infty [e^{\nu t} + e^{-\nu t}(-1)^\nu] e^{-z \sin ht} dt. \quad (10)$$

$$= -\frac{2(\frac{1}{2}x)^{-\nu}}{\sqrt{\pi}\Gamma(\frac{1}{2}-\nu)} \int_1^\infty \frac{\cos(xt) dt}{(t^2-1)^{\nu+1/2}} \quad (11)$$

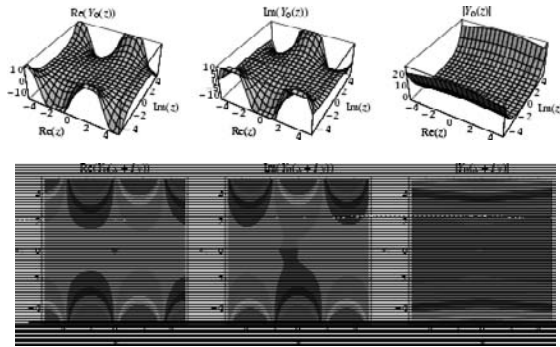
(Abramowitz and Stegun 1972, p. 360).

ASYMPTOTIC SERIES are

$$Y_m(x) \sim \begin{cases} \frac{2}{\pi} [\ln(\frac{1}{2}x) + \gamma] & m=0, x \ll 1 \\ -\frac{\Gamma(m)}{\pi} \left(\frac{2}{x}\right)^m & m \neq 0, x \ll 1 \end{cases} \quad (12)$$

$$Y_m(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \quad x \gg 1, \quad (13)$$

where $\Gamma(z)$ is a GAMMA FUNCTION.



For the special case $n=0$, $Y_0(x)$ is given by the series

$$Y_0(z) = \frac{2}{\pi} \left\{ [\ln(\frac{1}{2}z) + \gamma] J_0(z) + \sum_{k=1}^{\infty} (-1)^{k+1} H_k \frac{(\frac{1}{4}z^2)^k}{(k!)^2} \right\}, \quad (14)$$

(Abramowitz and Stegun 1972, p. 360), where γ is the EULER-MASCHERONI CONSTANT and H_n is a HARMONIC NUMBER.

See also BESSEL FUNCTION OF THE FIRST KIND, BOURGET'S HYPOTHESIS, HANKEL FUNCTION

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Bessel Function of the Third Kind

HANKEL FUNCTION

Bessel Polynomial

Krall and Find (1948) defined the Bessel polynomials as the function

$$y_n(x) = \sum_{k=0}^n \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k \quad (1)$$

which satisfies the differential equation

$$x^2 y'' + (2x+2)y' + n(n+1)y = 0. \quad (2)$$

Carlitz (1957) subsequently considered the related polynomials

$$p_n(x) = x^n y_{n-1} \left(\frac{1}{x}\right).$$

This polynomial forms an associated SHEFFER SEQUENCE with

$$f(t) = t - \frac{1}{2}t^2. \quad (3)$$

This gives the GENERATING FUNCTION

$$\sum_{k=0}^{\infty} \frac{p_k(x)}{k!} t^k = e^{x(1-\sqrt{1-2t})}. \quad (4)$$

The explicit formula is

$$p_n(x) = \sum_{k=1}^{\infty} \frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!} x^k. \quad (5)$$

The polynomials satisfy the recurrence formula

$$p_n''(x) - 2p_n'(x) + 2np_{n-1}(x) = 0. \quad (6)$$

The first few polynomials are

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x \\ p_2(x) &= x^2 + x \\ p_3(x) &= x^3 + 3x^2 + 3x \\ p_4(x) &= x^4 + 6x^3 + 15x^2 + 15x. \end{aligned}$$

See also BESSEL FUNCTION, SHEFFER SEQUENCE

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Bessel Transform

HANKEL TRANSFORM

Bessel's Correction

The factor $(N-1)/N$ in the relationship between the VARIANCE σ and the EXPECTATION VALUES of the SAMPLE VARIANCE,

$$\langle s^2 \rangle = \frac{N-1}{N} \sigma^2, \quad (1)$$

where

$$s^2 \equiv \langle x^2 \rangle - \langle x \rangle^2. \quad (2)$$

For two samples,

$$\hat{\sigma}^2 = \frac{N_1 s_1^2 + N_2 s_2^2}{N_1 + N_2 - 2}. \quad (3)$$

See also SAMPLE VARIANCE, VARIANCE

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Bessel's Finite Difference Formula

An INTERPOLATION formula also sometimes known as

$$\begin{aligned} f_p &= f_0 + p\delta_{1/2} + B_2(\delta_0^2 + \delta_1^2) + B_3\delta_{1/2}^3 + B_4(\delta_0^4 + \delta_1^4) \\ &\quad + B_5\delta_{1/2}^5 + \cdots, \end{aligned} \quad (1)$$

for $p \in [0, 1]$, where δ is the CENTRAL DIFFERENCE and

$$B_{2n} \equiv \frac{1}{2} G_{2n} \equiv \frac{1}{2} (E_{2n} + F_{2n}) \quad (2)$$

$$B_{2n+1} \equiv G_{2n+1} - \frac{1}{2} G_{2n} \equiv \frac{1}{2} (F_{2n} - E_{2n}) \quad (3)$$

$$E_{2n} \equiv G_{2n} - G_{2n+1} \equiv B_{2n} - B_{2n+1} \quad (4)$$

$$F_{2n} \equiv G_{2n+1} \equiv B_{2n} - B_{2n+1}, \quad (5)$$

where G_k are the COEFFICIENTS from GAUSS'S BACKWARD FORMULA and GAUSS'S FORWARD FORMULA and E_k and F_k are the COEFFICIENTS from EVERETT'S FORMULA. The B_k s also satisfy

$$B_{2n}(p) = B_{2n}(q) \quad (6)$$

$$B_{2n+1}(p) = -B_{2n+1}(q), \quad (7)$$

for

$$q \equiv 1 - p. \quad (8)$$

See also EVERETT'S FORMULA

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Bessel's First Integral

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - x \sin \theta) d\theta,$$

where $J_n(x)$ is a BESSEL FUNCTION OF THE FIRST KIND.

Bessel's Formula

BESSEL'S FINITE DIFFERENCE FORMULA, BESSEL'S INTERPOLATION FORMULA, BESSEL'S STATISTICAL FORMULA

Bessel's Inequality

If $f(x)$ is PIECEWISE CONTINUOUS and has a general FOURIER SERIES

$$\sum_i a_i \phi_i(x) \quad (1)$$

with WEIGHTING FUNCTION $w(x)$, it must be true that

$$\int \left[f(x) - \sum_i a_i \phi_i(x) \right]^2 w(x) dx \geq 0 \quad (2)$$

$$\int f^2(x) w(x) dx - 2 \sum_i a_i \int f(x) \phi_i(x) w(x) dx$$

$$+ \sum_i a_i^2 f \phi_i^2(x) w(x) dx \geq 0. \quad (3)$$

But the COEFFICIENT of the generalized FOURIER SERIES is given by

$$a_m \equiv \int f(x) \phi_m(x) w(x) dx, \quad (4)$$

so

$$\int f^2(x) w(x) dx - 2 \sum_i a_i^2 + \sum_i a_i^2 \geq 0 \quad (5)$$

$$\int f^2(x) w(x) dx \geq \sum_i a_i^2. \quad (6)$$

Equation (6) is an inequality if the functions ϕ_i are not COMPLETE. If they are COMPLETE, then the inequality (2) becomes an equality, so (6) becomes an equality and is known as PARSEVAL'S THEOREM. If $f(x)$ has a simple FOURIER SERIES expansion with COEFFICIENTS a_0, a_1, a_n, a_π and $b_1, \dots, b_n$, then

$$\frac{1}{2} a_0^2 + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx. \quad (7)$$

The inequality can also be derived from SCHWARZ'S INEQUALITY

$$|\langle f|g \rangle|^2 \leq \langle f|f \rangle \langle g|g \rangle \quad (8)$$

by expanding g in a superposition of EIGENFUNCTIONS of f , $g = \sum_i a_i f_i$. Then

$$\langle f|g \rangle = \sum_i a_i \langle f|f_i \rangle \leq \sum_i a_i \quad (9)$$

$$|\langle f|g \rangle|^2 \leq \left| \sum_i a_i \right|^2 = \left(\sum_i a_i \right) \left(\sum_i \bar{a}_i \right) = \sum_i a_i \bar{a}_i \leq \langle f|f \rangle \langle g|g \rangle, \quad (10)$$

where $\bar{f}$ is the COMPLEX CONJUGATE. If g is normalized, then $\langle g|g \rangle = 1$ and

$$\langle f|f \rangle \geq \sum_i a_i \bar{a}_i \quad (11)$$

See also SCHWARZ'S INEQUALITY, TRIANGLE INEQUALITY

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Bessel's Interpolation Formula

BESSEL'S FINITE DIFFERENCE FORMULA

Bessel's Second Integral

POISSON INTEGRAL

Bessel's Statistical Formula

Let $\bar{x}_1$ and s_1^2 be the observed mean and variance of a sample of N_1 drawn from a normal universe with unknown mean $\mu_{(1)}$ and let $\bar{x}_2$ and s_2^2 be the observed mean and variance of a sample of N_2 drawn from a normal universe with unknown mean $\mu_{(2)}$. Assume the two universes have a common variance σ^2 , and define

$$\bar{w} \equiv \hat{x}_1 - \bar{x}_2 \quad (1)$$

$$\omega \equiv \mu_{(1)} - \mu_{(2)} \quad (2)$$

$$N \equiv N_1 + N_2 \quad (3)$$

Then

$$t = \frac{\bar{w} - \omega}{\sigma_w / \sqrt{N}} = \frac{\bar{w} - \omega}{\sqrt{\frac{\sum_{i=1}^n (w_i - \bar{w})^2}{N(N-1)}}} \quad (4)$$

is distributed as STUDENT'S T -DISTRIBUTION $f_n(t)$ with $n = N - 2$.

See also STUDENT'S T -DISTRIBUTION

References

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BesselI

MODIFIED BESSEL FUNCTION OF THE FIRST KIND

BesselJ

BESSEL FUNCTION OF THE FIRST KIND

BesselK

MODIFIED BESSEL FUNCTION OF THE SECOND KIND

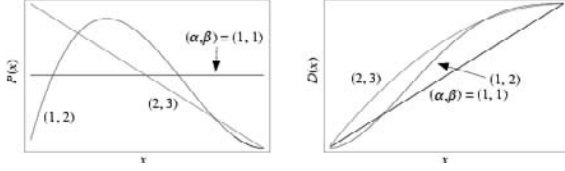
BesselY

BESSEL FUNCTION OF THE SECOND KIND

Beta

A financial measure of a fund's sensitivity to market movements which measures the relationship between a fund's excess return over Treasury Bills and the excess return of a benchmark index (which, by definition, has $\beta = 1$). A fund with a beta of β has performed $r \equiv (\beta - 1) \times 100\%$ better (or $|r|$ worse if $r < 0$) than its benchmark index (after deducting the T-bill rate) in up markets and $|r|$ worse (or $|r|$ better if $r < 0$) in down markets.

See also ALPHA, BETA DISTRIBUTION, BETA FUNCTION, BETA INTEGRAL, SHARPE RATIO

Beta Distribution

A general type of STATISTICAL DISTRIBUTION which is related to the GAMMA DISTRIBUTION. Beta distributions have two free parameters, which are labeled according to one of two notational conventions. The usual definition calls these α and β , and the other uses $\beta' \equiv \beta - 1$ and $\alpha' \equiv \alpha - 1$ (Beyer 1987, p. 534). The above plots are for various values of (α, β) . The domain is $[0, 1]$, and the probability function $P(x)$ and DISTRIBUTION FUNCTION $D(x)$ are given by

$$P(x) = \frac{(1-x)^{\beta-1} x^{\alpha-1}}{B(\alpha, \beta)} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} (1-x)^{\beta-1} x^{\alpha-1} \quad (1)$$

$$D(x) = I(x; \alpha, \beta), \quad (2)$$

where $B(a, b)$ is the BETA FUNCTION, $I(x; a, b)$ is the REGULARIZED BETA FUNCTION, and $\alpha, \beta > 0$. The distribution is normalized since

$$\int_0^1 P(x) dx = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \quad (3)$$

$$= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} B(\alpha, \beta) = 1. \quad (4)$$

The CHARACTERISTIC FUNCTION is

$$\begin{aligned} \phi(t) &= \mathcal{F} \left\{ \frac{x^{\alpha-1} (1-x)^{\beta-1}}{\beta(\alpha, \beta)} [\tfrac{1}{2} \operatorname{sgn}(1-x) + \operatorname{sgn} x] \right\} \\ &= {}_1F_1(\alpha, \alpha + \beta, it), \end{aligned} \quad (5)$$

where $\mathcal{F}[f]$ is a FOURIER TRANSFORM with parameters $\alpha = b = 1$ and ${}_1F_1(a; b; z)$ is a CONFLUENT HYPERGEOMETRIC FUNCTION.

The MEAN is

$$\begin{aligned} \mu &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} x dx \\ &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} B(\alpha + 1, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha + 1)\Gamma(\beta)}{\Gamma(\alpha + \beta + 1)} \\ &= \frac{\alpha}{\alpha + \beta}. \end{aligned} \quad (6)$$

The RAW MOMENTS are given by

$$\mu'_r = \int_0^1 P(x)(x - \mu)^r dx = \frac{\Gamma(\alpha + \beta)\Gamma(\alpha + r)}{\Gamma(\alpha + \beta + r)\Gamma(\alpha)} \quad (7)$$

(Papoulis 1984, p. 147), and the CENTRAL MOMENTS by

Beta Exponential Function

$$\mu_r = \left(-\frac{\alpha}{\alpha + \beta} \right)^r {}_2F_1 \left(-r, \alpha; \alpha + \beta; \frac{\alpha + \beta}{\alpha} \right), \quad (8)$$

where ${}_2F_1(a, b; c; x)$ is a HYPERGEOMETRIC FUNCTION. The VARIANCE, SKEWNESS, and KURTOSIS are therefore given by

$$\sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \quad (9)$$

$$\gamma_1 = \frac{2(\beta - \alpha)\sqrt{1 + \alpha + \beta}}{\sqrt{\alpha\beta}(2 + \alpha + \beta)} \quad (10)$$

$$\gamma_2 = \frac{6[\alpha^3 + \alpha^2(1 - 2\beta) + \beta^2(1 + \beta) - 2\alpha\beta(2 + \beta)]}{\alpha\beta(\alpha + \beta + 2)(\alpha + \beta + 3)}. \quad (11)$$

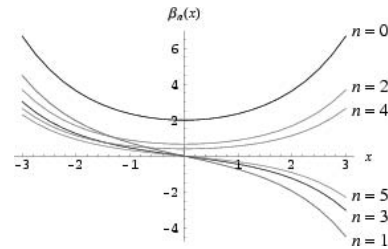
The MODE of a variate distributed as $\beta(\alpha, \beta)$ is

$$\hat{x} = \frac{\alpha - 1}{\alpha + \beta - 2}. \quad (12)$$

See also GAMMA DISTRIBUTION

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Beta Exponential Function

Another "BETA FUNCTION" defined in terms of an integral is the "exponential" beta function, given by

$$\beta_n(z) \equiv \int_{-1}^1 t^n e^{-zt} dt \quad (1)$$

$$= n! z^{-(n+1)} \left[e^z \sum_{k=0}^n \frac{(-1)^k z^k}{k!} - e^{-z} \sum_{k=0}^n \frac{z^k}{k!} \right]. \quad (2)$$

If n is an integer, then

$$\beta_n(z) = (-1)^{n+1} E_{-n}(-z) - E_{-n}(z), \quad (3)$$

where $E_n(z)$ is the EN -FUNCTION. The exponential beta function satisfies the RECURRENCE RELATION

$$z\beta_n(z) = (-1)^n e^z - e^{-z} + n\beta_{n-1}(z). \quad (4)$$

The values for $n = 0, 1$, and 2 are

$$\beta_0(z) = \frac{2 \sinh z}{z} \quad (5)$$

$$\beta_1(z) = \frac{2(\sinh z - z \cosh z)}{z^2}. \quad (6)$$

$$\beta_2(z) = \frac{2(2 + z^2) \sinh z - 4z \cosh z}{z^3}. \quad (7)$$

See also ALPHA FUNCTION, EN -FUNCTION

Beta Function

The beta function is the name used by Legendre and Whittaker and Watson (1990) for the BETA INTEGRAL (also called the Eulerian integral of the first kind). To derive the integral representation of the beta function, write the product of two FACTORIALS as

$$m!n! = \int_0^\infty e^{-u} u^m du \int_0^\infty e^{-v} v^n dv. \quad (1)$$

Now, let $u \equiv x^2$, $v \equiv y^2$, so

$$\begin{aligned} m!n! &= 4 \int_0^\infty e^{-x^2} x^{2m+1} dx \int_0^\infty e^{-y^2} y^{2n+1} dy \\ &= 4 \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)} x^{2m+1} y^{2n+1} dx dy. \end{aligned} \quad (2)$$

Transforming to POLAR COORDINATES with $x = r \cos \theta$, $y = r \sin \theta$

$$\begin{aligned} m!n! &= 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2} (r \cos \theta)^{2m+1} (r \sin \theta)^{2n+1} r dr d\theta \\ &= 4 \int_0^\infty e^{-r^2} r^{2m+2n+3} dr \int_0^{\pi/2} \cos^{2m+1} \theta \sin^{2n+1} \theta d\theta \\ &= 2(m+n+1)! \int_0^{\pi/2} \cos^{2m+1} \theta \sin^{2n+1} \theta d\theta. \end{aligned} \quad (3)$$

The beta function is then defined by

$$\begin{aligned} B(m+1, n+1) &= B(n+1, m+1) \\ &\equiv 2 \int_0^{\pi/2} \cos^{2m+1} \theta \sin^{2n+1} \theta d\theta = \frac{m!n!}{(m+n+1)!}. \end{aligned} \quad (4)$$

Rewriting the arguments,

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{(p-1)!(q-1)!}{(p+q-1)!}. \quad (5)$$

The general trigonometric form is

$$\int_0^{\pi/2} \sin^n x \cos^m x dx = \frac{1}{2} B\left(\frac{1}{2}(n+1), \frac{1}{2}(m+1)\right). \quad (6)$$

Equation (6) can be transformed to an integral over POLYNOMIALS by letting $u \equiv \cos^2 \theta$,

$$B(m+1, n+1) \equiv \frac{m!n!}{(m+n+1)!} = \int_0^1 u^m (1-u)^n du \quad (7)$$

$$B(m, n) \equiv \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^1 u^{m-1} (1-u)^{n-1} du. \quad (8)$$

The beta function is implemented in *Mathematica* as `Beta[a, b]`.

For any z_1, z_2 with $\Re[z_1], \Re[z_2] > 0$,

$$B(z_1, z_2) = B(z_2, z_1) \quad (9)$$

(Krantz 1999, p. 158).

The INCOMPLETE BETA FUNCTION $B(z; a, b)$, implemented in *Mathematica* as `Beta[z, a, b]`, is defined by the integral in (8) with an upper limit of z instead of 1. The REGULARIZED BETA FUNCTION $I(z; a, b)$, implemented in *Mathematica* as `BetaRegularized[z, a, b]` is defined by

$$I(z; a, b) = \frac{B(z; a, b)}{B(a, b)}. \quad (10)$$

To put it in a form which can be used to derive the LEGENDRE DUPLICATION FORMULA, let $x \equiv \sqrt{u}$, so $u = x^2$ and $du = 2x dx$, and

$$\begin{aligned} B(m, n) &= \int_0^1 x^{2(m-1)} (1-x^2)^{n-1} (2x dx) \\ &= 2 \int_0^1 x^{2m-1} (1-x^2)^{n-1} dx. \end{aligned} \quad (11)$$

To put it in a form which can be used to develop integral representations of the BESSEL FUNCTIONS and HYPERGEOMETRIC FUNCTION, let $u \equiv x/(1+x)$, so

$$B(m+1, n+1) = \int_0^\infty \frac{u^m du}{(1+u)^{m+n+2}}. \quad (12)$$

Derivatives of the beta function are given by

$$\frac{d}{da} B(a, b) = B(a, b) [\psi_0(a) - \psi_0(a+b)] \quad (13)$$

$$\frac{d}{db} B(a, b) = B(a, b) [\psi_0(b) - \psi_0(a+b)] \quad (14)$$

$$\frac{d^2}{da^2} B(a, b) = B(a, b)$$

$$\times \{[\psi_0(a) - \psi_0(a+b)]^2 + \psi_1(a) - \psi_1(a+b)\}, \quad (15)$$

$$\frac{d^2}{db^2} B(a, b) = B(a, b) \times \{[\psi_0(b) - \psi_0(a+b)]^2 + \psi_1(b) - \psi_1(a+b)\}, \quad (16)$$

$$\begin{aligned} \frac{d^2}{da db} B(a, b) \\ = B(a, b) \{[\psi_0(a) - \psi_0(a+b)][\psi_0(b) - \psi_0(a+b)] \\ - \psi_1(a+b)\} \end{aligned} \quad (17)$$

where $\psi_n(x)$ is the POLYGAMMA FUNCTION.

Various identities can be derived using the GAUSS MULTIPLICATION FORMULA

$$\begin{aligned} B(np, nq) &= \frac{\Gamma(np)\Gamma(nq)}{\Gamma[n(p+q)]} \\ &= n^{-nq} \frac{B(p, q)B\left(p + \frac{1}{n}, q\right) \cdots B\left(p + \frac{n-1}{n}, q\right)}{B(q, q)B(2q, q) \cdots B[(n-1]q, q)}. \end{aligned} \quad (18)$$

Additional identities include

$$\begin{aligned} B(p, q+1) &= \frac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)} = \frac{q}{p} \frac{\Gamma(p+1)\Gamma(q)}{\Gamma([p+1]q)} \\ &= \frac{q}{p} B(p+1, q) \end{aligned} \quad (19)$$

$$B(p, q) = B(p+1, q) + B(p, q+1) \quad (20)$$

$$B(p, q+1) = \frac{q}{p+q} B(p, q). \quad (21)$$

If n is a POSITIVE INTEGER, then

$$B(p, n+1) = \frac{1 \cdot 2 \cdots n}{p(p+1) \cdots (p+n)} \quad (22)$$

$$B(p, p)B(p + \frac{1}{2}, p + \frac{1}{2}) = \frac{\pi}{2^{4p-1}p} \quad (23)$$

$$B(p+q)B(p+q, r) = B(q, r)B(q+r, p). \quad (24)$$

Gosper gives the general formulas

$$\begin{aligned} \prod_{i=0}^{2n} B\left(\frac{i}{2n+1} + a, \frac{i}{2n+1} + b\right) \\ = \frac{(2n+1)^{(2n+1)/2} \pi^n B(n, \frac{1}{2}[(b+a)(2n+1)+1])B(a(2n+1), b(2n+1))}{(n-1)!} \end{aligned} \quad (25)$$

for ODD n , and

$$\begin{aligned} \prod_{i=0}^{2n-1} B\left(\frac{i}{2n} + a, \frac{i}{2n} + b\right) \\ = \frac{n^n \pi^n B(n, 2(a+b)n)B(2an, 2bn)}{2^{2(a+b)n} (n-1)!B((a+b)n, (a+b+1)n)}, \end{aligned} \quad (26)$$

which are an immediate consequence of the analogous identities for GAMMA FUNCTIONS. Plugging $n=1$ and $n=2$ into the above give the special cases

$$\begin{aligned} B(a, b)B(a + \frac{1}{3}, b + \frac{1}{3})B(a + \frac{2}{3}, b + \frac{2}{3}) \\ = \frac{6\pi\sqrt{3}B(3a, 3b)}{1 + 3(a+b)} \end{aligned} \quad (27)$$

$$\begin{aligned} B(a, b)B(a + \frac{1}{4}, b + \frac{1}{4})B(a + \frac{1}{2}, b + \frac{1}{2})B(a + \frac{3}{4}, b + \frac{3}{4}) \\ = \frac{2^{3-4(a+b)} \pi^2 B(4a, 4b)}{(a+b)[1 + 4(a+b)]B(2(a+b), 2(a+b+1))}. \end{aligned} \quad (28)$$

See also BETA INTEGRAL, CENTRAL BETA FUNCTION, DIRICHLET INTEGRALS, GAMMA FUNCTION, INCOMPLETE BETA FUNCTION, REGULARIZED BETA FUNCTION

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Beta Function (Exponential)

$$\mu_r = \left(-\frac{\alpha}{\alpha + \beta} \right)^r {}_2F_1 \left(-r, \alpha; \alpha + \beta; \frac{\alpha + \beta}{\alpha} \right),$$

Another "BETA FUNCTION" defined in terms of an integral is the "exponential" beta function, given by

$${}_2F_1(a, b; c; x) \theta_2 \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \quad (1)$$

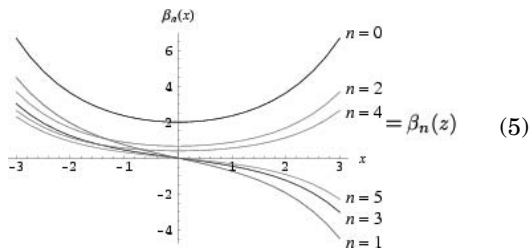
$$= \frac{2(\beta - \alpha)\sqrt{1 + \alpha + \beta}}{\sqrt{\alpha\beta}(2 + \alpha + \beta)} \quad (2)$$

The exponential beta function satisfies the RECURRENCE RELATION

$$\frac{6[\alpha^3 + \alpha^2(1 - 2\beta) + \beta^2(1 + \beta) - 2\alpha\beta(2 + \beta)]}{\alpha\beta(\alpha + \beta + 2)(\alpha + \beta + 3)}. \quad (3)$$

The first few integral values are

$$\beta(\alpha, \beta) = \hat{x} = \frac{\alpha - 1}{\alpha + \beta - 2}. \quad (4)$$



$$\int_{-1}^1 t^n e^{-zt} dt = n! z^{-(n+1)} \left[e^z \sum_{k=0}^n \frac{(-1)^k z^k}{k!} - e^{-z} \sum_{k=0}^n \frac{z^k}{k!} \right]. \quad (6)$$

See also ALPHA FUNCTION

Beta Integral

The integral

$$\int_0^1 x^p (1-x)^q dx$$

called the EULERIAN INTEGRAL OF THE FIRST KIND by Legendre and Whittaker and Watson (1990). The solution is the BETA FUNCTION $B(p+1, q+1)$.

See also BETA FUNCTION, EULERIAN INTEGRAL OF THE FIRST KIND, EULERIAN INTEGRAL OF THE SECOND KIND

References

Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, 1990.

Beta Prime Distribution

A distribution with probability function

$$P(x) = \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha, \beta)},$$

where B is a BETA FUNCTION. The MODE of a variate distributed as $\beta'(\alpha, \beta)$ is

$$\hat{x} = \frac{\alpha - 1}{\beta + 1}.$$

If x is a $\beta'(\alpha, \beta)$ variate, then $1/x$ is a $\beta'(\beta, \alpha)$ variate. If x is a $\beta(\alpha, \beta)$ variate, then $(1-x)/x$ and $x/(1-x)$ are $\beta'(\beta, \alpha)$ and $\beta'(\alpha, \beta)$ variates. If x and y are $\gamma(\alpha_1)$ and $\gamma(\alpha_2)$ variates, then x/y is a $\beta'(\alpha_1, \alpha_2)$ variate. If $x^2/2$ and $y^2/2$ are $\gamma(1/2)$ variates, then $z^2 \equiv (x/y)^2$ is a $\beta'(1/2, 1/2)$ variate.

BetaRegularized

REGULARIZED BETA FUNCTION

Bethe Lattice

CAYLEY TREE

Betrothed Numbers

QUASIAMICABLE PAIR

Betti Group

The free part of the HOMOLOGY GROUP with a domain of COEFFICIENTS in the GROUP OF INTEGERS (if this HOMOLOGY GROUP is finitely generated).

See also HOMOLOGY GROUP

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Betti Number

Betti numbers are topological objects which were proved to be invariants by Poincaré, and used by him to extend the POLYHEDRAL FORMULA to higher dimensional spaces. Informally, the Betti number is the maximum number of cuts that can be made without dividing a surface into two separate pieces (Gardner 1984, pp. 9-0). Formally, the n th Betti number is the rank of the n th HOMOLOGY GROUP of a TOPOLOGICAL SPACE. The following table gives the Betti number of some common surfaces.

SURFACE	Betti number
CROSS-CAP	1
CYLINDER	1
KLEIN BOTTLE	2
MÖBIUS STRIP	1
plane lamina	0
PROJECTIVE PLANE	1
SPHERE	0
TORUS	2

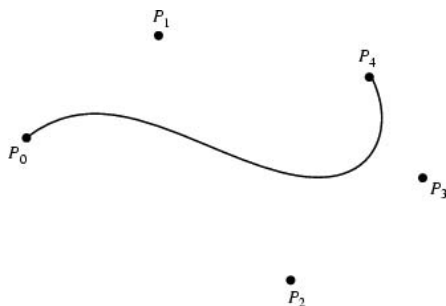
Let p_r be the RANK of the HOMOLOGY GROUP H_r of a TOPOLOGICAL SPACE K . For a closed, orientable surface of GENUS g , the Betti numbers are $p_0 = 1$, $p_1 = 2g$, and $p_2 = 1$. For a NONORIENTABLE SURFACE with k CROSS-CAPS, the Betti numbers are $p_0 = 1$, $p_1 = k - 1$, and $p_2 = 0$.

See also CHROMATIC NUMBER, EULER CHARACTERISTIC, GENUS (SURFACE), HOMOLOGY GROUP, POINCARÉ DUALITY, TOPOLOGICAL SPACE

References

Gardner, M. *The Sixth Book of Mathematical Games from Scientific American*. Chicago, IL: University of Chicago Press, pp. 9–1 and 15–6, 1984.

Bézier Curve



Given a set of $n + 1$ control points $P_0, P_1, \dots, P_n$, the corresponding Bézier curve (or Bernstein-Bézier curve) is given by

$$\mathbf{C}(t) = \sum_{i=0}^n \mathbf{P}_i B_{i,n}(t),$$

where $B_{i,n}(t)$ is a BERNSTEIN POLYNOMIAL and $t \in [0, 1]$.

A "rational" Bézier curve is defined by

$$\mathbf{C}(t) = \frac{\sum_{i=0}^n B_{i,p}(t) w_i \mathbf{P}_i}{\sum_{i=0}^n B_{i,p}(t) w_i},$$

where p is the order, $B_{i,p}$ are the BERNSTEIN POLYNOMIALS, $\mathbf{P}_i$ are control points, and the weight w_i of $\mathbf{P}_i$ is the last ordinate of the homogeneous point $\mathbf{P}_i^{\omega}$. These curves are CLOSED under perspective transformations, and can represent CONIC SECTIONS exactly.

The Bézier curve always passes through the first and last control points and lies within the CONVEX HULL of the control points. The curve is tangent to $\mathbf{P}_1 - \mathbf{P}_0$ and $\mathbf{P}_n - \mathbf{P}_{n-1}$ at the endpoints. The "variation diminishing property" of these curves is that no line can have more intersections with a Bézier curve than with the curve obtained by joining consecutive points with straight line segments. A desirable property of these curves is that the curve can be translated and rotated by performing these operations on the control points.

Undesirable properties of Bézier curves are their numerical instability for large numbers of control points, and the fact that moving a single control point changes the global shape of the curve. The former is sometimes avoided by smoothly patching together low-order Bézier curves. A generalization of the Bézier curve is the B-SPLINE.

See also B-SPLINE, NURBS CURVE

Bézier Spline

BÉZIER CURVE, SPLINE

Bézout Numbers

Integers (λ, μ) for a and b such that

$$\lambda a + \mu b = \text{GCD}(a, b).$$

For INTEGERS $a_1, \dots, a_n$, the Bézout numbers are a set of numbers $k_1, \dots, k_n$ such that

$$k_1 a_1 + k_2 a_2 + \dots + k_n a_n = d,$$

where d is the GREATEST COMMON DIVISOR of $a_1, \dots, a_n$.

See also GREATEST COMMON DIVISOR

Bézout's Theorem

In general, two algebraic curves of degrees m and n intersect in $m \cdot n$ points and cannot meet in more than $m \cdot n$ points unless they have a component in common (i.e., the equations defining them have a common factor). This can also be stated: if P and Q are two POLYNOMIALS with no roots in common, then there exist two other POLYNOMIALS A and B such that $AP + BQ = 1$. Similarly, given N POLYNOMIAL equations of degrees $n_1, n_2, \dots, n_N$ in N variables, there are in general $n_1 n_2 \dots n_N$ common solutions.

Séroul (2000, p. 10) uses the term Bézout's theorem for the following two theorems.

1. Let $a, b \in \mathbb{Z}$ be any two integers, then there exist $u, v \in \mathbb{Z}$ such that

$$au + bv = \text{GCD}(a, b).$$

2. Two integers a and b are RELATIVELY PRIME if there exist $u, v \in \mathbb{Z}$ such that

$$au + bv = 1.$$

See also BLANKINSHIP ALGORITHM, GREATEST COMMON DIVISOR, POLYNOMIAL

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Bhargava's Theorem

Let the n th composition of a function $f(x)$ be denoted $f^{(n)}(x)$, such that $f^{(0)}(x) = f(x)$ and $f^{(1)}(x) = f(x)$. Denote the COMPOSITION of f and g by $f \circ g(x) = f(g(x))$, and define

$$\begin{aligned} \sum F(a, b, c) \\ = F(a, b, c) + F(b, c, a) + F(c, b, a). \end{aligned} \quad (1)$$

Let

$$\mathbf{u} \equiv (a, b, c) \quad (2)$$

$$\|\mathbf{u}\| \equiv a + b + c \quad (3)$$

$$\|\mathbf{u}\| \equiv a^4 + b^4 + c^4, \quad (4)$$

and

$$f(\mathbf{u}) = (a(b - c), b(c - a), c(a - b)) \quad (5)$$

$$g(\mathbf{u}) = \left(\sum a^2 b, \sum ab^2, 3abc \right). \quad (6)$$

Then if $|\mathbf{u}| = 0$ (i.e., $c = -a - b$),

$$\begin{aligned} \|f^{(m)} \circ g^{(n)}(\mathbf{u})\| &= \|g^{(n)} \circ f^{(m)}(\mathbf{u})\| \\ &= 2(ab + bc + ca)^{2^{m+1}3^n}, \end{aligned} \quad (7)$$

where $m, n \in \{0, 1, \dots\}$ and COMPOSITION is done in terms of components.

See also DIOPHANTINE EQUATION–4TH POWERS, FORD'S THEOREM

References

- Berndt, B. C. *Ramanujan's Notebooks, Part IV*. New York: Springer-Verlag, pp. 97–00, 1994.
- Bhargava, S. "On a Family of Ramanujan's Formulas for Sums of Fourth Powers." *Ganita* **43**, 63–7, 1992.

Bhaskara-Brouckner Algorithm

SQUARE ROOT

Bialtitude

The common perpendicular to two opposite edges of a TETRAHEDRON.

See also ALTITUDE, BIMEDIAN, TETRAHEDRON

References

- Altshiller-Court, N. *Modern Pure Solid Geometry*. New York: Chelsea, p. 50, 1979.

Bianchi Identities

The RIEMANN TENSOR is defined by

$$\begin{aligned} R_{\lambda\mu\nu\kappa; \eta} &= \frac{1}{2} \frac{\partial}{\partial x^\eta} \\ &\times \left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right). \end{aligned} \quad (1)$$

Permuting ν, κ , and η (Weinberg 1972, pp. 146–47) gives the Bianchi identities

$$R_{\lambda\mu\nu\kappa; \eta} + R_{\lambda\mu\eta\nu; \kappa} + R_{\lambda\mu\kappa\eta; \nu} = 0, \quad (2)$$

which can be written concisely as

$$R^\alpha{}_{\beta[\lambda\mu; \nu]} = 0 \quad (3)$$

(Misner *et al.* 1973, p. 221), where $T_{[a_1 \dots a_n]}$ denoted the ANTISYMMETRIC TENSOR part. Wald (1984, p. 39) calls

$$\nabla_{[a} R_{bc]d}{}^e = 0 \quad (4)$$

the Bianchi identity, where ∇ is the COVARIANT DERIVATIVE, and $R_{abc}{}^{d'}$ is the RIEMANN TENSOR.

See also BIANCHI IDENTITIES (CONTRACTED), RIEMANN TENSOR

References

- Misner, C. W.; Thorne, K. S.; and Wheeler, J. A. *Gravitation*. San Francisco: W. H. Freeman, 1973.
- Wald, R. M. *General Relativity*. Chicago, IL: University of Chicago Press, 1984.
- Weinberg, S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. New York: Wiley, 1972.

Bianchi Identities (Contracted)

CONTRACTING λ with ν in the BIANCHI IDENTITIES

$$R_{\lambda\mu\nu\kappa; \eta} + R_{\lambda\mu\eta\nu; \kappa} + R_{\lambda\mu\kappa\eta; \nu} = 0 \quad (1)$$

gives

212 Bias (Estimator)

$$R_{\mu\kappa; \eta} - R_{\mu\eta; \kappa} + R_{\mu\kappa\eta; \nu}^{\nu} = 0. \quad (2)$$

CONTRACTING again,

$$R_{; \eta}^{\mu} - R_{\eta; \mu}^{\mu} - R_{\eta; \nu}^{\nu} = 0, \quad (3)$$

or

$$(R_{\eta}^{\mu} - \frac{1}{2} \delta_{\eta}^{\mu} R)_{; \mu} = 0, \quad (4)$$

or

$$(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R)_{; \mu} = 0. \quad (5)$$

Bias (Estimator)

The bias of an ESTIMATOR $\tilde{\theta}$ is defined as

$$B(\tilde{\theta}) \equiv \langle \tilde{\theta} \rangle - \theta.$$

It is therefore true that

$$\tilde{\theta} - \theta = (\tilde{\theta} - \langle \tilde{\theta} \rangle) + (\langle \tilde{\theta} \rangle - \theta) = (\tilde{\theta} - \langle \tilde{\theta} \rangle) + B(\tilde{\theta}).$$

An ESTIMATOR for which $B = 0$ is said to be UNBIASED ESTIMATOR.

See also BIASED ESTIMATOR, ESTIMATOR, UNBIASED ESTIMATOR

Bias (Series)

The bias of a SERIES is defined as

$$Q[a_i, a_{i+1}, a_{i+2}] \equiv \frac{a_i a_{i+2} - a_{i+1}^2}{a_1 a_{i+1} a_{i+2}}.$$

A SERIES is GEOMETRIC IFF $Q = 0$. A SERIES is ARTISTIC IFF the bias is constant.

See also ARTISTIC SEQUENCE, GEOMETRIC SEQUENCE

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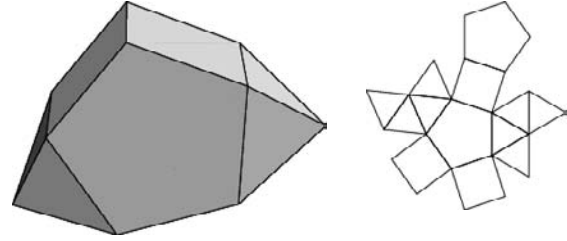
Biased Estimator

An ESTIMATOR which exhibits BIAS.

See also BIAS (ESTIMATOR), ESTIMATOR, UNBIASED ESTIMATOR

Biaugmented Truncated Cube

Biaugmented Pentagonal Prism



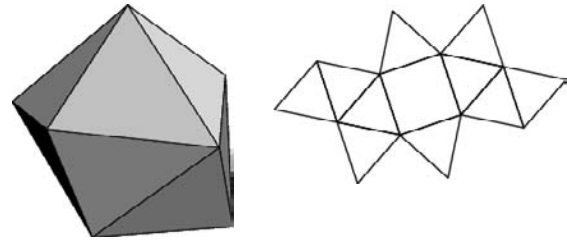
JOHNSON SOLID J_{53} .

References

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Weisstein, E. W. "Johnson Solid Netlib Database." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.DAT.

Biaugmented Triangular Prism



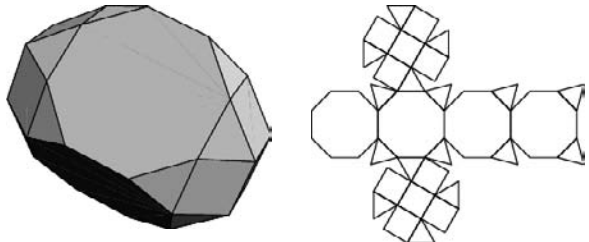
JOHNSON SOLID J_{50} .

References

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Weisstein, E. W. "Johnson Solid Netlib Database." MATHEMATICA NOTEBOOK JOHNSONSOLIDS.DAT.

Biaugmented Truncated Cube



JOHNSON SOLID J_{67} .

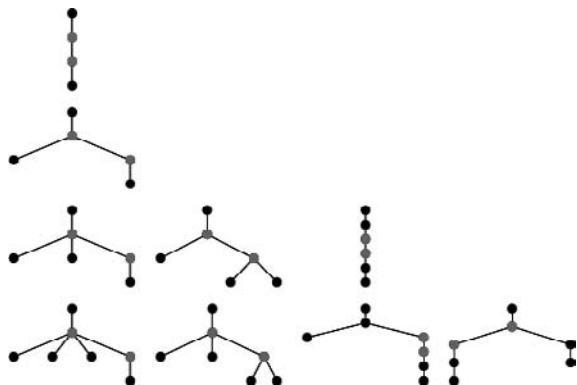
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BIBD

BLOCK DESIGN

Bicentered Tree



A TREE (also called a bicentral tree) having two nodes that are GRAPH CENTERS. The numbers of bicentered trees on $n = 1, 2, \dots$ nodes are 0, 1, 0, 1, 1, 3, 4, 11, 20, 51, 108 ... (Sloane's A000677).

See also CENTERED TREE, GRAPH CENTER, TREE

References

- Biggs, N. L.; Lloyd, E. K.; and Wilson, R. J. *Graph Theory* 1736–936. Oxford, England: Oxford University Press, p. 49, 1976.
 Cayley, A. "On the Analytical Forms Called Trees, with Application to the Theory of Chemical Combinations." *Reports Brit. Assoc. Advance. Sci.* **45**, 237–05, 1875. Reprinted in *Math Papers*, Vol. 9, pp. 427–60.
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Bicentral Tree

BICENTERED TREE

Bicentric Perspective

Bicentric perspective is the study of the projection of 3D space from a pair of fiducial points instead of a single one, the latter of which may be called "centric" or "natural" PERSPECTIVE by way of distinction.

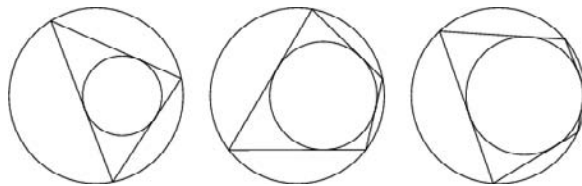
See also PERSPECTIVE, PROJECTION

References

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Paris, December 8–1, 1992 (Ed. A. Bensoussan and J.-P. Verjus). New York: Springer-Verlag, 233–51, 1992.

Bicentric Polygon



A POLYGON which has both a CIRCUMCIRCLE (which touches each vertex) and an INCIRCLE (which is tangent to each side). All TRIANGLES are bicentric with

$$R^2 - x^2 = 2Rr, \quad (1)$$

where R is the CIRCUMRADIUS, r is the INRADIUS, and x is the separation of centers. For BICENTRIC QUADRILATERALS (Fuss's problem), the CIRCLES satisfy

$$2r^2(R^2 + x^2) = (R^2 - x^2)^2 \quad (2)$$

(Dörrie 1965) or, in another form,

$$\frac{1}{(R - x)^2} + \frac{1}{(R + x)^2} = \frac{1}{r^2} \quad (3)$$

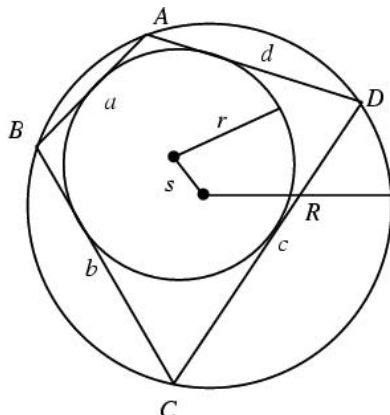
(Davis; Durège; Casey 1888, pp. 109–10; Johnson 1929; Dörrie 1965).

If the circles permit successive tangents around the INCIRCLE which close the POLYGON for one starting point on the CIRCUMCIRCLE, then they do so for all points on the CIRCUMCIRCLE, a result known as PONCELET'S PORISM.

See also BICENTRIC QUADRILATERAL, BICENTRIC TRIANGLE, CIRCUMCIRCLE, INCIRCLE, POLYGON, PONCELET'S PORISM, PONCELET TRANSVERSE, TANGENTIAL QUADRILATERAL, TRIANGLE, WEILL'S THEOREM

References

- Beyer, W. H. (Ed.). *CRC Standard Mathematical Tables*, 28th ed. Boca Raton, FL: CRC Press, p. 124, 1987.
 Casey, J. *A Sequel to the First Six Books of the Elements of Euclid, Containing an Easy Introduction to Modern Geometry with Numerous Examples*, 5th ed., rev. enl. Dublin: Hodges, Figgis, & Co., 1888.
 Dörrie, H. "Fuss' Problem of the Chord-Tangent Quadrilateral." §39 in *100 Great Problems of Elementary Mathematics: Their History and Solutions*. New York: Dover, pp. 188–93, 1965.
 Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*. Boston, MA: Houghton Mifflin, pp. 91–6, 1929.

Bicentric Quadrilateral

A 4-sided BICENTRIC POLYGON, also called a CYCLIC-INScriPTABLE QUADRILATERAL. The INRADIUS r , CIRCUMRADIUS R , and offset s are connected by the equation

$$\frac{1}{(R-s)^2} + \frac{1}{(R+s)^2} = \frac{1}{r^2} \quad (1)$$

(Davis; Durège; Casey 1888, pp. 109–10; Johnson 1929; Dörrie 1965; Coolidge 1971, p. 46). In addition

$$r = \frac{\sqrt{abcd}}{s} \quad (2)$$

$$R = \frac{1}{4} \sqrt{\frac{(ac+bd)(ad+bc)(ad+cd)}{abcd}} \quad (3)$$

(Beyer 1987), and

$$a+c = b+d. \quad (4)$$

The AREA of a bicentric quadrilateral is

$$A = \sqrt{abcd}. \quad (5)$$

See also BICENTRIC POLYGON, BICENTRIC TRIANGLE, CYCLIC QUADRILATERAL, PONCELET'S PORISM

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Durège, H. *Theorie der elliptischen Functionen: Versuch einer elementaren Darstellung*. Leipzig, Germany: Teubner, p. 185, 1861.

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*. Boston, MA: Houghton Mifflin, pp. 91–6, 1929.

Bicentric Triangle

All triangles are bicentric, i.e., possess both an INCIRCLE and a CIRCUMCIRCLE. This is not necessarily the case for polygons with four or more sides. The INRADIUS r and CIRCUMRADIUS R are connected by

$$\frac{1}{r+d} + \frac{1}{r-d} = \frac{1}{R},$$

where d is the distance between the INCENTER and CIRCUMCENTER (Coolidge 1971, p. 45).

See also BICENTRIC POLYGON, BICENTRIC QUADRILATERAL

References

- Coolidge, J. L. *A Treatise on the Geometry of the Circle and Sphere*. New York: Chelsea, 1971.

Bichromatic Graph

A GRAPH with EDGES of two possible "colors," usually identified as red and blue. For a bichromatic graph with R red EDGES and B blue EDGES,

$$R+B \geq 2.$$

See also BLUE-EMPTY GRAPH, EXTREMAL COLORING, EXTREMAL GRAPH, MONOCHROMATIC FORCED TRIANGLE, RAMSEY NUMBER

Bicollared

A SUBSET $X \subset Y$ is said to be bicollared in Y if there exists an embedding $b: X \times [-1, 1] \rightarrow Y$ such that $b(x, 0) = x$ when $x \in X$. The MAP b or its image is then said to be the bicollar.

References

- Rolfsen, D. *Knots and Links*. Wilmington, DE: Publish or Perish Press, pp. 34–5, 1976.

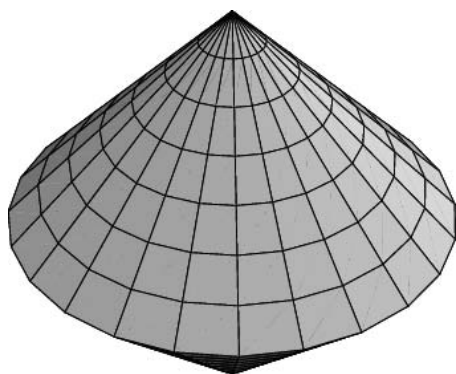
Biconditional

The CONNECTIVE in $A \Leftrightarrow B$ (also denoted $A \equiv B$) that returns a true result IFF A and B are either both true or both false. The biconditional is also called an EQUIVALENCE.

See also CONDITIONAL, EQUIVALENT

References

- Carnap, R. *Introduction to Symbolic Logic and Its Applications*. New York: Dover, p. 8, 1958.
 Mendelson, E. *Introduction to Mathematical Logic*, 4th ed. London: Chapman & Hall, p. 14, 1997.

Bicone

Two cones placed base-to-base.

See also DIPYRAMID, CONE, DOUBLE CONE, NAPPE, SPHERICON

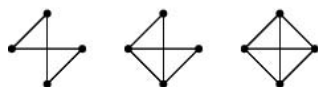
Bi-Connected Component

A maximal SUBGRAPH of an undirected graph such that any two edges in the SUBGRAPH lie on a common simple cycle.

See also STRONGLY CONNECTED COMPONENT

Biconnected Component

BLOCK

Biconnected Graph

A GRAPH with no ARTICULATION VERTICES is called biconnected (Skiena 1990, p. 175), block, or "nonseparable graph" (Harary 1994, p. 26). The numbers of biconnected simple graphs on $n = 1, 2, \dots$ nodes are 0, 1, 1, 3, 10, 56, 468, ... (Sloane's A002218). A graph can be tested for biconnectivity using BiconnectedQ[g] in the *Mathematica* add-on package Discrete-

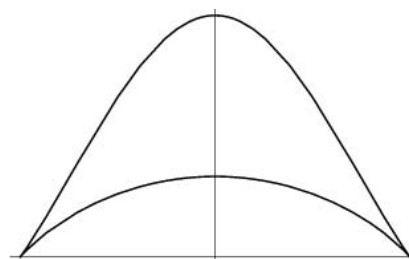
Math'Combinatorica' (which can be loaded with the command <<DiscreteMath').

Any graph containing a node of degree 1 cannot be biconnected. All HAMILTONIAN GRAPHS are biconnected (Skiena 1990, p. 177).

See also ARTICULATION VERTEX, BLOCK, CONNECTED GRAPH, K-CONNECTED GRAPH

References

- Harary, F. *Graph Theory*. Reading, MA: Addison-Wesley, 1994.
- Skiena, S. *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*. Reading, MA: Addison-Wesley, 1990.
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Bicorn

The bicorn is the name of a collection of QUARTIC CURVES studied by Sylvester in 1864 and Cayley in 1867 (MacTutor Archive). The bicorn is given by the PARAMETRIC EQUATIONS

$$x = a \sin t \quad (1)$$

$$y = \frac{a \cos^2 t (2 + \cos t)}{3 + \sin^2 t} \quad (2)$$

and Cartesian equation

$$y^2(a^2 - x^2) = (x^2 + 2ay - a^2)^2 \quad (3)$$

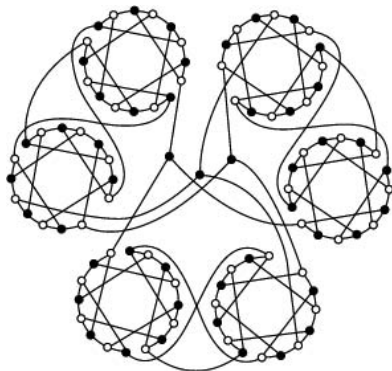
(MacTutor, with the final a squared instead of to the first power). The graph of the bicorn is similar to that of the COCKED HAT CURVE.

The CURVATURE is given by

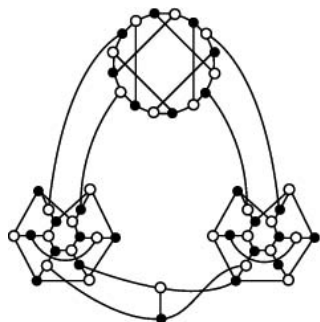
$$\kappa = \frac{6\sqrt{2}(\cos t - 2)^3(3 \cos t - 2) \sec t}{a[73 - 80 \cos t + 9 \cos(2t)]^{3/2}}. \quad (4)$$

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- Lawrence, J. D. *A Catalog of Special Plane Curves*. New York: Dover, pp. 147–49, 1972.
- MacTutor History of Mathematics Archive. "Bicorn." <http://www-groups.dcs.st-and.ac.uk/~history/Curves/Bicorn.html>.

Bicubic Graph

A BIPARTITE CUBIC GRAPH. Tutte (1971) conjectured that all 3-connected bicubic graphs are Hamiltonian (the TUTTE CONJECTURE). The Horton graph on 96 nodes provided the first counterexample (Bondy and Murty 1976, p. 240; illustrated above).



Horton subsequently found a counterexample on 92 nodes (Horton 1982). Two smaller (nonisomorphic) counterexamples on 78 nodes have since been found (Ellingham 1981, 1982b; Owens 1983). Ellingham and Horton (1983) subsequently found a nonhamiltonian 3-connected bicubic graph on 54 vertices, illustrated above.

See also BIPARTITE GRAPH, CUBIC GRAPH, TUTTE CONJECTURE

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- Tutte, W. T. "On the 2-Factors of Bicubic Graphs." *Discr. Math.* **1**, 203–08, 1971.

Bicubic Spline

A bicubic spline is a special case of bicubic interpolation which uses an interpolation function OF THE FORM

$$y(x_1, x_2) = \sum_{i=1}^4 \sum_{j=1}^4 c_{ij} t^{i-1} u^{j-1}$$

$$y_{x_1}(x_1, x_2) = \sum_{i=1}^4 \sum_{j=1}^4 (i-1) c_{ij} t^{i-2} u^{j-1}$$

$$y_{x_2}(x_1, x_2) = \sum_{i=1}^4 \sum_{j=1}^4 (j-1) c_{ij} t^{i-1} u^{j-2}$$

$$y_{x_1 x_2} = \sum_{i=1}^4 \sum_{j=1}^4 (i-1)(j-1) c_{ij} t^{i-2} u^{j-2},$$

where c_{ij} are constants and u and t are parameters ranging from 0 to 1. For a bicubic spline, however, the partial derivatives at the grid points are determined globally by 1-D SPLINES.

See also B-SPLINE, SPLINE

References

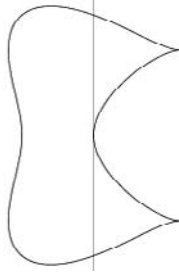
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Bicupola

Two adjoined CUPOLAS.

See also CUPOLA, ELONGATED GYROBICUPOLA, ELONGATED ORTHOBICUPOLA, GYROBICUPOLA, ORTHOBICUPOLA

Bicuspid Curve



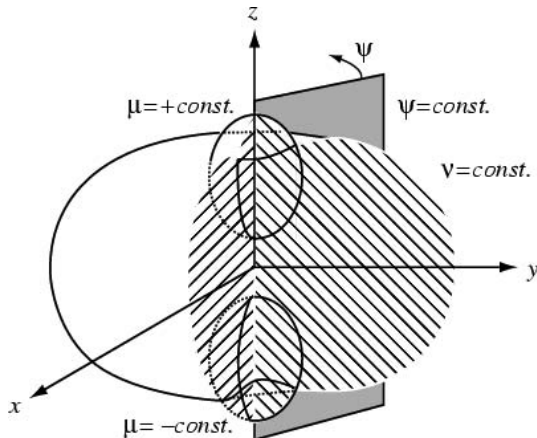
The PLANE CURVE given by the Cartesian equation

$$(x^2 - a^2)(x - a)^2 + (y^2 - a^2)^2 = 0.$$

Bi-Cyclide Coordinates

BICYCLIDE COORDINATES

Bicyclide Coordinates



A coordinate system which is similar to BISPHERICAL COORDINATES but having fourth-degree surfaces instead of second-degree surfaces for constant μ . The coordinates are given by the transformation equations

$$x = \frac{a}{\Lambda} \operatorname{cn} \mu \operatorname{dn} \mu \operatorname{sn} v \operatorname{cn} v \cos \psi \quad (1)$$

$$y = \frac{a}{\Lambda} \operatorname{cn} \mu \operatorname{dn} \mu \operatorname{sn} v \operatorname{cn} v \sin \psi \quad (2)$$

$$z = \frac{a}{\Lambda} \sin \mu \operatorname{dn} v, \quad (3)$$

where

$$\Lambda \equiv 1 - \operatorname{dn}^2 \mu \operatorname{sn}^2 v, \quad (4)$$

$\mu \in [0, K]$, $v \in [0, K']$, $\psi \in [0, 2\pi)$, and $\operatorname{cn} x$, $\operatorname{dn} x$, and

$\operatorname{sn} x$ are JACOBI ELLIPTIC FUNCTIONS. Surfaces of constant μ are given by the bicyclides

$$(x^2 + y^2 + z^2)^2 + \frac{a^2}{k^4} \frac{(1 - k^2)^2 - 2(1 - k^2) \operatorname{dn}^2 \mu + (1 + k^2) \operatorname{dn}^4 \mu}{\operatorname{dn}^2 \mu \operatorname{cn}^2 \mu} \times (x^2 + y^2) - a^2 \left(\operatorname{sn}^2 \mu + \frac{1}{k^2 \operatorname{sn}^2 \mu} \right) z^2 + \frac{a^4}{k^2} = 0, \quad (5)$$

surfaces of constant v by the cyclides of rotation

$$\left[\frac{\operatorname{cn}^2 v}{a^2 \operatorname{sn}^2 v} (x^2 + y^2) + \frac{\operatorname{dn}^2 v}{a^2} z^2 \right]^2 - \frac{2 \operatorname{cn}^2 v}{a^2 \operatorname{sn}^2 v} (x^2 + y^2) - \frac{2 \operatorname{dn}^2 v}{a^2} z^2 + 1 = 0, \quad (6)$$

and surfaces of constant ψ by the half-planes

$$\tan \psi = \frac{y}{x}. \quad (7)$$

See also BISPHERICAL COORDINATES, CAP-CYCLIDE COORDINATES, CYCLIDIC COORDINATES

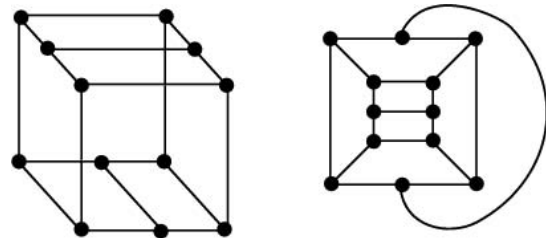
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Bicylinder

STEINMETZ SOLID

Bidiakis Cube



The 12-VERTEX graph consisting of a CUBE in which two opposite faces (say, top and bottom) have edges drawn across them which connect the centers of opposite sides of the faces in such a way that the orientation of the edges added on top and bottom are PERPENDICULAR to each other.

See also BISLIT CUBE, CUBE, CUBICAL GRAPH

Bieberbach Conjecture

The n th COEFFICIENT in the POWER SERIES of a UNIVALENT FUNCTION should be no greater than n . In other words, if

$$f(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n + \dots$$

is a CONFORMAL MAP of a UNIT DISK on any domain, then $|a_n| \leq n|a_1|$. In more technical terms, "geometric extremality implies metric extremality." An alternate formulation is that $|a_j| \leq j$ for any SCHLICHT FUNCTION f (Krantz 1999, p. 150).

The conjecture had been proven for the first six terms (the cases $n = 2, 3$, and 4 were done by Bieberbach, Löwner, and Garabedian and Schiffer, respectively), was known to be false for only a finite number of indices (Hayman 1954), and true for a convex or symmetric domain (Le Lionnais 1983). The general case was proved by Louis de Branges (1985). de Branges proved the MILIN CONJECTURE, which established the ROBERTSON CONJECTURE, which in turn established the Bieberbach conjecture (Stewart 1996).

author	result
Bieberbach (1916)	$ a_2 \leq 2$
Löwner (1923)	$ a_3 \leq 3$
Garabedian and Schiffer (1955)	$ a_4 \leq 4$
Pederson (1968), Ozawa (1969)	$ a_6 \leq 6$
Pederson and Schiffer (1972)	$ a_5 \leq 5$
de Branges (1985)	$ a_j \leq j$ for all j

The sum

$$\sum_{j=k}^n (-1)^{k+j} \binom{2j}{j-k} \binom{n+j+1}{n-j} e^{-jt}$$

was an essential tool in de Branges' proof (Koepf 1998, p. 29).

See also MILIN CONJECTURE, ROBERTSON CONJECTURE, SCHLICHT FUNCTION, UNIVALENT FUNCTION

References

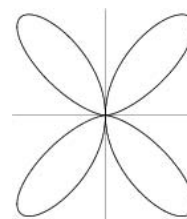
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Bienaymé-Chebyshev Inequality

CHEBYSHEV INEQUALITY

Bifoliate



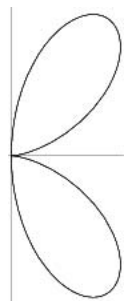
The PLANE CURVE given by the Cartesian equation

$$x^4 + y^4 = 2axy^2.$$

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Bifolium



A FOLIUM with $b = 0$. The bifolium is the PEDAL CURVE of the DELTOID, where the PEDAL POINT is the MIDPOINT of one of the three curved sides. The Cartesian equation is

$$(x^2 + y^2)^2 = 4axy^2$$

and the POLAR equation is

$$r = 4a \sin^2 \theta \cos \theta.$$

See also FOLIUM, QUADRIFOLIUM, TRIFOLIUM

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Bifurcation

A period doubling, quadrupling, etc., that accompanies the onset of CHAOS. It represents the sudden appearance of a qualitatively different solution for a nonlinear system as some parameter is varied. Bifurcations come in four basic varieties: FLIP BIFURCATION, FOLD BIFURCATION, PITCHFORK BIFURCATION, and TRANSCRITICAL BIFURCATION (Rasband 1990).

See also CODIMENSION, FEIGENBAUM CONSTANT, FEIGENBAUM FUNCTION, FLIP BIFURCATION, HOPF BIFURCATION, LOGISTIC MAP, PERIOD DOUBLING, PITCHFORK BIFURCATION, TANGENT BIFURCATION, TRANSCRITICAL BIFURCATION

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Bifurcation Theory

The study of the nature and properties of BIFURCATIONS.

See also CHAOS, DYNAMICAL SYSTEM

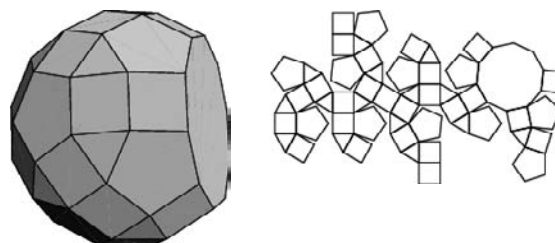
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Bigraph

BIPARTITE GRAPH

Bigyrate Diminished Rhombicosidodecahedron



JOHNSON SOLID J_{79} .

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Biharmonic Equation

The differential equation obtained by applying the BIHARMONIC OPERATOR and setting to zero.

$$\nabla^4 \phi = 0. \quad (1)$$

In CARTESIAN COORDINATES, the biharmonic equation is

$$\begin{aligned} \nabla^4 \phi &= \nabla^2 (\nabla^2 \phi) \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi \\ &= \frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial y^4} + \frac{\partial^4 \phi}{\partial z^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + 2 \frac{\partial^4 \phi}{\partial y^2 \partial z^2} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial z^2} \\ &= 0. \end{aligned} \quad (2)$$

In POLAR COORDINATES (Kaplan 1984, p. 148)

$$\begin{aligned} \nabla^4 \phi &= \phi_{rrrr} + \frac{2}{r^2} \phi_{rr\theta\theta} + \frac{1}{r^4} \phi_{\theta\theta\theta\theta} + \frac{2}{r} \phi_{rrr} - \frac{2}{r^3} \phi_{r\theta\theta} \\ &\quad - \frac{1}{r^2} \phi_{rr} + \frac{4}{r^4} \phi_{\theta\theta} + \frac{1}{r^3} \phi_r = 0. \end{aligned} \quad (3)$$

For a radial function $\phi(r)$, the biharmonic equation becomes

$$\begin{aligned} \nabla^4 \phi &= \frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) \right] \right\} \\ &= \phi_{rrrr} + \frac{2}{r} \phi_{rrr} - \frac{1}{r^2} \phi_{rr} + \frac{1}{r^3} \phi_r = 0. \end{aligned} \quad (4)$$

Writing the inhomogeneous equation as

$$\nabla^4 \phi = 64\beta, \quad (5)$$

we have

$$64\beta r \, dr = d \left\{ r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) \right] \right\} \quad (6)$$

$$32\beta r^2 + C_1 = r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) \right] \quad (7)$$

$$\left(32\beta r + \frac{C_1}{r} \right) dr = d \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) \right] \quad (8)$$

$$16\beta r^2 + C_1 \ln r + C_2 = \frac{1}{r} \frac{d}{dr} \left(r \frac{d\phi}{dr} \right) \quad (9)$$

$$(16\beta r^3 + C_1 r \ln r + C_2 r) dr = d \left(r \frac{d\phi}{dr} \right). \quad (10)$$

Now use

$$\int r \ln r \, dr = \frac{1}{2} r^2 \ln r - \frac{1}{4} r^2 \quad (11)$$

to obtain

$$4\beta r^4 + C_1 \left(\frac{1}{2} r^2 \ln r - \frac{1}{4} r^2 \right) + \frac{1}{2} C_2 r^2 + C_3 = r \frac{d\phi}{dr} \quad (12)$$

$$\left(4\beta r^3 + C_1' r \ln r + C_2' r + \frac{C_3}{r} \right) dr = d\phi \quad (13)$$

$$\begin{aligned} \phi(r) &= \beta r^4 + C_1' \left(\frac{1}{2} r^2 \ln r - \frac{1}{4} r^2 \right) + \frac{1}{2} C_2' r^2 + C_3 \ln r + C_4 \\ &= \beta r^4 + ar^2 + b + (cr^2 + d) \ln \left(\frac{r}{R} \right). \end{aligned} \quad (14)$$

The homogeneous biharmonic equation can be separated and solved in 2-D BIPOLAR COORDINATES.

See also BIHARMONIC OPERATOR, VON KÁRMÁN EQUATIONS

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Biharmonic Operator

Also known as the BILAPLACIAN.

$$\nabla^4 = (\nabla^2)^2.$$

In n -D space,

$$\nabla^4 \left(\frac{1}{r} \right) = \frac{3(15 - 8n + n^2)}{r^5}.$$

See also BIHARMONIC EQUATION, D'ALEMBERTIAN, LAPLACIAN, VON KÁRMÁN EQUATIONS

Biholomorphic Function

CONFORMAL MAPPING

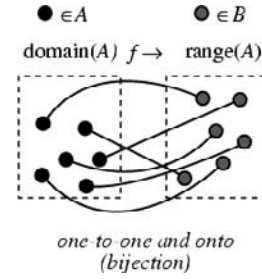
Biholomorphic Map

CONFORMAL MAPPING

Biholomorphic Transformation

CONFORMAL MAPPING

Bijection



A transformation which is ONE-TO-ONE and ONTO.

See also DOMAIN, ONE-TO-ONE, ONTO, PERMUTATION, RANGE (IMAGE)

Bilaplacian

BIHARMONIC OPERATOR

Bilinear Basis

A bilinear basis is a BASIS, which satisfies the conditions

$$(a\mathbf{x} + b\mathbf{y}) \cdot \mathbf{z} = a(\mathbf{x} \cdot \mathbf{z}) + b(\mathbf{y} \cdot \mathbf{z})$$

$$\mathbf{z} \cdot (a\mathbf{x} + b\mathbf{y}) = a(\mathbf{z} \cdot \mathbf{x}) + b(\mathbf{z} \cdot \mathbf{y}),$$

See also BASIS, BILINEAR FUNCTION, MULTILINEAR BASIS

Bilinear Form

A bilinear form on a REAL VECTOR SPACE is a function

$$b : V \times V \rightarrow \mathbb{R}$$

that satisfies the following axioms for any scalar α and any choice of vectors v, w, v_1, v_2, w_1 and w_2 .

1. $b(xv, w) = b(v, \alpha w) = \alpha b(v, w)$
2. $b(v_1 + v_2, w) = b(v_1, w) + b(v_2, w)$
3. $b(v, w_1 + w_2) = b(v, w_1) + b(v, w_2)$.

For example, the function $b((x_1, x_2), (y_1, y_2)) = x_1y_2 + x_2y_1$ is a bilinear form on $\mathbb{R}^2$.

On a COMPLEX VECTOR SPACE, a bilinear form takes values in the COMPLEX NUMBERS. In fact, a bilinear form can take values in any VECTOR SPACE, since the axioms make sense as long as VECTOR ADDITION and SCALAR MULTIPLICATION are defined.

See also BILINEAR FUNCTION, MULTILINEAR FORM, SYMMETRIC BILINEAR FORM, VECTOR SPACE

Bilinear Function

A function of two variables is bilinear if it is linear with respect to each of its variables. The simplest example is $f(x, y) = xy$.

See also BILINEAR BASIS, LINEAR FUNCTION, SYMMETRIC BILINEAR FORM

Billiard Table Problem

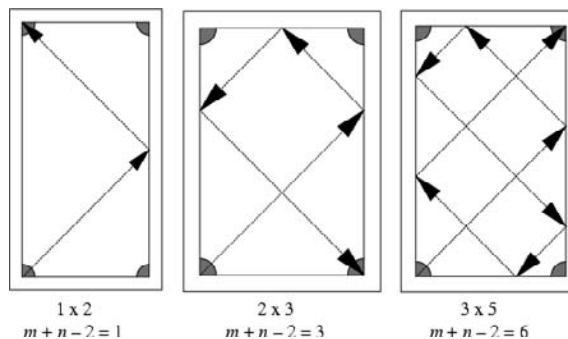
BILLIARDS

Billiards

The game of billiards is played on a RECTANGULAR table (known as a billiard table) upon which balls are placed. One ball (the "cue ball") is then struck with the end of a "cue" stick, causing it to bounce into other balls and REFLECT off the sides of the table. Real

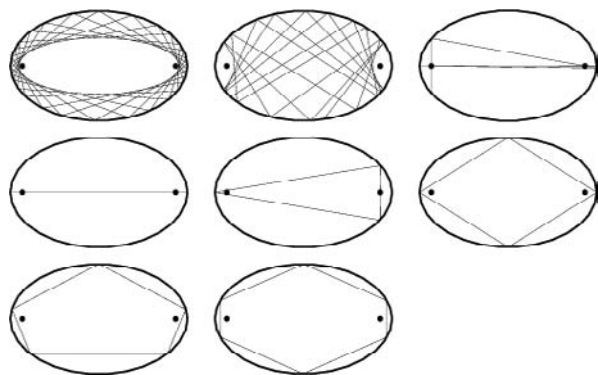
billiards can involve spinning the ball so that it does not travel in a straight LINE, but the mathematical study of billiards generally consists of REFLECTIONS in which the reflection and incidence angles are the same. However, strange table shapes such as CIRCLES and ELLIPSES are often considered.

Many interesting problems can arise in the detailed study of billiards trajectories. For example, any smooth plane convex set has at least two DOUBLE NORMALS, so there are always two distinct "to and fro" paths for any smoothly curved table. More amazingly, there are always $\phi(k)$ distinct k -gonal periodic orbits on smooth billiard table, where $\phi(k)$ is the TOTIENT FUNCTION (Croft *et al.* 1991, p. 16). This gives Steinhaus's result that there are always two distinct periodic triangular orbits (Croft and Swinnerton-Dyer 1963) as a special case. Analysis of billiards path can involve sophisticated use of ERGODIC THEORY and DYNAMICAL SYSTEMS.



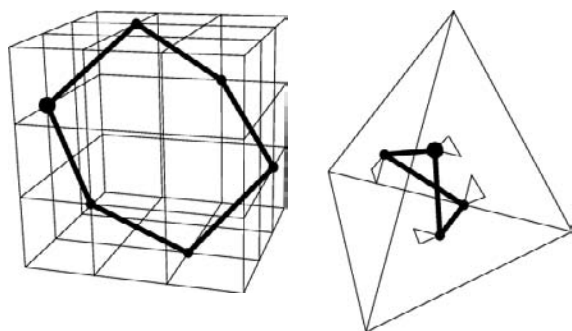
Given a rectangular billiard table with only corner pockets and sides of INTEGER lengths m and n (with m and n RELATIVELY PRIME), a ball sent at a 45° angle from a corner will be pocketed in another corner after $m + n - 2$ bounces (Steinhaus 1983, p. 63; Gardner 1984, pp. 211–14). Steinhaus (1983, p. 64) also gives a method for determining how to hit a billiard ball such that it caroms off all four sides before hitting a second ball (Knaster and Steinhaus 1946, Steinhaus 1948).

ALHAZEN'S BILLIARD PROBLEM seeks to find the point at the edge of a circular "billiards" table at which a cue ball at a given point must be aimed in order to carom once off the edge of the table and strike another ball at a second given point. It was not until 1997 that Neumann proved that the problem is insoluble using a COMPASS and RULER construction.



On an ELLIPTICAL billiard table, the ENVELOPE of a trajectory is a smaller ELLIPSE, a HYPERBOLA, a LINE through the FOCI of the ELLIPSE, or a closed polygon (Steinhaus 1983, pp. 239 and 241; Wagon 1991). The closed polygon case is related to PONCELET'S PORISM.

The only closed billiard path of a single circuit in an ACUTE TRIANGLE is the PEDAL TRIANGLE. There are an infinite number of multiple-circuit paths, but all segments are parallel to the sides of the PEDAL TRIANGLE. There exists a closed billiard path inside a CYCLIC QUADRILATERAL if its CIRCUMCENTER lies inside the quadrilateral (Wells 1991).



There are four identical closed billiard paths inside and touching each face of a CUBE such that each leg on the path has the same length (Hayward 1962; Steinhaus 1979; Steinhaus 1983; Gardner 1984, pp. 33–5; Wells 1991). This path is in the form of a chair-shaped hexagon, and each leg has length $\sqrt{3}/3$. For a unit cube, one such path has vertices $(0, 2/3, 2/3)$, $(1/3, 1, 1/3)$, $(2/3, 2/3, 0)$, $(1, 1/3, 1/3)$, $(2/3, 0, 2/3)$, $(1/3, 1/3, 1)$. Lewis Carroll (Charles Dodgson) also considered this problem (Weaver 1954).

There are three identical closed billiard paths inside and touching each face of a TETRAHEDRON such that each leg of the path has the same length (Gardner 1984, pp. 35–6; Wells 1991). These were discovered by J. H. Conway and independently by Hayward (1962). The vertices of the path are appropriately chosen vertices of equilateral triangles in each facial

plane which are scaled by a factor of $1/10$. For a tetrahedron with unit side lengths, each leg has length $\sqrt{10}/10$. For a tetrahedron with vertices $(0, 0, 0)$, $(0, \sqrt{2}/2, \sqrt{2}/2)$, $(\sqrt{2}/2, 0, \sqrt{2}/2)$, $(\sqrt{2}/2, \sqrt{2}/2, 0)$, the vertices of one such path are $(3\sqrt{2}/20, 7\sqrt{2}/20, \sqrt{2}/5)$, $(3\sqrt{2}/20, 3\sqrt{2}/20, 3\sqrt{2}/10)$, $(7\sqrt{2}/20, 3\sqrt{2}/20, \sqrt{2}/5)$, $(7\sqrt{2}/20, 7\sqrt{2}/20, 3\sqrt{2}/10)$.

Conway has shown that period orbits exist in all TETRAHEDRA, but it is not known if there are periodic orbits in every POLYHEDRON (Croft *et al.* 1991, p. 16).

See also ALHAZEN'S BILLIARD PROBLEM, BILLIARD TABLE PROBLEM, PONCELET'S PORISM, REFLECTION PROPERTY, SALMON'S THEOREM

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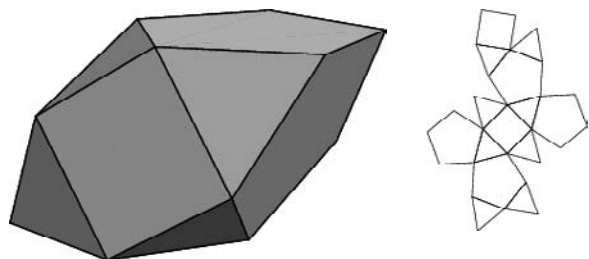
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Billion

The word billion denotes different numbers in American and British usage. In the American system, one billion equals 10^9 . In the British, French, and German systems, one billion equals 10^{12} . Fortunately, in recent years, the "American" system has become common in both the United States and Britain.

See also LARGE NUMBER, MILLIARD, MILLION, TRILLION

Bilunabirotunda



JOHNSON SOLID J_{91} .

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Bimagic Cube

A bimagic cube of order 25 is known.

See also MAGIC CUBE

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Bimagic Square

16	41	36	5	27	62	55	18
26	63	54	19	13	44	33	8
1	40	45	12	22	51	58	31
23	50	59	30	4	37	48	9
38	3	10	47	49	24	29	60
52	21	32	57	39	2	11	46
43	14	7	34	64	25	20	53
61	28	17	56	42	15	6	35

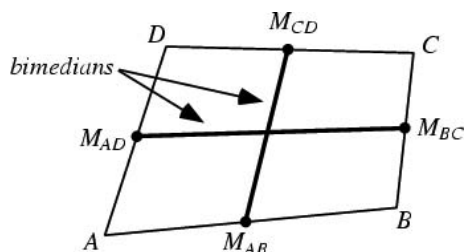
If replacing each number by its square in a MAGIC SQUARE produces another MAGIC SQUARE, the square is said to be a bimagic square. Bimagic squares are also called DOUBLY MAGIC SQUARES, and are 2-MULTI-MAGIC SQUARES.

The first known bimagic square (shown above) has order 8 with magic constant 260 for addition and 11,180 after squaring. It is believed that no bimagic squares of order less than 8 exists (Benson and Jacoby 1976), and Hendricks (1998) shows that a bimagic square of order 3 is impossible for *any* set of numbers except the trivial case of using the same number 9 times.

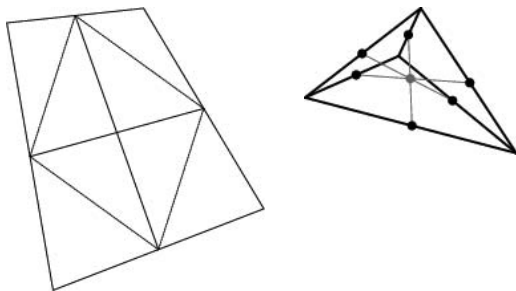
See also MAGIC SQUARE, MULTIMAGIC SQUARE, TRIMAGIC SQUARE

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Bimedian

A LINE SEGMENT joining the MIDPOINTS of opposite sides of a QUADRILATERAL or TETRAHEDRON.



VARIGNON'S THEOREM states that the bimedians of a QUADRILATERAL bisect each other (left figure). In addition, the three bimedians of a tetrahedron are CONCURRENT and bisect each other (right figure; Altshiller-Court 1979, p. 48).

See also COMMANDINO'S THEOREM, MEDIAN (TRIANGLE), VARIGNON'S THEOREM

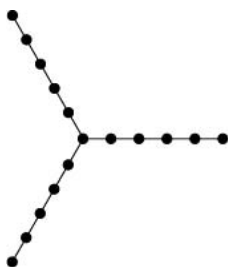
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Bimodal Distribution

A STATISTICAL DISTRIBUTION having two separated peaks.

See also UNIMODAL DISTRIBUTION

Bimonster

The wreathed product of the MONSTER GROUP by $\mathbb{Z}_2$.

The bimonster is a quotient of the COXETER GROUP with the above COXETER-DYCKIN DIAGRAM. This had been conjectured by Conway, but was proven around 1990 by Ivanov and Norton. If the parameters p , q , r in Coxeter's NOTATION $[3^p, q, r]$ are written side by side, the bimonster can be denoted by the BEAST NUMBER 666.

Bin

An interval into which a given data point does or does not fall.

See also BIN-PACKING PROBLEM, HISTOGRAM

Binary

The BASE 2 method of counting in which only the digits 0 and 1 are used. In this BASE, the number 1011 equals $1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 = 11$. This BASE is used in computers, since all numbers can be simply REPRESENTED AS a string of electrically pulsed ons and offs. The following table gives the binary equivalents of the first few decimal numbers.

1	1	11	1011	21	10101
2	10	12	1100	22	10110
3	11	13	1101	23	10111
4	100	14	1110	24	11000
5	101	15	1111	25	11001
6	110	16	10000	26	11010
7	111	17	10001	27	11011
8	1000	18	10010	28	11100
9	1001	19	10011	29	11101
10	1010	20	10100	30	11110

A NEGATIVE $-n$ is most commonly REPRESENTED AS the complement of the POSITIVE number $n - 1$, so $-11 = 00001011_2$ would be written as the complement of $10 = 00001010_2$, or 11110101 . This allows addition to be carried out with the usual carrying and the left-most digit discarded, so $17 - 1 = 6$ gives

00010001 17

11110101 -11

00000110 6

The number of times k a given binary number $b_n \dots b_2 b_1 b_0$ is divisible by 2 is given by the position of the first $b_k = 1$ counting from the right. For example, $12 = 1100$ is divisible by 2 twice, and $13 = 1101$ is divisible by 2 0 times.

The number of 1s $N(1, n)$ in the binary representation of a number is given by

$$N(1, n) = n - \text{gde}(n!, 2) = n - \sum_{k=1}^{\lfloor \log_2 n \rfloor} \left\lfloor \frac{n}{2^k} \right\rfloor, \quad (1)$$

where $\text{gde}(n!, 2)$ is the GREATEST DIVIDING EXPONENT of 2 with respect to $n!$. This is a special application of the general result that the POWER of a PRIME p dividing a FACTORIAL (Graham *et al.* 1990, Vardi 1991). Writing $a(n)$ for $N(1, n)$, the number of 1s is also given by the RECURRENCE RELATION

$$a(2n) = a(n) \quad (2)$$

$$a(2n+1) = a(n) + 1, \quad (3)$$

with $a(0) = 0$, and by

$$N(1, n) = 2n - \log_2(d), \quad (4)$$

where d is the DENOMINATOR of

$$\frac{1}{n!} \left[\frac{d^n}{dx^n} (1-x)^{-1/2} \right]_{x=0}. \quad (5)$$

For $n = 1, 2, \dots$, the first few values are 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, ... (Sloane's A000120; Smith 1966, Graham 1970, McIlroy 1974).

Unfortunately, the storage of binary numbers in computers is not entirely standardized. Because computers store information in 8-bit bytes (where a bit is a single binary digit), depending on the "word size" of the machine, numbers requiring more than 8 bits must be stored in multiple bytes. The usual FORTRAN77 integer size is 4 bytes long. However, a number REPRESENTED AS (byte1 byte2 byte3 byte4) in a VAX would be read and interpreted as (byte4 byte3 byte2 byte1) on a Sun. The situation is even worse for floating point (real) numbers, which are represented in binary as a MANTISSA and CHARACTERISTIC, and worse still for long (8-byte) reals!

Binary multiplication of single bit numbers (0 or 1) is equivalent to the AND operation, as can be seen in the following MULTIPLICATION TABLE.

×	0	1
0	0	0
1	0	1

See also BASE (NUMBER), BINARY CARRY SEQUENCE, DECIMAL, FACTORIAL, HEXADECIMAL, MOSER-DE BRUIJN SEQUENCE, NEGABINARY, OCTAL, QUATERNARY, RUDIN-SHAPIRO SEQUENCE, STOLARSKY-HARBORTH CONSTANT, TERNARY

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Binary Bracketing

A binary bracketing is a BRACKETING built up entirely of binary operations. The number of binary bracketings of n letters (CATALAN'S PROBLEM) are given by the CATALAN NUMBERS C_{n-1} , where

$$C_n \equiv \frac{1}{n+1} \binom{2n}{n} = \frac{1}{n+1} \frac{(2n)!}{n!^2} = \frac{(2n)!}{(n+1)!n!},$$

where $\binom{2n}{n}$ denotes a BINOMIAL COEFFICIENT and $n!$ is the usual FACTORIAL, as first shown by Catalan in 1838. For example, for the four letters a, b, c , and d there are five possibilities: $((ab)c)d$, $(a(bc))d$, $(ab)(cd)$, $a((bc)d)$, and $a(b(cd))$, written in shorthand as $((xx)x)x$, $(x(xx))x$, $(xx)(xx)$, $x((xx)x)$, and $x(x(xx))$.

See also BRACKETING, CATALAN NUMBER, CATALAN'S PROBLEM

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Binary Carry Sequence

The sequence $a(n)$ given by the exponents of the highest power of 2 dividing n , i.e., the number of trailing 0s in the BINARY representation of n . For $n = 1, 2, \dots$, the first few are 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, ... (Sloane's A007814). Amazingly, this corresponds to one less than the number of disk to be moved at n th step of optimal solution to TOWERS OF HANOI problem, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, ... (Sloane's A001511).

The anti-PARITY of this sequence is given by 1, 0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 1, ... (Sloane's A035263) which, amazingly, also corresponds to the ACCUMULATION POINT of 2^n cycles through successive bifurcations.

See also DOUBLE-FREE SET, TOWERS OF HANOI

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Binary Goldbach Conjecture

GOLDBACH CONJECTURE

Binary Heap

HEAP

Binary Matrix

(0,1)-MATRIX

Binary Operation

This entry contributed by J. BRAD WEATHERLY

A binary operation on a nonempty set A is a map $f : A \times A \rightarrow A$, such that f is defined for every element in A and the image of f is unique. Examples of binary operations on A from $A \times A$ to A include $+$ and $-$.

See also BINARY OPERATOR

Binary Operator

An OPERATOR defined on a set S which takes two elements from S as inputs and returns a single element of S . Binary operators are called compositions by Rosenfeld (1968). Sets possessing a binary multiplication operation include the GROUP, GROUPOID, MONOID, QUASIGROUP, and SEMIGROUP. Sets possessing both a binary multiplication and a binary addition operation include the DIVISION ALGEBRA, FIELD, RING, RINGOID, SEMIRING, and UNIT RING.

See also AND, BINARY OPERATION, BOOLEAN ALGEBRA, CLOSURE (SET), CONNECTIVE, DIVISION ALGEBRA, FIELD, GROUP, GROUPOID, MONOID, OPERATOR, OR, MONOID, NOT, QUASIGROUP, RING, RINGOID, SEMIGROUP, SEMIRING, XNOR, XOR, UNIT RING

References

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Binary Quadratic Form

A QUADRATIC FORM in two variables having the form

$$Q(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2. \quad (1)$$

Consider a binary quadratic form with real coefficients a_{11} , a_{12} , and a_{22} , determinant

$$D \equiv a_{11}a_{22} - a_{12}^2 = 1, \quad (2)$$

and $a_{11} > 0$. Then $Q(x, y)$ is POSITIVE DEFINITE. An important result states that exist two integers x and y not both 0 such that

$$Q(x, y) \leq \frac{2}{\sqrt{3}} \quad (3)$$

for all values of a_{ij} satisfying the above constraint (Hilbert and Cohn-Vossen 1999, p. 39).

See also PELL EQUATION, POSITIVE DEFINITE QUADRATIC FORM, QUADRATIC FORM, QUADRATIC INVARIANT

References

- Hilbert, D. and Cohn-Vossen, S. "The Minimum Value of Quadratic Forms." §6.2 in *Geometry and the Imagination*. New York: Chelsea, pp. 39–1, 1999.

Binary Relation

Given a set of objects S , a binary relation is a subset of the CARTESIAN PRODUCT $S \otimes S$.

See also RELATION

References

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Binary Remainder Method

AN ALGORITHM for computing a UNIT FRACTION (Stewart 1992).

References

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Binary Search

A SEARCHING algorithm which works on a sorted table by testing the middle of an interval, eliminating the half of the table in which the key cannot lie, and then repeating the procedure iteratively.

See also SEARCHING

References

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Binary Splitting**References**

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Binary Tree

A TREE with two BRANCHES at each FORK and with one or two LEAVES at the end of each BRANCH. (This definition corresponds to what is sometimes known as an "extended" binary tree.) The height of a binary tree is the number of levels within the TREE. For a binary tree of height H with n nodes,

$$H \leq n \leq 2^H - 1.$$

These extremes correspond to a balanced tree (each node except the LEAVES has a left and right CHILD, and all LEAVES are at the same level) and a degenerate tree (each node has only one outgoing BRANCH), respectively. For a search of data organized into a binary tree, the number of search steps $S(n)$ needed

to find an item is bounded by

$$\lg n \leq S(n) \leq n.$$

Partial balancing of an arbitrary tree into a so-called AVL binary search tree can improve search speed.

The number of binary trees with n internal nodes is the CATALAN NUMBER C_n (Sloane's A000108), and the number of binary trees of height b is given by Sloane's A001699. The numbers of binary trees on $n = 1, 2, \dots$ nodes (i.e., n -node trees having VERTEX DEGREE either 1 or 3; also called 3-Cayley trees, 3-valent trees, or boron trees) are 1, 1, 0, 1, 0, 1, 0, 1, 0, 2, 0, 2, 0, 4, 0, 6, 0, 11, ... (Sloane's A052120).

See also B-TREE, CAYLEY TREE, COMPLETE BINARY TREE, EXTENDED BINARY TREE, HEAP, QUADTREE, QUATERNARY TREE, RAMUS TREE, RED-BLACK TREE, SPLAY TREE, STERN-BROCOT TREE, WEAKLY BINARY TREE

References

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Binet Forms

The two RECURRENCE SEQUENCES

$$U_n = mU_{n-1} + U_{n-2} \quad (1)$$

$$V_n = mV_{n-1} + V_{n-2} \quad (2)$$

with $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = m$, can be solved for the individual U_n and V_n . They are given by

$$U_n = \frac{\alpha^n - \beta^n}{\Delta} \quad (3)$$

$$V_n = \alpha^n + \beta^n, \quad (4)$$

where

$$\Delta \equiv \sqrt{m^2 + 4} \quad (5)$$

$$\alpha \equiv \frac{m + \Delta}{2} \quad (6)$$

$$\beta \equiv \frac{m - \Delta}{2}. \quad (7)$$

A useful related identity is

$$U_{n-1} + U_{n+1} = V_n. \quad (8)$$

BINET'S FIBONACCI NUMBER FORMULA is a special case of the Binet form for U_n corresponding to $m = 1$.

See also BINET'S FIBONACCI NUMBER FORMULA, FIBONACCI Q-MATRIX

Binet's Fibonacci Number Formula

A special case of the U_n BINET FORM with $m = 1$, corresponding to the n th FIBONACCI NUMBER,

$$F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}.$$

It was derived by Binet in 1843, although the result was known to Euler and to Daniel Bernoulli more than a century earlier.

See also BINET FORMS, FIBONACCI NUMBER

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Binet's Log Gamma Formulas

Binet's first formula for $\ln \Gamma(z)$, where $\Gamma(z)$ is a GAMMA FUNCTION, is given by

$$\ln \Gamma(z) = (z - \tfrac{1}{2}) \ln z - z + \tfrac{1}{2} \ln(2\pi) + \int_0^\infty [(e^t - 1)^{-1} - t^{-1} + \tfrac{1}{2}] t^{-1} e^{-tz} dt$$

for $\Re[z] > 0$ (Erdélyi *et al.* 1981, p. 21). Binet's second formula is

$$\ln \Gamma(z) = (z - \tfrac{1}{2}) \ln z - z + \tfrac{1}{2} \ln(2\pi) + 2 \int_0^\infty \frac{\tan\left(\frac{t}{2}\right)}{e^{2\pi t} - 1} dt$$

for $\Re[z] > 0$ (Erdélyi *et al.* 1981, p. 22; Whittaker and Watson 1990, p. 251).

See also GAMMA FUNCTION, MALMSTÉN'S FORMULA

References

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Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, 1990.

Binet-Cauchy Identity

The algebraic identity

$$\left(\sum_{i=1}^n a_i c_i \right) \left(\sum_{i=1}^n b_i d_i \right) - \left(\sum_{i=1}^n a_i d_i \right) \left(\sum_{i=1}^n b_i c_i \right)$$

$$= \sum_{1 \leq i \leq j \leq n} (a_i b_j - a_j b_i)(c_i d_j - c_j d_i). \quad (1)$$

Letting $c_i = a_i$ and $d_i = b_i$ gives LAGRANGE'S IDENTITY.

The identity can be coded in *Mathematica* as follows.

```
<< DiscreteMath`Combinatorica`;  
BinetCauchyId[n_] := Module[{  
  aa = Array[a, n], bb = Array[b, n],  
  cc = Array[c, n], dd = Array[d, n]  
},  
  aa.cc bb.dd - aa.dd bb.cc ==  
    Plus @@ ((a[#1] b[#2] -  
  a[#2] b[#1]) (c[#1] d[#2] - c[#2] d[#1]) & @@@  
  KSubsets[Range[n], 2])  
]
```

The $n = 2$ case then gives

$$(a_1 c_1 + a_2 c_2)(b_1 d_1 + b_2 d_2) - (b_1 c_1 + b_2 c_2)(a_1 d_1 + a_2 d_2) \\ = (a_1 b_2 - a_2 b_1)(c_1 d_2 - c_2 d_1). \quad (2)$$

The $n = 3$ case is equivalent to the vector identity

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}), \quad (3)$$

where $\mathbf{A} \cdot \mathbf{B}$ is the DOT PRODUCT and $\mathbf{A} \times \mathbf{B}$ is the CROSS PRODUCT. Note that this identity itself is sometimes known as LAGRANGE'S IDENTITY.

See also LAGRANGE'S IDENTITY

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Bing's Theorem

If M^3 is a closed oriented connected 3-MANIFOLD such that every simple closed curve in M lies interior to a BALL in M , then M is HOMEOMORPHIC with the HYPERSPHERE, $\mathbb{S}^3$.

See also BALL, HYPERSPHERE

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Binomial

A POLYNOMIAL with 2 terms.

See also BINOMIAL COEFFICIENT, MONOMIAL, POLYNOMIAL, TRINOMIAL

Binomial Coefficient

The number of ways of picking n unordered outcomes from N possibilities, also known as a COMBINATION or combinatorial number. The symbols ${}_N C_n$ and $\binom{N}{n}$ are used to denote a binomial coefficient, and are sometimes read as " N CHOOSE n ." The value of the binomial coefficient is given by

$${}_NC_n \equiv \binom{N}{n} \equiv \frac{N!}{(N-n)!n!}, \quad (1)$$

where $n!$ denotes a FACTORIAL. Writing the FACTORIAL as a GAMMA FUNCTION $n! = \Gamma(n+1)$ allows the binomial coefficient to be generalized to non-integral arguments.

The binomial coefficients form the rows of PASCAL'S TRIANGLE, and the number of LATTICE PATHS from the ORIGIN $(0, 0)$ to a point (a, b) is the binomial coefficient $\binom{a+b}{a}$ (Hilton and Pedersen 1991).

For a POSITIVE INTEGER n , the BINOMIAL THEOREM gives

$$(x+a)^n = \sum_{k=0}^n \binom{n}{k} x^k a^{n-k}. \quad (2)$$

The FINITE DIFFERENCE analog of this identity is known as the CHU-VANDERMONDE IDENTITY. A similar formula holds for NEGATIVE INTEGERS,

$$(x+a)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k a^{-n-k}. \quad (3)$$

There are a number of elegant BINOMIAL SUMS.

The binomial coefficients satisfy the identities

$$\binom{n}{0} = \binom{n}{n} = 1 \quad (4)$$

$$\binom{n}{k} = \binom{n}{n-k} = (-1)^k \binom{k-n-1}{k} \quad (5)$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}. \quad (6)$$

As shown by Kummer in 1852, if p^k is the largest power of a PRIME p that divides $\binom{n+k}{k}$, where n and k are nonnegative integers, then k is the number of carries that occur when k is added to n in base p (Graham *et al.* 1989, Exercise 5.36, p. 245; Ribenboim 1989; Vardi 1991, p. 68). Kummer's result can also be stated in the form that the exponent of a PRIME p dividing $\binom{n}{m}$ is given by the number of integers $j \geq 0$ for which

$$\text{frac}(m/p^j) > \text{frac}(n/p^j), \quad (7)$$

where $\text{frac}(x)$ denotes the FRACTIONAL PART of x . This inequality may be reduced to the study of the exponential sums $\sum_n \Lambda(n)e(x/n)$, where $\Lambda(n)$ is the MANGOLDT FUNCTION. Estimates of these sums are given by Jutila (1974, 1975), but recent improvements have been made by Granville and Ramare (1996).

R. W. Gosper showed that

$$f(n) = \binom{n-1}{\frac{1}{2}(n-1)} \equiv (-1)^{(n-1)/2} \pmod{n} \quad (8)$$

for all PRIMES, and conjectured that it holds *only* for

PRIMES. This was disproved when Skiena (1990) found it also holds for the COMPOSITE NUMBER $n = 3 \times 11 \times 179$. Vardi (1991, p. 63) subsequently showed that $n = p^2$ is a solution whenever p is a WIEFERICH PRIME and that if $n = p^k$ with $k > 3$ is a solution, then so is $n = p^{k-1}$. This allowed him to show that the only solutions for COMPOSITE $n < 1.3 \times 10^7$ are 5907, 1093², and 3511², where 1093 and 3511 are WIEFERICH PRIMES.

Consider the binomial coefficients $f(n) = \binom{2n-1}{n}$, the first few of which are 1, 3, 10, 35, 126, ... (Sloane's A001700). The GENERATING FUNCTION is

$$\frac{1}{2} \left[\frac{1}{\sqrt{1-4x}} - 1 \right] = x + 3x^2 + 10x^3 + 35x^4 + \dots \quad (9)$$

These numbers are SQUAREFREE only for $n = 2, 3, 4, 6, 9, 10, 12, 36, \dots$ (Sloane's A046097), with no others known. It turns out that $f(n)$ is divisible by 4 unless n belongs to a 2-AUTOMATIC SET S_2 , which happens to be the set of numbers whose BINARY representations contain at most two 1s: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 17, 18, ... (Sloane's A048645). Similarly, $f(n)$ is divisible by 9 unless n belongs to a 3-AUTOMATIC SET S_3 , consisting of numbers n for which the representation of $2n$ in TERNARY consists entirely of 0s and 2s (except possibly for a pair of adjacent 1s; D. Wilson, A. Karttunen). The initial elements of S_3 are 1, 2, 3, 4, 6, 7, 9, 10, 11, 12, 13, 18, 19, 21, 22, 27, ... (Sloane's A051382). If $f(n)$ is squarefree, then n must belong to $S = S_2 \cap S_3$. It is very probable that S is finite, but no proof is known. Now, squares larger than 4 and 9 might also divide $f(n)$, but by eliminating these two alone, the only possible n for $n \leq 2^64$ are 1, 2, 3, 4, 6, 9, 10, 12, 18, 33, 34, 36, 40, 64, 66, 192, 256, 264, 272, 513, 514, 516, 576, 768, 1026, 1056, 2304, 16392, 65664, 81920, 532480, and 545259520. All of these but the last have been checked (D. Wilson), establishing that there are no other n such that $f(n)$ is squarefree for $n \leq 545, 259, 520$.

Erdos showed that the binomial coefficient $\binom{n}{k}$, with $3 \leq k \leq n/2$ is a POWER of an INTEGER for the single case $\binom{50}{3} = 140^2$ (Le Lionnais 1983, p. 48). Binomial coefficients $T_{n-1} = \binom{n}{2}$ are squares a^2 when a^2 is a TRIANGULAR NUMBER, which occur for $a = 1, 6, 35, 204, 1189, 6930, \dots$ (Sloane's A001109). These values of a have the corresponding values $n = 2, 9, 50, 289, 1682, 9801, \dots$ (Sloane's A052436).

The binomial coefficients $\binom{n}{\lfloor n/2 \rfloor}$ are called CENTRAL BINOMIAL COEFFICIENTS, where $\lfloor x \rfloor$ is the FLOOR FUNCTION, although the subset of coefficients $\binom{2n}{n}$ is sometimes also given this name. Erdos and Graham (1980, p. 71) conjectured that the CENTRAL BINOMIAL COEFFICIENT $\binom{2n}{n}$ is *never* SQUAREFREE for $n > 4$, and this is sometimes known as the ERDOS SQUAREFREE CONJECTURE. SÁRKOZY'S THEOREM (Sárközy 1985) provides a partial solution which states that the BINOMIAL COEFFICIENT $\binom{2n}{n}$ is never SQUAREFREE for

all sufficiently large $n \geq n_0$ (Vardi 1991). Granville and Ramare (1996) proved that the *only* SQUAREFREE values are $n = 2$ and 4. Sander (1992) subsequently showed that $\binom{2n+d}{n}$ are also never SQUAREFREE for sufficiently large n as long as d is not "too big."

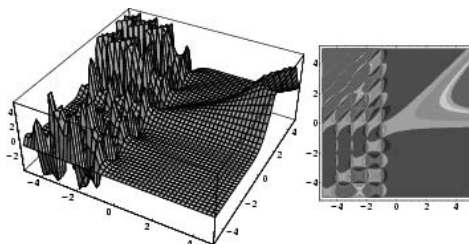
For p, q , and r distinct PRIMES, then the function (8) satisfies

$$f(pqr)f(p)f(q)f(r) \equiv f(pq)f(pr)f(qr) \pmod{pqr} \quad (10)$$

(Vardi 1991, p. 66).

Most binomial coefficients $\binom{n}{k}$ with $n \geq 2k$ have a prime factor $p \leq n/k$, and Lacampagne *et al.* (1993) conjecture that this inequality is true for all $n > 17.125k$, or more strongly that any such binomial coefficient has LEAST PRIME FACTOR $p \leq n/k$ or $p \leq 17$ with the exceptions $\binom{62}{6}$, $\binom{959}{56}$, $\binom{474}{66}$, $\binom{284}{28}$ for which $p = 19, 19, 23, 29$ (Guy 1994, p. 84).

The binomial coefficient $\binom{m}{n} \pmod{2}$ can be computed using the XOR operation $n \text{ XOR } m$, making PASCAL'S TRIANGLE mod 2 very easy to construct.



The binomial coefficient "function" can be defined as

$$C(x, y) \equiv \frac{x!}{y!(x-y)} \quad (11)$$

(Fowler 1996), shown above. It has a very complicated GRAPH for NEGATIVE x and y which is difficult to render using standard plotting programs.

See also APÉRY NUMBER, BALANCED BINOMIAL COEFFICIENT, BALLOT PROBLEM, BINOMIAL DISTRIBUTION, BINOMIAL IDENTITY, BINOMIAL SUMS, BINOMIAL THEOREM, CENTRAL BINOMIAL COEFFICIENT, CHOOSE, CHU-VANDERMONDE IDENTITY, COMBINATION, DEFICIENCY, ERDOS SQUAREFREE CONJECTURE, EXCEPTIONAL BINOMIAL COEFFICIENT, FACTORIAL, GAMMA FUNCTION, GAUSSIAN COEFFICIENT, GAUSSIAN POLYNOMIAL, GOOD BINOMIAL COEFFICIENT, KINGS PROBLEM, KLEE'S IDENTITY, LAH NUMBER, MULTICHOOSE, MULTINOMIAL COEFFICIENT, PERMUTATION, ROMAN COEFFICIENT, SÁRKOZY'S THEOREM, STANLEY'S IDENTITY, STAR OF DAVID THEOREM, STOLARSKY-HARBORTH CONSTANT, STREHL IDENTITIES, SZÉKELY IDENTITY, WOLSTENHOLME'S THEOREM

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Binomial Differential Equation

The ORDINARY DIFFERENTIAL EQUATION

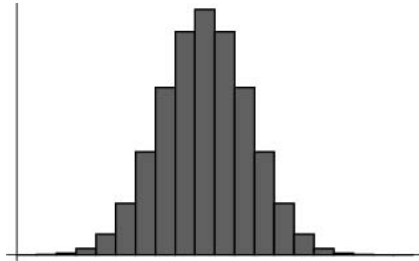
$$(y')^m = f(x, y)$$

(Hille 1969, p. 675; Zwillinger 1997, p. 120).

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Binomial Distribution



The binomial distribution gives the probability distribution $P_p(n|N)$ of obtaining exactly n successes out of N BERNOULLI TRIALS (where the result of each BERNOULLI TRIAL is true with probability p and false with probability $q = 1 - p$). The binomial distribution is therefore given by

$$P_p(n|N) = \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} p^n q^{N-n}. \quad (1)$$

The above plot shows the distribution of n successes out of $N = 20$ trials with $p = q = 1/2$. Steinhaus (1983, pp. 25–8) considers the expected number of squares

$S(n, N, s)$ containing a given number of grains n on board of size s after random distribution of N of grains,

$$S(n, N, s) = s P_{1/s}(n|N). \quad (2)$$

Taking $N = s = 64$ gives the results summarized in the following table.

S	n
0	23.3591
1	23.7299
2	11.8650
3	3.89221
4	0.942162
5	0.179459
6	0.0280109
7	0.0036840
8	4.16639×10^{-4}
9	4.11495×10^{-5}
10	3.59242×10^{-6}

The probability of obtaining *more* successes than the n observed in a binomial distribution is

$$P = \sum_{k=n+1}^N \binom{N}{k} p^k (1-p)^{N-k} = I_p(n+1, N-n), \quad (3)$$

where

$$I_x(a, b) \equiv \frac{B(x; a, b)}{B(a, b)}, \quad (4)$$

$B(a, b)$ is the BETA FUNCTION, and $B(x; a, b)$ is the incomplete BETA FUNCTION.

The CHARACTERISTIC FUNCTION for the binomial distribution is

$$\phi(t) = (q + pe^{it})^N \quad (5)$$

(Papoulis 1984, p. 154). The MOMENT-GENERATING FUNCTION M for the distribution is

$$\begin{aligned} M(t) &= \langle e^{tn} \rangle = \sum_{n=0}^N e^{tn} \binom{N}{n} p^n q^{N-n} \\ &= \sum_{n=0}^N \binom{N}{n} (pe^t)(1-p)^{N-n} = [pe^t + (1-p)]^N \end{aligned} \quad (6)$$

$$M'(t) = N[pe^t + (1-p)]^{N-1}(pe^t) \quad (7)$$

$$M''(t) = N(N-1)[pe^t + (1-p)]^{N-2}(pe^t)^2$$

$$+N[pe^t + (1-p)]^{N-1}(pe^t). \quad (8)$$

The MEAN is

$$\mu = M'(0) = N(p+1-p)p = Np. \quad (9)$$

The MOMENTS about 0 are

$$\mu'_1 = \mu = Np \quad (10)$$

$$\mu'_2 = Np(1-p+Np) \quad (11)$$

$$\mu'_3 = Np(1-3p+3Np+2p^2-3NP^2+N^2p^2) \quad (12)$$

$$\begin{aligned} \mu'_4 = Np(1-7p+7Np+12p^2-18Np^2+6N^2p^2-6p^3 \\ + 11Np^3-6N^2p^3+N^3p^3), \end{aligned} \quad (13)$$

so the MOMENTS about the MEAN are

$$\begin{aligned} \mu_2 = \sigma^2 = [N(N-1)p^2 + Np] - (Np)^2 \\ = N^2p^2 - Np^2 + Np - N^2p^2 = Np(1-p) = Npq \end{aligned} \quad (14)$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu_1)^3 = Np(1-p)(1-2p) \quad (15)$$

$$\begin{aligned} \mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu_1)^4 \\ = Np(1-p)[3p^2(2-N) + 3p(N-2) + 1]. \end{aligned} \quad (16)$$

The SKEWNESS and KURTOSIS are

$$\begin{aligned} \gamma_1 = \frac{\mu_3}{\sigma^3} = \frac{Np(1-p)(1-2p)}{[Np(1-p)]^{3/2}} = \frac{1-2p}{\sqrt{Np(1-p)}} \\ = \frac{q-p}{\sqrt{Npq}} \end{aligned} \quad (17)$$

$$\gamma_2 = \frac{\mu_4}{\sigma^4} - 3 = \frac{6p^2 - 6p + 1}{Np(1-p)} = \frac{1-6pq}{Npq}. \quad (18)$$

An approximation to the Bernoulli distribution for large N can be obtained by expanding about the value $\tilde{n}$ where $P(n)$ is a maximum, i.e., where $dP/dn = 0$. Since the LOGARITHM function is MONOTONIC, we can instead choose to expand the LOGARITHM. Let $n \equiv \tilde{n} + \eta$, then

$$\ln[P(n)] = \ln[P(\tilde{n})] + B_1\eta + \frac{1}{2}B_2\eta^2 + \frac{1}{3!}B_3\eta^3 + \dots, \quad (19)$$

where

$$B_k = \left[\frac{d^k \ln[P(n)]}{dn^k} \right]_{n=\tilde{n}}. \quad (20)$$

But we are expanding about the maximum, so, by definition,

$$B_1 = \left[\frac{d \ln[P(n)]}{dn} \right]_{n=\tilde{n}} = 0. \quad (21)$$

This also means that B_2 is negative, so we can write $B_2 = -|B_2|$. Now, taking the LOGARITHM of (1) gives

$$\begin{aligned} \ln[P(n)] = \ln N! - \ln n! - \ln(N-n)! + n \ln p \\ + (N-n) \ln q. \end{aligned} \quad (22)$$

For large n and $N-n$ we can use STIRLING'S APPROXIMATION

$$\ln(n!) \approx n \ln n - n, \quad (23)$$

so

$$\frac{d[\ln(n!)]}{dn} \approx (\ln n + 1) - 1 = \ln n \quad (24)$$

$$\begin{aligned} \frac{d[\ln(N-n)!]}{dn} &\approx \frac{d}{dn} [(N-n) \ln(N-n) - (N-n)] \\ &= \left[-\ln(N-n) + (N-n) \frac{-1}{N-n} + 1 \right] \\ &= -\ln(N-n), \end{aligned} \quad (25)$$

and

$$\frac{d \ln[P(n)]}{dn} \approx -\ln n + \ln(N-n) \ln p - \ln q. \quad (26)$$

To find $\tilde{n}$, set this expression to 0 and solve for n ,

$$\ln \left(\frac{N-\tilde{n}}{\tilde{n}} \frac{p}{q} \right) = 0 \quad (27)$$

$$\frac{N-\tilde{n}}{\tilde{n}} \frac{p}{q} = 1 \quad (28)$$

$$(N-\tilde{n})p = \tilde{n}q \quad (29)$$

$$\tilde{n}(q+p) = \tilde{n} = Np, \quad (30)$$

since $p+q=1$. We can now find the terms in the expansion

$$\begin{aligned} B_2 &= \left[\frac{d^2 \ln[P(n)]}{dn^2} \right]_{n=\tilde{n}} = -\frac{1}{\tilde{n}} - \frac{1}{N-\tilde{n}} \\ &= -\frac{1}{Np} - \frac{1}{N(1-p)} = -\frac{1}{N} \left(\frac{1}{p} + \frac{1}{q} \right) = -\frac{1}{N} \left(\frac{p+q}{pq} \right) \\ &= -\frac{1}{Npq} = -\frac{1}{N(1-p)} \end{aligned} \quad (31)$$

$$\begin{aligned} B_3 &\equiv \left[\frac{d^3 \ln[P(n)]}{dn^3} \right]_{n=\tilde{n}} = -\frac{1}{\tilde{n}^2} - \frac{1}{(N-\tilde{n})^2} = \frac{1}{N^2p^2} - \frac{1}{N^2q^2} \\ &= \frac{q^2 - p^2}{N^2p^2q^2} = \frac{(1-2p+p^2) - p^2}{N^2p^2(1-p)^2} \\ &= \frac{1-2p}{N^2p^2(1-p)^2} \end{aligned} \quad (32)$$

$$\begin{aligned}
B_4 &\equiv \left[\frac{d^4 \ln[P(n)]}{dn^4} \right]_{n=\tilde{n}} = -\frac{2}{\tilde{n}^3} - \frac{2}{(n-\tilde{n})^3} \\
&= -2 \left(\frac{1}{N^3 p^3} + \frac{1}{N^3 q^3} \right) = \frac{2(p^3 + q^3)}{N^3 p^3 q^3} \\
&= \frac{2(p^2 - pq + q^2)}{N^3 p^3 q^3} \\
&= \frac{2[p^2 - p(1-p) + (1-2p+p^2)]}{N^3 p^3 (1-p^3)} \\
&= \frac{2(3p^2 - 3p + 1)}{N^3 p^3 (1-p^3)}. \tag{33}
\end{aligned}$$

Now, treating the distribution as continuous,

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N P(n) \approx \int P(n) dn = \int_{-\infty}^{\infty} P(\tilde{n} + \eta) d\eta = 1. \tag{34}$$

Since each term is of order $1/N \sim 1/\sigma^2$ smaller than the previous, we can ignore terms higher than B_2 , so

$$P(n) = P(\tilde{n}) e^{-|B_2| \eta^2 / 2}. \tag{35}$$

The probability must be normalized, so

$$\int_{-\infty}^{\infty} P(\tilde{n}) e^{-|B_2| \eta^2 / 2} d\eta = P(\tilde{n}) \sqrt{\frac{2\pi}{|B_2|}} = 1, \tag{36}$$

and

$$\begin{aligned}
P(n) &= \sqrt{\frac{|B_2|}{2\pi}} e^{-|B_2|(n-\tilde{n})^2/2} \\
&= \frac{1}{\sqrt{2\pi Npq}} \exp \left[-\frac{(n - Np)^2}{2Npq} \right]. \tag{37}
\end{aligned}$$

Defining $\sigma^2 \equiv Npq$,

$$P(n) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[-\frac{(n - \tilde{n})^2}{2\sigma^2} \right], \tag{38}$$

which is a GAUSSIAN DISTRIBUTION. For $p \ll 1$, a different approximation procedure shows that the binomial distribution approaches the POISSON DISTRIBUTION. The first CUMULANT is

$$\kappa_1 = np, \tag{39}$$

and subsequent CUMULANTS are given by the RECURRENCE RELATION

$$\kappa_{r+1} = pq \frac{d\kappa_r}{dp}. \tag{40}$$

Let x and y be independent binomial RANDOM VARIABLES characterized by parameters n , p and m , p . The CONDITIONAL PROBABILITY of x given that $x + y = k$ is

$$\begin{aligned}
P(x=i|x+y=k) &= \frac{P(x=i, x+y=k)}{P(x+y=k)} \\
&= \frac{P(x=i, y=k-i)}{P(x+y=k)} = \frac{P(x=i)P(y=k-i)}{P(x+y=k)} \\
&= \frac{\binom{n}{i} p^i (1-p)^{n-i} \binom{m}{k-i} p^{k-i} (1-p)^{m-(k-i)}}{\binom{n+m}{k} p^k (1-p)^{n+m-k}} \\
&= \frac{\binom{n}{i} \binom{m}{k-i}}{\binom{n+m}{k}}. \tag{41}
\end{aligned}$$

Note that this is a HYPERGEOMETRIC DISTRIBUTION.

See also DE MOIVRE-LAPLACE THEOREM, HYPERGEOMETRIC DISTRIBUTION, NEGATIVE BINOMIAL DISTRIBUTION

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Binomial Expansion

BINOMIAL SERIES

Binomial Formula

BINOMIAL SERIES, BINOMIAL THEOREM

Binomial Identity

Roman (1984, p. 26) defines "the" binomial identity as the equation

$$p_n(x+y) = \sum_{k=0}^n \binom{n}{k} p_k(y) p_{n-k}(x). \tag{1}$$

IFF the sequence $p_n(x)$ satisfies this identity for all y in a FIELD C of characteristic 0, then $p_n(x)$ is an ASSOCIATED SEQUENCE known as a BINOMIAL-TYPE SEQUENCE.

In general, a binomial identity is a formula expressing products of factors as a sum over terms, each including a BINOMIAL COEFFICIENT $\binom{n}{k}$. The prototypical example is the BINOMIAL THEOREM

$$(x+a)^n = \sum_{k=0}^n \binom{n}{k} x^k a^{n-k} \quad (2)$$

for $n > 0$. Abel (1826) gave a host of such identities (Riordan 1979, Roman 1984), some of which include

$$\begin{aligned} & (x+y)(x+y-an)^{n-1} \\ &= \sum_{k=0}^n \binom{n}{k} xy(x-ak)^{k-1} [y-a(n-k)]^{n-k-1}, \end{aligned} \quad (3)$$

$$\begin{aligned} & x^{-1}(x+y-na)^n \\ &= \sum_{k=0}^n \sum_{k=0}^n \binom{n}{k} (x-ak)^{k-1} [y-a(n-k)]^{n-k} \end{aligned} \quad (4)$$

(Abel 1826, Riordan 1979, p. 18; Roman 1984, pp. 30 and 73), and

$$x^{-1}(x+y)^n = \sum_{k=0}^n \binom{n}{k} (x-ak)^{k-1} (y+ak)^{n-k} \quad (5)$$

(Saslaw 1989).

See also ABEL'S BINOMIAL THEOREM, ABEL POLYNOMIAL, BINOMIAL COEFFICIENT, DILCHER'S FORMULA, Q-ABEL'S THEOREM

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Binomial Number

A number OF THE FORM $a^n \pm b^n$, where a , b , and n are INTEGERS. They can be factored algebraically

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1}) \quad (1)$$

for all n ,

Binomial Number

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \dots - ab^{n-2} + b^{n-1}) \quad (2)$$

for n not a power of 2, and

$$\begin{aligned} a^{nm} - b^{nm} &= (a^m - b^m) \\ &\times [a^{m(n-1)} + a^{m(n-2)}b^m + \dots + b^{m(n-1)}]. \end{aligned} \quad (3)$$

for all positive integers m , n . For example,

$$a^2 - b^2 = (a-b)(a+b) \quad (4)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \quad (5)$$

$$a^4 - b^4 = (a-b)(a+b)(a^2 + b^2) \quad (6)$$

$$a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4) \quad (7)$$

$$a^6 - b^6 = (a-b)(a+b)(a^2 - ab + b^2)(a^2 + ab + b^2) \quad (8)$$

$$a^7 - b^7 = (a-b)(a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6) \quad (9)$$

$$a^8 - b^8 = (a-b)(a+b)(a^2 + b^2)(a^4 + b^4) \quad (10)$$

$$a^9 - b^9 = (a-b)(a^2 + ab + b^2)(a^6 + a^3b^3 + b^6) \quad (11)$$

$$\begin{aligned} a^{10} - b^{10} &= (a-b)(a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4) \\ &\times (a^4 + a^3b + a^2b^2 + ab^3 + b^4) \end{aligned} \quad (12)$$

and

$$a^2 + b^2 = a^2 + b^2 \quad (13)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \quad (14)$$

$$a^4 + b^4 = a^4 + b^4 \quad (15)$$

$$a^5 + b^5 = (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4) \quad (16)$$

$$a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4) \quad (17)$$

$$a^7 + b^7 = (a+b)(a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6) \quad (18)$$

$$a^8 + b^8 = a^8 + b^8 \quad (19)$$

$$a^9 + b^9 = (a+b)(a^2 - ab + b^2)(a^6 - a^3b^3 + b^6) \quad (20)$$

$$a^{10} + b^{10} = (a^2 + b^2)(a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8). \quad (21)$$

In 1770, Euler proved that if $(a, b) = 1$, then every FACTOR of

$$a^{2^n} + b^{2^n} \quad (22)$$

is either 2 or OF THE FORM $2^{n+1}K + 1$. (A number OF THE FORM $2^{2^n} + 1$ is called a FERMAT NUMBER.)

If p and q are PRIMES, then

$$\frac{(a^{pq} - 1)(a - 1)}{(a^p - 1)(a^q - 1)} - 1 \quad (23)$$

is DIVISIBLE by every PRIME FACTOR of a^{p-1} not dividing a^{q-1} .

See also CUNNINGHAM NUMBER, FERMAT NUMBER, MERSENNE NUMBER, RIESEL NUMBER, SIERPINSKI NUMBER OF THE SECOND KIND

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Binomial Polynomial

FALLING FACTORIAL

Binomial Series

For $|x| < 1$,

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad (1)$$

$$= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \binom{n}{2} x^2 + \dots \quad (2)$$

$$= 1 + \frac{n!}{1!(n-1)!} x + \frac{n!}{(n-2)!2!} x^2 + \dots \quad (3)$$

$$= 1 + nx + \frac{n(n-1)}{2} x^2 + \dots \quad (4)$$

The binomial series also has the CONTINUED FRACTION representation

$$(1+x)^n = \frac{1}{1 - \frac{nx}{1 + \frac{1 \cdot (1+n)}{1 \cdot 2} x}} \cdot \quad (5)$$

$$1 + \frac{\frac{1 \cdot (1-n)}{2 \cdot 3} x}{1 + \frac{2(2+n)}{3 \cdot 4} x}$$

$$1 + \frac{\frac{2(2-n)}{4 \cdot 5} x}{1 + \frac{3(3+n)}{5 \cdot 6} x}$$

$$1 + \frac{1}{1 + \dots}$$

See also BINOMIAL IDENTITY, BINOMIAL THEOREM, MULTINOMIAL SERIES, NEGATIVE BINOMIAL SERIES

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Binomial Sums

The important BINOMIAL THEOREM states that

$$\sum_{k=0}^n \binom{n}{k} r^k = (1+r)^n. \quad (1)$$

Sums of powers of BINOMIAL COEFFICIENTS

$$a_r(n) = \sum_{k=0}^n \binom{n}{k}^r \quad (2)$$

are given by

$$a_1(n) = 2^n \quad (3)$$

$$a_2(n) = \binom{2n}{n} \quad (4)$$

$a_1(n)$ and $a_2(n)$ obey the RECURRENCE RELATION

$$a_1(n+1) - 2a_1(n) = 0 \quad (5)$$

$$(n+1)a_2(n+1) - (4n+2)a_2(n) = 0. \quad (6)$$

Franel (1894, 1895) was the first to obtain recurrences for a_3n (Riordan 1948, p. 193) and $a_4(n)$,

$$(n+1)^2 a_3(n+1) - (7n^2 + 7n + 2)a_3(n) - 8n^2 a_3(n-1) = 0 \quad (7)$$

(Barrucand 1975, Cusick 1989, Jin and Dickinson 2000)

$$(n+1)^3 a_4(n+1) - 2(2n+1)(3n^2 + 3n + 1)a_4(n) - 4n(4n+1)(4n-1)a_4(n-1) = 0. \quad (8)$$

(Jin and Dickinson 2000). Therefore, a_3n are sometimes called FRANEL NUMBERS. The sequence for a_3n cannot be expressed as a fixed number of hypergeometric terms (Petkovsek *et al.* 1996, p. 160), and therefore has no closed-form hypergeometric expression. Perlstadt (1987) found recurrences of length 4 for $r=5$ and 6, while Schmidt and Yuan (1995) showed that the give recurrences for $r=3, 4, 5$, and 6 are minimal, are the minimal lengths for $r > 6$ are at least 3. The following table summarizes the first few values of $a_r(n)$ for small r .

k	Sloane	$a_k(n)$
1	A000079	1, 2, 4, 8, 16, 32, 54, ...
2	A000984	1, 2, 6, 20, 70, 252, 924, ...
3	A000172	1, 2, 10, 56, 346, 2252, ...
4	A005260	1, 2, 18, 164, 1810, 21252, ...
5	A005260	1, 2, 34, 488, 9826, 206252, ...

The corresponding alternating series is

$$b_r \equiv \sum_{k=0}^n (-1)^k \binom{n}{k}^r = 0. \quad (9)$$

The first few values are

$$b_1(n) = 0 \quad (10)$$

$$b_2(n) = \frac{2^n \sqrt{\pi}}{\Gamma(\frac{1}{2} - \frac{1}{2}n) \Gamma(1 + \frac{1}{2}n)}, \quad (11)$$

$$= \begin{cases} 0 & \text{for } n = 2k \\ (-1)^k \binom{n}{k} & \text{for } n = 2k-1 \end{cases} \quad (12)$$

$$b_3(n) = \frac{2^n \sqrt{\pi} \Gamma(1 + \frac{3}{2}n)}{n! \Gamma(\frac{1}{2}(1-n)) \Gamma(1 + \frac{1}{2}n)^2} \quad (13)$$

$$= \begin{cases} 0 & \text{for } n = 2k-1 \\ \frac{(-1)^k (3k)!}{(k!)^3} & \text{for } n = 2k, \end{cases} \quad (14)$$

where $\Gamma(z)$ is the GAMMA FUNCTION, and the odd terms of $b_3(n)$ are given by de Bruijn's $s(3, n)$ with alternating signs.

de Bruijn (1982) has considered the sum

$$s(m, n) = \sum_{k=0}^{2n} (-1)^{k+n} \binom{2n}{k}^m \quad (15)$$

for $m, n \geq 1$. This sum has closed form for $m = 1, 2$, and 3,

$$s(1, n) = 0 \quad (16)$$

$$s(2, n) = \frac{(2n)!}{(n!)^2}, \quad (17)$$

the CENTRAL BINOMIAL COEFFICIENT, giving 1, 2, 6, 20, 70, 252, 924, ... (Sloane's A000984), and

$$s(3, n) = \frac{(3n)!}{(n!)^3}, \quad (18)$$

giving 1, 6, 90, 1680, 36450, 756756, ... (Sloane's A006480; Aizenberg and Yuzhakov 1984). However, there is no similar formula for $m \geq 4$ (Finch). The first few terms of $s(4, n)$ are 1, 14, 786, 61340, 5562130, ... (Sloane's A050983), and for $s(5, n)$ are 1, 30, 5730, 1696800, 613591650, ... (Sloane's A050984).

An interesting generalization of $b_1(n)$ was found by Ruiz (1996),

$$\sum_{k=0}^{\infty} (-1)^k \binom{n}{k} (x-k)^n = n! \quad (19)$$

and

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (x-k)^n = n! \quad (20)$$

for positive integer n and all x .

The infinite sum of inverse binomial coefficients has the analytic form

$$\sum_{k=0}^{\infty} \frac{1}{\binom{n}{k}} = {}_2F_1(1, 1; -n; -1) \quad (21)$$

$$= -(n+1) \int_0^1 \frac{dx}{(1-x)^{n+2}(x+1)}, \quad (22)$$

where ${}_2F_1(a, b; c; x)$ is a HYPERGEOMETRIC FUNCTION. In fact, in general,

$$\sum_{k=0}^{\infty} \frac{1}{\binom{n}{k}^p} = {}_{p+1}F_p(\underbrace{1, \dots, 1}_{p+1}; \underbrace{-n, \dots, -n}_p; (-1)^k) \quad (23)$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\binom{n}{k}^p} = {}_{p+1}F_p(\underbrace{1, \dots, 1}_{p+1}; \underbrace{-n, \dots, -n}_p; (-1)^{k+1}). \quad (24)$$

A fascinating series of identities involving inverse central binomial coefficients times small powers are given by

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{1}{27}(2\pi\sqrt{3}+9) = 0.7363998587\dots \quad (25)$$

$$\sum_{n=1}^{\infty} \frac{1}{n \binom{2n}{n}} = \frac{1}{9}\pi\sqrt{3} = 0.6045997881\dots \quad (26)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} = \frac{1}{3}\zeta(2) = \frac{1}{8}\pi^2 \quad (27)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}} = \frac{17}{36}\zeta(4) = \frac{17}{3240}\pi^4 \quad (28)$$

(Comtet 1974, p. 89; Le Lionnais 1983, pp. 29, 30, 41, 36), which follow from the beautiful formula

$$\sum_{n=1}^{\infty} \frac{1}{n^k \binom{2n}{n}} = \frac{1}{2} {}_{k+1}F_k(\underbrace{1, \dots, 1}_{k+1}; \underbrace{\frac{3}{2}, 2, \dots, 2}_{k-1}; \frac{1}{4}) \quad (29)$$

for $k \geq 1$, where $mF_n(a_1, \dots, a_m; b_1, \dots, b_n; x)$ is a GENERALIZED HYPERGEOMETRIC FUNCTION. Additional sums of this type include

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}} = \frac{1}{18} \pi \sqrt{3} [\psi_1(\frac{1}{3}) - \psi_1(\frac{2}{3})] - \frac{4}{3} \zeta(3) \quad (30)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^5 \binom{2n}{n}} = \frac{1}{432} \pi \sqrt{3} [\psi_3(\frac{1}{3}) - \psi_3(\frac{2}{3})] - \frac{19}{3} \zeta(5) + \frac{1}{9} \zeta(3) \pi^2 \quad (31)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^7 \binom{2n}{n}} = \frac{11}{311040} \pi \sqrt{3} [\psi_5(\frac{1}{3}) - \psi_5(\frac{2}{3})] - \frac{493}{24} \zeta(7) + \frac{1}{3} \zeta(5) \pi^2 + \frac{17}{1620} \zeta(3) \pi^4, \quad (32)$$

where $\psi_n(x)$ is the POLYGAMMA FUNCTION and $\zeta(x)$ is the RIEMANN ZETA FUNCTION (Plouffe 1998).

Sums OF THE FORM

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^k \binom{2n}{n}} = \frac{1}{2} {}_{k+1}F_k \left(\underbrace{1, \dots, 1}_{k+1}; \frac{3}{2}, \underbrace{2, \dots, 2}_{k-1}; -\frac{1}{4} \right) \quad (33)$$

can also be simplified (Plouffe) to give the special cases

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \binom{2n}{n}} = \frac{2}{5} \sqrt{5} \sinh^{-1}(\frac{1}{2}) \quad (34)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 \binom{2n}{n}} = 2 [\sinh^{-1}(\frac{1}{2})]^2 \quad (35)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}} = \frac{2}{5} \zeta(3). \quad (36)$$

Other general identities include

$$\frac{(a+b)^n}{a} = \sum_{k=0}^n \binom{n}{k} (a-kc)^{k-1} (b+kc)^{n-k} \quad (37)$$

(Prudnikov *et al.* 1986), which gives the BINOMIAL THEOREM as a special case with $c=0$, and

$$\sum_{n=0}^{\infty} \binom{2n+s}{n} x^n = {}_2F_1(\frac{1}{2}(s+1), \frac{1}{2}(s+2); s+1, 4x) = \frac{2'}{(\sqrt{1-4x}+1)'\sqrt{1-4x}}, \quad (38)$$

where ${}_2F_1(a, b; c; z)$ is a HYPERGEOMETRIC FUNCTION (Abramowitz and Stegun 1972, p. 555; Graham *et al.* 1994, p. 203).

For NONNEGATIVE INTEGERS n and r with $r \leq n+1$,

$$\sum_{k=0}^n \frac{(-1)^k}{k+1} \binom{n}{k} \left[\sum_{j=0}^{r-1} (-1)^j \binom{n}{j} (r-j)^{n-k} \right]$$

$$+ \sum_{j=0}^{n-r} (-1)^j \binom{n}{j} (n+1-r-j)^{n-k} \Big] = n!. \quad (39)$$

Taking $n=2r-1$ gives

$$\sum_{k=0}^n \frac{(-1)^k}{K-1} \binom{n}{k} \sum_{j=0}^{r-1} \binom{n}{j} (r-j)^{n-k} = \frac{1}{2} n!. \quad (40)$$

Other identities are

$$\sum_{k=0}^n \binom{n+k}{k} [x^{n+1}(1-x)^k + (1-x)^{n+1}x^k] = 1 \quad (41)$$

(Gosper 1972) and

$$\sum_i \binom{n_i}{2} + \sum_{i>j} n_i n_j = \binom{n}{2}, \quad (42)$$

where

$$n \equiv \sum_i n_i. \quad (43)$$

The latter is the umbral analog of the multinomial theorem for n^2

$$\frac{(a+b+c)^2}{2} = \frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2} + ab + ac + bc \quad (44)$$

using the lower-factorial polynomial $(n)_2 = n(n-1)/2$, giving

$$\binom{a+b+c}{2} = \binom{a}{2} + \binom{b}{2} + \binom{c}{2} + ab + ac + bc. \quad (45)$$

The identity holds true not only for $(n)_2$ and $n^2/2$, but also for any quadratic polynomial OF THE FORM $n(n+a)/2$ (Dubuque).

See also APÉRY NUMBER, BINOMIAL COEFFICIENT, CENTRAL BINOMIAL COEFFICIENT, HYPERGEOMETRIC IDENTITY, HYPERGEOMETRIC SERIES, IDEMPOTENT NUMBER, JONAH FORMULA KLEE'S IDENTITY, LUCAS CORRESPONDENCE THEOREM, MARRIED COUPLES PROBLEM, MORLEY'S FORMULA, NEXUS NUMBER, STANLEY'S IDENTITY, STREHL IDENTITIES, SZÉKELY IDENTITY, WARING FORMULA, WORPITZKY'S IDENTITY

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Binomial Theorem

The theorem that, for POSITIVE INTEGERS n ,

$$(x + a)^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} x^k a^{n-k} = \sum_{k=0}^n \binom{n}{k} x^k a^{n-k},$$

the so-called BINOMIAL SERIES, where $\binom{n}{k}$ are BINOMIAL COEFFICIENTS. The theorem was known for the case $n = 2$ by Euclid around 300 BC, and stated in its modern form by Pascal in a posthumous pamphlet published in 1665. Newton (1676) showed that a similar formula (with INFINITE upper limit) holds for NEGATIVE INTEGERS n ,

$$(x + a)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k a^{-n-k},$$

the so-called NEGATIVE BINOMIAL SERIES, which converges for $|x| > |a|$.

See also BINOMIAL COEFFICIENT, BINOMIAL IDENTITY, BINOMIAL SERIES, CAUCHY BINOMIAL THEOREM, CHU-VANDERMONDE IDENTITY, LOGARITHMIC BINOMIAL

FORMULA, NEGATIVE BINOMIAL SERIES, q -BINOMIAL THEOREM, RANDOM WALK

References

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Binomial Transform

The binomial transform takes the sequence $a_0, a_1, a_2, \dots$ to the sequence $b_0, b_1, b_2, \dots$ via the transformation

$$b_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a_k.$$

The inverse transform is

$$a_n = \sum_{k=0}^n \binom{n}{k} b_k.$$

(Sloane and Plouffe 1995, pp. 13 and 22). The inverse binomial transform of $b_n = 1$ for prime n and $b_n = 0$ for composite n is 0, 1, 3, 6, 11, 20, 37, 70, ... (Sloane's A052467). The inverse binomial transform of $b_n = 1$ for even n and $b_n = 0$ for odd n is 0, 1, 2, 4, 8, 16, 32, 64, ... (Sloane's A000079). Similarly, the inverse binomial transform of $b_n = 1$ for odd n and $b_n = 0$ for even n is 1, 2, 4, 8, 16, 32, 64, ... (Sloane's A000079). The inverse binomial transform of the BELL NUMBERS 1, 1, 2, 5, 15, 52, 203, ... (Sloane's A000110) is a shifted version of the same numbers: 1, 2, 5, 15, 52, 203, ... (Bernstein and Sloane 1995, Sloane and Plouffe 1995, p. 22).

The CENTRAL and RAW MOMENTS of statistical distributions are also related by the binomial transform.

See also CENTRAL MOMENT, EULER TRANSFORM, EXPONENTIAL TRANSFORM, MÖBIUS TRANSFORM, RAW MOMENT

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Binomial Triangle

PASCAL'S TRIANGLE

Binomial-Type Sequence

A sequence of POLYNOMIALS p_n satisfying the identities

$$p_n(x+y) = \sum_{k \geq 0} \binom{n}{k} p_k(x) p_{n-k}(y).$$

See also BINOMIAL IDENTITY, SHEFFER SEQUENCE, UMBRAL CALCULUS

References

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Binormal Developable

A RULED SURFACE M is said to be a binormal developable of a curve $\mathbf{y}$ if M can be parameterized by $\mathbf{x}(u, v) = \mathbf{y}(u) + v\mathbf{B}(u)$, where $\mathbf{B}$ is the BINORMAL VECTOR.

See also NORMAL DEVELOPABLE, TANGENT DEVELOPABLE

References

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Binormal Vector

$$\hat{\mathbf{B}} \equiv \hat{\mathbf{T}} \times \hat{\mathbf{N}} \quad (1)$$

$$= \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|}, \quad (2)$$

where the unit TANGENT VECTOR $\mathbf{T}$ and unit "principal" NORMAL VECTOR $\mathbf{N}$ are defined by

$$\hat{\mathbf{T}} \equiv \frac{\mathbf{r}'(s)}{|\dot{\mathbf{r}}(s)|} \quad (3)$$

$$\hat{\mathbf{N}} \equiv \frac{\mathbf{r}''(s)}{|\mathbf{r}''(s)|} \quad (4)$$

Here, $\mathbf{r}$ is the RADIUS VECTOR, s is the ARC LENGTH, τ

is the TORSION, and κ is the CURVATURE. The binormal vector satisfies the remarkable identity

$$[\hat{\mathbf{B}}, \hat{\mathbf{B}}, \hat{\mathbf{B}}] = \tau^5 \frac{d}{ds} \left(\frac{\kappa}{\tau} \right). \quad (5)$$

See also FRENET FORMULAS, NORMAL VECTOR, TANGENT VECTOR

References

- Kreyszig, E. "Binormal. Moving Trihedron of a Curve." §13 in *Differential Geometry*. New York: Dover, pp. 36–7, 1991.

Bin-Packing Problem

The problem of packing a set of items into a number of bins such that the total weight, volume, etc. does not exceed some maximum value. A simple algorithm (the first-fit algorithm) takes items in the order they come and places them in the first bin in which they fit. In 1973, J. Ullman proved that this algorithm can differ from an optimal packing by as much as 70% (Hoffman 1998, p. 171). An alternative strategy first orders the items from largest to smallest, then places them sequentially in the first bin in which they fit. In 1973, D. Johnson showed that this strategy is never suboptimal by more than 22%, and furthermore that no efficient bin-packing algorithm can be guaranteed to do better than 22% (Hoffman 1998, p. 172).

There exist arrangements of items such that applying the packing algorithm after removing an item results in one more bin being required than the number obtained if the item is included (Hoffman 1998, pp. 172–73).

See also COOKIE-CUTTER PROBLEM, TILING PROBLEM

References

- Hoffman, P. *The Man Who Loved Only Numbers: The Story of Paul Erdos and the Search for Mathematical Truth*. New York: Hyperion, 1998.

Bioche's Theorem

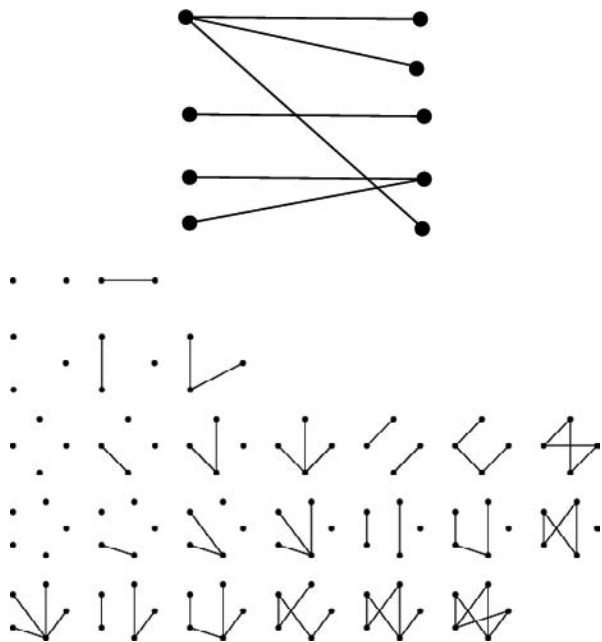
If two complementary PLÜCKER CHARACTERISTICS are equal, then each characteristic is equal to its complement except in four cases where the sum of order and class is 9.

References

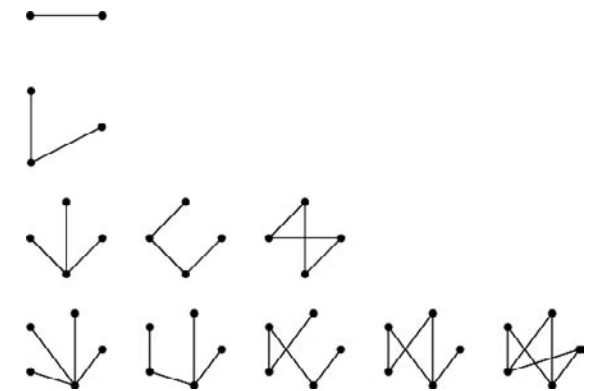
- Coolidge, J. L. *A Treatise on Algebraic Plane Curves*. New York: Dover, p. 101, 1959.

Biotic Potential

LOGISTIC EQUATION

Bipartite Graph

A set of VERTICES decomposed into two disjoint sets such that no two VERTICES within the same set are adjacent. A bigraph is a special case of a K -PARTITE GRAPH with $k = 2$. Bipartite graphs are equivalent to two-colorable graphs, and a graph is bipartite IFF all its cycles are of even length (Skiena 1990, p. 213). The numbers of bipartite graphs on $n = 1, 2, \dots$ nodes are 1, 2, 3, 7, 13, 35, 88, 303, ... (Sloane's A033995). A graph can be tested for bipartiteness using `BipartiteQ[g]` in the *Mathematica* add-on package `DiscreteMath`Combinatorica`` (which can be loaded with the command `<<DiscreteMath``).



The numbers of CONNECTED bipartite graphs on $n = 1, 2 \dots$ nodes are 1, 1, 1, 3, 5, 17, 44, 182, ... (Sloane's A005142).

All TREES are bipartite (Skiena 1990, p. 213).

See also BICUBIC GRAPH, COMPLETE BIPARTITE GRAPH, K -PARTITE GRAPH, KÖNIG-EGEVÁRY THEOREM

References

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Biplanar Double Point

ISOLATED SINGULARITY

Bipolar Coordinates

Bipolar coordinates are a 2-D system of coordinates. There are two commonly defined types of bipolar coordinates, the first of which is defined by

$$x = \frac{a \sinh v}{\cosh v - \cos u} \quad (1)$$

$$y = \frac{a \sin u}{\cosh v - \cos u}, \quad (2)$$

where $u \in [0, 2\pi)$, $v \in (-\infty, \infty)$. The following identities show that curves of constant u and v are CIRCLES in xy -space.

$$x^2 + (y - a \cot u)^2 = a^2 \csc^2 u \quad (3)$$

$$(x - a \coth v)^2 + y^2 = a^2 \operatorname{csch}^2 v. \quad (4)$$

The SCALE FACTORS are

$$h_u = \frac{a}{\cosh v - \cos u} \quad (5)$$

$$h_v = \frac{a}{\cosh v - \cos u} \quad (6)$$

The LAPLACIAN is

$$\nabla^2 = \frac{(\cosh v - \cos u)^2}{a^2} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right). \quad (7)$$

LAPLACE'S EQUATION is separable.

Two-center bipolar coordinates are two coordinates giving the distances from two fixed centers r_1 and r_2 , sometimes denoted r and r' . For two-center bipolar coordinates with centers at $(\pm c, 0)$,

$$r_1^2 = (x + c)^2 + y^2 \quad (8)$$

$$r_2^2 = (x - c)^2 + y^2. \quad (9)$$

Combining (8) and (9) gives

$$r_1^2 - r_2^2 = 4cx. \quad (10)$$

Solving for CARTESIAN COORDINATES x and y gives

$$x = \frac{r_1^2 - r_2^2}{4c} \quad (11)$$

$$y = \pm \frac{1}{4c} \sqrt{16c^2 r_1^2 - (r_1^2 - r_2^2 + 4c^2)^2}. \quad (12)$$

Solving for POLAR COORDINATES gives

$$r = \sqrt{\frac{r_1^2 + r_2^2 - 2c^2}{2}} \quad (13)$$

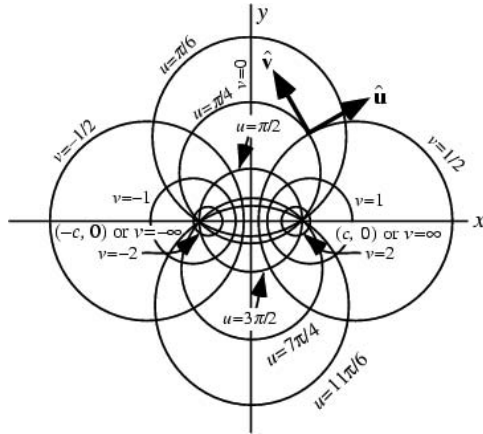
$$\theta = \tan^{-1} \left[\frac{\sqrt{r_2^4 - 2(4c^2 + r_1^2)r_2^2 - (4c^2 - r_1^2)^2}}{r_1^2 - r_2^2} \right]. \quad (14)$$

See also BIPOLAR CYLINDRICAL COORDINATES, POLAR COORDINATES

References

Lockwood, E. H. "Bipolar Coordinates." Ch. 25 in *A Book of Curves*. Cambridge, England: Cambridge University Press, pp. 186–90, 1967.

Bipolar Cylindrical Coordinates



A set of CURVILINEAR COORDINATES defined by

$$x = \frac{a \sinh v}{\cosh v - \cos u} \quad (1)$$

$$y = \frac{a \sin u}{\cosh v - \cos u} \quad (2)$$

$$z = z, \quad (3)$$

where $u \in [0, 2\pi)$, $v \in (-\infty, \infty)$, and $z \in (-\infty, \infty)$. There are several notational conventions, and whereas (u, v, z) is used in this work, Arfken (1970) prefers (η, ξ, z) . The following identities show that

curves of constant u and v are CIRCLES in xy -space.

$$x^2 + (y - a \cot u)^2 = a^2 \csc^2 u \quad (4)$$

$$(x - a \coth v)^2 + y^2 = a^2 \operatorname{csch}^2 v. \quad (5)$$

The SCALE FACTORS are

$$h_u = \frac{a}{\cosh v - \cos u} \quad (6)$$

$$h_v = \frac{a}{\cosh v - \cos u} \quad (7)$$

$$h_z = 1. \quad (8)$$

The LAPLACIAN is

$$\nabla^2 = \frac{(\cosh v - \cos u)^2}{a^2} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + \frac{\partial^2}{\partial z^2}. \quad (9)$$

LAPLACE'S EQUATION is not separable in BIPOLAR CYLINDRICAL COORDINATES, but it is in 2-D BIPOLAR COORDINATES.

See also BIPOLAR COORDINATES, POLAR COORDINATES

References

Arfken, G. "Bipolar Coordinates (ξ, η, z)." §2.9 in *Mathematical Methods for Physicists*, 2nd ed. Orlando, FL: Academic Press, pp. 97–02, 1970.

Bipolyhedral Group

The image of $A_5 \times A_5$ in the SPECIAL ORTHOGONAL GROUP $SO(4)$, where A_5 is the ICOSAHEDRAL GROUP.

See also ICOSAHEDRAL GROUP, SPECIAL ORTHOGONAL GROUP

References

Endraß, S. "The Sarti Surface." <http://enriques.mathematik.uni-mainz.de/kon/docs/Esarti.shtml>.

Biprism

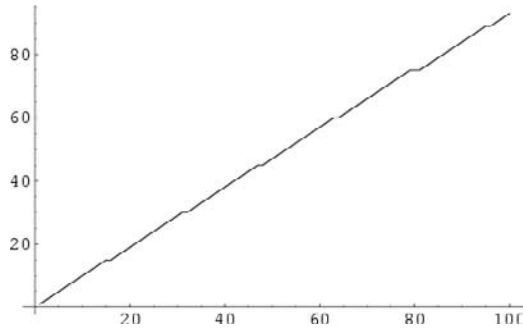
Two slant triangular PRISMS fused together.

See also PRISM, SCHMITT-CONWAY BIPRISM

Bipyramid

DIPYRAMID

Biquadratefree



A number is said to be biquadratafree (or quarticfree) if its PRIME FACTORIZATION contains no quadrupled factors. All PRIMES and PRIME POWERS p^n with $n \leq 3$ are therefore trivially biquadratafree. The biquadratafree numbers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, ... (Sloane's A046100). The biquadrataful numbers (i.e., those that contain at least one biquadrate) are 16, 32, 48, 64, 80, 81, 96, ... (Sloane's A046101). The number of biquadratafree numbers less than 10, 100, 1000, ... are 10, 93, 925, 9240, 92395, 923939, ..., and their asymptotic density is $1/\zeta(4) = 90/\pi^4 \approx 0.923938$, where $\zeta(n)$ is the RIEMANN ZETA FUNCTION.

See also CUBEFREE, PRIME NUMBER, RIEMANN ZETA FUNCTION, SQUAREFREE

References

Sloane, N. J. A. Sequences A046100 and A046101 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.

Biquadratic Equation

QUARTIC EQUATION

Biquadratic Number

A biquadratic number is a fourth POWER, n^4 . The first few biquadratic numbers are 1, 16, 81, 256, 625, ... (Sloane's A000583). The minimum number of biquadratic numbers needed to represent the numbers 1, 2, 3, ... are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 1, 2, 3, 4, 5, ... (Sloane's A002377), and the number of distinct ways to represent the numbers 1, 2, 3, ... in terms of biquadratic numbers are 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, ... A brute-force algorithm for enumerating the biquadratic permutations of n is repeated application of the GREEDY ALGORITHM.

Every POSITIVE integer is expressible as a SUM of (at most) $g(4) = 19$ biquadratic numbers (WARING'S PROBLEM). Davenport (1939) showed that $G(4) = 16$, meaning that all sufficiently large integers require only 16 biquadratic numbers. It is also known that every integer is a sum of at most 10 signed biquadrates ($eg(4) \leq 10$; although it is not known if 10 can

be reduced to 9). The following table gives the first few numbers which require 1, 2, 3, ..., 19 biquadratic numbers to represent them as a sum, with the sequences for 17, 18, and 19 being finite.

#	Sloane	Numbers
1	Sloane's A000290	1, 16, 81, 256, 625, 1296, 2401, 4096, ...
2	Sloane's A003336	2, 17, 32, 82, 97, 162, 257, 272, ...
3	Sloane's A003337	3, 18, 33, 48, 83, 98, 113, 163, ...
4	Sloane's A003338	4, 19, 34, 49, 64, 84, 99, 114, 129, ...
5	Sloane's A003339	5, 20, 35, 50, 65, 80, 85, 100, 115, ...
6	Sloane's A003340	6, 21, 36, 51, 66, 86, 96, 101, 116, ...
7	Sloane's A003341	7, 22, 37, 52, 67, 87, 102, 112, 117, ...
8	Sloane's A003342	8, 23, 38, 53, 68, 88, 103, 118, 128, ...
9	Sloane's A003343	9, 24, 39, 54, 69, 89, 104, 119, 134, ...
10	Sloane's A003344	10, 25, 40, 55, 70, 90, 105, 120, 135, ...
11	Sloane's A003345	11, 26, 41, 56, 71, 91, 106, 121, 136, ...
12	Sloane's A003346	12, 27, 42, 57, 72, 92, 107, 122, 137, ...
13	Sloane's A046044	13, 28, 43, 58, 73, 93, 108, 123, 138, ...
14	Sloane's A046045	14, 29, 44, 59, 74, 94, 109, 124, 139, ...
15	Sloane's A046046	15, 30, 45, 60, 75, 95, 110, 125, 140, ...
16	Sloane's A046047	31, 46, 61, 76, 111, 126, 141, 156, ...
17	Sloane's A046048	47, 62, 77, 127, 142, 157, 207, 222, ...
18	Sloane's A046049	63, 78, 143, 158, 223, 238, 303, 318, ...
19	Sloane's A046050	79, 159, 239, 319, 399

The following table gives the numbers which can be represented in n different ways as a sum of k biquadrates.

k	n	Sloane	Numbers
1	1	Sloane's A000290	1, 16, 81, 256, 625, 1296, 2401, 4096, ...
2	2	Sloane's A018786	635318657, 3262811042, 8657437697, ...

The numbers 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21, ... (Sloane's A046039) cannot be represented using distinct biquadrates.

See also CUBIC NUMBER, PARTITION, SQUARE NUMBER, WARING'S PROBLEM

References

Davenport, H. "On Waring's Problem for Fourth Powers." *Ann. Math.* **40**, 731–47, 1939.
 Hardy, G. H. and Wright, E. M. "The Representation of a Number by Two or Four Squares." Ch. 20 in *An Introduction to the Theory of Numbers*, 5th ed. Oxford, England: Clarendon Press, pp. 297–16, 1979.
 Sloane, N. J. A. Sequences A000290, A000583/M5004, A002377, A003336, A003337, A003338, A003339, A003340, A003341, A003342, A003343, A003344, A003345, A003346, A018786, and A046039 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.

Biquadratic Reciprocity Theorem

Gauss stated the reciprocity theorem for the case $n = 4$

$$x^4 \equiv q \pmod{p} \quad (1)$$

can be solved using the GAUSSIAN INTEGERS as

$$\left(\frac{\pi}{\sigma}\right)_4 \left(\frac{\sigma}{\pi}\right)_4 = (-1)^{[(N(\pi)-1)/4][(N(\sigma)-1)/4]}. \quad (2)$$

Here, π and σ are distinct GAUSSIAN INTEGER PRIMES, and

$$N(a + bi) = \sqrt{a^2 + b^2} \quad (3)$$

is the norm. The symbol $\left(\frac{\alpha}{\pi}\right)$ means

$$\left(\frac{\alpha}{\pi}\right)_4 = \begin{cases} 1 & \text{if } x^4 \equiv \alpha \pmod{\pi} \text{ is solvable} \\ -1, i, \text{ or } -i & \text{otherwise} \end{cases} \quad (4)$$

where "solvable" means solvable in terms of GAUSSIAN INTEGERS.

2 is a quartic residue $\pmod{p}$ IFF there are integers x, y such that

$$x^2 + 64y^2 = p. \quad (5)$$

This is a generalization of the GENUS THEOREM.

See also BIQUADRATIC RESIDUE, GENUS THEOREM, RECIPROCITY THEOREM

References

Ireland, K. and Rosen, M. "Cubic and Biquadratic Reciprocity." Ch. 9 in *A Classical Introduction to Modern Number Theory*, 2nd ed. New York: Springer-Verlag, pp. 108–37, 1990.

Biquadratic Residue

If there is an INTEGER x such that

$$x^4 \equiv q \pmod{p}, \quad (1)$$

then q is said to be a biquadratic residue $\pmod{p}$. If not, q is said to be a biquadratic nonresidue $\pmod{p}$.

See also BIQUADRATIC RECIPROCITY THEOREM, CUBIC RESIDUE, QUADRATIC RESIDUE

References

Nagell, T. *Introduction to Number Theory*. New York: Wiley, p. 115, 1951.

Biquaternion

A QUATERNION with COMPLEX coefficients. The ALGEBRA of biquaternions is isomorphic to a full matrix ring over the complex number field (van der Waerden 1985).

See also QUATERNION

References

Clifford, W. K. "Preliminary Sketch of Biquaternions." *Proc. London Math. Soc.* **4**, 381–95, 1873.
 Hamilton, W. R. *Lectures on Quaternions: Containing a Systematic Statement of a New Mathematical Method*. Dublin: Hodges and Smith, 1853.
 Study, E. "Von den Bewegung und Umlegungen." *Math. Ann.* **39**, 441–66, 1891.
 van der Waerden, B. L. *A History of Algebra from al-Khwarizmi to Emmy Noether*. New York: Springer-Verlag, pp. 188–89, 1985.

Birational Transformation

A transformation in which coordinates in two SPACES are expressed rationally in terms of those in another.

See also RIEMANN CURVE THEOREM, WEBER'S THEOREM

Birch Conjecture

SWINNERTON-DYER CONJECTURE

Birch-Swinnerton-Dyer Conjecture

SWINNERTON-DYER CONJECTURE

Birkhoff's Ergodic Theorem

Let T be an ergodic ENDOMORPHISM of the PROBABILITY SPACE X and let $f : X \rightarrow \mathbb{R}$ be a real-valued MEASURABLE FUNCTION. Then for ALMOST EVERY $x \in X$, we have

$$\frac{1}{n} \sum_{j=1}^n f \circ T^j(x) \rightarrow \int f \, dm \quad (1)$$

as $n \rightarrow \infty$. To illustrate this, take f to be the characteristic function of some SUBSET A of X so that

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \quad (2)$$

The left-hand side of (1) just says how often the orbit of x (that is, the points $x, Tx, T^2x, \dots$) lies in A , and the right-hand side is just the MEASURE of A . Thus, for an ergodic ENDOMORPHISM, "space-averages = time-averages almost everywhere." Moreover, if T is continuous and uniquely ergodic with BOREL PROBABILITY MEASURE m and f is continuous, then we can replace the ALMOST EVERYWHERE convergence in (1) with "everywhere."

See also BIRKHOFF'S THEOREM, ERGODIC THEORY

References

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Birkhoff-Khinchin Ergodic Theorem

BIRKHOFF'S ERGODIC THEOREM

Birkhoff-Witt Theorem

POINCARÉ-BIRKHOFF-WITT THEOREM

Berotunda

Two adjoined ROTUNDAS.

See also BILUNABIROTUNDA, CUPOLAROTUNDA, ELONGATED GYROCUPOLAROTUNDA, ELONGATED ORTHOCUPOLAROTUNDA, ELONGATED ORTHOBIROTUNDA, GYROCUPOLAROTUNDA, GYROELONGATED ROTUNDA, ORTHOBIROTUNDA, TRIANGULAR HEBESPHENOROTUNDA

Birthday Attack

Birthday attacks are a class of brute-force techniques used in an attempt to solve a class of CRYPTOGRAPHIC HASH FUNCTION problems. These methods take advantage of functions which, when supplied with a random input, return one of k equally likely values. By repeatedly evaluating the function for different inputs, the same output is expected to be obtained after about $1.2\sqrt{k}$ evaluations.

See also BIRTHDAY PROBLEM, CRYPTOGRAPHIC HASH FUNCTION

References

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Birthday Problem

Consider the probability $Q_1(n, d)$ that *no two people* out of a group of n will have matching birthdays out of d equally possible birthdays. Start with an arbitrary person's birthday, then note that the probability that the second person's birthday is different is $(d-1)/d$, that the third person's birthday is different from the first two is $[(d-1)/d][(d-2)/d]$, and so on, up through the n th person. Explicitly,

$$\begin{aligned} Q_1(n, d) &= \frac{d-1}{d} \frac{d-2}{d} \dots \frac{d-(n-1)}{d} \\ &= \frac{(d-1)(d-2) \dots [d-(n-1)]}{d^{n-1}}. \end{aligned} \quad (1)$$

But this can be written in terms of FACTORIALS as

$$Q_1(n, d) = \frac{d!}{(d-n)!d^n}, \quad (2)$$

so the probability $P_2(n, 365)$ that two people out of a group of n *do have the same birthday* is therefore

$$P_2(n, d) = 1 - Q_1(n, d) = 1 - \frac{d!}{(d-n)!d^n}. \quad (3)$$

If 365-day years have been assumed, i.e., the existence of leap days is ignored, then the number of people needed for there to be at least a 50% chance that two share birthdays is the smallest n such that $P_2(n, 365) \geq 1/2$. This is given by $n = 23$, since

$$\begin{aligned} P_2(23, 365) &= \frac{3809390470229739078524370829105639051888645406094}{7509188326851535012542620742522314756326980590820} \\ &\approx 0.507297. \end{aligned} \quad (4)$$

The number n of people needed to obtain $P_2(n, d) \geq 1/2$ for $d = 1, 2, \dots$, are 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, ... (Sloane's A033810).

The probability $P_2(n, d)$ can be estimated as

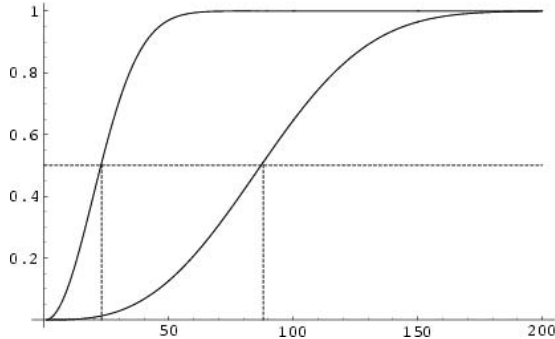
$$P_2(n, d) \approx 1 - e^{-n(n-1)/2d} \quad (5)$$

$$\approx 1 - \left(1 - \frac{n}{2d}\right)^{n-1}, \quad (6)$$

where the latter has error

$$\epsilon < \frac{n^3}{6(d-n+1)^2} \quad (7)$$

(Sayrafiezadeh 1994).



In general, let $Q_i(n, d)$ denote the probability that a birthday is shared by exactly i (and no more) people out of a group of n people. Then the probability that a birthday is shared by k or more people is given by

$$P_k(n, d) = 1 - \sum_{i=1}^{k-1} Q_i(n, d). \quad (8)$$

Q_2 can be computed explicitly as

$$\begin{aligned} Q_2(n, d) &= \frac{n!}{d^n} \sum_{i=2}^{\lfloor n/2 \rfloor} \frac{1}{2^i} \binom{d}{i} \binom{d-i}{n-2i} \\ &= \frac{n!}{d^n} \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{d!}{2^i i! (n-2i)! (d-n+i)!} \\ &= \frac{(-1)^n}{d^n} \left[2^{-n/2} \Gamma(1+n) P_n^{(-d)} \left(\frac{1}{2} \sqrt{2} \right) \right. \\ &\quad \left. - \frac{\Gamma(1+d)}{\Gamma(1+d-n)} \right], \end{aligned} \quad (9)$$

where $\binom{n}{m}$ is a BINOMIAL COEFFICIENT, $\Gamma(n)$ is a GAMMA FUNCTION, and $P_n^{(\lambda)}(x)$ is an ULTRASPHERICAL POLYNOMIAL. This gives the explicit formula for $P_3(n, d)$ as

$$\begin{aligned} P_3(n, d) &= 1 - Q_1(n, d) - Q_2(n, d) \\ &= 1 + \frac{(-1)^{n+1} \Gamma(n+1) P_n^{(-d)}(2^{-1/2})}{2^{n/2} d^n}. \end{aligned} \quad (10)$$

$Q_3(n, d)$ cannot be computed in entirely closed form, but a partially reduced form is

$$\begin{aligned} Q_3(n, d) &= \frac{\Gamma(d+1)}{d^n} \left[\frac{(-1)^n F(\frac{9}{8}) - F(-\frac{9}{8})}{\Gamma(d-n+1)} + (-1)^n \Gamma \right. \\ &\quad \left. \times (1+n) \sum_{i=1}^{\lfloor n/3 \rfloor} \frac{(-3)^{-i} 2^{(i-n)/2} P_{n-3i}^{(-d)}(\frac{1}{2} \sqrt{2})}{\Gamma(d-i+1) \Gamma(i+1)} \right], \end{aligned} \quad (11)$$

where

$$F = F(n, d, a) \equiv 1 - {}_3F_2 \left[\begin{matrix} \frac{1}{3}(1-n), \frac{1}{3}(2-n), -\frac{1}{3} \\ \frac{1}{2}(d-n+1), \frac{1}{2}(d-n+2) \end{matrix}; a \right] \quad (12)$$

and ${}_3F_2(a, b, c; d, e; z)$ is a GENERALIZED HYPERGEOMETRIC FUNCTION.

In general, $Q_k(n, d)$ can be computed using the RECURRENCE RELATION

$$\begin{aligned} Q_k(n, d) &= \sum_{i=1}^{\lfloor n/k \rfloor} \left[\frac{n! d!}{d^{ik} i! (k!)^i (n-ik)! (d-i)!} \right. \\ &\quad \left. \times \sum_{j=1}^{k-1} Q_j(n-k, d-i) \frac{(d-i)^{n-ik}}{d^{n-ik}} \right] \end{aligned} \quad (13)$$

(Finch). However, the time to compute this recursive function grows exponentially with k and so rapidly becomes unwieldy. The minimal number of people to give a 50% probability of having at least n coincident birthdays is 1, 23, 88, 187, 313, 460, 623, 798, 985, 1181, 1385, 1596, 1813, ... (Sloane's A014088; Diaconis and Mosteller 1989).

A good approximation to the number of people n such that $p = P_k(n, d)$ is some given value can be given by solving the equation

$$n e^{-n/(dk)} = \left[d^{k-1} k! \ln \left(\frac{1}{1-p} \right) \left(1 - \frac{n}{d(k+1)} \right) \right]^{1/k} \quad (14)$$

for n and taking $\lceil n \rceil$, where $\lceil n \rceil$ is the CEILING FUNCTION (Diaconis and Mosteller 1989). For $p = 0.5$ and $k = 1, 2, 3, \dots$, this formula gives $n = 1, 23, 88, 187, 313, 459, 622, 797, 983, 1179, 1382, 1592, 1809, \dots$ (Sloane's A050255), which differ from the true values by from 0 to 4. A much simpler but also poorer approximation for n such that $p = 0.5$ for $k < 20$ is given by

$$n = 47(k - 1.5)^{3/2} \quad (15)$$

(Diaconis and Mosteller 1989), which gives 86, 185, 307, 448, 606, 778, 965, 1164, 1376, 1599, 1832, ... for $k = 3, 4, \dots$ (Sloane's A050256).

The "almost" birthday problem, which asks the number of people needed such that two have a birthday within a day of each other, was considered by Abramson and Moser (1970), who showed that 14 people suffice. An approximation for the minimum number of people needed to get a 50-0 chance that two have a match within k days out of d possible is given by

$$n(k, d) = 1.2 \sqrt{\frac{d}{2k+1}} \quad (16)$$

(Sevast'yanov 1972, Diaconis and Mosteller 1989).

See also BIRTHDAY ATTACK, COINCIDENCE, SMALL WORLD PROBLEM, SULTAN'S DOWRY PROBLEM

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Bisected Perimeter Point

NAGEL POINT

Bisection Procedure

A simple procedure for iteratively converging on a solution which is known to lie inside some interval $[a, b]$. Let a_n and b_n be the endpoints at the n th iteration and r_n be the n th approximate solution. Then, the number of iterations required to obtain an error smaller than ϵ is found as follows.

$$b_n - a_n = \frac{1}{2^{n-1}}(b - a) \quad (1)$$

$$r_n \equiv \frac{1}{2}(a_n + b_n) \quad (2)$$

$$|r_n - r| \leq \frac{1}{2}(b_n - a_n) = 2^{-n}(b - a) < \epsilon \quad (3)$$

$$-n \ln 2 < \ln \epsilon - \ln(b - a), \quad (4)$$

so

$$n > \frac{\ln(b - a) - \ln \epsilon}{\ln 2}. \quad (5)$$

See also ROOT

References

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Bisector

Bisection is the division of a given curve or figure into two equal parts (halves).

See also ANGLE BISECTOR, BISECTION PROCEDURE, EXTERIOR ANGLE BISECTOR, HALF, HEMISPHERE, LINE BISECTOR, PERPENDICULAR BISECTOR, TRISECTION

Bishop's Inequality

Let $V(r)$ be the volume of a BALL of radius r in a complete n -D RIEMANNIAN MANIFOLD with RICCI CURVATURE $\geq (n-1)\kappa$. Then $V(r) \geq V_\kappa(r)$, where V_κ is the volume of a BALL in a space having constant SECTIONAL CURVATURE. In addition, if equality holds for some BALL, then this BALL is ISOMETRIC to the BALL of radius r in the space of constant SECTIONAL CURVATURE κ .

See also BALL, ISOMETRY

References

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Bishops Problem

B							
B							B
B							B
B							B
B							B
B							B
B							B
B							B

Find the maximum number of bishops $B(n)$ which can be placed on an $n \times n$ CHESSBOARD such that no two attack each other. The answer is $2n - 2$ (Dudeney 1970, Madachy 1979), giving the sequence 2, 4, 6, 8, ... (the EVEN NUMBERS) for $n = 2, 3, \dots$. One maximal solution for $n = 8$ is illustrated above. The number of distinct maximal arrangements of bishops for $n = 1, 2, \dots$ are 1, 4, 26, 260, 3368, ... (Sloane's A002465). The number of rotationally and reflectively distinct solutions on an $n \times n$ board for $n \geq 2$ is

$$B(n) = \begin{cases} 2^{(n-4)/2} [2^{(n-2)/2} + 1] & \text{for } n \text{ even} \\ 2^{(n-3)/2} [2^{(n-3)/2} + 1] & \text{for } n \text{ odd} \end{cases}$$

(Dudeney 1970, p. 96; Madachy 1979, p. 45; Pickover 1995). An equivalent formula is

$$B(n) = 2^{n-3} + 2^{\lfloor (n-1)/2 \rfloor - 1},$$

where $\lfloor n \rfloor$ is the FLOOR FUNCTION, giving the sequence for $n = 1, 2, \dots$ as 1, 1, 2, 3, 6, 10, 20, 36, ... (Sloane's A005418).

			B				
			B				
			B				
			B				
			B				
			B				
			B				
			B				

The minimum number of bishops needed to occupy or attack all squares on an $n \times n$ CHESSBOARD is n , arranged as illustrated above.

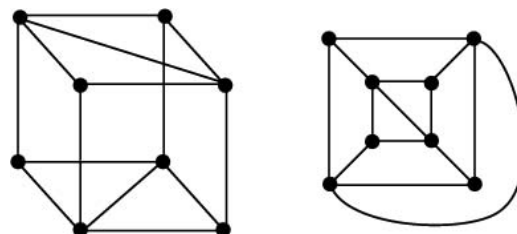
See also CHESS, KINGS PROBLEM, KNIGHTS PROBLEM, QUEENS PROBLEM, ROOKS PROBLEM

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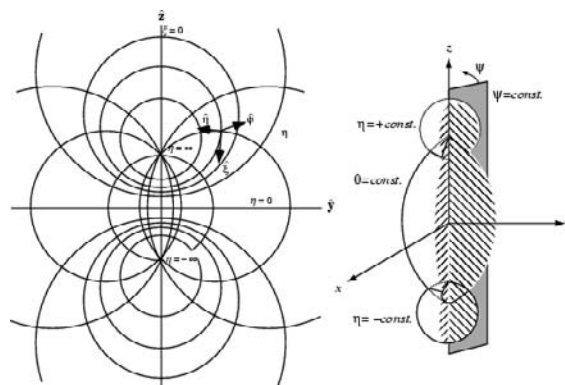
Bislit Cube



The 8-VERTEX graph consisting of a CUBE in which two opposite faces have DIAGONALS oriented PERPENDICULAR to each other.

See also BIDIAKIS CUBE, CUBE, CUBICAL GRAPH

Bispherical Coordinates



A system of CURVILINEAR COORDINATES variously denoted (ξ, η, ϕ) (Arfken 1970) or (θ, η, ψ) (Moon and Spencer 1988). Using the notation of Arfken, the bispherical coordinates are defined by

$$x = \frac{a \sin \xi \cos \phi}{\cosh \eta - \cos \xi} \quad (1)$$

$$y = \frac{a \sin \xi \sin \phi}{\cosh \eta - \cos \xi} \quad (2)$$

$$z = \frac{a \sinh \eta}{\cosh \eta - \cos \xi} \quad (3)$$

Surfaces of constant η are given by the spheres

$$x^2 + y^2 + (z - a \coth \eta)^2 = \frac{a^2}{\sinh^2 \eta}, \quad (4)$$

surfaces of constant ξ by the APPLES ($\xi < \pi/2$) or LEMONS ($\xi > \pi/2$)

$$x^2 + y^2 + z^2 - 2a\sqrt{x^2 + y^2} \cot \xi = a^2, \quad (5)$$

and surface of constant ψ by the half-planes

$$\tan \phi = y/x. \quad (6)$$

The SCALE FACTORS are

$$h_\xi = \frac{a}{\cos \eta - \cos \xi} \quad (7)$$

$$h_\eta = \frac{a}{\cosh \eta - \cos \xi} \quad (8)$$

$$h_\phi = \frac{a \sin \xi}{\cosh \eta - \cos \xi}. \quad (9)$$

The LAPLACIAN is given by

$$\begin{aligned} \nabla^2 f = & \frac{(\cosh \eta - \cos \xi)^2}{a^2 \sin^2 \xi} \\ & \times \left\{ \sin \xi \frac{\partial}{\partial \eta} \left(\frac{1}{\cosh \eta - \cos \xi} \frac{\partial f}{\partial \eta} \right) \right. \\ & + \frac{\partial}{\partial \xi} \left(\frac{\sin \xi}{\cosh \eta - \cos \xi} \frac{\partial f}{\partial \xi} \right) \Bigg\} \\ & + \frac{(\cosh \eta - \cos \xi)^2}{a^2 \sin^2 \xi} \frac{\partial^2 f}{\partial \phi^2}. \end{aligned}$$

In bispherical coordinates, LAPLACE'S EQUATION is separable (Moon and Spencer 1988), but the HELMHOLTZ DIFFERENTIAL EQUATION is not.

See also BICYCLIDE COORDINATES, LAPLACE'S EQUATION-BISPHERICAL COORDINATES, SPHERICAL COORDINATES, TOROIDAL COORDINATES

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Bisymmetric Matrix

A SQUARE MATRIX is called bisymmetric if it is both CENTROSYMMETRIC and either SYMMETRIC or SKEW SYMMETRIC (Muir 1960, p. 19).

See also CENTROSYMMETRIC MATRIX, SKEW SYMMETRIC MATRIX, SYMMETRIC MATRIX

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Bit Complexity

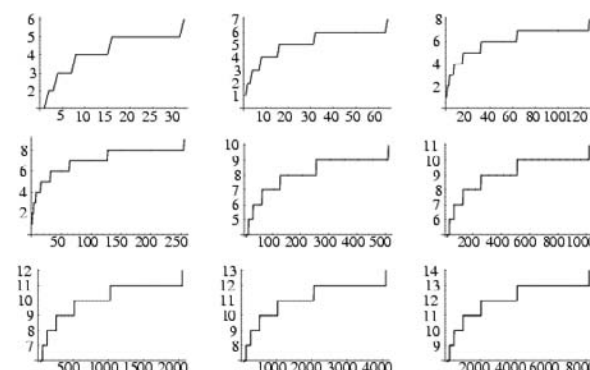
The number of single operations (of ADDITION, SUBTRACTION, and MULTIPLICATION) required to complete an algorithm.

See also STRASSEN FORMULAS

References

- Borodin, A. and Munro, I. *The Computational Complexity of Algebraic and Numeric Problems*. New York: American Elsevier, 1975.

Bit Length



The number of binary bits necessary to represent a number, given explicitly by

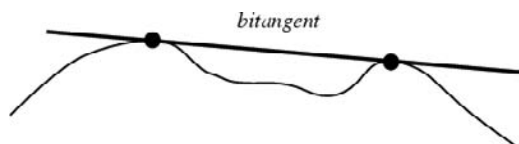
$$BL(n) = \lceil \lg n \rceil,$$

where $\lceil x \rceil$ is the CEILING FUNCTION and $\lg n$ is LG, the LOGARITHM to base 2. For $n = 0, 1, 2, \dots$, the first few values are 0, 1, 2, 2, 3, 3, 3, 3, 4, 4, ... (Sloane's A036377). The function is given by the *Mathematica* 4.0 function `BitLength[n]` in the Developer context.

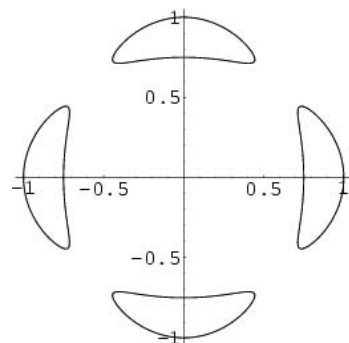
References

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Bitangent



A LINE which is TANGENT to a curve at two distinct points.



There exist plane QUARTIC CURVES

$$\sum_{i+j \leq 4} a_{ij} x^i y^j = 0$$

that have 28 real bitangents (Shioda 1995, Trott 1997), for example

$$12^2(x^4 + y^4) - 15^2(x^2 + y^2) + 350x^2y^2 + 81 = 0$$

(Trott 1997), illustrated above.

See also KLEIN'S EQUATION, PLÜCKER CHARACTERISTICS, SECANT LINE, SOLOMON'S SEAL LINES, TANGENT LINE

References

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Bitwin Chain

A bitwin chain of length one consists of two pairs of TWIN PRIMES with the property that they are related by being of the form:

$$(n-1, n+1) \text{ and } (2n-1, 2n+1).$$

In general a chain of length i consists of $i+1$ pairs of TWIN PRIMES,

$$(n-1, n+1), (2n-1, 2n+1), \dots, (2^i \cdot n-1, 2^i \cdot n+1).$$

Bitwin chains can also be viewed as consisting of two related CUNNINGHAM CHAINS of the first and second kinds,

$$(n-1, 2n-1, 4n-1, \dots) \text{ and}$$

$$(n+1, 2n+1, 4n+1, \dots).$$

P. Jobling (1999) found the largest known chain of length six,

$$337190719854678690 \cdot 2^n \pm 1,$$

where $n = 0$ to 6.

See also CUNNINGHAM CHAIN, TWIN PRIMES

References

Jobling, P. "A BiTwin chain of length 6 discovered." NMBRTHRY@listserv.nodak.edu posting, 4 Oct 1999.

Biunitary Divisor

A divisor d of a positive integer n is biunitary if the greatest common unitary divisor of d and n/d is 1. For a prime power p^y , the biunitary divisors are the powers $1, p, p^2, \dots, p^y$, except for $p^{y/2}$ when y is EVEN (Cohen 1990).

See also DIVISOR, K -ARY DIVISOR, UNITARY DIVISOR

References

Cohen, G. L. "On an Integer's Infinary Divisors." *Math. Comput.* **54**, 395-11, 1990.
Suryanarayana, D. "The Number of Bi-Unitary Divisors of an Integer." *The Theory of Arithmetic Functions (Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1971. New York: Springer-Verlag, pp. 273-82, 1972.*
Suryanarayana, D. and Rao, R. S. R. C. "The Number of Bi-Unitary Divisors of an Integer. II." *J. Indian Math. Soc.* **39**, 261-80, 1975.

Bivalent

Capable of taking on one out of two possible values.

See also EXCLUDED MIDDLE LAW, UNIVALENT

Bivalent Range

If the CROSS-RATIO κ of $\{AB, CD\}$ satisfy

$$\kappa^2 - \kappa + 1 = 0, \quad (1)$$

then the points are said to form a bivalent range, and

$$\{AB, CD\} = \{AC, DB\} = \{AD, BC\} = \kappa \quad (2)$$

$$\{AC, BD\} = \{AD, BC\} = \{AB, DC\} = -\kappa^2. \quad (3)$$

See also HARMONIC RANGE

References

Lachlan, R. *An Elementary Treatise on Modern Pure Geometry.* London: Macmillan, p. 268, 1893.

Bivariate Distribution

See also GAUSSIAN BIVARIATE DISTRIBUTION

Bivariate Normal Distribution

GAUSSIAN BIVARIATE DISTRIBUTION

Bivector

An antisymmetric TENSOR of second RANK (a.k.a. 2-form).

$$\mathbf{X} = X_{ab} \omega^a \wedge \omega^b,$$

where $\wedge$ is the WEDGE PRODUCT (or OUTER PRODUCT).

See also TENSOR, VECTOR

Biweight

TUKEY'S BIWEIGHT

Björling Curve

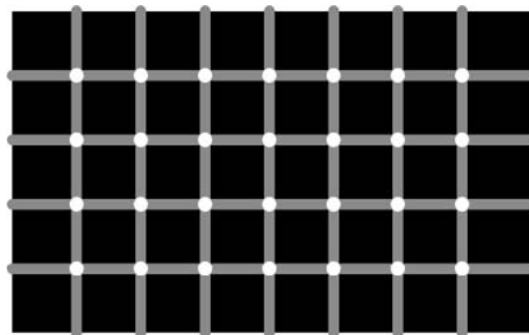
Let $\alpha(z)$, $\gamma(z) : (a, b) \rightarrow \mathbb{R}^3$ be curves such that $\|\gamma\| = 1$ and $\alpha \cdot \gamma = 0$, and suppose that α and γ have holomorphic extensions $\alpha, \gamma : (a, b) \times (c, d) \rightarrow \mathbb{C}^3$ such that $\|\gamma\| = 1$ and $\alpha \cdot \gamma = 0$ also for $z \in (a, b) \times (c, d)$. Fix $z_0 \in (a, b) \times (c, d)$. Then the Björling curve, defined by

$$B(z) = \alpha(z) - i \int_{z_0}^z \gamma(z) \times \alpha'(z) dz,$$

is a minimal curve (Gray 1997, p. 762).

References

- Björling, E. G. "In integrationem aequationis derivatarum partialum superficiei, cujus in puncto, unoquoque principales ambo radii curvaturae aequales sunt signoque contrario." *Arch. Math. Phys.* **4**, 290–15, 1844.
- Dierkes, U.; Hildebrand, S.; Küster, A.; and Wohlrab, O. *Minimal Surfaces*, 2 vols. New York: Springer-Verlag, pp. 120–35, 1992.
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- Nitsche, J. C. C. *Lectures on Minimal Surfaces, Vol. 1: Introduction, Fundamentals, Geometry and Basic Boundary Value Problems*. Cambridge, England: Cambridge University Press, pp. 139–45, 1989.
- Schwarz, H. A. *Gesammelte Mathematische Abhandlungen*, Vols. 1–. New York: Chelsea, pp. 179–89, 1972.

Black Dot Illusion

In the above illustration, black dots appear to form and vanish at the intersections of the gray horizontal and vertical lines. When focusing attention on a single white dot, some gray dots nearby and some black dots a little further away also seem to appear. More black dots seem to appear as the eye is scanned across the image (as opposed to focusing on a single point). Strangely, the effect seems to be reduced, but not eliminated, when the head is cocked at a 45° angle. The effect seems to exist only at intermediate distances; if the eye is moved very close to or very far away from the figure, the phantom black dots do not appear.

See also ILLUSION

References

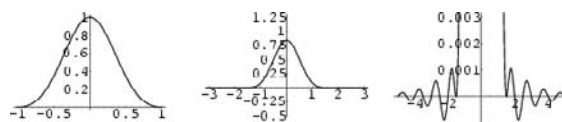
- Gephardt, J. "Find the Black Dot." <http://udel.edu/~jgephardt/fun2.htm>.

Black Spleenwort Fern

BARNSELY'S FERN

Blackboard Bold

DOUBLESTRUCK

Blackman Function

An APODIZATION FUNCTION given by

$$A(x) = 0.42 + 0.5 \cos\left(\frac{\pi x}{a}\right) + 0.08 \cos\left(\frac{2\pi x}{a}\right). \quad (1)$$

Its FULL WIDTH AT HALF MAXIMUM is $0.810957a$. The APPARATUS FUNCTION is

$$I(k) = \frac{a(0.84 - 0.36a^2k^2 - 2.17 \times 10^{-19}a^4k^4)\sin(2\pi ak)}{(1 - a^2k^2)(1 - 4a^2k^2)}. \quad (2)$$

The COEFFICIENTS are approximations in the general

expansion

$$A(x) = a_0 + 2 \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{b}\right), \quad (3)$$

to

$$a_0 = \frac{3969}{9304} \approx 0.42659 \quad (4)$$

$$a_1 = \frac{1155}{4652} \approx 0.24828 \quad (5)$$

$$a_2 = \frac{715}{18608} \approx 0.38424, \quad (6)$$

which produce zeros of $I(k)$ at $ka = 7/4$ and $ka = 9/4$.

See also APODIZATION FUNCTION

References

Blackman, R. B. and Tukey, J. W. "Particular Pairs of Windows." In *The Measurement of Power Spectra, From the Point of View of Communications Engineering*. New York: Dover, pp. 98–9, 1959.

Black-Scholes Theory

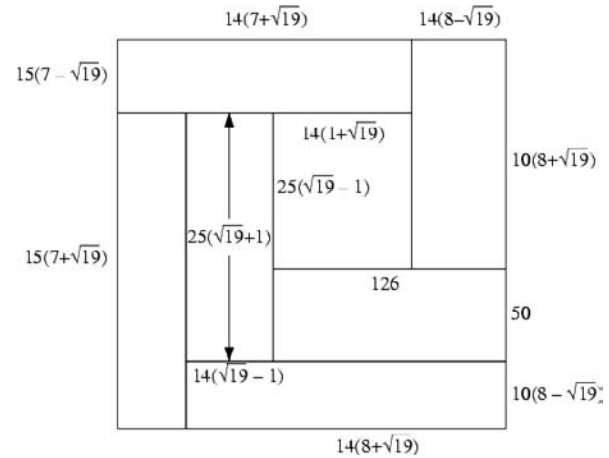
The theory underlying financial derivatives which involves "stochastic calculus" and assumes an uncorrelated LOG NORMAL DISTRIBUTION of continuously varying prices. A simplified "binomial" version of the theory was subsequently developed by Sharpe *et al.* (1995) and Cox *et al.* (1979). It reproduces many results of the full-blown theory, and allows approximation of options for which analytic solutions are not known (Price 1996).

See also GARMAN-KOHLHAGEN FORMULA

References

Black, F. and Scholes, M. S. "The Pricing of Options and Corporate Liabilities." *J. Political Econ.* **81**, 637–59, 1973.
Cox, J. C.; Ross, A.; and Rubenstein, M. "Option Pricing: A Simplified Approach." *J. Financial Economics* **7**, 229–63, 1979.
Price, J. F. "Optional Mathematics is Not Optional." *Not. Amer. Math. Soc.* **43**, 964–71, 1996.
Sharpe, W. F.; Alexander, G. J.; Bailey, J. V.; and Sharpe, W. C. *Investments*, 6th ed. Englewood Cliffs, NJ: Prentice-Hall, 1998.

Blanche's Dissection



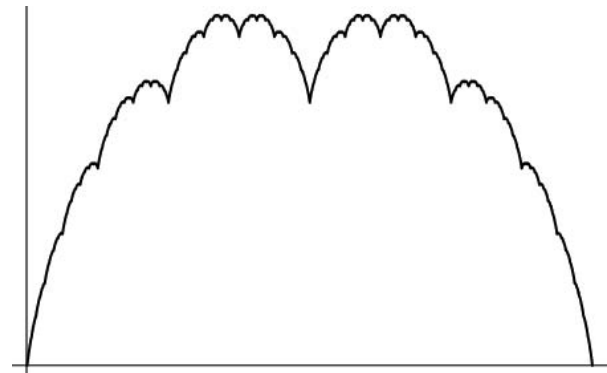
The simplest dissection of a SQUARE into rectangles of the same AREAS but different shapes, composed of the seven pieces illustrated above. The square is 210 units on a side, and each RECTANGLE has AREA $210^2/7 = 6300$.

See also PERFECT SQUARE DISSECTION, RECTANGLE

References

Descartes, B. "Division of a Square into Rectangles." *Eureka*, No. 34, 31–5, 1971.
Wells, D. *The Penguin Dictionary of Curious and Interesting Geometry*. London: Penguin, pp. 14–5, 1991.

Blancmange Function

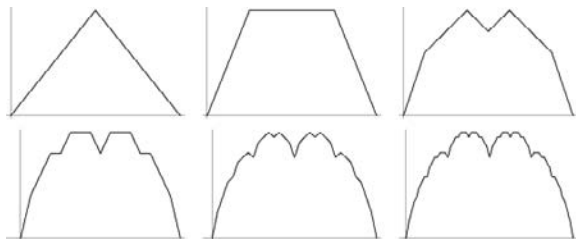


A CONTINUOUS FUNCTION which is nowhere DIFFERENTIABLE. The iterations towards the continuous function are BATRACHIONS resembling the HOFSTADTER-CONWAY \$10,000 SEQUENCE. The first six iterations are illustrated below. The d th iteration contains $N + 1$ points, where $N = 2^d$, and can be obtained by setting $b(0) = b(N) = 0$, letting

$$b(m + 2^{n-1}) = 2^n + \frac{1}{2}[b(m) + b(m + 2^n)],$$

and looping over $n = d$ to 1 by steps of -1 and $m = 0$

to $N - 1$ by steps of 2^n .



Peitgen and Saupe (1988) refer to this curve as the TAKAGI FRACTAL CURVE.

See also HOFSTADTER-CONWAY \$10,000 SEQUENCE, WEIERSTRASS FUNCTION

References

- Dixon, R. *Mathographics*. New York: Dover, pp. 175–76 and 210, 1991.
- Peitgen, H.-O. and Saupe, D. (Eds.). "Midpoint Displacement and Systematic Fractals: The Takagi Fractal Curve, Its Kin, and the Related Systems." §A.1.2 in *The Science of Fractal Images*. New York: Springer-Verlag, pp. 246–48, 1988.
- Takagi, T. "A Simple Example of the Continuous Function without Derivative." *Proc. Phys. Math. Japan* **1**, 176–77, 1903.
- Tall, D. O. "The Blancmange Function, Continuous Everywhere but Differentiable Nowhere." *Math. Gaz.* **66**, 11–2, 1982.
- Tall, D. "The Gradient of a Graph." *Math. Teaching* **111**, 48–2, 1985.
- Wells, D. *The Penguin Dictionary of Curious and Interesting Geometry*. London: Penguin, pp. 16–7, 1991.

Blankinship Algorithm

A method for finding solutions u and v to a linear congruence

$$au + bv = d$$

by constructing a matrix formed by adjoining a vector containing a and b with a UNIT MATRIX,

$$\mathbf{M} = \begin{bmatrix} a & 1 & 0 \\ b & 0 & 1 \end{bmatrix},$$

and applying the EUCLIDEAN ALGORITHM to the first column, while extending the operations to all rows. The algorithm terminates when the first column contains the GREATEST COMMON DIVISOR $\text{GCD}(a, b)$.

See also EUCLIDEAN ALGORITHM, GREATEST COMMON DIVISOR

References

- Blankinship, W. A. "A New Version of the Euclidean Algorithm." *Amer. Math. Monthly* **70**, 742–45, 1963.
- Sérout, R. "The Blankinship Algorithm." §8.2 in *Programming for Mathematicians*. Berlin: Springer-Verlag, pp. 161–63, 2000.

Blaschke Condition

If $\{a_j\} \subseteq D(0, 1)$ (with possible repetitions) satisfies

$$\sum_{j=1}^{\infty} (1 - |a_j|) \leq \infty,$$

where $D(0, 1)$ is the unit open disk, and no $a_j = 0$, then there is a bounded ANALYTIC FUNCTION on $D(0, 1)$ which has ZERO SET consisting precisely of the a_j s, counted according to their MULTIPLICITIES. More specifically, the INFINITE PRODUCT

$$\prod_{j=1}^{\infty} -\frac{\bar{a}_j}{|a_j|} B_{a_j}(z),$$

where $B_{a_j}(z)$ is a BLASCHKE FACTOR and $\bar{z}$ is the COMPLEX CONJUGATE, converges uniformly on compact subsets of $D(0, 1)$ to a bounded analytic function $B(z)$.

See also BLASCHKE FACTOR, BLASCHKE FACTORIZATION, BLASCHKE PRODUCT

References

- Krantz, S. G. "The Blaschke Condition." §9.1.5 in *Handbook of Complex Analysis*. Boston, MA: Birkhäuser, pp. 118–19, 1999.

Blaschke Conjecture

The only WIEDERSEHEN MANIFOLDS are the standard round spheres. The conjecture has been proven by combining the BERGER-KAZDAN COMPARISON THEOREM with A. Weinstein's results for n EVEN and C. T. Yang's for n ODD.

See also WIEDERSEHEN MANIFOLD

References

- Chavel, I. *Riemannian Geometry: A Modern Introduction*. New York: Cambridge University Press, 1994.

Blaschke Factor

If a is a point in the open UNIT DISK, then the Blaschke factor is defined by

$$B_a(z) = \frac{z - a}{1 - \bar{a}z},$$

where $\bar{a}$ is the COMPLEX CONJUGATE of a . Blaschke factors allow the manipulation of the zeros of a HOLOMORPHIC FUNCTION analogously to factors of $(z - a)$ for complex polynomials (Krantz 1999, p. 117).

See also BLASCHKE CONDITION, BLASCHKE FACTORIZATION

References

- Krantz, S. G. "Blaschke Factors." §9.1.1 in *Handbook of Complex Analysis*. Boston, MA: Birkhäuser, p. 117, 1999.

Blaschke Factorization

Let f be a bounded ANALYTIC FUNCTION on $D(0, 1)$ vanishing to order $m \geq 0$ at 0 and let $\{a_j\}$ be its other zeros, listed with multiplicities. Then

$$f(z) = z^m F(z) \prod_{j=1}^{\infty} -\frac{\bar{a}_j}{|a_j|} B_{a_j}(z),$$

where F is a bounded ANALYTIC FUNCTION on $D(0, 1)$, F is zerofree, $\bar{z}$ is the COMPLEX CONJUGATE, and

$$\sup_{z \in D(0, 1)} |f(z)| = \sup_{z \in D(0, 1)} |F(z)|.$$

See also BLASCHKE FACTOR

References

Krantz, S. G. "Blaschke Factorization." §9.1.7 in *Handbook of Complex Analysis*. Boston, MA: Birkhäuser, p. 119, 1999.

Blaschke Product

A Blaschke product is an expression of the form

$$B(z) = z^m \prod_{j=1}^{\infty} -\frac{\bar{a}_j}{|a_j|} B_{a_j}(z),$$

where m is a nonnegative integer and $\bar{z}$ is the COMPLEX CONJUGATE.

See also BLASCHKE FACTOR

References

Krantz, S. G. "Blaschke Products." §9.1.6 in *Handbook of Complex Analysis*. Boston, MA: Birkhäuser, p. 119, 1999.

Blaschke's Theorem

A convex planar domain in which the minimal GENERALIZED DIAMETER is > 1 always contains a CIRCLE of RADIUS $1/3$.

See also GENERALIZED DIAMETER

References

Le Lionnais, F. *Les nombres remarquables*. Paris: Hermann, p. 25, 1983.
Wells, D. *The Penguin Dictionary of Curious and Interesting Geometry*. London: Penguin, pp. 17–8, 1991.

Blasius Differential Equation

The third-order ORDINARY DIFFERENTIAL EQUATION

$$2y''' + yy'' = 0.$$

This equation arises in the theory of fluid boundary layers, and must be solved numerically (Rosenhead 1963; Schlichting 1979; Tritton 1989, p. 129). The velocity profile produced by this differential equation is known as the Blasius profile.

References

Meyer, G. H. *Initial Value Methods for Boundary Value Problems: Theory and Application of Invariant Imbedding*. New York: Academic Press, 1973.
Rosenhead, L. (Ed.). *Laminar Boundary Layers*. Oxford, England: Oxford University Press, 1963.
Schlichting, H. *Boundary Layer Theory*, 7th ed. New York: McGraw-Hill, 1979.
Tritton, D. J. *Physical Fluid Dynamics*, 2nd ed. Oxford, England: Clarendon Press, p. 129, 1989.
Zwillinger, D. *Handbook of Differential Equations*, 3rd ed. Boston, MA: Academic Press, p. 128, 1997.

Bleeksmith-Brillhart-Gerst Theorem

A generalization of SCHRÖTER'S FORMULA.

References

Berndt, B. C. *Ramanujan's Notebooks, Part III*. New York: Springer-Verlag, p. 73, 1985.

Blichfeldt's Lemma

BLICHFELDT'S THEOREM

Blichfeldt's Theorem

Any bounded planar region with POSITIVE AREA $> A$ placed in any position of the UNIT SQUARE LATTICE can be TRANSLATED so that the number of LATTICE POINTS inside the region will be at least $A + 1$ (Blichfeldt 1914, Steinhaus 1983) The theorem can be generalized to n -D.

See also LATTICE POINT, MINKOWSKI CONVEX BODY THEOREM, PICK'S THEOREM

References

Blichfeldt, H. F. "A New Principle in the Geometry of Numbers, with Some Applications." *Trans. Amer. Math. Soc.* **15**, 227–35, 1914.
Steinhaus, H. *Mathematical Snapshots*, 3rd ed. New York: Dover, pp. 97–9, 1999.

B-Line

A line which simultaneously bisects a triangle's perimeter and area.

See also CLEAVER, SPLITTER

References

Todd, A. "Bisecting a Triangle." *Pi Mu Epsilon J.* **11**, 31–7, Fall 1999.
Todd, A. "Bisecting a Triangle." <http://www.math.colostate.edu/~todd/triangle.html>.

BLM/Ho Polynomial

A 1-variable unoriented KNOT POLYNOMIAL $Q(x)$. It satisfies

$$Q_{\text{unknot}} = 1 \quad (1)$$

and the SKEIN RELATIONSHIP

$$Q_{L_+} + Q_{L_-} = x(Q_{L_0} + Q_{L_\infty}). \quad (2)$$

It also satisfies

$$Q_{L_1 \# L_2} = Q_{L_1} Q_{L_2}, \quad (3)$$

where $\#$ is the KNOT SUM and

$$Q_{L^*} = Q_L, \quad (4)$$

where L^* is the MIRROR IMAGE of L . The BLM/Ho polynomials of MUTANT KNOTS are also identical. Brandt *et al.* (1986) give a number of interesting properties. For any LINK L with ≥ 2 components, $Q_L - 1$ is divisible by $2(x - 1)$. If L has c components, then the lowest POWER of x in $Q_L(x)$ is $1 - c$, and

$$\lim_{x \rightarrow 0} x^{c-1} Q_L(x) = \lim_{(\ell, m) \rightarrow (1, 0)} (-m)^{c-1} P_L(\ell, m), \quad (5)$$

where P_L is the HOMFLY POLYNOMIAL. Also, the degree of Q_L is less than the CROSSING NUMBER of L . If L is a 2-BRIDGE KNOT, then

$$Q_L(z) = 2z^{-1} V_L(t) V_L(t^{-1} + 1 - 2z^{-1}), \quad (6)$$

where $z \equiv -t - t^{-1}$ (Kanenobu and Sumi 1993).

The POLYNOMIAL was subsequently extended to the 2-variable KAUFFMAN POLYNOMIAL F , which satisfies

$$Q(x) = F(1, x). \quad (7)$$

Brandt *et al.* (1986) give a listing of Q POLYNOMIALS for KNOTS up to 8 crossings and links up to 6 crossings.

References

- Brandt, R. D.; Lickorish, W. B. R.; and Millett, K. C. "A Polynomial Invariant for Unoriented Knots and Links." *Invent. Math.* **84**, 563–73, 1986.
- Ho, C. F. "A New Polynomial for Knots and Links--Preliminary Report." *Abstracts Amer. Math. Soc.* **6**, 300, 1985.
- Kanenobu, T. and Sumi, T. "Polynomial Invariants of 2-Bridge Knots through 22-Crossings." *Math. Comput.* **60**, 771–78 and S17-S28, 1993.
- Stoimenow, A. "Brandt-Lickorish-Millett-Ho Polynomials." <http://guests.mpim-bonn.mpg.de/alex/ptab/blmh10.html>.
- Weisstein, E. W. "Knots." MATHEMATICA NOTEBOOK KNOTS.M.

Bloch Constant

N.B. A detailed online essay by S. Finch was the starting point for this entry. Let F be the set of COMPLEX ANALYTIC FUNCTIONS f defined on an open region containing the CLOSURE of the UNIT DISK $D = \{z : |z| < 1\}$ satisfying $f(0) = 0$ and $df/dz(0) = 1$. For each f in F , let $b(f)$ be the SUPREMUM of all numbers r such that there is a disk S in D on which f is ONE-TO-ONE and such that $f(S)$ contains a disk of radius r . In 1925, Bloch (Conway 1978) showed that $b(f) \geq 1/72$. Define Bloch's constant by

$$B \equiv \inf \{b(f) : f \in F\}.$$

Ahlfors and Grunsky (1937) derived

$$0.433012701 \dots = \frac{1}{4}\sqrt{3} \leq B < \frac{1}{\sqrt{1+\sqrt{3}}} \frac{\Gamma(\frac{1}{3})\Gamma(\frac{11}{12})}{\Gamma(\frac{1}{4})} < 0.4718617.$$

They also conjectured that the upper limit is actually the value of B ,

$$\begin{aligned} B &= \frac{1}{\sqrt{1+\sqrt{3}}} \frac{\Gamma(\frac{1}{3})\Gamma(\frac{11}{12})}{\Gamma(\frac{1}{4})} \\ &= \sqrt{\pi} 2^{1/4} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{4})} \sqrt{\frac{\Gamma(\frac{11}{12})}{\Gamma(\frac{1}{12})}} \\ &= 0.4718617 \dots \end{aligned}$$

(Le Lionnais 1983).

See also LANDAU CONSTANT

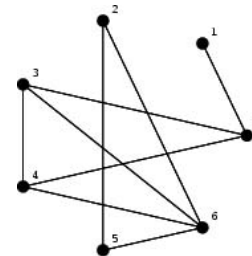
References

- Conway, J. B. *Functions of One Complex Variable I*, 2nd ed. New York: Springer-Verlag, 1989.
- Finch, S. "Favorite Mathematical Constants." <http://www.mathsoft.com/asolve/constant/bloch/bloch.html>.
- Le Lionnais, F. *Les nombres remarquables*. Paris: Hermann, p. 25, 1983.
- Minda, C. D. "Bloch Constants." *J. d'Analyse Math.* **41**, 54–4, 1982.

Bloch-Landau Constant

LANDAU CONSTANT

Block



A maximal BICONNECTED SUBGRAPH of a given GRAPH G . In the illustration above, the blocks are $\{2, 5, 6\}$, $\{3, 4, 6, 7\}$, and $\{1, 7\}$.

If a graph G is biconnected, then G itself is called a block (Harary 1994, p. 26) or a BICONNECTED GRAPH (Skiena 1990, p. 175).

See also BICONNECTED GRAPH, BLOCK DESIGN, DIGIT BLOCK, SQUARE POLYOMINO

References

- Aho, A. V.; Hopcroft, J. E.; and Ullman, J. D. *The Design and Analysis of Computer Algorithms*. Reading, MA: Addison-Wesley, 1974.
- Harary, F. *Graph Theory*. Reading, MA: Addison-Wesley, 1994.
- Skiena, S. "Biconnected Components." §5.1.4 in *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*. Reading, MA: Addison-Wesley, pp. 175–77, 1990.

Block (Group Action)

A GROUP ACTION $G \times \Omega \rightarrow \Omega$ might preserve a special kind of PARTITION of Ω called a system of blocks. A block is a SUBSET Δ of Ω such that for any group element g either

1. g preserves Δ , i.e., $g\Delta = \Delta$, or
2. g translates everything in Δ out of Δ , i.e., $g\Delta \cap \Delta = \emptyset$.

For example, the GENERAL LINEAR GROUP $GL(2, \mathbb{R})$ acts on the plane minus the origin, $\mathbb{R}^2 - (0, 0)$. The lines $A = \{(at, bt)\}$ are blocks because either a line is mapped to itself, or to another line. Of course, the points on the line may be rescaled, so the lines in A are minimal blocks.

In fact, if two blocks intersect then their intersection is also a block. Hence, the minimal blocks form a PARTITION of Ω . It is important to avoid confusion with the notion of a block in a BLOCK DESIGN, which is different.

See also GROUP, PRIMITIVE (GROUP ACTION), STEINER SYSTEM

References

Dixon, J. and Mortimer, B. *Permutation Groups*. New York: Springer-Verlag, 1996.

Block (Set)

One of the disjoint SUBSETS making up a SET PARTITION. A block containing n elements is called an n -block. The partitioning of sets into blocks can be denoted using a RESTRICTED GROWTH STRING.

See also BLOCK DESIGN, RESTRICTED GROWTH STRING, SET PARTITION

Block Design

An incidence system (v, k, λ, r, b) in which a set X of v points is partitioned into a family A of b subsets (blocks) in such a way that any two points determine λ blocks with k points in each block, and each point is contained in r different blocks. It is also generally required that $k < v$, which is where the "incomplete" comes from in the formal term most often encountered for block designs, BALANCED INCOMPLETE BLOCK DESIGNS (BIBD). The five parameters are not independent, but satisfy the two relations

$$vr = bk \quad (1)$$

$$\lambda(v-1) = r(k-1). \quad (2)$$

A BIBD is therefore commonly written as simply (v, k, λ) , since b and r are given in terms of v, k , and λ by

$$b = \frac{v(v-1)\lambda}{k(k-1)} \quad (3)$$

$$r = \frac{\lambda(v-1)}{k-1}. \quad (4)$$

A BIBD is called SYMMETRIC if $b = v$ (or, equivalently, $r = k$).

Writing $X = \{x_i\}_{i=1}^v$ and $A = \{A_j\}_{j=1}^b$ then the INCIDENCE MATRIX of the BIBD is given by the $v \times b$ MATRIX $\mathbf{M}$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if } x_i \in A_j \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

This matrix satisfies the equation

$$\mathbf{M}\mathbf{M}^T = (r - \lambda)\mathbf{I} + \lambda\mathbf{J}, \quad (6)$$

where $\mathbf{I}$ is a $v \times v$ IDENTITY MATRIX and $\mathbf{J}$ is the $v \times v$ UNIT MATRIX (Dinitz and Stinson 1992).

Examples of BIBDs are given in the following table.

Block Design	(v, k, λ)
AFFINE PLANE	$(n^2, n, 1)$
FANO PLANE	$(7, 3, 1)$
HADAMARD DESIGN	SYMMETRIC $(4n + 3, 2n + 1, n)$
PROJECTIVE PLANE	SYMMETRIC $(n^2 + n + 1, n + 1, 1)$
STEINER TRIPLE SYSTEM	$(v, 3, 1)$
UNITAL	$(q^3 + 1, q + 1, 1)$

See also AFFINE PLANE, DESIGN, FANO PLANE, HADAMARD DESIGN, PARALLEL CLASS, PROJECTIVE PLANE, RESOLUTION, RESOLVABLE, STEINER TRIPLE SYSTEM, SYMMETRIC BLOCK DESIGN, UNITAL

References

Dinitz, J. H. and Stinson, D. R. "A Brief Introduction to Design Theory." Ch. 1 in *Contemporary Design Theory: A Collection of Surveys* (Ed. J. H. Dinitz and D. R. Stinson). New York: Wiley, pp. 1–2, 1992.
 Ryser, H. J. "The (b, v, r, k, λ) -Configuration." §8.1 in *Combinatorial Mathematics*. Buffalo, NY: Math. Assoc. Amer., pp. 96–02, 1963.

Block Diagonal Matrix

A block diagonal matrix, also called a diagonal block matrix, is a SQUARE DIAGONAL MATRIX in which the diagonal elements are SQUARE MATRICES of any size (possibly even 1×1), and the off-diagonal elements are 0. A block diagonal matrix is therefore a BLOCK MATRIX in which the blocks off the diagonal are the ZERO MATRICES, and the diagonal matrices are SQUARE.

Block diagonal matrices can be constructed in *Mathematica* using the following code snippet.

```
<<LinearAlgebra`MatrixManipulation`
BlockDiagonal[a_List]:=
Module[{n=Length[a],lens=Length/@a,i,k,tmp},
k=Outer[List,lens,lens];
tmp=Map[ZeroMatrix[#1[[1]],#1[[2]]]&,k,{2}];
BlockMatrix@
ReplacePart[tmp,a,Table[{i,i},{i,Length[a]}],
Table[{i},{i,Length[a]}]]]
```

See also BLOCK MATRIX, CAYLEY-HAMILTON THEOREM, DIAGONAL MATRIX, DIRECT SUM, JORDAN CANONICAL FORM, LINEAR TRANSFORMATION, MATRIX, MATRIX DIRECT SUM

Block Growth

Let $(x_0x_1x_2\dots)$ be a sequence over a finite ALPHABET A (all the entries are elements of A). Define the block growth function $B(n)$ of a sequence to be the number of ADMISSIBLE words of length n . For example, in the sequence $aabaabaabaab\dots$, the following words are ADMISSIBLE

Length	Admissible Words
1	a, b
2	aa, ab, ba
3	aab, aba, baa
4	$aaba abaa, baab$

so $B(1)=2$, $B(2)=3$, $B(3)=3$, $B(4)=3$, and so on. Notice that $B(n) \leq B(n+1)$, so the block growth function is always nondecreasing. This is because any ADMISSIBLE word of length n can be extended rightwards to produce an ADMISSIBLE word of length $n+1$. Moreover, suppose $B(n)=B(n+1)$ for some n . Then each admissible word of length n extends to a *unique* ADMISSIBLE word of length $n+1$.

For a SEQUENCE in which each substring of length n uniquely determines the next symbol in the SEQUENCE, there are only finitely many strings of length n , so the process must eventually cycle and the SEQUENCE must be eventually periodic. This gives us the following theorems:

1. If the SEQUENCE is eventually periodic, with least period p , then $B(n)$ is strictly increasing until it reaches p , and $B(n)$ is constant thereafter.
2. If the SEQUENCE is not eventually periodic, then $B(n)$ is strictly increasing and so $B(n) \geq n+1$ for

all n . If a SEQUENCE has the property that $B(n) = n+1$ for all n , then it is said to have minimal block growth, and the SEQUENCE is called a STURMIAN SEQUENCE.

The block growth is also called the GROWTH FUNCTION or the COMPLEXITY of a SEQUENCE.

Block Matrix

A block matrix is a MATRIX that is defined using smaller matrices, called blocks. For example,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (1)$$

where A , B , C , and D are themselves matrices, is a block matrix. In the specific example

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad (2)$$

$$B = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \quad (3)$$

$$C = \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{bmatrix} \quad (4)$$

$$D = \begin{bmatrix} 5 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 5 \end{bmatrix}, \quad (5)$$

it is the matrix

$$\begin{bmatrix} 0 & 2 & 3 & 3 & 3 \\ 2 & 0 & 3 & 3 & 3 \\ 4 & 4 & 5 & 0 & 5 \\ 4 & 4 & 0 & 5 & 0 \\ 4 & 4 & 5 & 0 & 5 \end{bmatrix}. \quad (6)$$

Block matrices can be created using `BlockMatrix[blocks]` in the *Mathematica* add-on package `LinearAlgebra`MatrixMultiplication`` (which can be loaded with the command `<<LinearAlgebra``).

When two block matrices have the same shape and their diagonal blocks are square matrices, then they multiply similarly to MATRIX MULTIPLICATION. For example,

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{bmatrix}. \quad (7)$$

When the blocks are SQUARE MATRICES, the set of invertible block matrices form a group, which is a special case of the GENERAL LINEAR GROUP. In this case, it is $GL_2(R^*)$, the invertible two by two matrices with entries in the UNITS of a RING R , where here R is the ring of square matrices.

See also BLOCK DIAGONAL MATRIX, CAYLEY-HAMILTON THEOREM, MATRIX, RING

Blow-Up

A common mechanism which generates SINGULARITIES from smooth initial conditions.

See also BLOW-UP LEMMA

Blow-Up Lemma

The blow-up lemma essentially says that regular pairs in SZEMERÉDI'S REGULARITY LEMMA behave like COMPLETE BIPARTITE GRAPHS from the point of view of embedding bounded degree subgraphs.

In particular, given a graph R of order r , minimal VERTEX DEGREE δ and maximal VERTEX DEGREE Δ , then there exists an $\epsilon > 0$ such that the following holds. Let N be an arbitrary positive integer, and replace the vertices of R with pairwise disjoint N -sets $V_1, V_2, \dots, V_r$ (blowing up). Now construct two graphs on the same vertex set $V = \cup V_i$. The graph $R(N)$ is obtained by replacing all edges of R with copies of the complete bipartite graph $K_{N,N}$, and construct a sparser graph by replacing the edges of R with some (ϵ, δ) -superregular pair. If a graph H with $\Delta(H) \leq \Delta$ is embeddable into $R(N)$, then it is already embeddable into G (Komlós *et al.* 1998).

See also SZEMERÉDI'S REGULARITY LEMMA

References

- Komlós, J.; Sárközy, G. N.; and Szemerédi, E. "Blow-Up Lemma." *Combinatorica* **17**, 109–23, 1997.
 Komlós, J.; Sárközy, G. N.; and Szemerédi, E. "Proof of the Seymour Conjecture for Large Graphs." *Ann. Comb.* **2**, 43–0, 1998.

Blue-Empty Coloring

BLUE-EMPTY GRAPH

Blue-Empty Graph

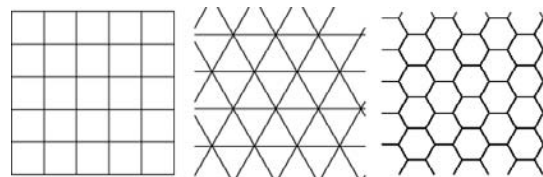
AN EXTREMAL GRAPH in which the forced TRIANGLES are all the same color. Call R the number of red MONOCHROMATIC FORCED TRIANGLES and B the number of blue MONOCHROMATIC FORCED TRIANGLES, then a blue-empty graph is an EXTREMAL GRAPH with $B = 0$. For EVEN n , a blue-empty graph can be achieved by coloring red two COMPLETE SUBGRAPHS of $n/2$ points (the RED NET method). There is no blue-empty coloring for ODD n except for $n = 7$ (Lorden 1962).

See also COMPLETE GRAPH, EXTREMAL GRAPH, MONOCHROMATIC FORCED TRIANGLE, RED NET

References

- Lorden, G. "Blue-Empty Chromatic Graphs." *Amer. Math. Monthly* **69**, 114–20, 1962.
 Sauvé, L. "On Chromatic Graphs." *Amer. Math. Monthly* **68**, 107–11, 1961.

Board



A board is a subset of the polygons determined by a number of (usually regularly spaced and oriented) lines. These polygons form the spaces on which "pieces" can be placed and move in many games (called board games). The simplest division the plane is into equal squares. The 3×3 square board is used in TIC-TAC-TOE. The 8×8 square board is used in CHECKERS and CHESS. Hexagonal boards are used in some games. Chinese checkers uses a board in the space of a pentagram with spaces at the vertices of a regular triangular tiling.

See also CHECKERS, CHESS, CHESSBOARD, GRID, ROOK NUMBER, TIC-TAC-TOE

References

- Bell, R. C. *Board and Table Games from Many Civilizations*. New York: Dover, 1980.
 Gardner, M. "Four Unusual Board Games." Ch. 5 in *The Sixth Book of Mathematical Games from Scientific American*. Chicago, IL: University of Chicago Press, pp. 39–7, 1984.
 Murray, H. J. R. *A History of Board-Games Other than Chess*. New York: Oxford University Press, 1952.
 Parlett, D. *The Oxford History of Board Games*. Oxford, England: Oxford University Press, 1999.
 Steinhaus, H. *Mathematical Snapshots*, 3rd ed. New York: Dover, p. 10, 1999.

Boatman's Knot

CLOVE HITCH

Bôcher Equation

A second-order ORDINARY DIFFERENTIAL EQUATION OF THE FORM

$$y'' + \frac{1}{2} \left[\frac{m_1}{x - a_1} + \dots + \frac{m_{n-1}}{x - a_{n-1}} \right] y' + \frac{1}{4} \left[\frac{A_0 + A_1 x + \dots + A_1 x^1}{(x - a_1)^{m_1} (x - a_2)^{m_2} \dots (x - a_{n-1})^{m_{n-1}}} \right] y = 0.$$

References

- Moon, P. and Spencer, D. E. "Differential Equations." §6 in *Field Theory Handbook, Including Coordinate Systems, Differential Equations, and Their Solutions*, 2nd ed. New York: Springer-Verlag, pp. 144–62, 1988.
 Zwillinger, D. (Ed.). *CRC Standard Mathematical Tables and Formulae*. Boca Raton, FL: CRC Press, p. 413, 1995.

Bochner Identity

For a smooth HARMONIC MAP $u : M \rightarrow N$,

$$\Delta(|\nabla u|^2) = |\nabla(du)|^2 + \langle \text{Ric}_M \nabla u, \nabla u \rangle - \langle \text{Riem}_N(u)(\nabla u, \nabla u) \nabla u, \nabla u \rangle,$$

where ∇ is the GRADIENT, Ric is the RICCI TENSOR, and Riem is the RIEMANN TENSOR.

References

Eels, J. and Lemaire, L. "A Report on Harmonic Maps." *Bull. London Math. Soc.* **10**, 1–8, 1978.

Bochner's Theorem

Among the continuous functions on $\mathbb{R}^n$, the POSITIVE DEFINITE FUNCTIONS are those functions which are the FOURIER TRANSFORMS of finite measures.

Bode's Rule

Let the values of a function $f(x)$ be tabulated at points x_i equally spaced by $h = x_{i+1} - x_i$, so $f_1 = f(x_1)$, $f_2 = f(x_2)$, ..., $f_5 = f(x_5)$. Then Bode's rule approximating the integral of $f(x)$ is given by the NEWTON-COTES-like formula

$$\int_{x_1}^{x_5} f(x) dx = \frac{2}{45}h(7f_1 + 32f_2 + 12f_3 + 32f_4 + 7f_5) - \frac{8}{945}h^7 f^{(6)}(\xi).$$

See also HARDY'S RULE, NEWTON-COTES FORMULAS, SIMPSON'S 3/8 RULE, SIMPSON'S RULE, TRAPEZOIDAL RULE, WEDDLE'S RULE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover, p. 886, 1972.

Bogdanov Map

A 2-D MAP which is conjugate to the HÉNON MAP in its nondissipative limit. It is given by

$$\begin{aligned} x' &= x + y' \\ y' &= y + \epsilon y + kx(x - 1) + \mu xy. \end{aligned}$$

See also HÉNON MAP

References

Arrowsmith, D. K.; Cartwright, J. H. E.; Lansbury, A. N.; and Place, C. M. "The Bogdanov Map: Bifurcations, Mode Locking, and Chaos in a Dissipative System." *Int. J. Bifurcation Chaos* **3**, 803–42, 1993.
Bogdanov, R. "Bifurcations of a Limit Cycle for a Family of Vector Fields on the Plane." *Selecta Math. Soviet* **1**, 373–88, 1981.

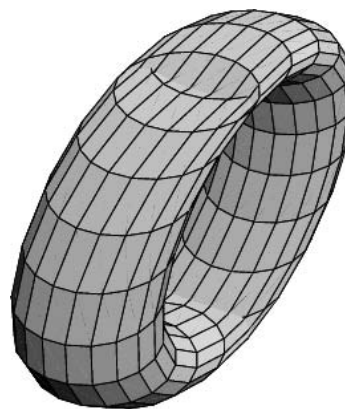
Bogomolov-Miyaoka-Yau Inequality

Relates invariants of a curve defined over the INTEGERS. If this inequality were proven true, then FERMAT'S LAST THEOREM would follow for sufficiently large exponents. Miyaoka claimed to have proven this inequality in 1988, but the proof contained an error.

See also FERMAT'S LAST THEOREM

References

Cox, D. A. "Introduction to Fermat's Last Theorem." *Amer. Math. Monthly* **101**, 3–4, 1994.

Bohemian Dome

A QUARTIC SURFACE which can be constructed as follows. Given a CIRCLE C and PLANE E PERPENDICULAR to the PLANE of C , move a second CIRCLE K of the same RADIUS as C through space so that its CENTER always lies on C and it remains PARALLEL to E . Then K sweeps out the Bohemian dome. It can be given by the PARAMETRIC EQUATIONS

$$x = a \cos u$$

$$y = b \cos v + a \sin u$$

$$z = c \sin v$$

where $u, v \in [0, 2\pi)$. In the above plot, $a = 0.5$, $b = 1.5$, and $c = 1$.

See also QUARTIC SURFACE

References

Fischer, G. (Ed.). *Mathematical Models from the Collections of Universities and Museums*. Braunschweig, Germany: Vieweg, pp. 19–0, 1986.
Fischer, G. (Ed.). Plate 50 in *Mathematische Modelle / Mathematical Models, Bildband / Photograph Volume*. Braunschweig, Germany: Vieweg, p. 50, 1986.
Gray, A. *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2nd ed. Boca Raton, FL: CRC Press, p. 389, 1997.
Nordstrand, T. "Bohemian Dome." <http://www.uib.no/people/nfytn/bodtxt.htm>.

Bohr Matrix

A finite or infinite SQUARE MATRIX with RATIONAL entries. (If the matrix is infinite, all but a finite number of entries in each row must be 0.) The sum or product of two Bohr matrices is another Bohr matrix.

References

Apostol, T. M. "Bohr Matrices." §8.4 in *Modular Functions and Dirichlet Series in Number Theory*, 2nd ed. New York: Springer-Verlag, pp. 167–68, 1997.

Bohr-Favard Inequalities

If f has no spectrum in $[-\lambda, \lambda]$, then

$$\|f\|_{\infty} \leq \frac{\pi}{2\lambda} \|f'\|_{\infty}$$

(Bohr 1935). A related inequality states that if A_k is the class of functions such that

$$f(x) = f(x + 2\pi), f(x), f'(x), \dots, f^{(k-1)}(x)$$

are absolutely continuous and $\int_0^{2\pi} f(x) dx = 0$, then

$$\|f\|_{\infty} \leq \frac{4}{\pi} \sum_{v=0}^{\infty} \frac{(-1)^{v(k+1)}}{(2v+1)^{k+1}} \|f^{(k)}(x)\|_{\infty}$$

(Northcott 1939). Further, for each value of k , there is always a function $f(x)$ belonging to A_k and not identically zero, for which the above inequality becomes an equality (Favard 1936). These inequalities are discussed in Mitrinovic *et al.* (1991).

References

- Bohr, H. "Ein allgemeiner Satz über die Integration eines trigonometrischen Polynoms." *Prace Matem.-Fiz.* **43**, 1935.
- Favard, J. "Application de la formule sommatoire d'Euler à la démonstration de quelques propriétés extrémales des intégrales des fonctions périodiques ou presquepériodiques." *Mat. Tidsskr. B*, 81–4, 1936. Reviewed in *Zentralblatt f. Math.* **16**, 58–9, 1939.
- Mitrinovic, D. S.; Pecaric, J. E.; and Fink, A. M. *Inequalities Involving Functions and Their Integrals and Derivatives*. Dordrecht, Netherlands: Kluwer, pp. 71–2, 1991.
- Northcott, D. G. "Some Inequalities Between Periodic Functions and Their Derivatives." *J. London Math. Soc.* **14**, 198–02, 1939.
- Tikhomirov, V. M. "Approximation Theory." In *Analysis II. Convex Analysis and Approximation Theory* (Ed. R. V. Gamkrelidze). New York: Springer-Verlag, pp. 93–55, 1990.

Bohr-Mollerup Theorem

If a function $\varphi : (0, \infty) \rightarrow (0, \infty)$ satisfies

1. $\ln[\varphi(x)]$ is convex,
2. $\varphi(x+1) = x\varphi(x)$ for all $x > 0$, and
3. $\varphi(1) = 1$,

then $\varphi(x)$ is the GAMMA FUNCTION $\Gamma(x)$. Therefore, by ANALYTIC CONTINUATION, $\Gamma(z)$ is the only MEROMORPHIC FUNCTION on $\mathbb{C}$ satisfying the functional

equation

$$z\Gamma(z) = \Gamma(z+1)$$

with $\Gamma(1) = 1$ and which is logarithmically convex on the positive REAL AXIS.

See also GAMMA FUNCTION

References

Krantz, S. G. "The Bohr-Mollerup Theorem." §13.1.10 in *Handbook of Complex Analysis*. Boston, MA: Birkhäuser, p. 157, 1999.

Bolyai-Gerwein Theorem

WALLACE-BOLYAI-GERWEIN THEOREM

Bolza Problem

Given the functional

$$U = \int_{t_0}^{t_1} f(y_1, \dots, y_n; y'_1, \dots, y'_n) dt + G(y_{10}, \dots, y_{nr}; y_{11}, \dots, y_{n1}), \quad (1)$$

find in a class of arcs satisfying p differential and q finite equations

$$\phi_{\alpha}(y_1, \dots, y_n; y'_1, \dots, y'_n) = 0 \quad \text{for } \alpha = 1, \dots, p \quad (2)$$

$$\psi_{\beta}(y_1, \dots, y_n) = 0 \quad \text{for } \beta = 1, \dots, q \quad (3)$$

as well as the r equations on the endpoints

$$\chi_{\gamma}(y_{10}, \dots, y_{nr}; y_{11}, \dots, y_{n1}) = 0 \quad (4)$$

for $\gamma = 1, \dots, r$,

one which renders U a minimum.

References

Goldstine, H. H. *A History of the Calculus of Variations from the 17th through the 19th Century*. New York: Springer-Verlag, p. 374, 1980.

Bolzano Theorem

BOLZANO-WEIERSTRASS THEOREM

Bolzano-Weierstrass Theorem

Every BOUNDED infinite set in $\mathbb{R}^n$ has an ACCUMULATION POINT.

For $n = 1$, an infinite subset of a closed bounded set S has an ACCUMULATION POINT in S . For instance, given a bounded SEQUENCE a_n , with $-C \leq a_n \leq C$ for all n , it must have a MONOTONIC subsequence a_{n_k} . The SUBSEQUENCE a_{n_k} must converge because it is monotonic and bounded. Because S is closed, it contains the limit of a_{n_k} .

The Bolzano-Weierstrass theorem is closely related to the HEINE-BOREL THEOREM and CANTOR'S INTERSECTION THEOREM, each of which can be easily derived from either of the other two.

See also ACCUMULATION POINT, CANTOR'S INTERSECTION THEOREM, HEINE-BOREL THEOREM, INTERMEDIATE VALUE THEOREM

References

- Jeffreys, H. and Jeffreys, B. S. §1.034 in *Methods of Mathematical Physics, 3rd ed.* Cambridge, England: Cambridge University Press, pp. 9–0, 1988.
Knopp, K. *Theory of Functions Parts I and II, Two Volumes Bound as One, Part I.* New York: Dover, p. 7, 1996.

Bombieri Inner Product

For HOMOGENEOUS POLYNOMIALS P and Q of degree n ,

$$[P, Q] \equiv \sum_{i_1, \dots, i_n \geq 0} (i_1! \dots i_n!) (a_{i_1, \dots, i_n} b_{i_1, \dots, i_n}).$$

Bombieri Norm

This entry contributed by KEVIN O'BRYANT

The Bombieri p -norm of a polynomial

$$Q(x) = \sum_{i=0}^n a_i x^i \quad (1)$$

is defined by

$$[Q]_p \equiv \left[\sum_{i=0}^n \binom{n}{i}^{1-p} |a_i|^p \right]^{1/p}, \quad (2)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. The most remarkable feature of Bombieri's norm is that given polynomials R and S such that $RS = Q$, then BOMBIERI'S INEQUALITY

$$[R]_2 [S]_2 \leq \binom{n}{m}^{1/2} [Q]_2 \quad (3)$$

holds, where n is the degree of Q , and m is the degree of either R or S . This theorem captures the heuristic that if R and S have big coefficients, then so does RS , i.e., there can't be too much cancellation.

See also NORM, BOMBIERI'S INEQUALITY, POLYNOMIAL NORM

References

- Beauzamy, B.; Bombieri, E.; Enflo, P.; and Montgomery, H. L. "Products of Polynomials in Many Variables." *J. Number Th.* **36**, 219–45, 1990.
Borwein, P. and Erdélyi, T. "Bombieri's Norm." §5.3.E.7 in *Polynomials and Polynomial Inequalities*. New York: Springer-Verlag, p. 274, 1995.
Reznick, B. "An Inequality for Products of Polynomials." *Proc. Amer. Math. Soc.* **117**, 1063–073, 1993.

Bombieri's Inequality

For HOMOGENEOUS POLYNOMIALS P and Q of degree m and n , then

Bombieri's Theorem

$$[P \cdot Q]_2 \geq \sqrt{\frac{m!n!}{(m+n)!}} [P]_2 [Q]_2,$$

where $[P \cdot Q]_2$ is the BOMBIERI NORM. If $m = n$, this becomes

$$[P \cdot Q]_2 \geq [P]_2 [Q]_2,$$

See also BOMBIERI NORM, BEAUZAMY AND DÉGOT'S IDENTITY, REZNIK'S IDENTITY

References

- Borwein, P. and Erdélyi, T. "Bombieri's Norm." §5.3.E.7 in *Polynomials and Polynomial Inequalities*. New York: Springer-Verlag, p. 274, 1995.

Bombieri's Theorem

Define

$$E(x; q, a) \equiv \psi(x; q, a) - \frac{x}{\phi(q)}, \quad (1)$$

where

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n) \quad (2)$$

(Davenport 1980, p. 121), $\Lambda(n)$ is the MANGOLDT FUNCTION, and $\phi(q)$ is the TOTIENT FUNCTION. Now define

$$E(x; q) = \max_{\substack{a \\ (a, q)=1}} |E(x; q, a)| \quad (3)$$

where the sum is over a RELATIVELY PRIME to q , $(a, q) = 1$, and

$$E^*(x, q) = \max_{y \leq x} E(y, q). \quad (4)$$

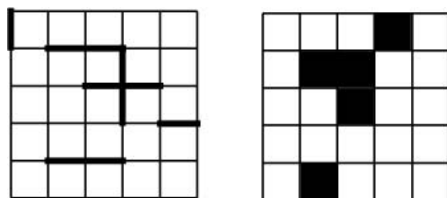
Bombieri's theorem then says that for fixed $A > 0$,

$$\sum_{q \leq Q} E^*(x, q) \ll \sqrt{x} Q (\ln x)^5, \quad (5)$$

provided that $\sqrt{x} (\ln x)^{-4} < Q < \sqrt{x}$.

References

- Bombieri, E. "On the Large Sieve." *Mathematika* **12**, 201–25, 1965.
Davenport, H. "Bombieri's Theorem." Ch. 28 in *Multiplicative Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 161–68, 1980.

Bond Percolation

bond percolation

site percolation

A PERCOLATION which considers the lattice edges as the relevant entities (left figure).

See also PERCOLATION THEORY, SITE PERCOLATION

Bonferroni Correction

The Bonferroni correction is a multiple-comparison correction used when several independent STATISTICAL TESTS are being performed simultaneously (since while a given ALPHA VALUE α may be appropriate for each individual comparison, it is not for the set of *all* comparisons). In order to avoid a lot of spurious positives, the ALPHA VALUE needs to be lowered to account for the number of comparisons being performed.

The simplest and most conservative approach is the Bonferroni correction, which sets the ALPHA VALUE for the entire set of n comparisons equal to α by taking the ALPHA VALUE for *each* comparison equal to α/n . Explicitly, given n tests T_i for hypotheses H_i ($1 \leq i \leq n$) under the assumption H_0 that all hypotheses H_i are false, and if the individual test critical values are $\leq \alpha/n$, then the experiment-wide critical value is $\leq \alpha$. In equation form, if

$$P(T_i \text{ passes} | H_0) \leq \frac{\alpha}{n}$$

for $1 \leq i \leq n$, then

$$P(\text{some } T_i \text{ passes} | H_0) \leq \alpha,$$

which follows from BONFERRONI'S INEQUALITIES.

Another correction instead uses $1 - (1 - \alpha)^{1/n}$. While this choice is applicable for two-sided hypotheses, multivariate normal statistics, and positive orthant dependent statistics, it is not, in general, correct (Shaffer 1995).

See also ALPHA VALUE, HYPOTHESIS TESTING, STATISTICAL TEST

References

- Bonferroni, C. E. "Il calcolo delle assicurazioni su gruppi di teste." In *Studi in Onore del Professore Salvatore Ortu Carboni*. Rome: Italy, pp. 13–0, 1935.
- Bonferroni, C. E. "Teoria statistica delle classi e calcolo delle probabilità." *Pubblicazioni del R Istituto Superiore di Scienze Economiche e Commerciali di Firenze* 8, 3–2, 1936.

Dewey, M. "Carlo Emilio Bonferroni: Life and Works." <http://www.nottingham.ac.uk/~mhmd/life.html>.

Miller, R. G. Jr. *Simultaneous Statistical Inference*. New York: Springer-Verlag, 1991.

Perneger, T. V. "What's Wrong with Bonferroni Adjustments." *Brit. Med. J.* 316, 1236–238, 1998.

Shaffer, J. P. "Multiple Hypothesis Testing." *Ann. Rev. Psych.* 46, 561–84, 1995.

Bonferroni Test

BONFERRONI CORRECTION

Bonferroni's Inequalities

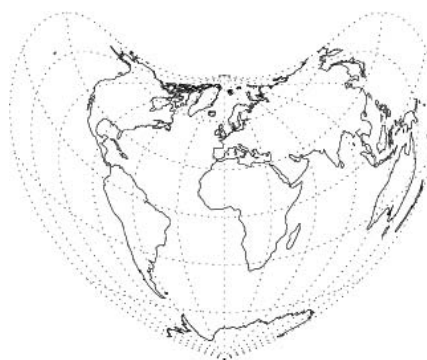
Let $P(E_i)$ be the probability that E_i is true, and $P(\bigcup_{i=1}^n E_i)$ be the probability that at least one of $E_1, E_2, \dots, E_n$ is true. Then

$$P\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n P(E_i).$$

A slightly wider class of inequalities are also known as "Bonferroni inequalities."

References

- Comtet, L. "Bonferroni Inequalities." §4.7 in *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, rev. enl. ed. Dordrecht, Netherlands: Reidel, pp. 193–94, 1974.
- Galambos, J.; and Simonelli, I. *Bonferroni-Type Inequalities with Applications*. New York: Springer-Verlag, 1996.

Bonne Projection

A MAP PROJECTION which resembles the shape of a heart. Let ϕ_1 be the standard parallel, λ_0 the central meridian, ϕ be the LATITUDE, and λ the LONGITUDE on a UNIT SPHERE. Then

$$x = \rho \sin E \quad (1)$$

$$y = \cot \phi_1 - \rho \cos E, \quad (2)$$

where

$$\rho = \cot \phi_1 + \phi_1 - \phi \quad (3)$$

$$E = \frac{(\lambda - \lambda_0) \cos \phi}{\rho}. \quad (4)$$

The inverse FORMULAS are

$$\phi = \cot \phi_1 + \phi_1 - \rho \quad (5)$$

$$\lambda = \lambda_0 + \frac{\rho}{\cos \phi} \tan^{-1} \left(\frac{x}{\cot \phi_1 - y} \right), \quad (6)$$

where

$$\rho = \pm \sqrt{x^2 + (\cot \phi_1 - y)^2}. \quad (7)$$

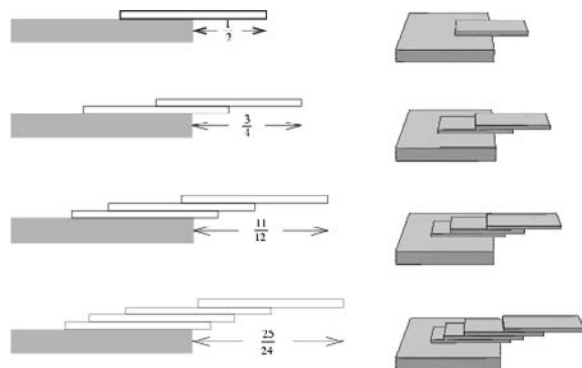
The WERNER PROJECTION is a special case of the Bonne projection.

See also MAP PROJECTION, WERNER PROJECTION

References

- MathWorks. "Mapping Toolbox: Bonne Projection." <http://www.mathworks.com/access/helpdesk/help/toolbox/map/bonneprojection.shtml>.
 Snyder, J. P. *Map Projections--A Working Manual*. U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 138–40, 1987.

Book Stacking Problem



How far can a stack of n books protrude over the edge of a table without the stack falling over? It turns out that the maximum overhang possible d_n for n books (in terms of book lengths) is half the n th partial sum of the HARMONIC SERIES, given explicitly by

$$d_n = \frac{1}{2} \sum_{k=1}^n \frac{1}{k} = \frac{1}{2} [\gamma + \Psi(1+n)]$$

where $\Psi(z)$ is the DIGAMMA FUNCTION and γ is the EULER-MASCHERONI CONSTANT. The first few values

are

$$d_1 = \frac{1}{2} = 0.5$$

$$d_2 = \frac{3}{4} = 0.75$$

$$d_3 = \frac{11}{12} \approx 0.91667$$

$$d_4 = \frac{25}{24} \approx 1.04167,$$

(Sloane's A001008 and A002805).

In order to find the number of stacked books required to obtain d book-lengths of overhang, solve the d_n equation for d , and take the CEILING FUNCTION. For $n = 1, 2, \dots$ book-lengths of overhang, 4, 31, 227, 1674, 12367, 91380, 675214, 4989191, 36865412, 272400600, ... (Sloane's A014537) books are needed.

References

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Boole

IVERSON BRACKET

Boole Polynomial

Polynomials $s_k(x; \lambda)$ which form a SHEFFER SEQUENCE with

$$g(t) = 1 + e^{\lambda t} \quad (1)$$

$$f(t) = e^t - 1 \quad (2)$$

and have GENERATING FUNCTION

$$\sum_{k=0}^{\infty} \frac{s_k(x; \lambda)}{k!} t^k = \frac{(1+t)^x}{1+(1+t)^\lambda}. \quad (3)$$

The first few are

$$s_0(x; \lambda) = \frac{1}{2}$$

$$s_1(x; \lambda) = \frac{1}{4}(2x - \lambda)t$$

$$s_2(x; \lambda) = \frac{1}{4}[2x(x - \lambda - 1) + \lambda].$$

Jordan (1950) considers the related polynomials $r_n(x)$

which form a SHEFFER SEQUENCE with

$$g(t) = \frac{1}{2}(1 + e^t) \quad (4)$$

$$f(t) = e^t - 1. \quad (5)$$

These polynomials have GENERATING FUNCTION

$$\sum_{k=0}^{\infty} \frac{r_n(x)}{k!} t^k = \frac{2(1+t)^x}{2+t}. \quad (6)$$

The first few are

$$\begin{aligned} r_0(x) &= 1 \\ r_1(x) &= \frac{1}{2}(2x - 1) \\ r_2(x) &= \frac{1}{2}(2x^2 - 4x + 1) \\ r_3(x) &= \frac{1}{4}(4x^3 - 18x^2 + 20x - 3). \end{aligned}$$

The PETERS POLYNOMIALS are a generalization of the Boole polynomials.

See also PETERS POLYNOMIAL

References

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Boole's Inequality

Let $P(E_i)$ be the probability of an event E_i occurring. Then

$$P\left(\bigcup_{i=1}^N E_i\right) \leq \sum_{i=1}^N P(E_i),$$

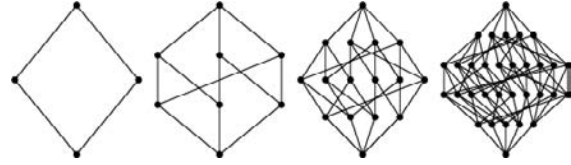
where $\cup$ denotes the UNION. If E_i and E_j are DISJOINT SETS for all i and j , then the INEQUALITY becomes an equality.

See also DISJOINT SETS, UNION

Boolean Algebra

A mathematical structure which is similar to a BOOLEAN RING, but which is defined using the meet and join operators instead of the usual addition and multiplication operators. Explicitly, a Boolean algebra is the PARTIAL ORDER on subsets defined by inclusion (Skiena 1990, p. 207), i.e., the Boolean algebra $b(A)$ of a set A is the set of subsets of A that can be obtained by means of a finite number of the set operations UNION (OR), INTERSECTION (AND), and COMPLEMENTATION (NOT) (Comtet 1974, p. 185). A Boolean algebra also forms a LATTICE (Skiena 1990, p. 170), and each of the elements of $b(A)$ is called a BOOLEAN FUNCTION. There are 2^{2^n} BOOLEAN FUNCTIONS in a Boolean algebra of order n (Comtet 1974, p. 186).

In 1938, Shannon proved that a two-valued Boolean algebra (whose members are most commonly denoted 0 and 1, or false and true) can describe the operation of two-valued electrical switching circuits. In modern times, Boolean algebra and BOOLEAN FUNCTIONS are therefore indispensable in the design of computer chips and integrated circuits.



Boolean algebras have a recursive structure apparent in the HASSE DIAGRAMS illustrated above for Boolean algebras of orders $n = 2, 3, 4$, and 5 . These figures illustrate the partition between left and right halves of the lattice, each of which is the Boolean algebra on $n - 1$ elements (Skiena 1990, pp. 169–70).

A Boolean algebra can be formally defined as a SET B of elements $a, b, \dots$ with the following properties:

1. B has two binary operations, $\wedge$ (logical AND, or "WEDGE") and $\vee$ (logical OR, or "VEE"), which satisfy the IDEMPOTENT laws

$$a \wedge a = a \vee a = a, \quad (1)$$

the COMMUTATIVE laws

$$a \wedge b = b \wedge a \quad (2)$$

$$a \vee b = b \vee a, \quad (3)$$

and the ASSOCIATIVE laws

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c \quad (4)$$

$$a \vee (b \vee c) = (a \vee b) \vee c. \quad (5)$$

2. The operations satisfy the ABSORPTION LAW

$$a \wedge (a \vee b) = a \vee (a \wedge b) = a. \quad (6)$$

3. The operations are mutually distributive

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad (7)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \quad (8)$$

4. B contains universal bounds $\emptyset$ and I which satisfy

$$\emptyset \wedge a = \emptyset \quad (9)$$

$$\emptyset \vee a = a \quad (10)$$

$$I \wedge a = a \quad (11)$$

$$I \vee a = I. \quad (12)$$

5. B has a unary operation $a \rightarrow a'$ of complementation which obeys the laws

$$a \wedge a' = \emptyset \quad (13)$$

$$a \vee a' = I \quad (14)$$

(Birkhoff and Mac Lane 1965).

In the slightly archaic terminology of (Bell 1937, p. 444), a Boolean algebra can be defined as a set B of elements $a, b, \dots$ with BINARY OPERATORS $\vee$ (or $+$; logical OR) and $\wedge$ (or $\cdot$; logical AND) such that

- 1a. If a and b are in the set B , then $a \vee b$ is in the set B .
- 1b. If a and b are in the set B , then $a \wedge b$ is in the set B .
- 2a. There is an element Z (zero) such that $a \vee Z = a$ for every element a .
- 2b. There is an element U (unity) such that $a \wedge U = a$ for every element a .
- 3a. $a \vee b = b \vee a$.
- 3b. $a \wedge b = b \wedge a$.
- 4a. $a \vee b \wedge c = (a \vee b) \wedge (a \vee c)$.
- 4b. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
5. For every element a there is an element a' such that $a \vee a' = U$ and $a \wedge a' = Z$.
6. There are at least two distinct elements in the set B .

Huntington (1933ab) presented the following basis for Boolean algebra:

1. Commutativity. $x \vee y = y \vee x$.
2. Associativity. $(x \vee y) \vee z = x \vee (y \vee z)$.
3. HUNTINGTON AXIOM. $!(x \vee y) \vee !(x \vee !y) = x$.

H. Robbins then conjectured that the HUNTINGTON AXIOM could be replaced with the simpler ROBBINS AXIOM,

$$!(x \vee y) \vee (x \vee !y) = x \quad (15)$$

The ALGEBRA defined by commutativity, associativity, and the ROBBINS AXIOM is called ROBBINS ALGEBRA. Computer theorem proving demonstrated that every ROBBINS ALGEBRA satisfies the second WINKLER CONDITION, from which it follows immediately that all ROBBINS ALGEBRAS are Boolean (McCune, Kolata 1996).

See also BOOLEAN FUNCTION, BOOLEANS, HUNTINGTON AXIOM, MAXIMAL IDEAL THEOREM, ROBBINS ALGEBRA, ROBBINS AXIOM, WINKLER CONDITIONS, WOLFRAM AXIOM

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Boolean Connective

One of the LOGIC operators AND $\wedge$, OR $\vee$, and NOT $\neg$.

See also QUANTIFIER

Boolean Function

Consider a Boolean algebra of subsets $b(A)$ generated by a set A , which is the set of subsets of A that can be obtained by means of a finite number of the set operations union, intersection, and complementation. Then each of the elements of $b(A)$ is called a Boolean function generated by A (Comtet 1974, p. 185). Each Boolean function has a unique representation (up to order) as a union of COMPLETE PRODUCTS. It follows that there are 2^{2^p} inequivalent Boolean functions for a set A with cardinality p (Comtet 1974, p. 187).

In 1938, Shannon proved that a two-valued Boolean algebra (whose members are most commonly denoted 0 and 1, or false and true) can describe the operation of two-valued electrical switching circuits. The following table gives the TRUTH TABLE for the $2^{2^2} = 16$ possible Boolean functions of two binary variables.

A	B	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

A	B	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1

1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

The names and symbols for these functions are given in the following table (Simpson 1987, p. 539).

operation	symbol	name
F_0	0	FALSE
F_1	$A \wedge B$	AND
F_2	$A \wedge !B$	A AND NOT B
F_3	A	A
F_4	$!A \wedge B$	NOT A AND B
F_5	B	B
F_6	$A \vee B$	XOR
F_7	$A \vee B$	OR
F_8	$A \nabla B$	NOR
F_9	$A \text{ XNOR } B$	XNOR
F_{10}	$!B$	NOT B
F_{11}	$A \vee !B$	A OR NOT B
F_{12}	$!A$	NOT A
F_{13}	$!A \vee B$	NOT A OR B
F_{14}	$A \nabla B$	NAND
F_{15}	1	TRUE

Determining the number of *monotone* Boolean functions of n variables is known as DEDEKIND'S PROBLEM and is equivalent to the number of ANTICHAINS on the n -set $\{1, 2, \dots, n\}$. Boolean functions can also be thought of as colorings of a Boolean n -cube. The numbers of inequivalent monotone Boolean functions in $n = 1, 2, \dots$ variables are given by 2, 3, 5, 10, 30, ... (Sloane's A003182).

Let $M(n, k)$ denote the number of distinct monotone Boolean functions of n variables with k MINCUTS. Then

$$M(n, 0) = 1$$

$$M(n, 1) = 2^n$$

$$M(n, 2) = 2^{n-1}(2^n - 1) - 3^n + 2^n$$

$$M(n, 3) = \frac{1}{6}(2^n)(2^n - 1)(2^n - 2) - 6^n + 5^n + 4^n - 3^n.$$

See also ANTICHAIN, BOOLEAN ALGEBRA, BOOLEANS,

COMPLETE PRODUCT, CONJUNCTION, DEDEKIND'S PROBLEM, MINCUT, MONOTONE FUNCTION

References

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Boolean Representation Theorem

Every BOOLEAN ALGEBRA is isomorphic to the BOOLEAN ALGEBRA of sets. It is equivalent to the MAXIMAL IDEAL THEOREM, which can be proved without using the AXIOM OF CHOICE (Mendelson 1997, p. 121).

See also BOOLEAN ALGEBRA, MAXIMAL IDEAL THEOREM

References

- Mendelson, E. *Introduction to Mathematical Logic*, 4th ed. London: Chapman & Hall, p. 121, 1997.
- Stone, M. "The Representation Theorem for Boolean Algebras." *Trans. Amer. Math. Soc.* **40**, 37–11, 1936.

Boolean Ring

A RING with a unit element in which every element is IDEMPOTENT.

See also BOOLEAN ALGEBRA

Booleans

The domain of Booleans, sometimes denoted $\mathbb{B}$, consisting of the elements TRUE and FALSE, implemented in *Mathematica* as Booleans. In *Mathematica*, a quantity can be tested to determine if it is in the domain of Booleans using `Element[e, Booleans]`.

See also BOOLEAN ALGEBRA, BOOLEAN FUNCTION, FALSE, TRUE

Boomeron Equation

The system of PARTIAL DIFFERENTIAL EQUATIONS

$$u_t = \mathbf{b} \cdot \mathbf{v}_x$$

$$\mathbf{b}_{xt} = u_{xx} \mathbf{b} + \mathbf{a} \times \mathbf{v}_x - 2\mathbf{v} \times (\mathbf{v} \times \mathbf{b}).$$

References

- Calogero, F. and Degasperis, A. *Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations*. New York: North-Holland, p. 57, 1982.
- Zwillinger, D. *Handbook of Differential Equations*, 3rd ed. Boston, MA: Academic Press, p. 137, 1997.

Boosting

See also RESAMPLING STATISTICS

Bootstrap Methods

A set of methods that are generally superior to ANOVA for small data sets or where sample distributions are non-normal.

See also ANOVA, JACKKNIFE, PERMUTATION TESTS, RESAMPLING STATISTICS

References

- Chernick, M. R. *Bootstrap Methods: A Practitioner's Guide*. New York: Wiley, 1999.
- Davison, A. C. and Hinkley, D. V. *Bootstrap Methods and Their Application*. Cambridge, England: Cambridge University Press, 1997.
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Borchardt-Pfaff Algorithm

ARCHIMEDES ALGORITHM

Border Square

40	1	2	3	42	41	46
38	31	13	14	32	35	12
39	30	26	21	28	20	11
43	33	27	25	23	17	7
6	16	22	29	24	34	44
5	15	37	36	18	19	45
4	49	48	47	8	9	10

31	13	14	32	35
30	26	21	28	20
33	27	25	23	17
16	22	29	24	34
15	37	36	18	19

26	21	28
27	25	23
22	29	24

A MAGIC SQUARE that remains magic when its border is removed. A nested magic square remains magic after the border is successively removed one ring at a time. An example of a nested magic square is the order 7 square illustrated above (i.e., the order 7, 5, and 3 squares obtained from it are all magic).

See also MAGIC SQUARE

References

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- Kraitchik, M. "Border Squares." §7.7 in *Mathematical Recreations*. New York: W. W. Norton, pp. 167–70, 1942.

Bordism

A relation between COMPACT boundaryless MANIFOLDS (also called closed MANIFOLDS). Two closed MANIFOLDS are bordant IFF their disjoint union is the boundary of a compact $(n+1)$ -MANIFOLD. Roughly, two MANIFOLDS are bordant if together they form the boundary of a MANIFOLD. The word

bordism is now used in place of the original term COBORDISM.

References

- Budney, R. "The Bordism Project." <http://www.math.cornell.edu/~rybu/bordism/bordism.html>.

Bordism Group

There are bordism groups, also called COBORDISM GROUPS or COBORDISM RINGS, and there are singular bordism groups. The bordism groups give a framework for getting a grip on the question, "When is a compact boundaryless MANIFOLD the boundary of another MANIFOLD?" The answer is, precisely when all of its STIEFEL-WHITNEY CLASSES are zero. Singular bordism groups give insight into STEENROD'S REALIZATION PROBLEM: "When can homology classes be realized as the image of fundamental classes of manifolds?" That answer is known, too.

The machinery of the bordism group winds up being important for HOMOTOPY THEORY as well.

References

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Borel Algebra

See also BOREL SIGMA ALGEBRA, BOREL SUBALGEBRA

Borel Determinacy Theorem

Let T be a TREE defined on a metric over a set of paths such that the distance between paths p and q is $1/n$, where n is the number of nodes shared by p and q . Let A be a BOREL SET of paths in the topology induced by this metric. Suppose two players play a game by choosing a path down the tree, so that they alternate and each time choose an immediate successor of the previously chosen point. The first player wins if the chosen path is in A . Then one of the players has a winning STRATEGY in this GAME.

See also GAME THEORY, TREE

Borel Field

If a FIELD has the property that, if the sets $A_n, \dots, A_n, \dots$ belong to it, then so do the sets $A_1 + \dots + A_n + \dots$ and $A_1 \dots A_n \dots$, then the field is called a Borel field (Papoulis 1984, p. 29).

See also FIELD

References

- Papoulis, A. *Probability, Random Variables, and Stochastic Processes, 2nd ed.* New York: McGraw-Hill, 1984.

Borel Measure

If F is the BOREL SIGMA ALGEBRA on some TOPOLOGICAL SPACE, then a MEASURE $m : F \rightarrow \mathbb{R}$ is said to be a Borel measure (or BOREL PROBABILITY MEASURE). For a Borel measure, all continuous functions are MEASURABLE.

Borel Probability Measure

BOREL MEASURE

Borel Set

A Borel set is an element of a BOREL SIGMA ALGEBRA. Roughly speaking, Borel sets are the sets that can be constructed from open or closed sets by repeatedly taking countable unions and intersections. Formally, the class B of Borel sets in Euclidean $\mathbb{R}^n$ is the smallest collection of sets that includes the open and closed sets such that if $E, E_1, E_2, \dots$ are in B , then so are $\bigcup_{i=1}^{\infty} E_i$, $\bigcap_{i=1}^{\infty} E_i$, and $\mathbb{R}^n \setminus E$, where $F \setminus E$ is a SET DIFFERENCE (Croft *et al.* 19991).

The set of rational numbers is a Borel set, as is the CANTOR SET.

See also CLOSED SET, OPEN SET, STANDARD SPACE

References

Croft, H. T.; Falconer, K. J.; and Guy, R. K. *Unsolved Problems in Geometry*. New York: Springer-Verlag, p. 3, 1991.

Borel Sigma Algebra

A SIGMA ALGEBRA which is related to the TOPOLOGY of a SET. The Borel σ -algebra is defined to be the SIGMA ALGEBRA generated by the OPEN SETS (or equivalently, by the CLOSED SETS).

See also BOREL ALGEBRA, BOREL MEASURE, BOREL SUBALGEBRA

Borel Space

A SET equipped with a SIGMA ALGEBRA of SUBSETS.

Borel Subalgebra

See also BOREL ALGEBRA, BOREL SIGMA ALGEBRA

Borel's Expansion

Let $\phi(t) = \sum_{n=0}^{\infty} A_n t^n$ be any function for which the integral

$$I(x) \equiv \int_0^{\infty} e^{-tx} t^p \phi(t) dt$$

converges. Then the expansion

$$I(x) \frac{\Gamma(p+1)}{x^{p+1}} \left[A_0 + (p+1) \frac{A_1}{x} + (p+1)(p+2) \frac{A_2}{x^2} + \dots \right],$$

where $\Gamma(z)$ is the GAMMA FUNCTION, is usually an ASYMPTOTIC SERIES for $I(x)$.

Borel-Cantelli Lemma

Let $\{A_n\}_{n=0}^{\infty}$ be a SEQUENCE of events occurring with a certain probability distribution, and let A be the event consisting of the occurrence of a finite number of events A_n , $n = 1, \dots$. Then if

$$\sum_{n=1}^{\infty} P(A_n) < \infty,$$

then

$$P(A) = 1.$$

References

Hazewinkel, M. (Managing Ed.). *Encyclopaedia of Mathematics: An Updated and Annotated Translation of the Soviet "Mathematical Encyclopaedia"*. Dordrecht, Netherlands: Reidel, pp. 435–36, 1988.

Borel-Weyl Theorem

Let $G = SL(n, \mathbb{C})$. If $\lambda \in Z^n$ is the highest weight of an irreducible holomorphic representation V of G , (i.e., λ is a dominant integral weight), then the G -map $\phi : V^* \rightarrow \Gamma(\lambda)$ defined by $\alpha \mapsto F_{\alpha}$, where $F_{\alpha}(g) = \langle \alpha, gv \rangle$, is an ISOMORPHISM. Thus, $V \cong \Gamma(\lambda)^*$.

References

Huang, J.-S. "The Borel-Weyl Theorem." §8.7 in *Lectures on Representation Theory*. Singapore: World Scientific, pp. 105–07, 1999.

Born-Infeld Equation

The PARTIAL DIFFERENTIAL EQUATION

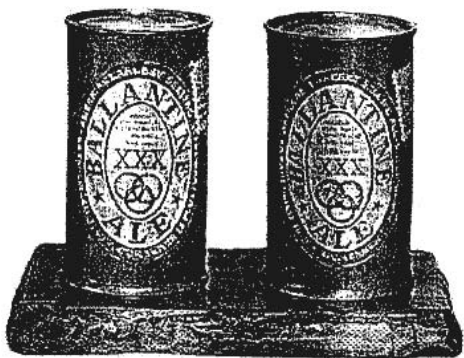
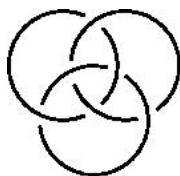
$$(1 - u_t^2)u_{xx} + 2u_x u_t u_{xt} - (1 + u_x^2)u_{tt} = 0.$$

References

Whitham, G. B. *Linear and Nonlinear Waves*. New York: Wiley, p. 617, 1974.
Zwillinger, D. *Handbook of Differential Equations, 3rd ed.* Boston, MA: Academic Press, p. 132, 1997.

Boron Tree

BINARY TREE

Borromean Rings

Three mutually interlocked rings, named after the Italian Renaissance family who used them on their coat of arms. The configuration of rings is also known as a "Ballantine," and a brand of beer (illustrated above) has been brewed under this name. In the Borromean rings, no two rings are linked, so if any one of the rings is cut, all three rings fall apart. Any number of rings can be linked in an analogous manner (Steinhaus 1983, Wells 1991).

The Borromean rings have LINK symbol $06-3-2$, BRAID WORD $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$, and are also the simplest BRUNNIAN LINK.

See also BRUNNIAN LINK, CIRCLE-CIRCLE INTERSECTION, TRIQUETRA, VENN DIAGRAM

References

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Borrow

$$\begin{array}{r}
 1 \ 2 \ 3 \ 4 \\
 - \ 7 \ 8 \ 9 \\
 \hline
 4 \ 4 \ 5
 \end{array}
 \qquad
 \begin{array}{r}
 1 \ 2 \ 1 \\
 1 \ 2 \ 1 \\
 - \ 7 \ 8 \ 9 \\
 \hline
 4 \ 4 \ 5
 \end{array}$$

← borrows

The procedure used in SUBTRACTION to "borrow" 10 from the next higher DIGIT column in order to obtain a POSITIVE DIFFERENCE in the column in question.

See also CARRY

Borsuk's Conjecture

Borsuk conjectured that it is possible to cut an n -D shape of GENERALIZED DIAMETER 1 into $n + 1$ pieces each with diameter smaller than the original. It is true for $n = 2, 3$ and when the boundary is "smooth." However, the minimum number of pieces required has been shown to increase as $\sim 1.1^{\sqrt{n}}$. Since $1.1^{\sqrt{n}} > n + 1$ at $n = 9162$, the conjecture becomes false at high dimensions. In fact, the conjecture is false for every $n > 561$.

See also GENERALIZED DIAMETER, KELLER'S CONJECTURE, LEBESGUE MINIMAL PROBLEM

References

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Borsuk-Ulam Theorem

Every continuous map $f: \mathbb{S}^n \rightarrow \mathbb{R}^n$ must identify a pair of ANTIPODAL POINTS.

References

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Borwein Conjectures

Use the definition of the Q -SERIES

$$(a; q)_n \equiv \prod_{j=0}^{n-1} (1 - aq^j) \quad (1)$$

and define

$$\left[\begin{matrix} N \\ M \end{matrix} \right] \equiv \frac{(q^{N-M+1}; q)_M}{(q; q)_m}. \quad (2)$$

Then P. Borwein has conjectured that (1) the POLYNOMIALS $A_n(q)$, $B_n(q)$, and $C_n(q)$ defined by

$$(q; q^3)_n (q^2; q^3)_n = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3) \quad (3)$$

have NONNEGATIVE COEFFICIENTS, (2) the POLYNOMIALS $A_n^*(q)$, $B_n^*(q)$, and $C_n^*(q)$ defined by

$$(q; q^3)_n^2 (q^2; q^3)_n^2 = A_n^*(q^3) - qB_n^*(q^3) - q^2C_n^*(q^3) \quad (4)$$

have NONNEGATIVE COEFFICIENTS, (3) the POLYNOMIALS $A_n^*(q)$, $B_n^*(q)$, $C_n^*(q)$, $D_n^*(q)$, and $E_n^*(q)$ defined by

$$(q; q^5)_n (q^2; q^5)_n (q^3; q^5)_n (q^4; q^5)_n = A_n^*(q^5) - qB_n^*(q^5) - q^2C_n^*(q^5) - q^3D_n^*(q^5) - q^4E_n^*(q^5) \quad (5)$$

have NONNEGATIVE COEFFICIENTS, (4) the POLYNOMIALS $A_n^\dagger(m, n, t, q)$, $B_n^\dagger(m, n, t, q)$, and $C_n^\dagger(m, n, t, q)$ defined by

$$(q; q^3)_m (q^2; q^3)_m (zq; q^3)_n (zq^2; q^3)_n = \sum_{t=0}^{2m} z^t [A^\dagger(m, n, t, q^3) - qB^\dagger(m, n, t, q^3) - q^2C^\dagger(m, n, t, q^3)] \quad (6)$$

have NONNEGATIVE COEFFICIENTS, (5) for k ODD and $1 \leq a \leq k/2$, consider the expansion

$$(q^a; q^k)_m (q^{k-a}; q^k)_n = \sum_{v=(1-k)/2}^{(k-1)/2} (-1)^v q^{k(v^2+v)/2 - av} F_v(q^k) \quad (7)$$

with

$$F_v(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(k^2j+2kv+k-2a)/2} \left[\begin{matrix} m+n \\ m+v+Kj \end{matrix} \right], \quad (8)$$

then if a is RELATIVELY PRIME to k and $m = n$, the COEFFICIENTS of $F_v(q)$ are NONNEGATIVE, and (6) given $\alpha + \beta < 2K$ and $-K + \beta \leq n - m \leq K - \alpha$, consider

$$G(\alpha, \beta, K; q) = \sum_q (-1)^j q^{j[K(\alpha+\beta)j+K(\alpha+\beta)]/2} \left[\begin{matrix} m+n \\ m+Kj \end{matrix} \right], \quad (9)$$

the GENERATING FUNCTION for partitions inside an $m \times n$ rectangle with hook difference conditions specified by α , β , and K . Let α and β be POSITIVE RATIONAL NUMBERS and $k > 1$ an INTEGER such that

αk and βk are integers. then if $1 \leq \alpha + \beta \leq 2k - 1$ (with strict inequalities for $k = 2$) and $-k + \beta \leq n - m \leq k - \alpha$, then $g(\alpha, \beta, k; q)$ has NONNEGATIVE COEFFICIENTS.

See also Q -SERIES

References

- Andrews, G. E. *et al.* "Partitions with Prescribed Hook Differences." *Europ. J. Combin.* **8**, 341–50, 1987.
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Bott Periodicity Theorem

Define

$$O = \varinjlim O(n), \quad F = \mathbb{R} \quad (1)$$

$$U = \varinjlim U(n), \quad F = \mathbb{C} \quad (2)$$

$$Sp = \varinjlim Sp(n), \quad F = \mathbb{H}. \quad (3)$$

Then

$$\Omega^2 BU \cong BU \times \mathbb{Z} \quad (4)$$

$$\Omega^4 BO \cong BSp \times \mathbb{Z} \quad (5)$$

$$\Omega^4 BSp \cong BO \times \mathbb{Z}. \quad (6)$$

References

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Bott, R. "The Stable Homotopy of the Classical Groups." *Ann. Math.* **70**, 313–37, 1959.
Dodson, C. T. J. and Parker, P. E. *A User's Guide to Algebraic Topology*. Dordrecht, Netherlands: Kluwer, p. 229, 1997.
Milnor, J. W. *Morse Theory*. Princeton, NJ: Princeton University Press, 1963.

Bottle Imp Paradox

In Robert Louis Stevenson's "bottle imp paradox," you are offered the opportunity to buy, for whatever price you wish, a bottle containing a genie who will fulfill your every desire. The only catch is that the bottle must thereafter be resold for a price smaller than what you paid for it, or you will be condemned to live out the rest of your days in excruciating torment. Obviously, no one would buy the bottle for 1c since he would have to give the bottle away, but no one would accept the bottle knowing he would be unable to get rid of it. Similarly, no one would buy it for 2c, and so on. However, for some reasonably large amount, it will always be possible to find a next buyer, so the bottle will be bought (Paulos 1995).

See also UNEXPECTED HANGING PARADOX

References

- Erickson, G. W. and Fossa, J. A. *Dictionary of Paradox*. Lanham, MD: University Press of America, pp. 25–7, 1998.
- Paulos, J. A. *A Mathematician Reads the Newspaper*. New York: BasicBooks, p. 97, 1995.

Bouligand Dimension

MINKOWSKI-BOULIGAND DIMENSION

Bound

GREATEST LOWER BOUND, INFIMUM, LEAST UPPER BOUND, SUPREMUM

Bound Variable

An occurrence of a variable in a LOGIC which is not FREE. Bound variables are also called DUMMY VARIABLES.

See also DUMMY VARIABLE, SENTENCE

References

- Comtet, L. "Bound Variables." §1.11 in *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, rev. enl. ed. Dordrecht, Netherlands: Reidel, pp. 30–4, 1974.

Boundary

The set of points, known as BOUNDARY POINTS, which are members of the CLOSURE of a given set S and the CLOSURE of its complement set. The boundary is sometimes called the FRONTIER.

See also BOUNDARY CONDITIONS, BOUNDARY MAP, BOUNDARY POINT, BOUNDARY SET, NATURAL BOUNDARY, SURGERY

Boundary Conditions

There are several types of boundary conditions commonly encountered in the solution of PARTIAL DIFFERENTIAL EQUATIONS.

1. DIRICHLET BOUNDARY CONDITIONS specify the value of the function on a surface $T = f(\mathbf{r}, t)$.
2. NEUMANN BOUNDARY CONDITIONS specify the normal derivative of the function on a surface,

$$\frac{\partial T}{\partial n} = \hat{\mathbf{n}} \cdot \nabla T = f(\mathbf{r}, y).$$

3. CAUCHY BOUNDARY CONDITIONS specify a weighted average of first and second kinds.
4. ROBIN BOUNDARY CONDITIONS. For an elliptic partial differential equation in a region Ω , Robin boundary conditions specify the sum of αu and the normal derivative of $u = f$ at all points of the boundary of Ω , with α and f being prescribed.

See also BOUNDARY VALUE PROBLEM, DIRICHLET BOUNDARY CONDITIONS, GOURSAT PROBLEM, INITIAL

VALUE PROBLEM, NEUMANN BOUNDARY CONDITIONS, PARTIAL DIFFERENTIAL EQUATION, ROBIN BOUNDARY CONDITIONS

References

- Arfken, G. *Mathematical Methods for Physicists*, 3rd ed. Orlando, FL: Academic Press, pp. 502–04, 1985.
- Morse, P. M. and Feshbach, H. "Boundary Conditions and Eigenfunctions." Ch. 6 in *Methods of Theoretical Physics, Part I*. New York: McGraw-Hill, pp. 495–98 and 676–90, 1953.

Boundary Map

The map $H_n(X, A) \rightarrow H_{n-1}(A)$ appearing in the LONG EXACT SEQUENCE OF A PAIR AXIOM.

See also LONG EXACT SEQUENCE OF A PAIR AXIOM

Boundary Point

A point which is a member of the CLOSURE of a given set S and the CLOSURE of its complement set. If A is a subset of $\mathbb{R}^n$, then a point $\mathbf{x} \in \mathbb{R}^n$ is a boundary point of A if every NEIGHBORHOOD of $\mathbf{x}$ contains at least one point in A and at least one point not in A .

See also BOUNDARY

Boundary Set

A (symmetrical) boundary set of RADIUS r and center $\mathbf{x}_0$ is the set of all points $\mathbf{x}$ such that

$$|\mathbf{x} - \mathbf{x}_0| = r.$$

Let $\mathbf{x}_0$ be the ORIGIN. In $\mathbb{R}^1$, the boundary set is then the pair of points $x = r$ and $x = -r$. In $\mathbb{R}^2$, the boundary set is a CIRCLE. In $\mathbb{R}^3$, the boundary set is a SPHERE.

See also CIRCLE, COMPACT SET, DISK, OPEN SET, SPHERE

References

- Croft, H. T.; Falconer, K. J.; and Guy, R. K. *Unsolved Problems in Geometry*. New York: Springer-Verlag, p. 2, 1991.

Boundary Value Problem

A boundary value problem is a problem, typically an ORDINARY DIFFERENTIAL EQUATION or a PARTIAL DIFFERENTIAL EQUATION, which has values assigned on the physical boundary of the DOMAIN in which the problem is specified. For example,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f & \text{in } \Omega \\ u(0, t) = u_1 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial t}(0, t) = u_2 & \text{on } \partial\Omega, \end{cases}$$

where $\partial\Omega$ denotes the boundary of Ω , is a boundary problem.

See also BOUNDARY CONDITIONS, INITIAL VALUE PROBLEM

References

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- Powers, D. L. *Boundary Value Problems, 4th ed.* San Diego, CA: Academic Press, 1999.
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Bounded

A mathematical object (such as a set or function) is said to be bounded if it possesses a BOUND, i.e., a value which all members of the set, functions, etc., are less than.

See also BOUNDED SET

Bounded Set

A SET in a METRIC SPACE (X, d) is bounded if it has a FINITE GENERALIZED DIAMETER, i.e., there is an $R < \infty$ such that $d(x, y) \leq R$ for all $x, y \in X$. A SET in $\mathbb{R}^n$ is bounded if it is contained inside some BALL $x_1^2 + \dots + x_n^2 \leq R^2$ of FINITE RADIUS R (Adams 1994).

See also BOUND, FINITE

References

- Adams, R. A. *Calculus: A Complete Course*. Reading, MA: Addison-Wesley, p. 707, 1994.
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- Jeffreys, H. and Jeffreys, B. S. "Bounded, Unbounded, Convergent, Oscillatory." §1.041 in *Methods of Mathematical Physics, 3rd ed.* Cambridge, England: Cambridge University Press, pp. 11–2, 1988.

Bounded Variation

A FUNCTION $f(x)$ is said to have bounded variation if, over the CLOSED INTERVAL $x \in [a, b]$, there exists an M such that

$$|f(x_i) - f(a)| + |f(x_2) - f(x_1)| + \dots + |f(b) - f(x_n - 1)| \leq M \quad (1)$$

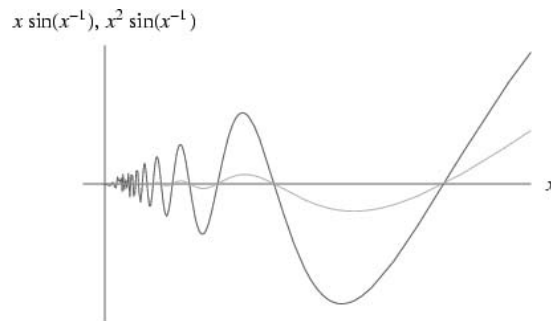
for all $a < x_1 < x_2 < \dots < x_{n-1} < b$.

The space of functions of bounded variation is denoted "BV," and has the SEMINORM

$$\Phi(f) = \sup \int f \frac{d\phi}{dx}, \quad (2)$$

where ϕ ranges over all COMPACTLY SUPPORTED functions bounded by -1 and 1. The seminorm is equal to the SUPREMUM over all sums above, and is also equal to $\int |df/dx| dx$ (when this expression

makes sense).



On the interval $[0, 1]$, the function $x^2 \sin(1/x)$ (purple) is of bounded variation, but $x \sin 1/x$ (red) is not. More generally, a function f is locally of bounded variation in a domain U if f is LOCALLY INTEGRABLE, $f \in L^1_{\text{loc}}$, and for all open subsets W , with COMPACT CLOSURE in U , and all SMOOTH VECTOR FIELDS g COMPACTLY SUPPORTED in W ,

$$\int_W f \operatorname{div} g dx \leq c(W) \sup |g|, \quad (3)$$

div denotes DIVERGENCE and c is a constant which only depends on the choice of W and f .

Such functions form the space $BV_{\text{loc}}(U)$. They may not be DIFFERENTIABLE, but by the RIESZ REPRESENTATION THEOREM, the derivative of a BV_{loc} -function f is a REGULAR BOREL MEASURE Df . Functions of bounded variation also satisfy a compactness theorem.

Given a sequence f_n of functions in $BV_{\text{loc}}(U)$, such that

$$\sup_n \left(\|f_n\|_{L^1(W)} + \int_W |Df_n| dx \right) < \infty,$$

that is the TOTAL VARIATION of the functions is bounded, in any COMPACTLY SUPPORTED open subset W , there is a SUBSEQUENCE f_{n_k} which converges to a function $f \in BV_{\text{loc}}$ in the topology of L^1_{loc} . Moreover, the limit satisfies

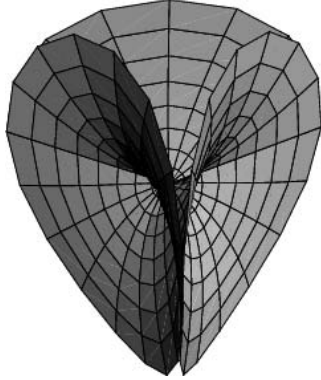
$$\int_W |Df| dx \leq \liminf \int_W |Df_{n_k}| dx. \quad (4)$$

They also satisfy a version of POINCARÉ'S LEMMA.

See also DIFFERENTIABLE, WEAKLY DIFFERENTIABLE

References

- Jeffreys, H. and Jeffreys, B. S. "Functions of Bounded Variation." §1.09 in *Methods of Mathematical Physics, 3rd ed.* Cambridge, England: Cambridge University Press, pp. 24–6, 1988.
- Simon, L. §2.6 in *Lectures on Geometric Measure Theory* Canberra: Centre for Mathematical Analysis, Australian National University, 1984.

Bour's Minimal Surface

Gray (1997) defines Bour's minimal curve over complex z by

$$x' = \frac{z^{m-1}}{m-1} - \frac{z^{m+1}}{m+1} \quad (1)$$

$$y' = i \left(\frac{z^{m-1}}{m-1} + \frac{z^{m+1}}{m+1} \right) \quad (2)$$

$$z' = \frac{2z^m}{m}, \quad (3)$$

and then derives a family of MINIMAL SURFACES.

The order three Bour surface resembles a CROSS-CAP and is given using ENNEPER-WEIERSTRASS PARAMETERIZATION by

$$f = 1 \quad (4)$$

$$g = \sqrt{z} \quad (5)$$

or explicitly by the PARAMETRIC EQUATIONS

$$x = r \cos \theta - \frac{1}{2} r^2 \cos(2\theta) \quad (6)$$

$$y = -r \sin \theta - \frac{1}{2} r^2 \sin(2\theta), \quad (7)$$

$$z = \frac{4}{3} r^{3/2} \cos(\frac{3}{2}\theta) \quad (8)$$

(Maeder 1997). The coefficients of the FIRST FUNDAMENTAL FORM are given by

$$E = 1 + r^2 \quad (9)$$

$$F = 0 \quad (10)$$

$$G = r^2(r^2 + 1) \quad (11)$$

and the coefficients of the SECOND FUNDAMENTAL FORM by

$$e = -r^{-1/2} \cos(\frac{3}{2}\phi) \quad (12)$$

$$f = \sqrt{r} \sin(\frac{3}{2}\phi) \quad (13)$$

Bourget's Hypothesis

$$g = r^{3/2} \cos(\frac{3}{2}\phi). \quad (14)$$

The AREA ELEMENT is

$$dA = r(r+1)^2 dr \wedge d\phi. \quad (15)$$

The GAUSSIAN and MEAN CURVATURES are given by

$$K = -\frac{1}{r(r+1)^4} \quad (16)$$

$$H = 0. \quad (17)$$

See also CROSS-CAP, ENNEPER-WEIERSTRASS PARAMETERIZATION, MINIMAL SURFACE

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2nd ed. Boca Raton, FL: CRC Press, pp. 732–33, 1997.
Maeder, R. *Programming in Mathematica*, 3rd ed. Reading, MA: Addison-Wesley, pp. 29–0, 1997.

Bourget Function

The function defined by the CONTOUR INTEGRAL

$$J_{n,k}(z) = \frac{1}{2\pi i} \int_{(0+)}^{(0+)} t^{-n-1} \left(t + \frac{1}{t}\right)^k \exp\left[\frac{1}{2}z\left(t - \frac{1}{t}\right)\right] dt,$$

where $\int_{(0+)}$ denotes the CONTOUR encircling the point $z = 0$ once in a counterclockwise direction. It is equal to

$$J_{n,k}(z) = \frac{1}{\pi} \int_0^\pi (2 \cos \theta)^k \cos(n\theta - z \sin \theta) d\theta$$

(Watson 1966, p. 326).

See also BESSEL FUNCTION OF THE FIRST KIND

References

Bourget, J. "Mémoire sue les nombres de Cauchy et leur application à divers problèmes de mécanique céleste." *J. de Math.* **6**, 33–4, 1861.
Giuliani, G. "Alcune osservazioni sopra le funzioni sferiche di ordine superiore al secondo e sopra altre funzioni che se ne possono dedurre (April, 1888)." *Giornale di Mat.* **26**, 155–71, 1888.
Hazewinkel, M. (Managing Ed.). *Encyclopaedia of Mathematics: An Updated and Annotated Translation of the Soviet "Mathematical Encyclopaedia."* Dordrecht, Netherlands: Reidel, p. 465, 1988.
Watson, G. N. "The Functions of Bourget and Giuliani." §10.31 in *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge, England: Cambridge University Press, pp. 326–27, 1966.

Bourget's Hypothesis

When n is an INTEGER ≥ 0 , then $J_n(z)$ and $J_{n+m}(z)$ have no common zeros other than at $z = 0$ for m an

INTEGER ≥ 1 , where $J_n(z)$ is a BESSEL FUNCTION OF THE FIRST KIND. The theorem has been proved true for $m = 1, 2, 3$, and 4 .

References

Watson, G. N. *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge, England: Cambridge University Press, 1966.

Bourque-Ligh Conjecture

Bourque and Ligh (1992) conjectured that the LEAST COMMON MULTIPLE MATRIX on a GCD-CLOSED SET S is nonsingular. This conjecture was shown to be false by Hong (1999).

See also GCD-CLOSED SET, LEAST COMMON MULTIPLE MATRIX

References

Bourque, K. and Ligh, S. "On GCD and LCM Matrices." *Linear Algebra Appl.* **174**, 65–4, 1992.
Hong, S. "On the Bourque-Ligh Conjecture of Least Common Multiple Matrices." *J. Algebra* **218**, 216–28, 1999.

Boussinesq Equation

The linear Boussinesq equation is the PARTIAL DIFFERENTIAL EQUATION

$$u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{xxtt} \quad (1)$$

(Whitham 1974, p. 9; Zwillinger 1997, p. 129). The nonlinear Boussinesq equation is

$$u_{tt} - u_{xx} - u_{xxx} + 3(u^2)_{xx} = 0 \quad (2)$$

(Calogero and Degasperis 1982; Zwillinger 1997, p. 130). The modified Boussinesq equation is

$$\frac{1}{3} u_{tt} - u_t u_{xx} - \frac{3}{2} u_x^2 u_{xx} + u_{xxxx} = 0 \quad (3)$$

(Clarkson 1986; Zwillinger 1997, p. 132).

References

Calogero, F. and Degasperis, A. *Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations*. New York: North-Holland, 1982.
Clarkson, P. A. "The Painlevé Property, a Modified Boussinesq Equation and a Modified Kadomtsev-Petviashvili Equation." *Physica D* **19**, 447–50, 1986.
Whitham, G. B. *Linear and Nonlinear Waves*. New York: Wiley, 1974.
Zwillinger, D. *Handbook of Differential Equations*, 3rd ed. Boston, MA: Academic Press, pp. 129–30, 1997.

Boustrophedon Transform

The boustrophedon ("ox-plowing") transform $\mathbf{b}$ of a sequence $\mathbf{a}$ is given by

$$b_n = \sum_{k=0}^n \binom{n}{k} a_k E_{n-k} \quad (1)$$

$$a_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} b_k E_{n-k} \quad (2)$$

for $n \geq 0$, where E_n is a SECANT NUMBER OR TANGENT NUMBER defined by

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec x + \tan x. \quad (3)$$

The exponential generating functions of $\mathbf{a}$ and $\mathbf{b}$ are related by

$$\mathcal{B}(x) = (\sec x + \tan x) \mathcal{A}(x), \quad (4)$$

where the exponential generating function is defined by

$$\mathcal{A}(x) = \sum_{n=0}^{\infty} A_n \frac{x^n}{n!}. \quad (5)$$

See also ALTERNATING PERMUTATION, ENTRINGER NUMBER, SECANT NUMBER, SEIDEL-ENTRINGER-ARNOLD TRIANGLE, TANGENT NUMBER

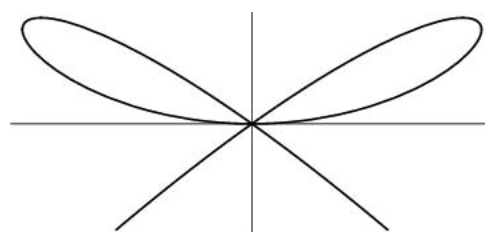
References

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Bovinum Problema

ARCHIMEDES' CATTLE PROBLEM

Bow



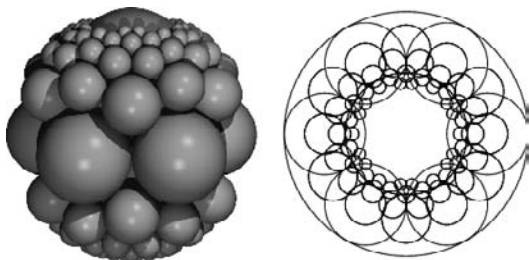
$$x^4 = x^2 y - y^3.$$

References

Cundy, H. and Rollett, A. *Mathematical Models*, 3rd ed. Stradbroke, England: Tarquin Pub., p. 72, 1989.

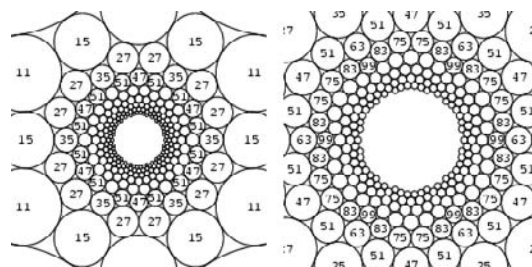
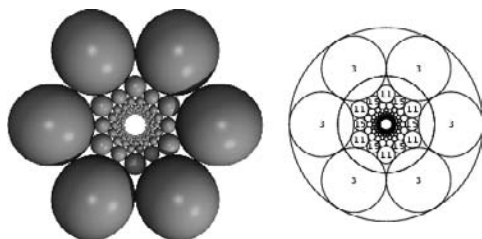
Bowditch Curve

LISSAJOUS CURVE

Bowl of Integers

Place two solid spheres of radius $1/2$ inside a hollow sphere of radius 1 so that the two smaller circles touch each other at the center of the large circle and are tangent to the large circle on the extremities of one of its diameters. This arrangement is called the "bowl of integers" (Soddy 1937) since the BEND of each of the infinite chain of spheres that can be packed into it such that each successive sphere is tangent to its neighbors is an integer. The first few bends are then $-1, 2, 5, 6, 9, 11, 14, 15, 18, 21, 23, \dots$ (Sloane's A046160). The sizes and positions of the first few rings of spheres are given in the table below.

n	κ_n	z_n	R_n	ϕ_n
1	-1	0	0	—
2	2	$\frac{1}{2}$	0	—
3	5	$\frac{2}{5}$	$\frac{2}{5}\sqrt{3}$	$\frac{1}{6}\pi$
4	6	$\frac{1}{2}$	$\frac{2}{3}$	0
5	9	$\frac{2}{3}$	$\frac{2}{9}\sqrt{7}$	$\pm \tan^{-1}(\frac{1}{2}\sqrt{3})$
6	11	$\frac{8}{11}$	$\frac{6}{11}$	0
7	14	$\frac{11}{14}$	$\frac{2}{7}\sqrt{3}$	$\frac{1}{6}\pi$
8	15	$\frac{4}{5}$	$\frac{2}{15}\sqrt{13}$	$\pm \tan^{-1}(2\sqrt{3})$
9	18	$\frac{5}{6}$	$\frac{4}{9}$	0
10	21	$\frac{6}{7}$	$\frac{1}{21}\sqrt{19}$	$\pm \tan^{-1}(\frac{3}{7}\sqrt{3})$
11	23	$\frac{20}{23}$	$\frac{2}{23}\sqrt{21}$	$\pm \tan^{-1}(\frac{1}{9}\sqrt{3})$
12	27	$\frac{8}{9}$	$\frac{10}{27}$	$0, \pm \tan^{-1}(\frac{1}{3}\sqrt{3})$
13	30	$\frac{9}{10}$	$\frac{2}{15}\sqrt{7}$	$\pm \tan^{-1}(\frac{1}{5}\sqrt{3})$
14	33	$\frac{10}{11}$	$\frac{2}{33}\sqrt{31}$	$\pm \tan^{-1}(\frac{1}{11}\sqrt{3})$
15	38	$\frac{35}{38}$	$\frac{6}{19}$	0

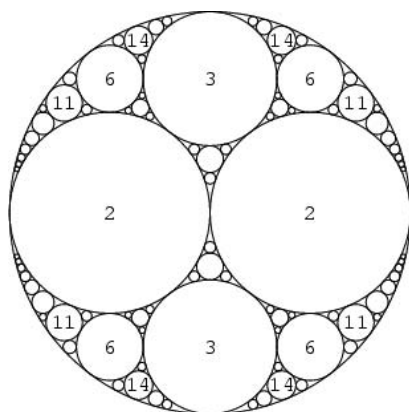


Spheres can also be packed along the plane tangent to the two spheres of radius 2 (Soddy 1937). The sequence of integers for can be found using the equation of five TANGENT SPHERES. Letting $\kappa_3 = \kappa_4 = 2$ gives

$$\kappa(\kappa_1, \kappa_2) = \frac{1}{2}(4 + \kappa_1 + \kappa_2 + \sqrt{3[\kappa_2(8 - \kappa_2) + 2\kappa_1(\kappa_2 + 4) - 3\kappa_1^2]}).$$

For example, $\kappa(3, 3) = 11$, $\kappa(3, 11) = 15$, $\kappa(11, 15) = 27$, $\kappa(15, 27) = 35$, $\kappa(27, 35) = 47$, and so on, giving the sequence $-1, 2, 3, 11, 15, 27, 35, 47, 51, 63, 75, 83, \dots$ (Sloane's A046159). The sizes and positions of the first few rings of spheres are given in the table below.

n	κ_n	R_n	ϕ_n
1	-1	0	—
2	2	0	—
3	3	$\frac{2}{3}$	0
4	11	$\frac{2}{11}\sqrt{3}$	$\frac{1}{6}\pi$
5	15	$\frac{4}{15}$	0
6	27	$\frac{2}{27}\sqrt{7}$	$\pm \tan^{-1}(3\sqrt{3})$
7	35	$\frac{6}{35}$	0
8	47	$\frac{4}{47}\sqrt{3}$	$\frac{1}{6}\pi$
9	51	$\frac{2}{51}\sqrt{13}$	$\pm \tan^{-1}(\frac{3}{5}\sqrt{3})$
10	63	$\frac{8}{63}$	0
11	75	$\frac{2}{75}\sqrt{19}$	$\pm \tan^{-1}(5\sqrt{3})$
12	83	$\frac{2}{83}\sqrt{21}$	$\pm \tan^{-1}(\frac{5}{3}\sqrt{3})$
13	99	$\frac{10}{99}$	0
14	107	$\frac{6}{107}\sqrt{3}$	$\frac{1}{6}\pi$
15	111	$\frac{4}{111}\sqrt{7}$	$\pm \tan^{-1}(\frac{1}{2}\sqrt{3})$
16	123	$\frac{2}{123}\sqrt{31}$	$\pm \tan^{-1}(\frac{5}{7}\sqrt{3})$
17	143	$\frac{12}{143}$	0
18	147	$\frac{2}{147}\sqrt{37}$	$\pm \tan^{-1}(7\sqrt{3})$
19	155	$\frac{2}{155}\sqrt{39}$	$\pm \tan^{-1}(\frac{1}{6}\sqrt{3})$
20	171	$\frac{2}{171}\sqrt{43}$	$\pm \tan^{-1}(\frac{7}{5}\sqrt{3})$



The analogous problem of placing two circles of bend 2 inside a circle of bend -1 and then constructing chains of mutually tangent circles was considered by B. L. Galebach and A. R. Wilks. The circle have integral bends given by -1, 2, 3, 6, 11, 14, 15, 18, 23, 26, 27, 30, 35, 38, ... (Sloane's A042944). Of these, the only known numbers congruent to 2, 3, 6, 11 (mod 12) missing from this sequence are 78, 159, 207, 243, 246, 342, ... (Sloane's A042945), a sequence which is conjectured to be finite.

See also APOLLONIAN GASKET, BEND (CURVATURE), COXETER'S LOXODROMIC SEQUENCE OF TANGENT CIRCLES, HEXLET, SPHERE, TANGENT SPHERES

References

- Borkovec, M.; de Paris, W.; and Peikert, R. "The Fractal Dimension of the Apollonian Sphere Packing." *Fractals* **2**, 521–26, 1994.
- Sloane, N. J. A. Sequences A042944, A042945, A046159, and A046160 in "An On-Line Version of the Encyclopedia of Integer Sequences." <http://www.research.att.com/~njas/sequences/eisonline.html>.
- Soddy, F. "The Bowl of Integers and the Hexlet." *Nature* **139**, 77–9, 1937.

Bowley Index

The statistical INDEX

$$P_B \equiv \frac{1}{2}(P_L + P_P),$$

where P_L is LASPEYRES' INDEX and P_P is PAASCHE'S INDEX.

See also INDEX

References

- Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics*, Pt. 1, 3rd ed. Princeton, NJ: Van Nostrand, p. 66, 1962.

Bowley Skewness

Also known as QUARTILE SKEWNESS COEFFICIENT,

$$\frac{(Q_3 - Q_2) - (Q_2 - Q_1)}{Q_3 - Q_1} = \frac{Q_1 - 2Q_2 + Q_3}{Q_3 - Q_1},$$

where the Q s denote the INTERQUARTILE RANGES.

See also INTERQUARTILE RANGE, SKEWNESS

References

- Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics*, Pt. 1, 3rd ed. Princeton, NJ: Van Nostrand, p. 102, 1962.

Bowling

Bowling is a game played by rolling a heavy ball down a long narrow track and attempting to knock down ten pins arranged in the form of a TRIANGLE with its vertex oriented towards the bowler. The number 10 is, in fact, the TRIANGULAR NUMBER $T_4 = 4(4 + 1)/2 = 10$.

Two "bowls" are allowed per "frame." If all the pins are knocked down in the two bowls, the score for that frame is the number of pins knocked down. If some or none of the pins are knocked down on the first bowl, then all the pins knocked down on the second, it is called a "spare," and the number of points tallied is 10 plus the number of pins knocked down on the bowl of the next frame. If all of the pins are knocked down on the first bowl, the number of points tallied is 10 plus the number of pins knocked down on the next two bowls. Ten frames are bowled, unless the last frame is a strike or spare, in which case an additional bowl is awarded.

The maximum number of points possible, corresponding to knocking down all 10 pins on every bowl, is 300.

References

- Cooper, C. N. and Kennedy, R. E. "A Generating Function for the Distribution of the Scores of All Possible Bowling Games." In *The Lighter Side of Mathematics* (Ed. R. K. Guy and R. E. Woodrow). Washington, DC: Math. Assoc. Amer., 1994.
- Cooper, C. N. and Kennedy, R. E. "Is the Mean Bowling Score Awful?" In *The Lighter Side of Mathematics* (Ed. R. K. Guy and R. E. Woodrow). Washington, DC: Math. Assoc. Amer., 1994.

Box

CUBOID

Box Counting Dimension

CAPACITY DIMENSION

Box Fractal



A FRACTAL also called the anticross-stitch curve which can be constructed using STRING REWRITING by creating a matrix with 3 times as many entries as

the current matrix using the rules

line 1: "*" → "* *'", " " → " "
 line 2: "*" → " *'", " " → " "
 line 3: "*" → " * *'", " " → " "

Let N_n be the number of black boxes, L_n the length of a side of a white box, and A_n the fractional AREA of black boxes after the n th iteration.

$$N_n = 5^n \quad (1)$$

$$L_n = \left(\frac{1}{3}\right)^n = 3^{-n} \quad (2)$$

$$A_n = L_n^2 N_n = \left(\frac{5}{9}\right)^n. \quad (3)$$

The CAPACITY DIMENSION is therefore

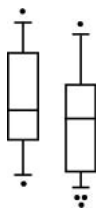
$$\begin{aligned} d_{\text{cap}} &= -\lim_{n \rightarrow \infty} \frac{\ln N_n}{\ln L_n} = -\lim_{n \rightarrow \infty} \frac{\ln(5^n)}{\ln(3^{-n})} \\ &= \frac{\ln 5}{\ln 3} = 1.464973521 \dots \end{aligned} \quad (4)$$

See also CANTOR DUST, CROSS-STITCH CURVE, SIERPINSKI CARPET, SIERPINSKI SIEVE

References

Weisstein, E. W. "Fractals." MATHEMATICA NOTEBOOK FRACTAL.M.

Box-and-Whisker Plot

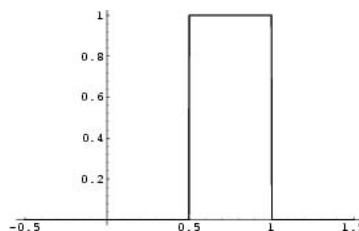


A HISTOGRAM-like method of displaying data invented by J. Tukey (1977). Draw a box with ends at the QUANTILES Q_1 and Q_3 . Draw the MEDIAN as a horizontal line in the box. Extend the "whiskers" to the farthest points. For every point that is more than $3/2$ times the INTERQUARTILE RANGE from the end of a box, draw a dot on the corresponding top or bottom of the whisker. If two dots have the same value, draw them side by side.

References

Tukey, J. W. *Explanatory Data Analysis*. Reading, MA: Addison-Wesley, pp. 39–1, 1977.

Boxcar Function



The function

$$B_e(a, b) = c[H(x - a) - H(x - b)]$$

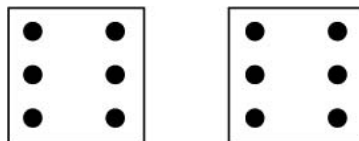
which is equal to c for $a \leq x \leq b$ and 0 otherwise. Here $H(x)$ is the HEAVISIDE STEP FUNCTION. The special case $B_1(-1/2, 1/2)$ gives the unit RECTANGLE FUNCTION.

See also HEAVISIDE STEP FUNCTION, RECTANGLE FUNCTION

References

von Seggern, D. *CRC Standard Curves and Surfaces*. Boca Raton, FL: CRC Press, p. 324, 1993.

Boxcars



A roll of two 6s (the highest roll possible) on a pair of 6-sided DICE. The probability of rolling boxcars in a single roll of two dice is $1/36$, or 2.777...%. In order to have a 50% chance of obtaining at least one boxcars in n rolls of two dice, it must be true that

$$1 - \left(\frac{35}{36}\right)^n = \frac{1}{2}, \quad (1)$$

so solving for n gives

$$n = \frac{\ln 2}{\ln 36 - \ln 35} = 24.605 \dots \quad (2)$$

In fact, rolling two dice 25 times gives a probability of

$$1 - \left(\frac{35}{36}\right)^{25} \approx 0.505532 \quad (3)$$

that at least once boxcars will occur.

See also DICE, DE MÉRÉ'S PROBLEM, SNAKE EYES

Box-Counting Dimension

CAPACITY DIMENSION

Box-Muller Transformation

A transformation which transforms from a 2-D continuous UNIFORM DISTRIBUTION to a 2-D GAUSSIAN BIVARIATE DISTRIBUTION (or COMPLEX GAUSSIAN DISTRIBUTION). If x_1 and x_2 are uniformly and independently distributed between 0 and 1, then z_1 and z_2 as defined below have a GAUSSIAN DISTRIBUTION with MEAN $\mu = 0$ and VARIANCE $\sigma^2 = 1$.

$$z_1 = \sqrt{-2 \ln x_1} \cos(2\pi x_2) \quad (1)$$

$$z_2 = \sqrt{-2 \ln x_1} \sin(2\pi x_2). \quad (2)$$

This can be verified by solving for x_1 and x_2 ,

$$x_1 = e^{-(z_1^2 + z_2^2)/2} \quad (3)$$

$$x_2 = \frac{1}{2\pi} \tan^{-1} \left(\frac{z_2}{z_1} \right). \quad (4)$$

Taking the JACOBIAN yields

$$\begin{aligned} \frac{\partial(x_1, x_2)}{\partial(z_1, z_2)} &= \begin{vmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{vmatrix} \\ &= - \left[\frac{1}{\sqrt{2\pi}} e^{-z_1^2/2} \right] \left[\frac{1}{\sqrt{2\pi}} e^{-z_2^2/2} \right]. \end{aligned} \quad (5)$$

See also GAUSSIAN BIVARIATE DISTRIBUTION, GAUSSIAN DISTRIBUTION, NORMAL DEVIATES

References

Box, G. E. P. and Muller, M. E. "A Note on the Generation of Random Normal Deviates." *Ann. Math. Stat.* **28**, 610–611, 1958.

Box-Packing Theorem

The number of "prime" boxes is always finite, where a set of boxes is prime if it cannot be built up from one or more given configurations of boxes.

See also CONWAY PUZZLE, CUBOID, DE BRUIJN'S THEOREM, KLARNER'S THEOREM, SLOTHOUBER-GRAATSMA PUZZLE

References

Honsberger, R. *Mathematical Gems II*. Washington, DC: Math. Assoc. Amer., p. 74, 1976.

Boy Surface

A NONORIENTABLE SURFACE which is one of the three possible SURFACES obtained by sewing a MÖBIUS STRIP to the edge of a DISK. The other two are the CROSS-CAP and ROMAN SURFACE. The Boy surface is a model of the PROJECTIVE PLANE without singularities and is a SEXTIC SURFACE. The Boy surface can be described using the general method for NONORIENTA-

BLE SURFACES, but this was not known until the analytic equations were found by Apéry (1986). Based on the fact that it had been proven impossible to describe the surface using quadratic polynomials, Hopf had conjectured that quartic polynomials were also insufficient (Pinkall 1986). Apéry's IMMERSION proved this conjecture wrong, giving the equations explicitly in terms of the standard form for a NONORIENTABLE SURFACE,

$$f_1(x, y, z) = \frac{1}{2}[(2x^2 - y^2 - z^2)(x^2 + y^2 + z^2) + 2yz(y^2 - z^2) + zx(x^1 - z^2) + xy(y^2 - x^2)] \quad (1)$$

$$f_2(x, y, z) = \frac{1}{2}\sqrt{3}[(y^2 - z^2)(x^2 + y^2 + z^2) + zx(z^2 - x^2) + xy(y^2 - x^2)] \quad (2)$$

$$f_3(x, y, z) = \frac{1}{8}(x + y + z) \times [(x + y + z)^3 + 4(y - x)(z - y)(x - z)]. \quad (3)$$



Plugging in

$$x = \cos u \sin v \quad (4)$$

$$y = \sin u \sin v \quad (5)$$

$$z = \cos v \quad (6)$$

and letting $u \in [0, \pi]$ and $v \in [0, \pi]$ then gives the Boy surface, three views of which are shown above.

The $\mathbb{R}^3$ parameterization can also be written as

$$x = \frac{\sqrt{2} \cos^2 v \cos(2u) + \cos u \sin(2v)}{2 - \sqrt{2} \sin(3u) \sin(2v)} \quad (7)$$

$$y = \frac{\sqrt{2} \cos^2 v \cos(2u) + \cos u \sin(2v)}{2 - \sqrt{2} \sin(3u) \sin(2v)} \quad (8)$$

$$z = \frac{2 \cos^2 v}{2 - \sqrt{2} \sin(3u) \sin(2v)} \quad (9)$$

(Nordstrand) for $u \in [-\pi/2, \pi/2]$ and $v \in [0, \pi]$.



Three views of the surface obtained using this parameterization are shown above.

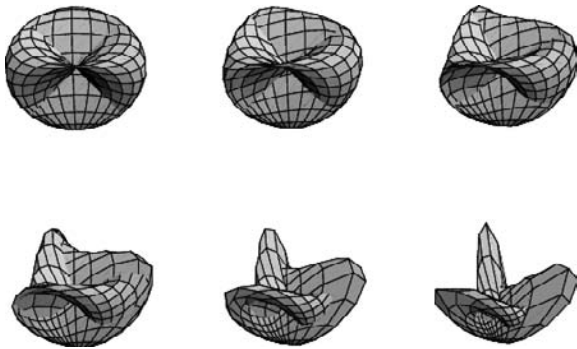
In fact, a HOMOTOPY (smooth deformation) between the ROMAN SURFACE and Boy surface is given by the equations

$$x(u, v) = \frac{\sqrt{2} \cos(2u) \cos^2 v + \cos u \sin(2v)}{2 - \alpha\sqrt{2} \sin(3u) \sin(2v)} \quad (10)$$

$$y(u, v) = \frac{\sqrt{2} \sin(2u) \cos^2 v - \sin u \sin(2v)}{2 - \alpha\sqrt{2} \sin(3u) \sin(2v)} \quad (11)$$

$$z(u, v) = \frac{3 \cos^2 v}{2 - \alpha\sqrt{2} \sin(3u) \sin(2v)} \quad (12)$$

as α varies from 0 to 1, where $\alpha = 0$ corresponds to the ROMAN SURFACE and $\alpha = 1$ to the Boy surface (Wang), shown below.



In $\mathbb{R}^4$, the parametric representation is

$$x_0 = 3[(u^2 + v^2 + w^2)(u^2 + v^2) - \sqrt{2}vw(3u^2 - v^2)] \quad (13)$$

$$x_1 = \sqrt{2}(u^2 + v^2)(u^2 - v^2 + \sqrt{2}uw) \quad (14)$$

$$x_2 = \sqrt{2}(u^2 + v^2)(2uv - \sqrt{2}vw) \quad (15)$$

$$x_3 = 3(u^2 + v^2)^2, \quad (16)$$

and the algebraic equation is

$$\begin{aligned} &64(x_0 - x_3)^3 x_3^3 - 48(x_0 - x_3)^2 x_3^2 (3x_1^2 + 3x_2^2 + 2x_3^2) \\ &+ 12(x_0 - x_3)x_3 [27(x_1^2 + x_2^2)^2 - 24x_3^2(x_1^2 + x_2^2) \\ &+ 36\sqrt{2}x_2x_3(x_2^2 - 3x_1^2) + x_3^4] + (9x_1^2 + 9x_2^2 - 2x_3^2) \\ &\times [-81(x_1^2 + x_2^2)^2 - 72x_3^2(x_1^2 + x_2^2) \\ &+ 108\sqrt{2}x_1x_3(x_1^2 - 3x_2^2) + 4x_3^4] = 0 \end{aligned} \quad (17)$$

(Apéry 1986). Letting

$$x_0 = 1 \quad (18)$$

$$x_1 = x \quad (19)$$

$$x_2 = y \quad (20)$$

$$x_3 = z \quad (21)$$

gives another version of the surface in $\mathbb{R}^3$.

See also CROSS-CAP, IMMERSION, MÖBIUS STRIP, NONORIENTABLE SURFACE, REAL PROJECTIVE PLANE, ROMAN SURFACE, SEXTIC SURFACE

References

- Apéry, F. "The Boy Surface." *Adv. Math.* **61**, 185–266, 1986.
 Apéry, F. *Models of the Real Projective Plane: Computer Graphics of Steiner and Boy Surfaces*. Braunschweig, Germany: Vieweg, 1987.
 Boy, W. "Über die Curvatura integra und die Topologie geschlossener Flächen." *Math. Ann.* **57**, 151–184, 1903.
 Brehm, U. "How to Build Minimal Polyhedral Models of the Boy Surface." *Math. Intell.* **12**, 51–56, 1990.
 Carter, J. S. "On Generalizing Boy Surface--Constructing a Generator of the 3rd Stable Stem." *Trans. Amer. Math. Soc.* **298**, 103–122, 1986.
 Fischer, G. (Ed.). Plates 115–120 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume*. Braunschweig, Germany: Vieweg, pp. 110–115, 1986.
 Hilbert, D. and Cohn-Vossen, S. §46–47 in *Geometry and the Imagination*. New York: Chelsea, 1999.
 Nordstrand, T. "Boy's Surface." <http://www.uib.no/people/nfytn/boytxt.htm>.
 Petit, J.-P. and Souriau, J. "Une représentation analytique de la surface de Boy." *C. R. Acad. Sci. Paris Sér. I Math* **293**, 269–272, 1981.
 Pinkall, U. *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 64–65, 1986.
 Stewart, I. *Game, Set and Math*. New York: Viking Penguin, 1991.

Bp-Theorem

If $O_p(G) = 1$ and if x is a p -element of G , then

$$L_p(C_G(x) \leq E(C_G(x)),$$

where L_p is the P -LAYER.

Bra

A (COVARIANT) 1-VECTOR denoted $\langle \psi |$. The bra is DUAL to the CONTRAVARIANT KET, denoted $|\psi\rangle$. Taken together, the bra and KET form an ANGLE BRACKET (bra + ket = bracket). The bra is commonly encountered in quantum mechanics.

See also ANGLE BRACKET, BRACKET PRODUCT, COVARIANT VECTOR, DIFFERENTIAL K -FORM, KET, ONE-FORM

References

- Dirac, P. A. M. "Bra and Ket Vectors." §6 in *Principles of Quantum Mechanics, 4th ed.* Oxford, England: Oxford University Press, pp. 18–22, 1982.

Brace

One of the symbols { and } used in many different contexts in mathematics. Braces are used

1. To denote grouping of mathematical terms, usually as the outermost delimiter in a complex expression such as $\{a + b[c + d(e + f)]\}$,
2. To delineate a SET, as in $\{a_1, \dots, a_n\}$,
3. Using a left bracket only, to denote different cases for an expression, such as

$$p(n) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd,} \end{cases}$$

4. Using a single horizontal underbrace, to indicate the number of items in a list with not all elements shown explicitly, as in $\underbrace{1, 1, \dots, 1}$.

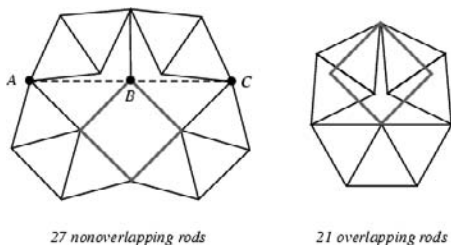
5. As an alternate notation to the FRACTIONAL PART function, $\{x\} = \text{frac } x$.

See also ANGLE BRACKET, PARENTHESIS, SQUARE BRACKET

References

Bringhurst, R. *The Elements of Typographic Style*, 2nd ed. Point Roberts, WA: Hartley and Marks, p. 273, 1997.

Braced Square



The braced square problem asks: given a hinged SQUARE composed of four equal rods (indicated by the thick lines above), how many more hinged rods must be added in the same plane (with no two rods crossing) so that the original square is rigid in the plane. The best solution known, illustrated in the left figure above, uses a total of 27 rods, where A , B , and C are COLLINEAR. If rods are allowed to cross, the best known solution, discovered by E. Friedman in Jan. 2000, requires 21 rods, as illustrated in the right figure above.

Friedman has also considered the minimum number of rods needed to construct RIGID regular n -gons (with overlapping permitted). The best known solutions for $n = 3, 4, \dots$ are 3, 21, 69, 11, 45, 99, 51, ...

See also HINGED TESSELLATION, RIGID GRAPH, SQUARE

References

Friedman, E. "Problem of the Month (January 2000)." <http://www.stetson.edu/~efriedma/mathmagic/0100.html>.
Gardner, M. "The Rigid Square." §6.1 in *The Sixth Book of Mathematical Games from Scientific American*. Chicago, IL: University of Chicago Press, pp. 48–49 and 54–55, 1984.
Wells, D. *The Penguin Dictionary of Curious and Interesting Geometry*. London: Penguin, p. 19, 1991.

Brachistochrone Problem

Find the shape of the CURVE down which a bead sliding from rest and ACCELERATED by gravity will slip (without friction) from one point to another in the least time. The term derives from the Greek

$\beta\rho\alpha\chi\iota\sigma\tau\omicron\varsigma$ (*brachistos*) "the shortest" and $\chi\rho\omicron\nu\omicron\varsigma$ (*chronos*) "time, delay."

The brachistochrone problem was one of the earliest problems posed in the CALCULUS OF VARIATIONS. The solution, a segment of a CYCLOID, was found by Leibniz, L'Hospital, Newton, and the two Bernoullis. Johann Bernoulli solved the problem using the analogous one of considering the path of light refracted by transparent layers of varying density (Mach 1893, Gardner 1984, Courant and Robbins 1996). Note that bead may actually travel uphill along the cycloid for a distance, but the path is nonetheless faster than a straight line or any other line.

The time to travel from a point P_1 to another point P_2 is given by the INTEGRAL

$$t_{12} = \int_1^2 \frac{ds}{v}, \quad (1)$$

The VELOCITY at any point is given by a simple application of energy conservation equating kinetic energy to gravitational potential energy,

$$\frac{1}{2}mv^2 = mgy, \quad (2)$$

so

$$v = \sqrt{2gy}. \quad (3)$$

Plugging this into (1) then gives

$$t_{12} = \int_1^2 \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx = \int_1^2 \sqrt{\frac{1+y'^2}{2gy}} dx. \quad (4)$$

The function to be varied is thus

$$f = (1+y'^2)^{1/2}(2gy)^{-1/2}, \quad (5)$$

To proceed, one would normally have to apply the full-blown EULER-LAGRANGE DIFFERENTIAL EQUATION

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0. \quad (6)$$

However, the function $f(y, y', x)$ is particularly nice since x does not appear explicitly. Therefore, $\partial f / \partial x = 0$, and we can immediately use the BELTRAMI IDENTITY

$$f - y' \frac{\partial f}{\partial y'} = C. \quad (7)$$

Computing

$$\frac{\partial f}{\partial y'} = y'(1+y'^2)^{-1/2}(2gy)^{-1/2}, \quad (8)$$

subtracting $y'(\partial f / \partial y')$ from f , and simplifying then gives

$$\frac{1}{\sqrt{2gy}\sqrt{1+y'^2}} = C. \quad (9)$$

Squaring both sides and rearranging slightly results in

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right] y = \frac{1}{2gC^2} = k^2, \quad (10)$$

where the square of the old constant C has been expressed in terms of a new (POSITIVE) constant k^2 . This equation is solved by the PARAMETRIC EQUATIONS

$$x = \frac{1}{2}k^2(\theta - \sin \theta) \quad (11)$$

$$y = \frac{1}{2}k^2(1 - \cos \theta), \quad (12)$$

which are—lo and behold—the equations of a CYCLOID. If kinetic friction is included, the problem can also be solved analytically, although the solution is significantly messier. In that case, terms corresponding to the normal component of weight and the normal component of the ACCELERATION (present because of path CURVATURE) must be included. Including both terms requires a constrained variational technique (Ashby *et al.* 1975), but including the normal component of weight only gives an elementary solution. The TANGENT and NORMAL VECTORS are

$$\mathbf{T} = \frac{dx}{ds} \hat{\mathbf{x}} + \frac{dy}{ds} \hat{\mathbf{y}} \quad (13)$$

$$\mathbf{N} = -\frac{dy}{ds} \hat{\mathbf{x}} + \frac{dx}{ds} \hat{\mathbf{y}}, \quad (14)$$

gravity and friction are then

$$\mathbf{F}_{\text{gravity}} = mg\hat{\mathbf{y}} \quad (15)$$

$$\mathbf{F}_{\text{friction}} = -\mu(\mathbf{F}_{\text{gravity}} \cdot \mathbf{N})\mathbf{T} = -\mu mg \frac{dy}{ds} \mathbf{T}, \quad (16)$$

and the components along the curve are

$$\mathbf{F}_{\text{gravity}} \cdot \mathbf{T} = mg \frac{dy}{ds} \quad (17)$$

$$\mathbf{F}_{\text{friction}} \cdot \mathbf{T} = -\mu mg \frac{dx}{ds}, \quad (18)$$

so Newton's Second Law gives

$$m \frac{dv}{dt} = mg \frac{dy}{ds} - \mu mg \frac{dx}{ds}. \quad (19)$$

But

$$\frac{dv}{dt} = v \frac{dv}{ds} = \frac{1}{2} \frac{d}{ds}(v^2) \quad (20)$$

$$\frac{1}{2}v^2 = g(y - \mu x) \quad (21)$$

$$v = \sqrt{2g(y - \mu x)}, \quad (22)$$

so

$$t = \int \sqrt{\frac{1 + (y')^2}{2g(y - \mu x)}} dx. \quad (23)$$

Using the EULER-LAGRANGE DIFFERENTIAL EQUATION gives

$$[1 + y'^2](1 + \mu y') + 2(y - \mu x)y'' = 0. \quad (24)$$

This can be reduced to

$$\frac{1 + (y')^2}{(1 + \mu y')^2} = \frac{C}{y - \mu x}. \quad (25)$$

Now letting

$$y' = \cot(\tfrac{1}{2}\theta), \quad (26)$$

the solution is

$$x = \frac{1}{2}k^2[(\theta - \sin \theta) + \mu(1 - \cos \theta)] \quad (27)$$

$$y = \frac{1}{2}k^2[(1 - \cos \theta) + \mu(\theta - \sin \theta)]. \quad (28)$$

See also CALCULUS OF VARIATIONS, CYCLOID, TAUTOCHRON PROBLEM

References

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Bracket

Mathematicians often use the term "bracket" to mean "COMMUTATOR," which is denoted using SQUARE BRACKETS.

See also ANGLE BRACKET, BRA, BRACE, BRACKET POLYNOMIAL, BRACKET PRODUCT, IVERSON BRACKET,

KET, LAGRANGE BRACKET, POISSON BRACKET, SQUARE BRACKET

Bracket Polynomial

A one-variable KNOT POLYNOMIAL related to the JONES POLYNOMIAL. The bracket polynomial, however, is *not* a topological invariant, since it is changed by type I REIDEMEISTER MOVES. However, the SPAN of the bracket polynomial is a knot invariant. The bracket polynomial is occasionally given the grandiose name REGULAR ISOTOPY INVARIANT. It is defined by

$$\langle L \rangle(A, B, d) \equiv \sum_{\sigma} \langle L | \sigma \rangle d^{||\sigma||}, \quad (1)$$

where A and B are the "splitting variables," σ runs through all "states" of L obtained by SPLITTING the LINK, $\langle L | \sigma \rangle$ is the product of "splitting labels" corresponding to σ , and

$$||\sigma|| \equiv N_L - 1, \quad (2)$$

where N_L is the number of loops in σ . Letting

$$B = A^{-1} \quad (3)$$

$$d = -A^2 - A^{-2} \quad (4)$$

gives a KNOT POLYNOMIAL which is invariant under REGULAR ISOTOPY, and normalizing gives the KAUFFMAN POLYNOMIAL X which is invariant under AMBI-ENT ISOTOPY. The bracket POLYNOMIAL of the UNKNOT is 1. The bracket POLYNOMIAL of the MIRROR IMAGE K^* is the same as for K but with A replaced by A^{-1} . In terms of the one-variable KAUFFMAN POLYNOMIAL X , the two-variable KAUFFMAN POLYNOMIAL F and the JONES POLYNOMIAL V ,

$$X(A) = (-A^3)^{-w(L)} \langle L \rangle, \quad (5)$$

$$\langle L \rangle(A) = F(-A^3, A + A^{-1}) \quad (6)$$

$$\langle L \rangle(A) = V(A^{-4}), \quad (7)$$

where $w(L)$ is the WRITHE of L .

See also JONES POLYNOMIAL, SQUARE BRACKET POLYNOMIAL

References

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 Weisstein, E. W. "Knots and Links." MATHEMATICA NOTEBOOK KNOTS.M.

Bracket Product

L_2 -INNER PRODUCT

Bracketing

Take x itself to be a bracketing, then recursively define a bracketing as a sequence $B = (B_1, \dots, B_k)$ where $k \geq 2$ and each B_i is a bracketing. A bracketing can be REPRESENTED AS a parenthesized string of x s, with parentheses removed from any single letter x for clarity of notation (Stanley 1997). Bracketings built up of binary operations only are called BINARY BRACKETINGS. For example, four letters have 11 possible bracketings:

$$\begin{array}{cccc} xxxx & (xx)xx & x(xx)x & xx(xx) \\ (xxx)x & x(xxx) & ((xx)x)x & (x(xx))x \\ (xx)(xx) & x((xx)x) & x(x(xx)) & \end{array}$$

the last five of which are binary.

The number of bracketings on n letters is given by the GENERATING FUNCTION

$$\frac{1}{4}(1 + x - \sqrt{1 - 6x + x^2}) = x + x^2 + 3x^3 + 11x^4 + 45x^5$$

(Schröder 1870, Stanley 1997) and the RECURRENCE RELATION

$$s_n = \frac{3(2n - 3)s_{n-1} - (n - 3)s_{n-2}}{n}$$

(Sloane), giving the sequence for s_n as 1, 1, 3, 11, 45, 197, 903, ... (Sloane's A001003). The numbers are also given by

$$s_n = \sum_{i_1 + \dots + i_k = n} s(i_1) \cdots s(i_k)$$

for $n \geq 2$ (Stanley 1997).

The first PLUTARCH NUMBER 103,049 is equal to s_{10} (Stanley 1997), suggesting that Plutarch's problem of ten compound propositions is equivalent to the number of bracketings. In addition, Plutarch's second number 310,954 is given by $(s_{10} + s_{11})/2 = 310,954$ (Habsieger *et al.* 1998).

See also BINARY BRACKETING, PLUTARCH NUMBERS

References

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 Stanley, R. P. "Hipparchus, Plutarch, Schröder, and Hough." *Amer. Math. Monthly* **104**, 344–350, 1997.

Bradley's Theorem

Let

$$S(\alpha, \beta, m; z)$$

$$\equiv m \sum_{j=0}^{\infty} \frac{\Gamma(m+j(z+1))\Gamma(\beta+1+jz)}{\Gamma(m+jz+1)\Gamma(\alpha+\beta+1+j(z+1))} \frac{(\alpha)_j}{j!},$$

where $(\alpha)_j$ is a POCHHAMMER SYMBOL, and let α be a NEGATIVE INTEGER. Then

$$S(\alpha, \beta, m; z) = \frac{\Gamma(\beta+1-m)}{\Gamma(\alpha+\beta+1-m)},$$

where $\Gamma(z)$ is the GAMMA FUNCTION.

References

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Bradley, D. "On a Claim by Ramanujan about Certain Hypergeometric Series." *Proc. Amer. Math. Soc.* **121**, 1145–1149, 1994.

Brahmagupta Identity

Let

$$\beta \equiv \det B = x^2 - ty^2,$$

where B is the BRAHMAGUPTA MATRIX, then

$$\begin{aligned} \det[B(x_1, y_1)B(x_2, y_2)] &= \det[B(x_1, y_1)] \det[B(x_2, y_2)] \\ &= \beta_1 \beta_2. \end{aligned}$$

References

Suryanarayan, E. R. "The Brahmagupta Polynomials." *Fib. Quart.* **34**, 30–39, 1996.

Brahmagupta Matrix

$$B(x, y) = \begin{bmatrix} x & y \\ \pm ty & \pm x \end{bmatrix}.$$

It satisfies

$$B(x_1, y_1)B(x_2, y_2) = B(x_1x_2 \pm ty_1y_2, x_1y_2 \pm y_1x_2).$$

Powers of the matrix are defined by

$$B^n = \begin{bmatrix} x & y \\ ty & x \end{bmatrix}^n = \begin{bmatrix} x_n & y_n \\ ty_n & x_n \end{bmatrix} \equiv B_n.$$

The x_n and y_n are called BRAHMAGUPTA POLYNOMIALS. The Brahmagupta matrices can be extended to NEGATIVE INTEGERS

$$B^{-n} = \begin{bmatrix} x & y \\ ty & x \end{bmatrix}^{-n} = \begin{bmatrix} x_{-n} & y_{-n} \\ ty_{-n} & x_{-n} \end{bmatrix} \equiv B_{-n}.$$

See also BRAHMAGUPTA IDENTITY

References

Suryanarayan, E. R. "The Brahmagupta Polynomials." *Fib. Quart.* **34**, 30–39, 1996.

Brahmagupta Polynomial

One of the POLYNOMIALS obtained by taking POWERS of the BRAHMAGUPTA MATRIX. They satisfy the RECURRENCE RELATION

$$x_{n+1} = xx_n + ty y_n \quad (1)$$

$$y_{n+1} = xy_n + yx_n. \quad (2)$$

A list of many others is given by Suryanarayan (1996). Explicitly,

$$x_n = x^n + t \binom{n}{2} x^{n-2} y^2 + t^2 \binom{n}{4} x^{n-4} y^4 + \dots \quad (3)$$

$$y_n = nx^{n-1}y + t \binom{n}{3} x^{n-3} y^3 + t^2 \binom{n}{5} x^{n-5} y^5 + \dots \quad (4)$$

The Brahmagupta POLYNOMIALS satisfy

$$\frac{\partial x_n}{\partial x} = \frac{\partial y_n}{\partial y} = nx_{n-1} \quad (5)$$

$$\frac{\partial x_n}{\partial y} = t \frac{\partial y_n}{\partial y} = nty_{n-1}. \quad (6)$$

The first few POLYNOMIALS are

$$x_0 = 0$$

$$x_1 = x$$

$$x_2 = x^2 + ty^2$$

$$x_3 = x^3 + 3txy^2$$

$$x_4 = x^4 + 6tx^2y^2 + t^2y^4$$

and

$$y_0 = 0$$

$$y_1 = y$$

$$y_2 = 2xy$$

$$y_3 = 3x^2y + ty^3$$

$$y_4 = 4x^3y + 4txy^3.$$

Taking $x = y = 1$ and $t = 2$ gives y_n equal to the PELL NUMBERS and x_n equal to half the Pell-Lucas numbers. The Brahmagupta POLYNOMIALS are related to the MORGAN-VOYCE POLYNOMIALS, but the relationship given by Suryanarayan (1996) is incorrect.

References

Suryanarayan, E. R. "The Brahmagupta Polynomials." *Fib. Quart.* **34**, 30–39, 1996.

Brahmagupta's Formula

For a QUADRILATERAL with sides of length a, b, c , and d , the AREA K is given by

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cos^2[\frac{1}{2}(A+B)]}, \quad (1)$$

where

$$s \equiv \frac{1}{2}(a+b+c+d) \quad (2)$$

is the SEMIPERIMETER, A is the ANGLE between a and d , and B is the ANGLE between b and c . For a CYCLIC QUADRILATERAL (i.e., a QUADRILATERAL inscribed in a CIRCLE), $A+B=\pi$, so

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)} \quad (3)$$

$$= \frac{\sqrt{(bc+ad)(ac+bd)(ab+cd)}}{4R}, \quad (4)$$

where R is the RADIUS of the CIRCUMCIRCLE. If the QUADRILATERAL is INSCRIBED in one CIRCLE and CIRCUMSCRIBED on another, then the AREA FORMULA simplifies to

$$K = \sqrt{abcd}. \quad (5)$$

See also BRETSCHNEIDER'S FORMULA, HERON'S FORMULA, QUADRILATERAL

References

- Brown, K. S. "Heron's Formula and Brahmagupta's Generalization." <http://www.seanet.com/~ksbrown/kmath196.htm>.
 Coxeter, H. S. M. and Greitzer, S. L. "Cyclic Quadrangles; Brahmagupta's Formula." §3.2 in *Geometry Revisited*. Washington, DC: Math. Assoc. Amer., pp. 56–60, 1967.
 Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*. Boston, MA: Houghton Mifflin, pp. 81–82, 1929.

Brahmagupta's Problem

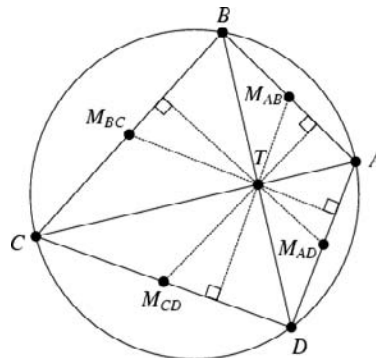
Solve the PELL EQUATION

$$x^2 - 92y^2 = 1$$

in INTEGERS. The smallest solution is $x = 1151$, $y = 120$.

See also DIOPHANTINE EQUATION, PELL EQUATION

Brahmagupta's Theorem



In a CYCLIC QUADRILATERAL $ABCD$ having perpendicular diagonals $AC \perp BD$, the perpendiculars to the sides through point T of intersection of the diagonals (the ANTICENTER) always bisects the opposite side (so M_{AB} , M_{BC} , M_{CD} , and M_{DA} are the MIDPOINTS of the corresponding sides of the QUADRILATERAL).

See also ANTICENTER, CYCLIC QUADRILATERAL, MID-POINT

References

- Honsberger, R. *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*. Washington, DC: Math. Assoc. Amer., p. 37, 1995.

Braid

An intertwining of strings attached to top and bottom "bars" such that each string never "turns back up." In other words, the path of each string in a braid could be traced out by a falling object if acted upon only by gravity and horizontal forces.

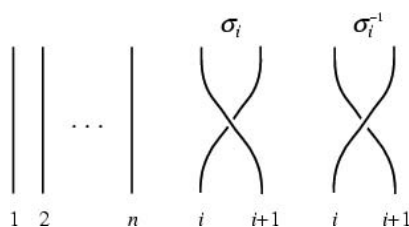
See also BRAID GROUP

References

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Braid Group

Also called ARTIN BRAID GROUPS. Consider n strings, each oriented vertically from a lower to an upper "bar." If this is the least number of strings needed to make a closed braid representation of a LINK, n is called the BRAID INDEX. Now enumerate the possible braids in a group, denoted B_n . A general n -braid is constructed by iteratively applying the σ_i ($i = 1, \dots, n-1$) operator, which switches the lower endpoints of the i th and $(i+1)$ th strings—keeping the upper endpoints fixed—with the $(i+1)$ th string brought *above* the i th string. If the $(i+1)$ th string passes *below* the i th string, it is denoted σ_i^{-1} .



Topological equivalence for different representations of a BRAID WORD $\Pi_i \sigma_i$ and $\Pi_i \sigma_i'$ is guaranteed by the conditions

$$\begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i-j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for all } i \end{cases}$$

as first proved by E. Artin. Any n -braid is expressed as a BRAID WORD, e.g., $\sigma_1 \sigma_2 \sigma_3 \sigma_2^{-1} \sigma_1$ is a BRAID WORD for the braid group B_3 . When the opposite ends of the braids are connected by nonintersecting lines, KNOTS are formed which are identified by their braid group and BRAID WORD. The BURAU REPRESENTATION gives a matrix representation of the braid groups.

References

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- Christy, J. "Braids." <http://www.mathsource.com/cgi-bin/msitem?0202-228>.
- Jones, V. F. R. "Hecke Algebra Representations of Braid Groups and Link Polynomials." *Ann. Math.* **126**, 335–388, 1987.
- Murasugi, K. and Kurpita, B. I. *A Study of Braids*. Dordrecht, Netherlands: Kluwer, 1999.
- Weisstein, E. W. "Knots and Links." MATHEMATICA NOTEBOOK KNOTS.M.

Braid Index

The least number of strings needed to make a closed braid representation of a LINK. The braid index is equal to the least number of SEIFERT CIRCLES in any projection of a KNOT (Yamada 1987). Also, for a nonsplittable LINK with CROSSING NUMBER $c(L)$ and braid index $i(L)$,

$$c(L) \geq 2[i(L) - 1]$$

(Ohyama 1993). Let E be the largest and e the smallest POWER of ℓ in the HOMFLY POLYNOMIAL of an oriented LINK, and i be the braid index. Then the MORTON-FRANKS-WILLIAMS INEQUALITY holds,

$$i \geq \frac{1}{2}(E - e) + 1$$

(Franks and Williams 1987). The inequality is sharp for all PRIME KNOTS up to 10 crossings with the exceptions of 09-042, 09-049, 10-132, 10-150, and 10-156.

References

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Braikenridge-Maclaurin Construction

- Jones, V. F. R. "Hecke Algebra Representations of Braid Groups and Link Polynomials." *Ann. Math.* **126**, 335–388, 1987.
- Ohyama, Y. "On the Minimal Crossing Number and the Brad Index of Links." *Canad. J. Math.* **45**, 117–131, 1993.
- Yamada, S. "The Minimal Number of Seifert Circles Equals the Braid Index of a Link." *Invent. Math.* **89**, 347–356, 1987.

Braid Word

Any n -braid is expressed as a braid word, e.g., $\sigma_1 \sigma_2 \sigma_3 \sigma_2^{-1} \sigma_1$ is a braid word for the BRAID GROUP B_3 . By ALEXANDER'S THEOREM, any LINK is representable by a closed braid, but there is no general procedure for reducing a braid word to its simplest form. However, MARKOV'S THEOREM gives a procedure for identifying different braid words which represent the same LINK.

Let b_+ be the sum of POSITIVE exponents, and b_- the sum of NEGATIVE exponents in the BRAID GROUP B_n . If

$$b_+ - 3b_- \geq n,$$

then the closed braid b is not AMPHICHIRAL (Jones 1985).

See also BRAID GROUP

References

- Jones, V. F. R. "A Polynomial Invariant for Knots via von Neumann Algebras." *Bull. Amer. Math. Soc.* **12**, 103–111, 1985.
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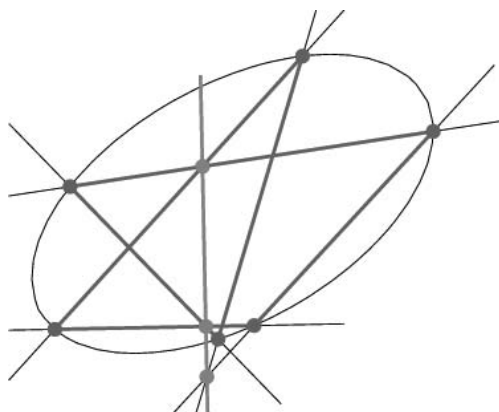
Braikenridge-Maclaurin Construction

Let A_n, B_2, C_1, A_2 , and B_1 be five points determining a CONIC. Then the CONIC is the LOCUS of the point

$$C_2 = A_1(L \cdot C_1 A_2) \cdot B_1(L \cdot C_1 B_2),$$

where L is a line through the point $A_1 B_2 \cdot B_1 A_2$.

See also BRAIKENRIDGE-MACLAURIN THEOREM, CONIC SECTION

Braikenridge-Maclaurin Theorem

The converse of PASCAL'S THEOREM, which states that if the three pairs of opposite sides of (an irregular) HEXAGON meet at three COLLINEAR points, then the six vertices lie on a conic, which may degenerate into a pair of lines (Coxeter and Greitzer 1967, p. 76).

See also BRAIKENRIDGE-MACLAURIN CONSTRUCTION, CONIC SECTION, PASCAL'S THEOREM

References

- Coxeter, H. S. M. *Projective Geometry*, 2nd ed. New York: Springer-Verlag, p. 85, 1987.
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited*. Washington, DC: Math. Assoc. Amer., p. 76, 1967.

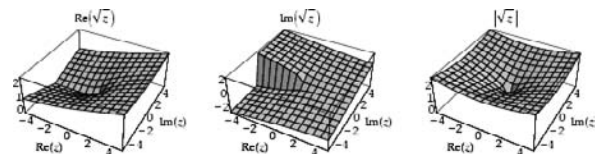
Branch

A branch at a point u in a TREE is a maximal SUBTREE containing u as an ENDPOINT (Harary 1994, p. 35).

See also FORK, LEAF (TREE), LIMB, TREE

References

- Harary, F. *Graph Theory*. Reading, MA: Addison-Wesley, 1994.
Lu, T. "The Enumeration of Trees with and without Given Limbs." *Disc. Math.* **154**, 153–165, 1996.
Schwenk, A. "Almost All Trees are Cospectral." In *New Directions in the Theory of Graphs* (Ed. F. Harary). New York: Academic Press, pp. 275–307, 1973.

Branch Cut

A line in the COMPLEX PLANE across which a MULTI-

VALUED FUNCTION is discontinuous. Some functions have a relatively simple branch cut structure, but branch cuts for some functions are extremely complicated. The illustrations above show the single branch cut present in the definition of the square root function in the complex plane. In general, branch cuts are not unique, but are chosen by convention to give simple analytic properties. An alternative to branch cuts is the use of RIEMANN SURFACES.

function	branch cut(s)
$\cos^{-1} z$	$(-\infty, -1)$ and $(1, \infty)$
$\cosh^{-1}$	$(-\infty, 1)$
$\cot^{-1} z$	$(-i, i)$
$\coth^{-1}$	$[-1, 1]$
$\csc^{-1} z$	$(-1, 1)$
$\csch^{-1}$	$(-i, i)$
$\ln z$	$(-\infty, 0]$
$\sec^{-1} z$	$(-1, 1)$
sech^{-1}	$(\infty, 0]$ and $(1, \infty)$
$\sin^{-1} z$	$(-\infty, -1)$ and $(1, \infty)$
$\sinh^{-1}$	$(-i\infty, -i)$ and $(i, i\infty)$
$\sqrt{z}$	$(-\infty, 0)$
$\tan^{-1} z$	$(-i\infty, -i)$ and $(i, i\infty)$
$\tanh^{-1}$	$(-\infty, -1)$ and $[1, \infty)$
$z^n, n \notin \mathbb{Z}$	$(-\infty, 0)$ for $\Re[n] \leq 0$; $(-\infty, 0]$ for $\Re[n] > 0$

See also BRANCH POINT, CUT, MULTIVALUED FUNCTION, RIEMANN SURFACE

References

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Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I*. New York: McGraw-Hill, pp. 399–401, 1953.

Branch Line

BRANCH CUT

Branch Point

An argument at which identical points in the COMPLEX PLANE are mapped to different points. For example, consider

$$f(z) = z^a.$$

Then $f(e^{0i}) = f(1) = 1$, but $f(e^{2\pi i}) = e^{2\pi ia}$, despite the fact that $e^{i0} = e^{2\pi i}$. PINCH POINTS are also called branch points.

See also BRANCH CUT, PINCH POINT

References

- Arfken, G. *Mathematical Methods for Physicists*, 3rd ed. Orlando, FL: Academic Press, pp. 397–399, 1985.
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I*. New York: McGraw-Hill, pp. 391–392 and 399–401, 1953.

Brauer Chain

A Brauer chain is an ADDITION CHAIN in which each member uses the previous member as a summand. A number n for which a shortest chain exists which is a Brauer chain is called a BRAUER NUMBER.

See also ADDITION CHAIN, BRAUER NUMBER, HANSEN CHAIN

References

- Guy, R. K. "Addition Chains. Brauer Chains. Hansen Chains." §C6 in *Unsolved Problems in Number Theory*, 2nd ed. New York: Springer-Verlag, pp. 111–113, 1994.

Brauer Group

The GROUP of classes of finite dimensional central simple ALGEBRAS over k with respect to a certain equivalence.

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- Hazewinkel, M. (Managing Ed.). *Encyclopaedia of Mathematics: An Updated and Annotated Translation of the Soviet "Mathematical Encyclopaedia"*. Dordrecht, Netherlands: Reidel, p. 479, 1988.

Brauer Number

A number n for which a shortest chain exists which is a BRAUER CHAIN is called a Brauer number. There are infinitely many non-Brauer numbers.

See also BRAUER CHAIN, HANSEN NUMBER

References

- Guy, R. K. "Addition Chains. Brauer Chains. Hansen Chains." §C6 in *Unsolved Problems in Number Theory*, 2nd ed. New York: Springer-Verlag, pp. 111–113, 1994.

Brauer's Theorem

If, in the GERSGORIN CIRCLE THEOREM for a given m ,

$$|a_{ij} - a_{mm}| > \Lambda_j + \Lambda_m$$

for all $j \neq m$, then exactly one EIGENVALUE of A lies in the DISK Γ_m .

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- Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products*, 6th ed. San Diego, CA: Academic Press, p. 1121, 2000.

Brauer-Severi Variety

An ALGEBRAIC VARIETY over a FIELD K that becomes ISOMORPHIC to a PROJECTIVE SPACE.

References

- Hazewinkel, M. (Managing Ed.). *Encyclopaedia of Mathematics: An Updated and Annotated Translation of the Soviet "Mathematical Encyclopaedia"*. Dordrecht, Netherlands: Reidel, pp. 480–481, 1988.

Braun's Conjecture

Let $B = \{b_1, b_2, \dots\}$ be an INFINITE ABELIAN SEMI-GROUP with linear order $b_1 < b_2 < \dots$ such that b_1 is the unit element and $a < b$ IMPLIES $ac < bc$ for $a, b, c \in B$. Define a MÖBIUS FUNCTION μ on B by $\mu(b_1) = 1$ and

$$\sum_{b_d | b_n} \mu(b_d) = 0$$

for $n = 2, 3, \dots$. Further suppose that $\mu(b_n) = \mu(n)$ (the true MÖBIUS FUNCTION) for all $n \geq 1$. Then Braun's conjecture states that

$$b_{mn} = b_m b_n$$

for all $m, n \geq 1$.

See also MÖBIUS PROBLEM

References

- Flath, A. and Zulauf, A. "Does the Möbius Function Determine Multiplicative Arithmetic?" *Amer. Math. Monthly* **102**, 354–256, 1995.

Breadth-First Traversal

A search algorithm of a GRAPH which explores all nodes adjacent to the current node before moving on. For cyclic graphs, care must be taken to make sure that no nodes are repeated. When properly implemented, all nodes in a given connected component are explored.

See also DEPTH-FIRST TRAVERSAL

References

- Skiena, S. "Breadth-First and Depth-First Search." §3.2.5 in *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*. Reading, MA: Addison-Wesley, pp. 95–97, 1990.

Breeder

A pair of POSITIVE INTEGERS (a_1, a_2) such that the equations

$$a_1 + a_2 x = \sigma(a_1) = \sigma(a_2)(x + 1)$$

have a POSITIVE INTEGER solution x , where $\sigma(n)$ is the DIVISOR FUNCTION. If x is PRIME, then $(a_1, a_2 x)$ is an AMICABLE PAIR (te Riele 1986). (a_1, a_2) is a "special" breeder if

$$a_1 = au$$

$$a_2 = a,$$

where a and u are RELATIVELY PRIME, $(a, u) = 1$. If regular amicable pairs of type $(i, 1)$ with $i \geq 2$ are OF THE FORM (au, ap) with p PRIME, then (au, a) are special breeders (te Riele 1986).

See also AMICABLE PAIR

References

te Riele, H. J. J. "Computation of All the Amicable Pairs Below 10^{10} ." *Math. Comput.* **47**, 361–368 and S9-S35, 1986.

Brelaz's Heuristic Algorithm

AN ALGORITHM which can be used to find a good, but not necessarily minimal, EDGE or VERTEX COLORING for a GRAPH. However, the algorithm does minimally color COMPLETE K -PARTITE GRAPH.

See also CHROMATIC NUMBER, EDGE COLORING, VERTEX COLORING

References

Brelaz, D. "New Methods to Color the Vertices of a Graph." *Comm. ACM* **22**, 251–256, 1979.
Skiena, S. "Finding a Vertex Coloring." §5.5.3 in *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*. Reading, MA: Addison-Wesley, pp. 214–215, 1990.

Brent's Factorization Method

A modification of the POLLARD RHO FACTORIZATION METHOD which uses

$$x_{i+1} = x_i^2 - c \pmod{n}.$$

References

Brent, R. "An Improved Monte Carlo Factorization Algorithm." *Nordisk Tidskrift for Informationsbehandling (BIT)* **20**, 176–184, 1980.

Brent's Method

A ROOT-finding ALGORITHM which combines root bracketing, bisection, and INVERSE QUADRATIC INTERPOLATION. It is sometimes known as the VAN WIJNGAARDEN-DEKER-BRENT METHOD.

Brent's method uses a LAGRANGE INTERPOLATING POLYNOMIAL of degree 2. Brent (1973) claims that this method will always converge as long as the values of the function are computable within a given region containing a ROOT. Given three points x_1 , x_2 , and x_3 , Brent's method fits x as a quadratic function of y , then uses the interpolation formula

$$x = \frac{[y - f(x_1)][y - f(x_2)]x_3}{[f(x_3) - f(x_1)][f(x_3) - f(x_2)]} + \frac{[y - f(x_2)][y - f(x_3)]x_1}{[f(x_1) - f(x_2)][f(x_1) - f(x_3)]} + \frac{[y - f(x_3)][y - f(x_1)]x_2}{[f(x_2) - f(x_3)][f(x_2) - f(x_1)]}. \quad (1)$$

Subsequent root estimates are obtained by setting $y = 0$, giving

$$x = x_2 + \frac{P}{Q}, \quad (2)$$

where

$$P = S[R(R - T)(x_3 - x_2) - (1 - R)(x_2 - x_1)] \quad (3)$$

$$Q = (T - 1)(R - 1)(S - 1) \quad (4)$$

with

$$R \equiv \frac{f(x_2)}{f(x_3)} \quad (5)$$

$$S \equiv \frac{f(x_2)}{f(x_1)} \quad (6)$$

$$T \equiv \frac{f(x_1)}{f(x_3)} \quad (7)$$

(Press *et al.* 1992).

References

Brent, R. P. Ch. 3–4 in *Algorithms for Minimization Without Derivatives*. Englewood Cliffs, NJ: Prentice-Hall, 1973.
Forsythe, G. E.; Malcolm, M. A.; and Moler, C. B. §7.2 in *Computer Methods for Mathematical Computations*. Englewood Cliffs, NJ: Prentice-Hall, 1977.
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Van Wijngaarden-Dekker-Brent Method." §9.3 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing*, 2nd ed. Cambridge, England: Cambridge University Press, pp. 352–355, 1992.

Brent-Salamin Formula

A formula which uses the ARITHMETIC-GEOMETRIC MEAN to compute π . It has quadratic convergence and is also called the GAUSS-SALAMIN FORMULA and SALAMIN FORMULA. Let

$$a_{n+1} = \frac{1}{2}(a_n + b_n) \quad (1)$$

$$b_{n+1} = \sqrt{a_n b_n} \quad (2)$$

$$c_{n+1} = \frac{1}{2}(a_n - b_n) \quad (3)$$

$$d_n \equiv a_n^2 - b_n^2, \quad (4)$$

and define the initial conditions to be $a_0 = 1$, $b_0 = 1/\sqrt{2}$. Then iterating a_n and b_n gives the ARITHMETIC-

GEOMETRIC MEAN, and π is given by

$$\pi = \frac{4[M(1, 2^{-1/2})]^2}{1 - \sum_{j=1}^{\infty} 2^{j+1}d_j} \quad (5)$$

$$= \frac{4[M(1, 2^{-1/2})]^2}{1 - \sum_{j=1}^{\infty} 2^{j+1}c_j^2}. \quad (6)$$

King (1924) showed that this formula and the LEGENDRE RELATION are equivalent and that either may be derived from the other.

See also ARITHMETIC-GEOMETRIC MEAN, π

References

- Borwein, J. M. and Borwein, P. B. *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity*. New York: Wiley, pp. 48–51, 1987.
- Castellanos, D. "The Ubiquitous π . Part II." *Math. Mag.* **61**, 148–163, 1988.
- King, L. V. *On the Direct Numerical Calculation of Elliptic Functions and Integrals*. Cambridge, England: Cambridge University Press, 1924.
- Lord, N. J. "Recent Calculations of π : The Gauss-Salamin Algorithm." *Math. Gaz.* **76**, 231–242, 1992.
- Salamin, E. "Computation of π Using Arithmetic-Geometric Mean." *Math. Comput.* **30**, 565–570, 1976.

Bretschneider's Formula

Given a general QUADRILATERAL with sides of lengths a , b , c , and d (Beyer 1987), the AREA is given by

$$A_{\text{quadrilateral}} = \frac{1}{4} \sqrt{4p^2q^2 - (b^2 + d^2 - a^2 - c^2)^2},$$

where p and q are the diagonal lengths.

See also BRAHMAGUPTA'S FORMULA, HERON'S FORMULA

References

- Beyer, W. H. (Ed.). *CRC Standard Mathematical Tables*, 28th ed. Boca Raton, FL: CRC Press, p. 123, 1987.

Brianchon Point

The point of CONCURRENCE of the joins of the VERTICES of a TRIANGLE and the points of contact of a CONIC SECTION INSCRIBED in the TRIANGLE. A CONIC INSCRIBED in a TRIANGLE has an equation OF THE FORM

$$\frac{f}{u} + \frac{g}{v} + \frac{h}{w} = 0,$$

so its Brianchon point has TRILINEAR COORDINATES $(1/f, 1/g, 1/h)$. For KIEPERT'S PARABOLA, the Brianchon point has TRIANGLE CENTER FUNCTION

$$\alpha = \frac{1}{a(b^2 - c^2)},$$

which is the STEINER POINT.

See also HEPTAGON THEOREM, KIEPERT'S PARABOLA, STEINER POINTS

References

- Evelyn, C. J. A.; Money-Coutts, G. B.; and Tyrrell, J. A. "The Heptagon Theorem." §2.1 in *The Seven Circles Theorem and Other New Theorems*. London: Stacey International, pp. 8–11, 1974.

Brianchon's Theorem



The DUAL of PASCAL'S THEOREM (Casey 1888, p. 146). It states that, given a HEXAGON CIRCUMSCRIBED on a CONIC SECTION, the lines joining opposite VERTICES (DIAGONALS) meet in a single point.

See also DUALITY PRINCIPLE, PASCAL'S THEOREM

References

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Brick

A RECTANGULAR PARALLELEPIPED.

See also CANONICAL BRICK, EULER BRICK, HARMONIC BRICK, RECTANGULAR PARALLELEPIPED

Bride's Chair

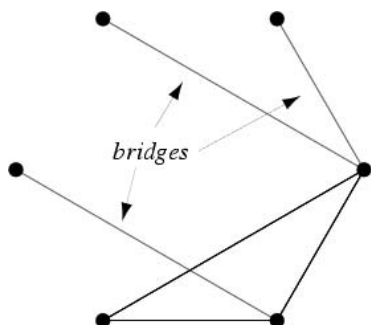
One name for the figure used by Euclid to prove the PYTHAGOREAN THEOREM.

See also PEACOCK'S TAIL, WINDMILL

References

Wells, D. *The Penguin Dictionary of Curious and Interesting Geometry*. London: Penguin, p. 203, 1991.

Bridge



The bridges of a CONNECTED GRAPH are the EDGES whose removal disconnects the GRAPH (Chartrand 1985, p. 45; Skiena 1990, p. 177). More generally, a bridge is an edge of a GRAPH G whose removal increases the number of components of G (Harary 1994, p. 26). An edge of a CONNECTED GRAPH is a bridge IFF it does not lie on any cycle. The bridges of a graph can be found using `Bridges[g]` in the *Mathematica* add-on package `DiscreteMath`Combinatorica`` (which can be loaded with the command `<<DiscreteMath``).

Every edge of a TREE is a bridge. A CUBIC GRAPH contains a bridge IFF it contains an ARTICULATION VERTEX (Skiena 1990, p. 177).

See also ARTICULATION VERTEX, BLOCK

References

Chartrand, G. "Cut-Vertices and Bridges." §2.4 in *Introductory Graph Theory*. New York: Dover, pp. 45–49, 1985.
Harary, F. *Graph Theory*. Reading, MA: Addison-Wesley, 1994.
Skiena, S. *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*. Reading, MA: Addison-Wesley, pp. 171 and 177, 1990.

Bridge Card Game

Bridge is a CARD game played with a normal deck of 52 cards. The number of possible distinct 13-card hands is

$$N = \binom{52}{13} = 635,013,559,600.$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. While the chances of being dealt a hand of 13 CARDS (out of 52) of the same suit are

$$\frac{\binom{4}{52}}{\binom{13}{13}} = \frac{1}{158,753,389,900},$$

the chance that one of four players will receive a hand of a single suit is

$$\frac{1}{39,688,347,497}.$$

There are special names for specific types of hands. A ten, jack, queen, king, or ace is called an "honor." Getting the three top cards (ace, king, and queen) of three suits and the ace, king, and queen, and jack of the remaining suit is called 13 top honors. Getting all cards of the same suit is called a 13-card suit. Getting 12 cards of same suit with ace high and the 13th card *not* an ace is called 2-card suit, ace high. Getting *no* honors is called a Yarborough.

The probabilities of being dealt 13-card bridge hands of a given type are given below. As usual, for a hand with probability P , the ODDS against being dealt it are $(1/P) - 1 : 1$.

Hand	Exact Probability	Probability	ODDS
13 top honors	$\frac{4}{N} = \frac{1}{158,753,389,900}$	6.30×10^{-12}	158,753,389,899:1
13-card suit	$\frac{4}{N} = \frac{1}{158,753,389,900}$	6.30×10^{-12}	158,753,389,899:1
12-card suit, ace high	$\frac{4 \cdot 12 \cdot 36}{N \cdot 4} = \frac{1,469,938,705}{1,469,938,705}$	2.72×10^{-9}	367,484,697.8:1
Yarborough	$\frac{\binom{32}{13}}{N} = \frac{5,394}{9,860,459}$	5.47×10^{-4}	1,827.0:1
four aces	$\frac{\binom{48}{9}}{N} = \frac{11}{4,165}$	2.64×10^{-3}	377.6:1
nine honors	$\frac{\binom{20}{9} \binom{32}{4}}{N} = \frac{888,212}{93,384,347}$	9.51×10^{-3}	104.1:1

See also CARDS, POKER

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 48–49, 1987.
Kraitchik, M. "Bridge Hands." §6.3 in *Mathematical Recreations*. New York: W. W. Norton, pp. 119–121, 1942.
Reese, T. *Bridge for Bright Beginners*. New York: Dover, 1973.
Rubens, J. *The Secrets of Winning Bridge*. New York: Dover, 1981.

Bridge Index

A numerical KNOT invariant. For a TAME KNOT K , the bridge index is the least BRIDGE NUMBER of all planar representations of the KNOT. The bridge index of the UNKNOT is defined as 1.

See also BRIDGE NUMBER, CROOKEDNESS

References

- Rolfsen, D. *Knots and Links*. Wilmington, DE: Publish or Perish Press, p. 114, 1976.
 Schubert, H. "Über eine numerische Knotteninvariante." *Math. Z.* **61**, 245–288, 1954.

Bridge Knot

An n -bridge knot is a knot with BRIDGE NUMBER n . The set of 2-bridge knots is identical to the set of rational knots. If L is a 2-BRIDGE KNOT, then the BLM/HO POLYNOMIAL Q and JONES POLYNOMIAL V satisfy

$$Q_L(z) = 2z^{-1}V_L(t)V_L(t^{-1} + 1 - 2z^{-1}),$$

where $z \equiv -t - t^{-1}$ (Kanenobu and Sumi 1993). Kanenobu and Sumi also give a table containing the number of distinct 2-bridge knots of n crossings for $n = 10$ to 22, both not counting and counting MIRROR IMAGES as distinct.

n	K_n	$K_n + K_n^*$
3	0	0
4	0	0
5		
6		
7		
8		
9		
10	45	85
11	91	182
12	176	341
13	352	704
14	693	1365
15	1387	2774
16	2752	5461
17	5504	11008
18	10965	21845
19	21931	43862
20	43776	87381
21	87552	175104
22	174933	349525

References

- Kanenobu, T. and Sumi, T. "Polynomial Invariants of 2-Bridge Links through 20 Crossings." *Adv. Studies Pure Math.* **20**, 125–145, 1992.
 Kanenobu, T. and Sumi, T. "Polynomial Invariants of 2-Bridge Knots through 22-Crossings." *Math. Comput.* **60**, 771–778 and S17-S28, 1993.
 Schubert, H. "Knotten mit zwei Brücken." *Math. Z.* **65**, 133–170, 1956.

Bridge Number

The least number of unknotted arcs lying above the plane in any projection. The knot 05–002 has bridge number 2. Such knots are called 2-BRIDGE KNOTS. There is a one-to-one correspondence between 2-BRIDGE KNOTS and rational knots. The knot 08–010 is a 3-bridge knot. A knot with bridge number b is an n -EMBEDDABLE KNOT where $n \leq b$.

See also BRIDGE INDEX

References

- Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. New York: W. H. Freeman, pp. 64–67, 1994.
 Rolfsen, D. *Knots and Links*. Wilmington, DE: Publish or Perish Press, p. 115, 1976.

Bridge of Königsberg

KÖNIGSBERG BRIDGE PROBLEM

Brightness

The area of the SHADOW of a body on a plane, also called the "outer quermass."

See also INNER QUERMAS, SHADOW

References

- Blaschke, W. *Kreis und Kugel*. New York: Chelsea, p. 140, 1949.
 Bonnesen, T. "Om Minkowski's ulighed for konvexer legemer." *Mat. Tidsskr.* **B**, 80, 1926.
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 Chakerian, G. D. "Is a Body Spherical If All Its Projections Have the Same I.Q.?" *Amer. Math. Monthly* **77**, 989–992, 1970.
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 Firey, W. J. "Blaschke Sum of Convex Bodies and Mixed Bodies." In *Proceedings of the Colloquium on Convexity* (Ed. W. Fenchel). Copenhagen, Denmark: Københavns Univ. Math. Inst., pp. 94–101, 1967.

Brill-Noether Theorem

If the total group of the canonical series is divided into two parts, the difference between the number of points in each part and the double of the dimension of the complete series to which it belongs is the same.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves*. New York: Dover, p. 263, 1959.

Bring Quintic Form

A TSCHIRNHAUSEN TRANSFORMATION can be used to take a general QUINTIC EQUATION to the form

$$x^5 - x - a = 0,$$

where a may be COMPLEX.

See also BRING-JERRARD QUINTIC FORM, QUINTIC EQUATION

References

Bring, E. S. *Quart. J. Math.* **6**, 1864.
 Grunert, J. A. "VIII. Miscellen von dem Herausgeber." *Archiv der Math. Phys.* **41**, 105–112, 1864.
 Harley, R. "A Contribution to the History of the Problem of the Reduction of the General Equation of the Fifth Degree to a Trinomial Form." *Quart. J. Math.* **6**, 38–47, 1864.
 Ruppert, W. M. "On the Bring Normal Form of a Quintic in Characteristic 5." *Arch. Math.* **58**, 44–46, 1992.
 Tortolini, B. "Rivista bibliografica sopra a trasformazione del Sig. Jerrard per l'equazioni di quinto grado." *Annali di Mat. pura appl.* **6**, 33–42, 1864.

Bring-Jerrard Quintic Form

A TSCHIRNHAUSEN TRANSFORMATION can be used to algebraically transform a general QUINTIC EQUATION to the form

$$z^5 + c_1 z + c_0 = 0. \quad (1)$$

In practice, the general quintic is first reduced to the PRINCIPAL QUINTIC FORM

$$y^5 + b_2 y^2 + b_1 y + b_0 = 0 \quad (2)$$

before the transformation is done. Then, we require that the sum of the third POWERS of the ROOTS vanishes, so $s_3(y_i) = 0$. We assume that the ROOTS z_i of the Bring-Jerrard quintic are related to the ROOTS y_i of the PRINCIPAL QUINTIC FORM by

$$z_i = \alpha y_i^4 + \beta y_i^3 + \gamma y_i^2 + \delta y_i + \epsilon. \quad (3)$$

In a similar manner to the PRINCIPAL QUINTIC FORM transformation, we can express the COEFFICIENTS c_j in terms of the b_j .

See also BRING QUINTIC FORM, PRINCIPAL QUINTIC FORM, QUINTIC EQUATION

References

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Brioschi Formula

For a curve with METRIC

$$ds^2 = E du^2 + F du dv + G dv^2, \quad (1)$$

where E , F , and G is the first FUNDAMENTAL FORM, the GAUSSIAN CURVATURE is

$$K = \frac{M_1 + M_2}{(EG - F^2)^2}, \quad (2)$$

where

$$M_1 \equiv \begin{vmatrix} -\frac{1}{2}E_{uv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} \quad (3)$$

$$M_2 \equiv \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}, \quad (4)$$

which can also be written

$$K = -\frac{1}{\sqrt{EG}} \left[\frac{\partial}{\partial u} \left(\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) \right] \quad (5)$$

$$= -\frac{1}{\sqrt{EG}} \left[\frac{\partial}{\partial u} \left(\frac{G_u}{\sqrt{EG}} \right) + \frac{\partial}{\partial v} \left(\frac{E_v}{\sqrt{EG}} \right) \right]. \quad (6)$$

See also FUNDAMENTAL FORMS, GAUSSIAN CURVATURE

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2nd ed. Boca Raton, FL: CRC Press, pp. 504–507, 1997.

Briot-Bouquet Equation

An ORDINARY DIFFERENTIAL EQUATION OF THE FORM

$$x^m y' = f(x, y),$$

where m is a POSITIVE INTEGER, f is ANALYTIC at $x = y = 0$, $f(0, 0) = 0$, and $f'_y(0, 0) \neq 0$.

Zwillinger (1997, p. 120), citing Ince (1956, p. 295), define the Briot-Bouquet equation as

$$xy' - \lambda y = a_{10}x + a_{20}x^2 + a_{11}yx + a_{02}y^2 + \dots$$

References

Briot and Bouquet. "Propriétés des fonctions définie par des équations différentielles." *J. l'Ecole Polytechnique*, Cah. 36.
 Hazewinkel, M. (Managing Ed.). *Encyclopaedia of Mathematics: An Updated and Annotated Translation of the Soviet "Mathematical Encyclopaedia"*. Dordrecht, Netherlands: Reidel, pp. 481–482, 1988.

Ince, E. L. *Ordinary Differential Equations*. New York: Dover, 1956.

Zwillinger, D. *Handbook of Differential Equations*, 3rd ed. Boston, MA: Academic Press, p. 120, 1997.

Brjuno Number

Let p_n/q_n be the sequence of CONVERGENTS of the CONTINUED FRACTION of a number α . Then a Brjuno number is an IRRATIONAL NUMBER such that

$$\sum_{n=0}^{\infty} \frac{\ln q_{n+1}}{q_n} < \infty$$

(Marmi *et al.* 1999). Brjuno numbers arise in the study of one-dimensional analytic small divisors problems, and Brjuno (1971, 1972) proved that all "germs" with linear part $\lambda = e^{2\pi i \alpha}$ are linearizable if α is a Brjuno number. Yoccoz (1995) proved that this condition is also NECESSARY.

References

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Brjuno, A. D. "Analytical Form of Differential Equations. II." *Trans. Moscow Math. Soc.* **26**, 199–239, 1972.

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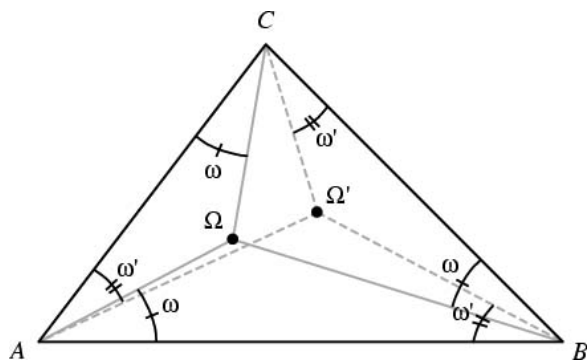
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Yoccoz, J.-C. "Théorème de Siegel, nombres de Bruno et polynômes quadratiques." *Astérisque* **231**, 3–88, 1995.

Broadcasting

GOSSIPING

Brocard Angle



Define the first BROCARD POINT as the interior point Ω of a TRIANGLE for which the ANGLES $\angle \Omega AB$, $\angle \Omega BC$, and $\angle \Omega CA$ are equal to an angle ω . Similarly, define the second BROCARD POINT as the interior point Ω' for which the ANGLES $\angle \Omega' AC$, $\angle \Omega' CB$, and $\angle \Omega' BA$ are

equal to an angle ω' . Then $\omega = \omega'$, and this angle is called the Brocard angle.

The Brocard angle ω of a TRIANGLE $\Delta A_1 A_2 A_3$ is given by the formulas

$$\cot \omega = \cot A_1 + \cot A_2 + \cot A_3 \quad (1)$$

$$= \left(\frac{a_1^2 + a_2^2 + a_3^2}{4\Delta} \right) \quad (2)$$

$$= \frac{1 + \cos \alpha_1 \cos \alpha_2 \cos \alpha_3}{\sin \alpha_1 \sin \alpha_2 \sin \alpha_3} \quad (3)$$

$$= \frac{\sin^2 \alpha_1 + \sin^2 \alpha_2 + \sin^2 \alpha_3}{2 \sin \alpha_1 \sin \alpha_2 \sin \alpha_3} \quad (4)$$

$$= \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2 + a_3 \sin \alpha_3}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2 + a_3 \cos \alpha_3} \quad (5)$$

$$\csc^2 \omega = \csc^2 \alpha_1 + \csc^2 \alpha_2 + \csc^2 \alpha_3 \quad (6)$$

$$\sin \omega = \frac{2\Delta}{\sqrt{a_1^2 a_2^2 + a_2^2 a_3^2 + a_3^2 a_1^2}} \quad (7)$$

where Δ is the TRIANGLE AREA, A , B , and C are ANGLES, and a , b , and c are side lengths (Johnson 1929), where (6) is due to Neuberg (Tucker 1883).

If an ANGLE α of a TRIANGLE is given, the maximum possible Brocard angle is given by

$$\cot \omega = \frac{3}{2} \tan(\frac{1}{2} \alpha) + \frac{1}{2} \cos(\frac{1}{2} \alpha) \quad (8)$$

(Johnson 1929, p. 289). If ω is specified, that the largest possible value $\alpha_{\max}$ and minimum possible value $\alpha_{\min}$ of any possible triangle having Brocard angle ω are given by

$$\cot(\frac{1}{2} \alpha_{\max}) = \cot \omega - \sqrt{\cot^2 \omega - 3} \quad (9)$$

$$\cot(\frac{1}{2} \alpha_{\min}) = \cot \omega + \sqrt{\cot^2 \omega - 3}, \quad (10)$$

where the square rooted quantity is the radius of the corresponding NEUBERG CIRCLE (Johnson 1929, p. 288). The maximum possible Brocard angle for any triangle is 30° (Honsberger 1995, pp. 102–103). Let a TRIANGLE have ANGLES A , B , and C . Then

$$\sin A \sin B \sin C \leq kABC, \quad (11)$$

where

$$k = \left(\frac{3\sqrt{3}}{2\pi} \right)^3 \quad (12)$$

(Le Lionnais 1983). This can be used to prove that

$$8\omega^3 < ABC \quad (13)$$

(Abi-Khuzam 1974).

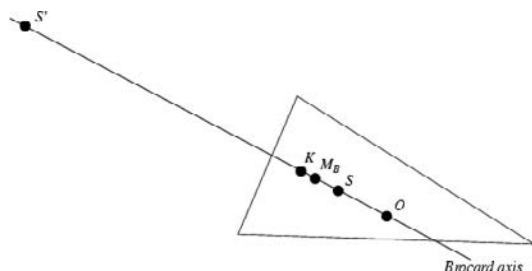
See also BROCARD CIRCLE, BROCARD LINE, EQUI-

BROCARD CENTER, FERMAT POINTS, NEUBERG CIRCLE

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Brocard Axis



The LINE KO passing through the SYMMEDIAN POINT K and CIRCUMCENTER O of a TRIANGLE. The distance $\overline{OK}$ is called the BROCARD DIAMETER. The Brocard axis is PERPENDICULAR to the LEMOINE AXIS and is the ISOGONAL CONJUGATE of KIEPERT'S HYPERBOLA. It has equations

$$\sin(B - C)\alpha + \sin(C - A)\beta + \sin(A - B)\gamma = 0$$

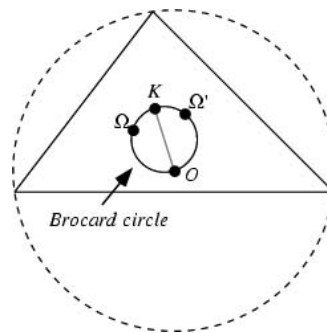
$$bc(b^2 - c^2)\alpha + ca(c^2 - a^2)\beta + ab(a^2 - b^2)\gamma = 0.$$

The SYMMEDIAN POINT K , CIRCUMCENTER O , ISODYNAMIC POINTS S and S' , and BROCARD MIDPOINT M_B all lie along the Brocard axis.

Note that the Brocard axis is *not* equivalent to the BROCARD LINE.

See also BROCARD CIRCLE, BROCARD DIAMETER, BROCARD LINE

Brocard Circle



The CIRCLE passing through the first and second BROCARD POINTS Ω and Ω' , the LEMOINE POINT K , and the CIRCUMCENTER O of a given TRIANGLE. The BROCARD POINTS Ω and Ω' are symmetrical about the LINE $\overline{KO}$, which is called the BROCARD LINE. The LINE SEGMENT $\overline{KO}$ is called the BROCARD DIAMETER, and it has length

$$\overline{OK} = \frac{\overline{O\Omega}}{\cos \omega} = \frac{R\sqrt{1 - 4 \sin^2 \omega}}{\cos \omega},$$

where R is the CIRCUMRADIUS and ω is the BROCARD ANGLE. The distance between either of the BROCARD POINTS and the SYMMEDIAN POINT is

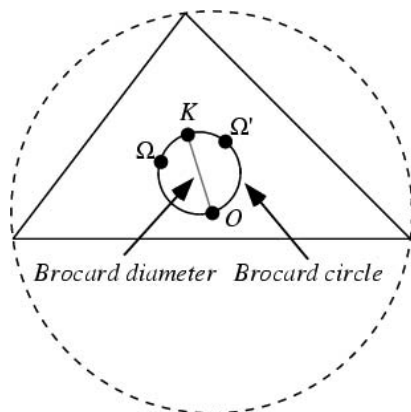
$$\overline{OK} = \overline{O'K} = \overline{O\Omega} \tan \omega.$$

The Brocard circle and LEMOINE CIRCLE are concentric.

See also BROCARD ANGLE, BROCARD DIAMETER, BROCARD POINTS

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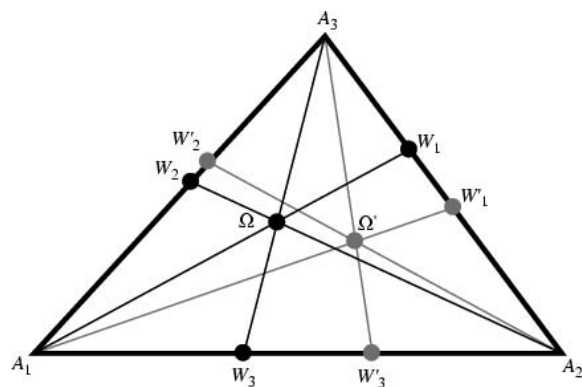
Brocard Diameter

The LINE SEGMENT $\overline{KO}$ joining the SYMMEDIAN POINT K and CIRCUMCENTER O of a given TRIANGLE. It is the DIAMETER of the TRIANGLE'S BROCARD CIRCLE, and lies along the BROCARD AXIS. The Brocard diameter has length

$$\overline{OK} = \frac{\overline{O\Omega}}{\cos \omega} = \frac{R\sqrt{1 - 4 \sin^2 \omega}}{\cos \omega},$$

where Ω is the first BROCARD POINT, R is the CIRCUMRADIUS, and ω is the BROCARD ANGLE.

See also BROCARD AXIS, BROCARD CIRCLE, BROCARD LINE, BROCARD POINTS

Brocard Line

A LINE from any of the VERTICES A_i of a TRIANGLE to the first Ω or second Ω' BROCARD POINT. Let the ANGLE at a VERTEX A_i also be denoted A_i , and denote the intersections of $A_1\Omega$ and $A_1\Omega'$ with A_2A_3 as W_1 and W_2 . Then the ANGLES involving these points are

$$\angle A_1\Omega W_3 = A_1 \quad (1)$$

$$\angle W_3\Omega A_2 = A_3 \quad (2)$$

$$\angle A_2\Omega W_1 = A_2 \quad (3)$$

Distances involving the points W_i and W'_i are given by

$$\overline{A_2\Omega} = \frac{a_3}{\sin A_2} \sin \omega \quad (4)$$

$$\frac{\overline{A_2\Omega}}{\overline{A_3\Omega}} = \frac{a_3^2}{a_1a_2} = \frac{\sin(A_3 - \omega)}{\sin \omega} \quad (5)$$

$$\frac{\overline{W_3A_1}}{\overline{W_3A_2}} = \frac{a_2 \sin \omega}{a_1 \sin(A_3 - \omega)} = \left(\frac{a_2}{a_3}\right)^2, \quad (6)$$

where ω is the BROCARD ANGLE (Johnson 1929, pp. 267–268).

The Brocard line, MEDIAN M , and SYMMEDIAN POINT K are concurrent, with $A_1\Omega_1$, A_2K , and A_3M meeting at a point P . Similarly, $A_1\Omega'$, A_2M , and A_3K meet at a point which is the ISOGONAL CONJUGATE point of P (Johnson 1929, pp. 268–269).

See also BROCARD AXIS, BROCARD DIAMETER, BROCARD POINTS, ISOGONAL CONJUGATE, SYMMEDIAN POINT, MEDIAN (TRIANGLE)

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Brocard Midpoint

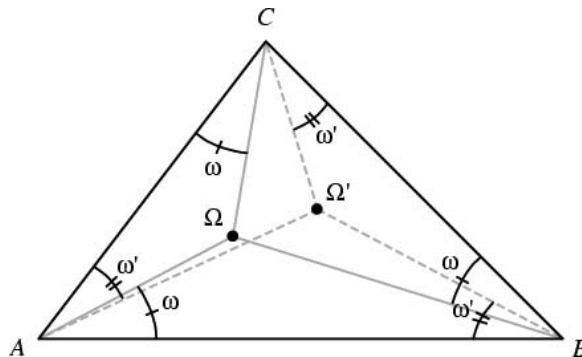
The MIDPOINT of the BROCARD POINTS. It has TRIANGLE CENTER FUNCTION

$$\alpha = a(b^2 + c^2) = \sin(A + \omega),$$

where ω is the BROCARD ANGLE. It lies on the BROCARD AXIS.

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Brocard Points

The first Brocard point is the interior point Ω (or τ_1 or

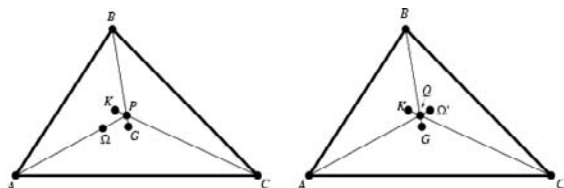
Z_1) of a TRIANGLE for which the ANGLES $\angle \Omega AB$, $\angle \Omega BC$, and $\angle \Omega CA$ are equal to an angle ω . The second Brocard point is the interior point Ω' (or τ_2 or Z_2) for which the ANGLES $\angle \Omega' AC$, $\angle \Omega' CB$, and $\angle \Omega' BA$ are equal to an angle ω' . The two angles $\omega = \omega'$ are equal, and this angle is called the BROCARD ANGLE,

$$\omega = \angle \Omega AB = \angle \Omega BC = \angle \Omega CA$$

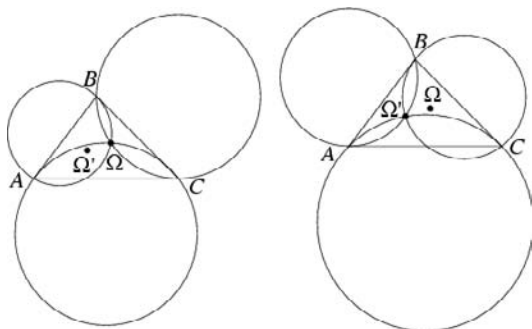
$$= \angle \Omega' AC = \angle \Omega' CB = \angle \Omega' BA.$$

The first two Brocard points are ISOGONAL CONJUGATES (Johnson 1929, p. 266). They were described by French army officer Henri Brocard in 1875, although they had previously been investigated by Jacobi and, in 1816, Crelle (Wells 1991; Honsberger 1995, p. 98). They satisfy $\Omega O = \Omega' O$ and $\angle \Omega O \Omega' = 2\omega$, where O is the CIRCUMCENTER and ω is the BROCARD ANGLE (Honsberger 1995, p. 106).

If three dogs start at the vertices of a triangle and chase either their left or right neighbor at a constant speed, that the three will meet at either Ω or Ω' (Wells 1991).

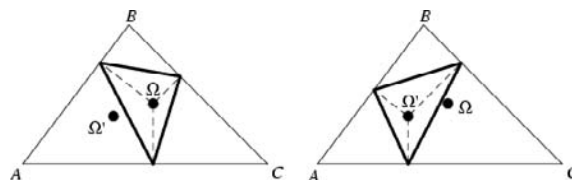


One BROCARD LINE, MEDIAN, and SYMMEDIAN (out of the three of each) are CONCURRENT, with $A\Omega$, CK , and BG meeting at a point, where G is the CENTROID and K is the SYMMEDIAN POINT. Similarly, $A\Omega'$, BG , and CK meet at a point which is the ISOGONAL CONJUGATE of the first (Johnson 1929, pp. 268–269; Honsberger 1995, pp. 121–124).



Let C_{BC} be the CIRCLE which passes through the vertices B and C and is TANGENT to the line AC at C , and similarly for C_{AB} and C_{AC} . Then the CIRCLES C_{AB} , C_{BC} , and C_{AC} intersect in the first Brocard point Ω . Similarly, let C'_{BC} be the CIRCLE which passes through the vertices B and C and is TANGENT to the line AB at B , and similarly for C'_{AB} and C'_{AC} . Then the CIRCLES C'_{AB} , C'_{BC} , and C'_{AC} intersect in the second Brocard points Ω' (Johnson 1929, pp. 264–265; Honsberger

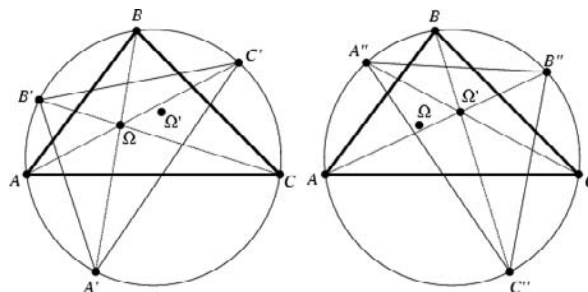
1995, pp. 99–100).



The PEDAL TRIANGLES of Ω and Ω' are congruent, and SIMILAR to the TRIANGLE ΔABC (Johnson 1929, p. 269). Lengths involving the Brocard points include

$$\overline{O\Omega} = \overline{O\Omega'} = R\sqrt{1 - 4\sin^2 \omega} \quad (1)$$

$$\overline{\Omega\Omega'} = 2R \sin \omega \sqrt{1 - 4\sin^2 \omega}. \quad (2)$$



Extend the segments $A\Omega$, $B\Omega$, and $C\Omega$ to the CIRCUMCIRCLE of ΔABC to form $\Delta C'A'B'$, and the segments $A\Omega'$, $B\Omega'$, and $C\Omega'$ to form $\Delta B''C''A''$. Then $\Delta A'B'C'$ and $\Delta A''B''C''$ are congruent to ΔABC (Honsberger 1995, pp. 104–106).

Brocard's third point is related to a given TRIANGLE by the TRIANGLE CENTER FUNCTION

$$\alpha = a^{-3} \quad (3)$$

(Casey 1893, Kimberling 1994). The third Brocard point Ω'' (or τ_3 or Z_3) is COLLINEAR with the SPIEKER CENTER and the ISOTOMIC CONJUGATE POINT of its TRIANGLE'S INCENTER.

See also BROCARD ANGLE, BROCARD MIDPOINT, EQUI-BROCARD CENTER, YFF POINTS

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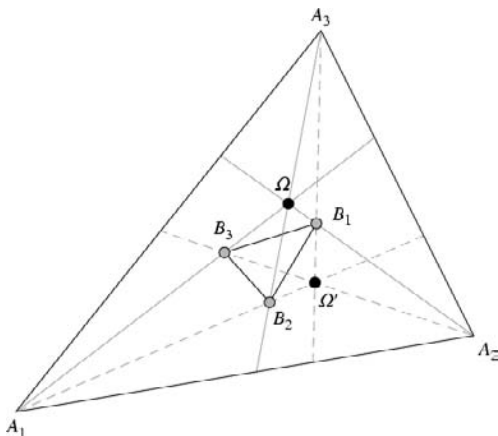
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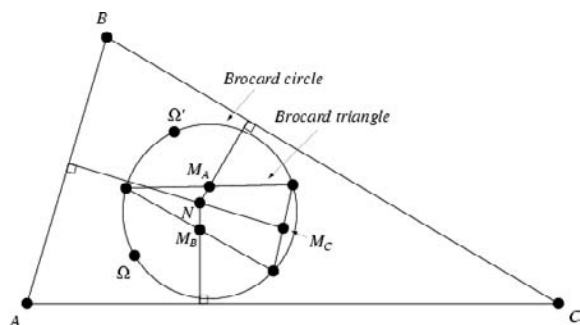
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Brocard Triangles



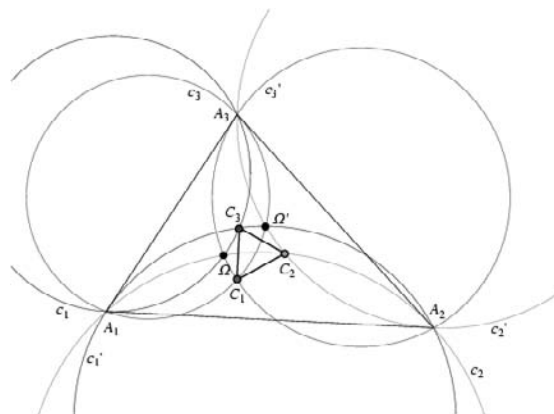
Given TRIANGLE $\Delta A_1A_2A_3$, let the point of intersection of $A_2\Omega$ and $A_3\Omega'$ be B_1 , where Ω and Ω' are the BROCARD POINTS, and similarly define B_2 and B_3 . Then $B_1B_2B_3$ is called the first Brocard triangle, and is INVERSELY SIMILAR to $A_1A_2A_3$ (Honsberger 1995, p. 112). It is inscribed in the BROCARD CIRCLE drawn with OK as the DIAMETER.

The triangles $B_1A_2A_3$, $B_2A_3A_1$, and $B_3A_1A_2$ are ISOSCELES TRIANGLES with base angles ω , where ω is the BROCARD ANGLE. The sum of the areas of the ISOSCELES TRIANGLES is Δ , the AREA of TRIANGLE $A_1A_2A_3$. The first Brocard triangle is in perspective with the given TRIANGLE, with A_1B_1 , A_2B_2 , and A_3B_3 CONCURRENT. The CENTROID of the first brocard triangle is the CENTROID G of the original triangle (Honsberger 1995, pp. 112–116).

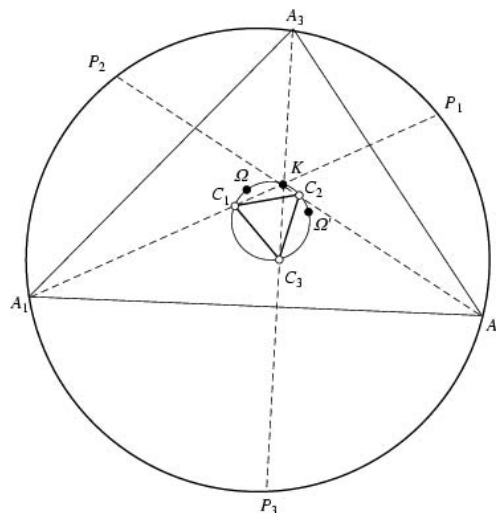


Let perpendiculars be drawn from the midpoints M_A , M_B , and M_C of each side of the first Brocard triangle

to the opposite sides of the triangle ΔABC . Then the extensions of these lines CONCUR in the NINE-POINT CENTER (Honsberger 1995, pp. 116–118).



Let c_1 , c_2 , and c_3 be the CIRCLES through the vertices A_2 and A_3 , A_1 and A_3 , and A_1 and A_2 , respectively, which intersect in the first BROCARD POINT Ω . Similarly, define c'_1 , c'_2 , and c'_3 with respect to the second BROCARD POINT Ω' . Let the two circles c_1 and c'_1 tangent at A_1 to A_1A_2 and A_1A_3 , and passing respectively through A_3 and A_2 , meet again at C_1 , and similarly for C_2 and C_3 . Then the triangle $\Delta C_1C_2C_3$ is called the second Brocard triangle.



The second Brocard triangle is also the triangle obtained as the intersections of the lines A_1K , A_2K , and A_3K with the BROCARD CIRCLE, where K is the SYMMEDIAN POINT. Let P_1 , P_2 , and P_3 be the intersections of the lines A_1K , A_2K , and A_3K with the CIRCUMCIRCLE of $\Delta A_1A_2A_3$. Then C_1 , C_2 , and C_3 are the midpoints of A_1P_1 , A_2P_2 , and A_3P_3 , respectively (Lachlan 1893).

The two Brocard triangles are in PERSPECTIVE at M .

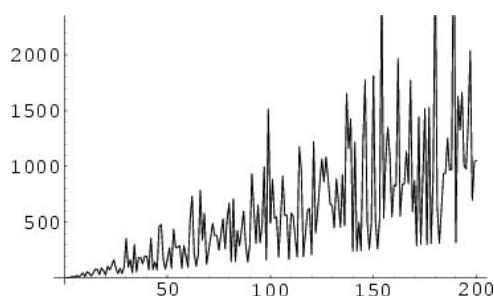
See also BROCARD CIRCLE, CIRCLE-CIRCLE INTERSEC-

TION, McCAY CIRCLE, NINE-POINT CENTER, STEINER POINTS, TARRY POINT

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Brocard's Conjecture



$$\pi(p_{n+1}^2) - \pi(p_n^2) \geq 4$$

for $n \geq 2$ where $\pi(n)$ is the PRIME COUNTING FUNCTION and p_n is the n th PRIME. For $n = 1, 2, \dots$, the first few values are 2, 5, 6, 15, 9, 22, 11, 27, 47, 16, ... (Sloane's A050216).

See also ANDRICA'S CONJECTURE

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Brocard's Problem

Find the values of n for which $n! + 1$ is a SQUARE NUMBER m^2 , where $n!$ is the FACTORIAL (Brocard 1876, 1885). Pairs of numbers (m, n) are called BROWN NUMBERS. The only known solutions are $n = 4, 5$, and 7, and there are no other solutions with $n \leq 10^7$ (Wells 1986, p. 70; D. Wilson). It is virtually certain that there are no more solutions (Guy 1994). In fact, Dabrowski (1996) has shown that $n! + A = k^2$ has only finitely many solutions for general A , although this result requires assumption of a weak form of the ABC CONJECTURE if A is SQUARE).

Wilson has also computed the least k such that $n! + k^2$ is square starting at $n = 4$, giving 1, 1, 3, 1, 9, 27, 15, 18, 288, 288, 420, 464, 1856, ... (Sloane's A038202).

See also BROWN NUMBERS, FACTORIAL, SQUARE NUMBER

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Bromwich Integral

The inverse of the LAPLACE TRANSFORM, given by

$$F(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\pi i f(s)} ds,$$

where γ is a vertical CONTOUR in the COMPLEX PLANE chosen so that all singularities of $f(s)$ are to the left of it.

See also LAPLACE TRANSFORM

References

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Brooks' Theorem

The CHROMATIC NUMBER of a graph is at most the maximum VERTEX DEGREE Δ , unless the graph is COMPLETE or an odd cycle.

See also CHROMATIC NUMBER

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Brothers

A PAIR of consecutive numbers.

See also PAIR, SMITH BROTHERS, TWINS

Brouwer Fixed Point Theorem

Any continuous FUNCTION $G: \mathbb{B} \rightarrow \mathbb{B}^n$ has a FIXED POINT, where

$$\mathbb{B}^n = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$$

is the unit n -BALL.

See also BALL, FIXED POINT THEOREM

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Browkin's Theorem

For every POSITIVE INTEGER n , there exists a SQUARE in the plane with exactly n LATTICE POINTS in its interior. This was extended by Schinzel and Kuliowski to *all* plane figures of a given shape. The generalization of the SQUARE in 2-D to the CUBE in 3-D was also proved by Browkin.

See also CUBE, SCHINZEL'S THEOREM, SQUARE

References

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Brown Function

For a FRACTAL PROCESS with values $y(t - \Delta t)$ and $y(t + \Delta t)$, the correlation between these two values is given by the Brown function

$$r = 2^{2H-1} - 1,$$

also known as the BACHELIER FUNCTION, LÉVY FUNCTION, or WIENER FUNCTION.

Brown Numbers

Brown numbers are PAIRS (m, n) of INTEGERS satisfying the condition of BROCARD'S PROBLEM, i.e., such that

$$n! + 1 = m^2$$

where $n!$ is the FACTORIAL and m^2 is a SQUARE NUMBER. Only three such PAIRS of numbers are known: (5, 4), (11, 5), (71, 7), and Erdos conjectured that these are the only three such PAIRS.

See also BROCARD'S PROBLEM, FACTORIAL, SQUARE NUMBER, WILSON PRIME

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Brown's Criterion

A SEQUENCE $\{v_i\}$ of nondecreasing POSITIVE INTEGERS is COMPLETE IFF

1. $v_1 = 1$.
2. For all $k = 2, 3, \dots$,

$$s_{k-1} = v_1 + v_2 + \cdots + v_{k-1} \geq v_k - 1.$$

A corollary states that a SEQUENCE for which $v_1 = 1$ and $v_{k+1} \leq 2v_k$ is COMPLETE (Honsberger 1985).

See also COMPLETE SEQUENCE

References

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Broyden's Method

An extension of the SECANT METHOD of root finding to higher dimensions.

See also SECANT METHOD

References

- Broyden, C. G. "A Class of Methods for Solving Nonlinear Simultaneous Equations." *Math. Comput.* **19**, 577–593, 1965.
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Bruck-Ryser Theorem

BRUCK-RYSER-CHOWLA THEOREM

Bruck-Ryser-Chowla Theorem

If $n \equiv 1, 2 \pmod{4}$, and the SQUAREFREE part of n is divisible by a PRIME $p \equiv 3 \pmod{4}$, then no DIFFERENCE SET of ORDER n exists. Equivalently, if a PROJECTIVE PLANE of order n exists, and $n = 1$ or $2 \pmod{4}$, then n is the sum of two SQUARES.

Dinitz and Stinson (1992) give the theorem in the following form. If a symmetric (v, k, λ) -BLOCK DESIGN exists, then

1. If v is EVEN, then $k - \lambda$ is a SQUARE NUMBER,
2. If v is ODD, then the DIOPHANTINE EQUATION

$$x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2$$

has a solution in integers, not all of which are 0.

See also BLOCK DESIGN, DIFFERENCE SET, FISHER'S BLOCK DESIGN INEQUALITY

References

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Bruhat Order

References

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Brun's Constant

The number obtained by adding the reciprocals of the odd TWIN PRIMES,

$$B \equiv \left(\frac{1}{3} + \frac{1}{5}\right) + \left(\frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{11} + \frac{1}{13}\right) + \left(\frac{1}{17} + \frac{1}{19}\right) + \cdots, \quad (1)$$

By BRUN'S THEOREM, the constant converges to a definite number as $p \rightarrow \infty$. Any finite sum underestimates B . Shanks and Wrench (1974) used all the TWIN PRIMES among the first 2 million numbers. Brent (1976) calculated all TWIN PRIMES up to 100 billion and obtained (Ribenoim 1989, p. 146)

$$B \approx 1.90216054, \quad (2)$$

assuming the truth of the first HARDY-LITTLEWOOD CONJECTURE. Using TWIN PRIMES up to 10^{14} , Nicely (1996) obtained

$$B \approx 1.9021605778 \pm 2.1 \times 10^{-9} \quad (3)$$

(Cipra 1995, 1996), in the process discovering a bug in Intel's® Pentium™ microprocessor. Using TWIN PRIMES up to 2.55^{15} , Nicely subsequently obtained the result

$$B \approx 1.9021605820 \pm 2.4 \times 10^{-9}. \quad (4)$$

(Note that the value given by Le Lionnais 1983 is incorrect)

Segal (1930) proved that Brun-type sums B_d of $1/p$ over consecutive primes separated by d are finite (Halberstam and Richert 1983, p. 92). Wolf suggests that B_d is roughly equal to $4/d$ which, in the $d = 2$ case of twin primes, gives $B_2 \approx 2$ instead of 1.902.... Wolf also considers the "COUSIN PRIMES" Brun's constant B_4 .

See also COUSIN PRIMES, TWIN PRIMES, TWIN PRIME

CONJECTURE, TWIN PRIMES CONSTANT

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Brun's Sieve

See also SIEVE

References

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Brun's Sum

BRUN'S CONSTANT

Brun's Theorem

The series producing BRUN'S CONSTANT CONVERGES even if there are an infinite number of TWIN PRIMES. Proved in 1919 by V. Brun.

Brunnian Link

A Brunnian link is a set of n linked loops such that each proper sublink is trivial, so that the removal of any component leaves a set of trivial unlinked UNKNOTS. The BORROMEAN RINGS are the simplest example and have $n = 3$.

See also BORROMEAN RINGS

References

Rolfsen, D. *Knots and Links*. Wilmington, DE: Publish or Perish Press, 1976.

Brunn-Minkowski Inequality

The n th root of the CONTENT of the set sum of two sets in Euclidean n -space is greater than or equal to the sum of the n th roots of the CONTENTS of the individual sets.

See also TOMOGRAPHY

References

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Brusselator Equations

The system of ordinary differential equations

$$u' = A + u^2v - (B + 1)u \quad (1)$$

$$v' = Bu - u^2v \quad (2)$$

(Hairer *et al.* 1987, p. 112; Zwillinger 1997, p. 136). The so-called full Brusselator equations are given by

$$u' = 1 + u^2v - (w + 1)u \quad (3)$$

$$v' = uw - u^2v \quad (4)$$

$$w' = -uw + \alpha \quad (5)$$

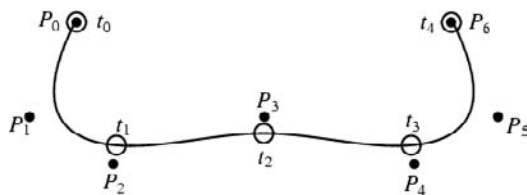
(Hairer *et al.* 1987, p. 114; Zwillinger 1997, p. 136).

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Zwillinger, D. *Handbook of Differential Equations*, 3rd ed. Boston, MA: Academic Press, p. 136, 1997.

Brute Force Factorization

DIRECT SEARCH FACTORIZATION

B-Spline

A generalization of the BÉZIER CURVE. Let a vector known as the KNOT VECTOR be defined

$$\mathbf{T} = \{t_0, t_1, \dots, t_m\}, \quad (1)$$

where $\mathbf{T}$ is a nondecreasing SEQUENCE with $t_i \in [0, 1]$,

and define control points $\mathbf{P}_0, \dots, \mathbf{P}_n$. Define the degree as

$$p \equiv m - n - 1. \quad (2)$$

The "knots" $t_{p+1}, \dots, t_{m-p-1}$ are called INTERNAL KNOTS.

Define the basis functions as

$$N_{i,0}(t) = \begin{cases} 1 & \text{if } t_i \leq t < t_{i+1} \text{ and } t_i < t_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

$$N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t). \quad (4)$$

Then the curve defined by

$$\mathbf{C}(t) = \sum_{i=0}^n \mathbf{P}_i N_{i,p}(t) \quad (5)$$

is a B-spline. Specific types include the nonperiodic B-spline (first $p + 1$ knots equal 0 and last $p + 1$ equal to 1) and uniform B-spline (INTERNAL KNOTS are equally spaced). A B-spline with no INTERNAL KNOTS is a BÉZIER CURVE.

A curve is $p - k$ times differentiable at a point where k duplicate knot values occur. The knot values determine the extent of the control of the control points.

See also BÉZIER CURVE, NURBS CURVE

B-Tree

B-trees were introduced by Bayer (1972) and McCreight. They are a special m -ary balanced tree used in databases because their structure allows records to be inserted, deleted, and retrieved with guaranteed worst-case performance. An n -node *B*-tree has height $\mathcal{O}(\lg 2)$, where LG is the LOGARITHM to base 2. The Apple[®] Macintosh[®] (Apple Computer, Cupertino, CA) HFS filing system uses *B*-trees to store disk directories (Benedict 1995). A *B*-tree satisfies the following properties:

1. The ROOT is either a LEAF (TREE) or has at least two CHILDREN.
2. Each node (except the ROOT and LEAVES) has between $\lceil m/2 \rceil$ and m CHILDREN, where $\lceil x \rceil$ is the CEILING FUNCTION.
3. Each path from the ROOT to a LEAF (TREE) has the same length.

Every 2- TREE is a *B*-tree of order 3. The number of *B*-trees of order-3 with $n = 1, 2, \dots$ leaves are 1, 1, 1, 1, 2, 2, 3, 4, 5, 8, 14, 23, 32, 43, 63, ... (Ruskey, Sloane's A014535). The number of order-4 *B*-trees with $n = 1$,

2, ... leaves are 1, 1, 1, 2, 2, 4, 5, 9, 15, 28, 45, ... (Sloane's A037026).

See also RED-BLACK TREE, TREE

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Bubble

A bubble is a minimal-energy surface of the type that is formed by soap film. The simplest bubble is a single SPHERE, illustrated above (courtesy of J. M. Sullivan). More complicated forms occur when multiple bubbles are joined together. The simplest example is the DOUBLE BUBBLE, and beautiful configurations can form when three or more bubbles are conjoined (Sullivan).

An outstanding problem involving bubbles is the determination of the arrangements of bubbles with the smallest SURFACE AREA which enclose and separate n given volumes in space.

See also DOUBLE BUBBLE, PLATEAU'S LAWS, PLATEAU'S PROBLEM, SPHERE

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Buchberger's Algorithm

The algorithm for the construction of a GRÖBNER BASIS from an arbitrary ideal basis.

See also GRÖBNER BASIS

References

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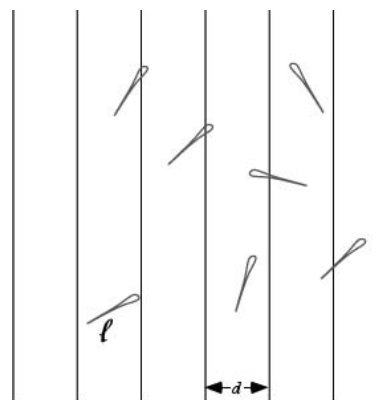
Buchowski Paradox

A paradox arising in the use of comparative adjectives. Suppose you have exactly two brothers, both of whom are older than you are. Then the following apparently false statement is actually true: "My younger brother is older than I am."

Buckminster Fuller Dome

GEODESIC DOME

Buffon's Needle Problem



Find the probability $P(\ell, d)$ that a needle of length ℓ will land on a line, given a floor with equally spaced PARALLEL LINES a distance d apart. The problem was first posed by the French naturalist Buffon in 1733, and reproduced with the solution by Buffon in 1777. For $\ell \leq d$,

$$P(\ell, d) = \int_0^{2\pi} \frac{\ell |\cos \theta|}{d} \frac{d\theta}{2\pi} = \frac{\ell}{2\pi d} 4 \int_0^{\pi/2} \cos \theta d\theta$$

$$= \frac{2\ell}{\pi d} [\sin \theta]_0^{\pi/2} = \frac{2\ell}{\pi d}. \quad (1)$$

For $\ell \geq d$, the solution is slightly more complicated,

$$P(\ell, d) = \frac{1}{\pi d} \left\{ d \left[\pi - 2 \sin^{-1} \left(\frac{d}{\ell} \right) \right] + 2\ell \left(1 - \sqrt{1 - \frac{d^2}{\ell^2}} \right) \right\} \quad (2)$$

(Uspensky 1937, p. 252; Kunkel).

Several attempts have been made to experimentally determine π by needle-tossing. For a discussion of the relevant statistics and a critical analysis of one of the more accurate (and least believable) needle-tossings, see Badger (1994). Uspensky (1937, pp. 112–113) discusses experiments conducted with 2520, 3204, and 5000 trials. An asymptotically unbiased estimator for π from the needle-tossing experiment is

$$\hat{\pi} = \frac{2rn}{N}, \quad (3)$$

where $r = \ell/d$, n is the number of throws, and N is the number of line crossings, which has asymptotic variance

$$\text{var}(\hat{\pi}) = \frac{\pi^2}{n} \left(\frac{1}{2}\pi - 1 \right) \approx \frac{5.63}{n} \quad (4)$$

(Mantel 1953; Solomon 1978, p. 7).

If the needle is longer than the distance between two lines, then the probability that it intersects at least one line is

$$P(\ell) = \frac{2\ell}{\pi d} (1 - \sin \phi_0) + \frac{2\phi_0}{\pi}, \quad (5)$$

where $\cos \phi_0 = d/\ell$ (Uspensky 1937, p. 258).

The problem can be extended to a "needle" in the shape of a CONVEX POLYGON with GENERALIZED DIAMETER less than d . The probability that the boundary of the polygon will intersect one of the lines is given by

$$P = \frac{p}{\pi d}, \quad (6)$$

where p is the PERIMETER of the polygon (Uspensky 1937, p. 253; Solomon 1978, p. 18). A further generalization obtained by throwing a needle on a board ruled with two sets of perpendicular lines is called the BUFFON-LAPLACE NEEDLE PROBLEM.

See also BUFFON-LAPLACE NEEDLE PROBLEM

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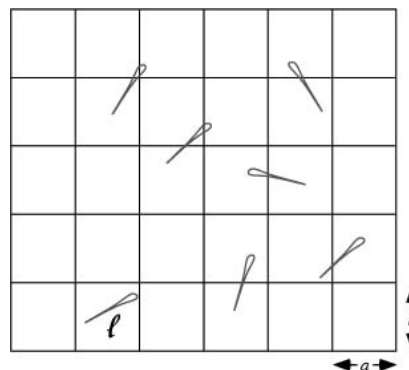
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Buffon-Laplace Needle Problem



Find the probability $P(\ell, a, b)$ that a needle of length ℓ will land on a line, given a floor with a grid of equally spaced PARALLEL LINES distances a and b apart, with $\ell < a, b$. The position of the needle can be specified with points (x, y) and its orientation with coordinate ϕ . By symmetry, we can consider a single rectangle of the grid, so $0 < x < a$ and $0 < y < b$. In

addition, since opposite orientations are equivalent, we can take $-\pi/2 < \phi < \pi/2$. The probability is given by

$$P(\ell; a, b) = 1 - \frac{\int_{-\pi/2}^{\pi/2} F(\phi) d\phi}{\pi ab}, \quad (1)$$

where

$$F(\phi) = ab - b\ell \cos \phi - \ell a |\sin \phi| + \frac{1}{2}\ell^2 |\sin(2\phi)| \quad (2)$$

(Uspensky 1937, p. 256; Solomon 1978, p. 4), giving

$$P(\ell; a, b) = \frac{2\ell(a + b) - \ell^2}{\pi ab}. \quad (3)$$

If the plane is instead tiled with congruent triangles with sides a , b , c , and a needle with length ℓ less than the shortest altitude is thrown, the probability that the needle is contained entirely within one of the triangles is given by

$$P = 1 + \frac{(Aa^2 + Bb^2 + Cc^2)\ell^2}{2\pi K^2} - \frac{(4a + 4b + 4c - 3\ell)\ell}{2\pi K}, \quad (4)$$

where A , B , and C are the angles opposite a , b , and c , respectively, and K is the AREA of the triangle. For equilateral triangles, this simplifies to

$$P = 1 + \frac{2}{3} \left(\frac{\ell}{a} \right)^2 - \frac{\ell\sqrt{3}}{\pi a} \left(4 - \frac{\ell}{a} \right) \quad (5)$$

(Uspensky 1937, p. 258).

See also BUFFON'S NEEDLE PROBLEM

References

- Schuster, E. F. "Buffon's Needle Experiment." *Amer. Math. Monthly* **81**, 26–29, 1974.
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Bug Problem

MICE PROBLEM

Building

A highly structured geometric object used to study GROUPS which act upon them.

See also COXETER GROUP, GROUP

References

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Bulirsch-Stoer Algorithm

An algorithm which finds RATIONAL FUNCTION extrapolations OF THE FORM

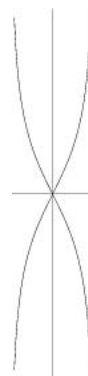
$$R_{i(i+1)\dots(i+m)} = \frac{P_\mu(x)}{P_\nu(x)} = \frac{p_0 + p_1x + \dots + p_\mu x^\mu}{q_0 + q_1x + \dots + q_\nu x^\nu}$$

and can be used in the solution of ORDINARY DIFFERENTIAL EQUATIONS.

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Bullet Nose



A plane curve with implicit equation

$$\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1. \quad (1)$$

In parametric form,

$$x = a \cos t \quad (2)$$

$$y = b \cot t. \quad (3)$$

The CURVATURE is

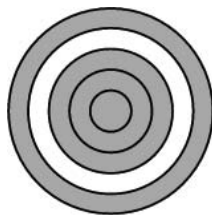
$$\kappa = \frac{3ab \cot t \csc t}{(b^2 \csc^4 t + a^2 \sin^2 t)^{3/2}} \quad (4)$$

and the TANGENTIAL ANGLE is

$$\phi = \tan^{-1} \left(\frac{b \csc^3 t}{a} \right). \quad (5)$$

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Bullseye Illusion

Although the inner shaded region has the same area as the outer shaded ANNULUS, it appears to be larger. Since the rings are equally spaced,

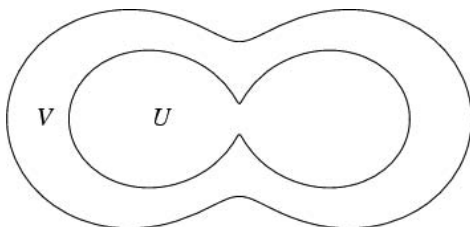
$$A_{\text{inner}} = \pi \cdot 3^2 = 9\pi$$

$$A_{\text{outer}} = \pi \cdot 5^2 - \pi \cdot 4^2 = 9\pi.$$

See also ILLUSION

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Bump Function

Given any OPEN SET U in $\mathbb{R}^n$ with COMPACT CLOSURE $K = \bar{U}$, there exists SMOOTH FUNCTIONS which are identically one on U and vanish arbitrarily close to U . One way to express this more precisely is that for any OPEN SET V containing K , there is a SMOOTH FUNCTION f such that

1. $f(x) = 1$ for all $x \in U$ and
2. $f(x) = 0$ for all $x \notin V$.



A function f that satisfies (1) and (2) is called a bump function. If $\int f = 1$ then by rescaling f , namely $f_k(x) = k^n f(kx)$, one gets a sequence of smooth functions which converges to the DELTA FUNCTION.

See also COMPACT SUPPORT, CONVOLUTION, DIRAC DISTRIBUTION, SMOOTH FUNCTION

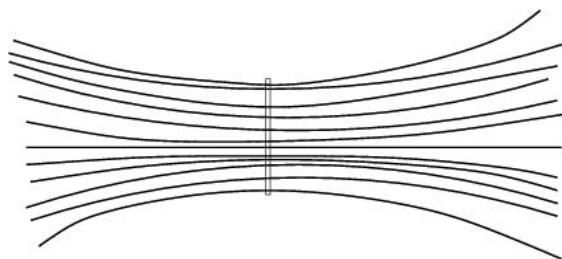
Bumping Algorithm

Given a PERMUTATION $\{p_1, p_2, \dots, p_n\}$ of $\{1, \dots, n\}$, the bumping algorithm constructs a standard YOUNG TABLEAU by inserting the p_i one by one into an already constructed YOUNG TABLEAU. To apply the bumping algorithm, start with $\{\{p_1\}\}$, which is a YOUNG TABLEAU. If p_1 through p_k have already been inserted, then in order to insert p_{k+1} , start with the first line of the already constructed YOUNG TABLEAU and search for the first element of this line which is greater than p_{k+1} . If there is no such element, append p_{k+1} to the first line and stop. If there is such an element (say, p_p), exchange p_p for p_{k+1} , search the second line using p_p , and so on.

See also TABLEAU CLASS, YOUNG TABLEAU

References

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Bundle

The term "bundle" is an abbreviated form of the full term FIBER BUNDLE. Depending on context, it may mean one of the special cases of FIBER BUNDLES, such as a VECTOR BUNDLE or a PRINCIPAL BUNDLE. Bundles are so named because they contain a collection of objects which, like a bundle of hay, are held together in a special way. All of the fibers line up—or at least they line up to nearby fibers.

LOCALLY, a bundle looks like a PRODUCT MANIFOLD in a TRIVIALIZATION. The graph of a function f sits inside the product as $(x, f(x))$. The SECTIONS of a bundle generalize functions in this way. It is necessary to use bundles when the range of a function only makes sense locally, as in the case of a VECTOR FIELD on the SPHERE.

Bundles are a special kind of SHEAF.

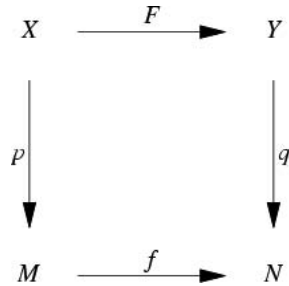
See also FIBER BUNDLE, JET BUNDLE, LINE BUNDLE, PRINCIPAL BUNDLE, SHEAF, TANGENT BUNDLE, VECTOR BUNDLE

Bundle Map

A bundle map is a map between bundles along with a compatible map between the BASE MANIFOLDS. Suppose $p : X \rightarrow M$ and $q : Y \rightarrow N$ are two BUNDLES, then

$$F : X \rightarrow Y$$

is a bundle map if there is a map $f : M \rightarrow N$ such that $q(F(x)) = f(p(x))$ for all $x \in X$. In particular, the FIBER of X over a point $m \in M$, gets mapped to the fiber of Y over $f(m) \in N$.



In the language of CATEGORY THEORY, the above diagram COMMUTES. To be more precise, the induced map between fibers has to be a map in the category of the fiber. For instance, in a bundle map between VECTOR BUNDLES the fiber over $m \in M$ is mapped to the fiber over $f(m) \in M$ by a LINEAR TRANSFORMATION.

For example, when $f : M \rightarrow N$ is a SMOOTH MAP between SMOOTH MANIFOLDS then $df : TM \rightarrow TN$ is the differential, which is a bundle map between the tangent bundles. Over any point in $m \in M$, the tangent vectors at m get mapped to tangent vectors at $f(m) \in N$ by the JACOBIAN.

See also BUNDLE, COMMUTATIVE DIAGRAM, FIBER (BUNDLE), JACOBIAN, PRINCIPAL BUNDLE, VECTOR BUNDLE

Buniakowsky Inequality

SCHWARZ'S INEQUALITY

Burali-Forti Paradox

In the theory of transfinite ORDINAL NUMBERS,

1. Every WELL ORDERED SET has a unique ORDINAL NUMBER,
2. Every segment of ordinals (i.e., any set of ordinals arranged in natural order which contains all the predecessors of each of its elements) has an ORDINAL NUMBER which is greater than any ordinal in the segment, and
3. The set B of all ordinals in natural order is well ordered.

Then by statements (3) and (1), B has an ordinal β . Since β is in B , it follows that $\beta < \beta$ by (2), which is a contradiction.

See also ORDINAL NUMBER

References

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Curry, H. B. *Foundations of Mathematical Logic*. New York: Dover, p. 5, 1977.

Erickson, G. W. and Fossa, J. A. *Dictionary of Paradox*. Lanham, MD: University Press of America, pp. 29–30, 1998.

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Burau Representation

Gives a MATRIX representation b_i of a BRAID GROUP in terms of $(n-1) \times (n-1)$ MATRICES. A $-t$ always appears in the (i, i) position.

$$b_1 = \begin{bmatrix} -t & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \dots & 1 \end{bmatrix} \quad (1)$$

$$b_i = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & -t & 0 & \dots & 0 \\ 0 & \dots & -t & 0 & \dots & 0 \\ 0 & \dots & -1 & 1 & \dots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix} \quad (2)$$

$$b_{n-1} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -t \\ 0 & 0 & \dots & 0 & -t \end{bmatrix} \quad (3)$$

Let Ψ be the MATRIX PRODUCT of BRAID WORDS, then

$$\frac{\det(1 - \Psi)}{1 + t + \dots + t^{n-1}} = \Delta_L, \quad (4)$$

where Δ_L is the ALEXANDER POLYNOMIAL and $\det$ is the DETERMINANT.

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Burgers' Equation

The PARTIAL DIFFERENTIAL EQUATION

$$u_t + uu_x = \nu u_{xx}$$

(Benton and Platzman 1972; Zwillinger 1995, p. 417; Zwillinger 1997, p. 130). The so-called nonplanar Burgers equation is given by

$$u_t + uu_x + \frac{Ju}{2t} = \frac{1}{2} \delta u_x x$$

(Sachdev and Nair 1987; Zwillinger 1997, p. 131).

References

- Benton, E. R. and Platzman, G. W. "A Table of Solutions of the of the One-Dimensional Burgers Equation." *Quart. Appl. Math.*, 195–212, Jul. 1972.
- Sachdev, P. L. and Nair, K. R. C. "Generalized Burgers Equations and Euler-Painlevé Transcendents. II." *J. Math. Phys.* **28**, 997–1004, 1987.
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Burkhardt Quartic

The VARIETY which is an invariant of degree four and is given by the equation

$$y_0^4 - y_0(y_1^3 + y_2^3 + y_3^3 + y_4^3) + 3y_1y_2y_3y_4 = 0.$$

See also QUARTIC EQUATION

References

- Burkhardt, H. "Untersuchungen aus dem Gebiet der hyperelliptischen Modulfunctionen. II." *Math. Ann.* **38**, 161–224, 1890.
- Burkhardt, H. "Untersuchungen aus dem Gebiet der hyperelliptischen Modulfunctionen. III." *Math. Ann.* **40**, 313–343, 1892.
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Bürmann's Theorem

Bürmann's theorem deals with the expansion of functions in powers of another function. Let $\phi(z)$ be a function of z which is analytic in a closed region S , of which a is an interior point, and let $\phi(a) = b$. Suppose also that $\phi'(a) \neq 0$. Then TAYLOR'S THEOREM furnishes the expansion

$$\phi(z) - b = \phi'(a)(z - a) + \frac{\phi''(a)}{2!}(z - a)^2 + \dots, \quad (1)$$

and if it is legitimate to revert this series, we obtain

$$z - a = \frac{\phi(z) - b}{\phi'(a)} - \frac{1}{2} \frac{\phi''(a)}{[\phi'(a)]^3} [\phi(z) - b]^2 + \dots, \quad (2)$$

which expresses z as an ANALYTIC FUNCTION of the variable $\phi(z) - b$ for sufficiently small values of $|z - a|$. If then $f(z)$ is analytic near $z = a$, it follows that $f(z)$ is an ANALYTIC FUNCTION of $\phi(z) - b$ when $|z - a|$ is sufficiently small, and so there will be an expansion in the form

$$f(z) = f(a) + a_1[\phi(z) - b] + \frac{a_2}{2!}[\phi(z) - b]^2 + \frac{a_3}{3!}[\phi(z) - b]^3 + \dots \quad (3)$$

The actual coefficients in the expansion are given by the following theorem, which is generally known as Bürmann's theorem. Let $\psi(z)$ be a function of z

defined by the equation

$$\psi(z) = \frac{z - a}{\phi(z) - b}. \quad (4)$$

Then an ANALYTIC FUNCTION $f(z)$ can, in a certain domain of values of z , be expanded in the form

$$f(z) = f(a) + \sum_{m=1}^{n-1} \frac{[\phi(z) - b]^m}{m!} \frac{d^{m-1}}{da^{m-1}} \{f'(a)[\psi(a)]^m\} + R_n, \quad (5)$$

where the remainder term is

$$R_n = \frac{1}{2\pi i} \int_a^x \int_\gamma \left[\frac{\phi(z) - b}{\phi(t) - b} \right]^{n-1} \frac{f'(t)\phi'(z) dt dz}{\phi(t) - \phi(z)}, \quad (6)$$

and γ is a CONTOUR in the t -plane enclosing the points a and z such that if ζ is any point inside γ , the equation $\phi(t) = \phi(\zeta)$ has no roots on or inside the CONTOUR except a simple root $t = \zeta$.

TEIXEIRA'S THEOREM is extended form of Bürmann's theorem. The LAGRANGE EXPANSION gives another such extension.

See also DARBOUX'S FORMULA, LAGRANGE EXPANSION, LAGRANGE INVERSION THEOREM, TAYLOR SERIES, TEIXEIRA'S THEOREM

References

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Burnside Problem

A problem originating with W. Burnside (1902), who wrote, "A still undecided point in the theory of discontinuous groups is whether the ORDER of a GROUP may be not finite, while the order of every operation it contains is finite." This question would now be phrased as "Can a finitely generated group be infinite while every element in the group has finite order?" (Vaughan-Lee 1990). This question was answered by Golod (1964) when he constructed finitely generated infinite P -GROUP. These GROUPS, however, do not have a finite exponent.

Let F_r be the FREE GROUP of RANK r and let N be the NORMAL SUBGROUP generated by the set of n th POWERS $\{g^n | g \in F_r\}$. Then N is a normal subgroup of F_r . We define $B(r, n) = F_r/N$ to be the QUOTIENT GROUP. We call $B(r, n)$ the r -generator Burnside group of exponent n . It is the largest r -generator group of exponent n , in the sense that every other

such group is a HOMOMORPHIC image of $B(r, n)$. The Burnside problem is usually stated as: "For which values of r and n is $B(r, n)$ a FINITE GROUP?"

An answer is known for the following values. For $r = 1$, $B(1, n)$ is a CYCLIC GROUP of ORDER n . For $n = 2$, $B(r, 2)$ is an elementary ABELIAN 2-group of ORDER 2^r . For $n = 3$, $B(r, 3)$ was proved to be finite by Burnside. The ORDER of the $B(r, 3)$ groups was established by Levi and van der Waerden (1933), namely 3^a where

$$a \equiv r + \binom{r}{2} + \binom{r}{3}, \quad (1)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. For $n = 4$, $B(r, 4)$ was proved to be finite by Sanov (1940). Groups of exponent four turn out to be the most complicated for which a POSITIVE solution is known. The precise nilpotency class and derived length are known, as are bounds for the ORDER. For example,

$$|B(2, 4)| = 2^{12} \quad (2)$$

$$|B(3, 4)| = 2^{69} \quad (3)$$

$$|B(4, 4)| = 2^{422} \quad (4)$$

$$|B(5, 4)| = 2^{2728}, \quad (5)$$

while for larger values of r the exact value is not yet known. For $n = 6$, $B(r, 6)$ was proved to be finite by Hall (1958) with ORDER $2^a 3^b$, where

$$a \equiv 1 + (r-1)3^c \quad (6)$$

$$b \equiv 1 + (r-1)2^r \quad (7)$$

$$c \equiv r + \binom{r}{2} + \binom{r}{3}. \quad (8)$$

No other Burnside groups are known to be finite. On the other hand, for $r > 2$ and $n \geq 665$, with n ODD, $B(r, n)$ is infinite (Novikov and Adjan 1968). There is a similar fact for $r > 2$ and n a large POWER of 2.

E. Zelmanov was awarded a FIELDS MEDAL in 1994 for his solution of the "restricted" Burnside problem.

See also FREE GROUP

References

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Burnside's Conjecture

This entry contributed by NICOLAS BRAY

In Note M, Burnside (1955) states, "The contrast that these results shew between groups of odd and of even order suggests inevitably that simple groups of odd order do not exist." Of course, SIMPLE GROUPS of prime order do exist, namely the groups $\mathbb{Z}_p$ for any prime p . Therefore, Burnside conjectured that every FINITE SIMPLE GROUP of non-prime order must have even order. The conjecture was proven true by Feit and Thompson (1963).

See also ABELIAN GROUP, FEIT-THOMPSON CONJECTURE, FEIT-THOMPSON THEOREM, SIMPLE GROUP

References

- Burnside, W. *Theory of Groups of Finite Order*, 2nd ed. New York: Dover, 1955.
- Feit, W. and Thompson, J. G. "Solvability of Groups of Odd Order." *Pacific J. Math.* **13**, 775–1029, 1963.

Burnside's Lemma

CAUCHY-FROBENIUS LEMMA

Bushman Transform

The INTEGRAL TRANSFORM defined by

$$(K\phi)(x) = \int_{-\infty}^{\infty} (x^2 - t^2)_+^{\lambda/2} P_\nu^\lambda\left(\frac{t}{x}\right) \phi(t) dt,$$

where y_+^a is the TRUNCATED POWER FUNCTION and $P_\nu^\lambda(x)$ is an associated LEGENDRE POLYNOMIAL.

References

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Busemann-Petty Problem

If the section function of a centered convex body in Euclidean n -space ($n \geq 3$) is smaller than that of another such body, is its volume also smaller?

References

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Busy Beaver

A busy beaver is an n -state, 2-symbol, 5-tuple TURING MACHINE which writes the maximum possible number $BB(n)$ of 1s on an initially blank tape before halting. For $n = 0, 1, 2, \dots$, $BB(n)$ is given by 0, 1, 4, 6, 13,

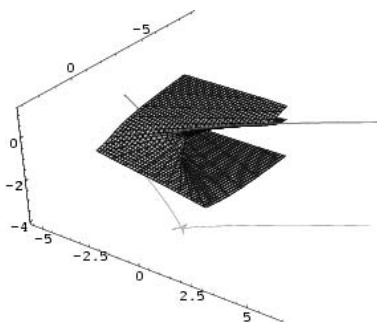
$\geq 4098, \geq 136612, \dots$ The busy beaver sequence is also known as RADO'S SIGMA FUNCTION.

See also HALTING PROBLEM, TURING MACHINE

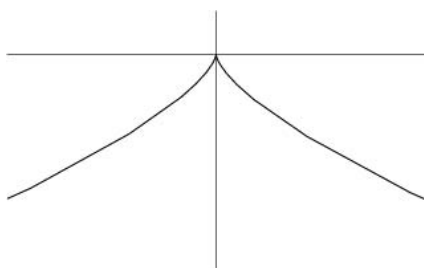
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Butterfly Catastrophe



A CATASTROPHE which can occur for four control factors and one behavior axis. The butterfly catastrophe is the universal unfolding of the singularity $f(x) = x^6$ of codimension 4, i.e., with four unfolding parameters. It has the form $F(x, u, v, w, t) = x^6 + ux^4 + vx^3 + wx^2 + tx$.



The equations

$$x = c(8at^3 + 24t^5)$$

$$y = c(-6at^2 - 15t^4)$$

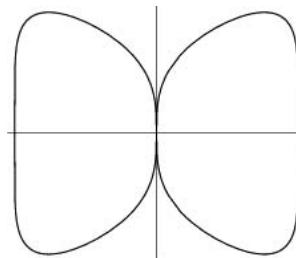
display such a catastrophe (von Seggern 1993).

References

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von Seggern, D. *CRC Standard Curves and Surfaces*. Boca Raton, FL: CRC Press, p. 94, 1993.

Butterfly Curve



A PLANE CURVE given by the implicit equation

$$y^6 = (x^2 - x^6).$$

See also DUMBBELL CURVE, EIGHT CURVE, PIRIFORM

References

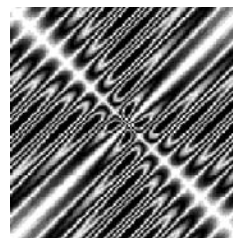
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Butterfly Effect

Due to nonlinearities in weather processes, a butterfly flapping its wings in Tahiti can, in theory, produce a tornado in Kansas. This strong dependence of outcomes on very slightly differing initial conditions is a hallmark of the mathematical behavior known as CHAOS.

See also CHAOS, LORENZ SYSTEM

Butterfly Fractal



The FRACTAL-like curve generated by the 2-D function

$$f(x, y) = \frac{(x^2 - y^2) \sin\left(\frac{x + y}{a}\right)}{x^2 + y^2}.$$

Butterfly Polyiamond

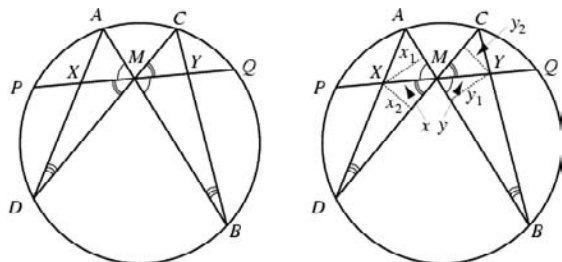


A 6-POLYIAMOND.

References

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Butterfly Theorem



Given a CHORD PQ of a CIRCLE, draw any other two CHORDS AB and CD passing through its MIDPOINT. Call the points where AD and BC meet PQ X and Y . Then M is also the MIDPOINT of XY . There are a number of proofs of this theorem, including those by W. G. Horner, Johnson (1929, p. 78), and Coxeter (1987, pp. 78 and 144). The latter concise proof employs PROJECTIVE GEOMETRY.

The following proof is given by Coxeter and Greitzer (1967, p. 46). In the figure at right, drop perpendiculars x_1 and y_1 from X and Y to AB , and x_2 and y_2

from X and Y to CD . Write $a = PM = MQ$, $x = XM$, and $y = MY$, and then note that by SIMILAR TRIANGLES

$$\frac{x}{y} = \frac{x_1}{y_1} = \frac{x_2}{y_2} \quad (1)$$

$$\frac{x_1}{y_1} = \frac{AX}{CY} \quad (2)$$

$$\frac{x_2}{y_2} = \frac{XD}{YB}, \quad (3)$$

so

$$\begin{aligned} \frac{x^2}{y^2} &= \frac{x_1}{y_1} \cdot \frac{x_2}{y_2} = \frac{x_1}{y_1} \cdot \frac{x_2}{y_1} = \frac{AX}{CY} \cdot \frac{XD}{YB} = \frac{PX}{PY} \cdot \frac{XQ}{YQ} \\ &= \frac{(a-x)(a+x)}{(a+y)(a-y)} = \frac{a^2 - x^2}{a^2 - y^2} = \frac{a^2}{a^2} = 1, \end{aligned} \quad (4)$$

so $x = y$. Q.E.D.

See also CHORD, CIRCLE, CYCLIC QUADRILATERAL, MIDPOINT, QUADRILATERAL

References

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C

Cable

TENSEGRITY

Cable Knot

Let K_1 be a TORUS KNOT. Then the SATELLITE KNOT with COMPANION KNOT K_2 is a cable knot on K_2 .

See also SATELLITE KNOT

References

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Cactus Fractal

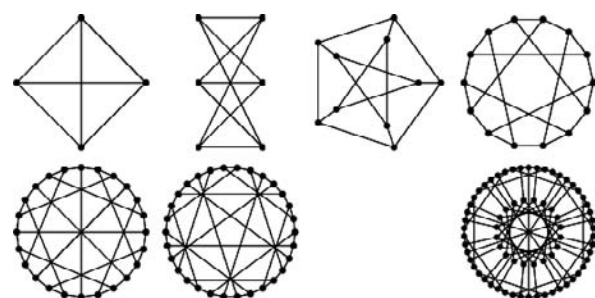


A MANDELBROT SET-like FRACTAL obtained by iterating the map

$$z_{n+1} = z_n^3 + (z_0 - 1)z_n - z_0.$$

See also FRACTAL, JULIA SET, MANDELBROT SET

Cage Graph



A 3-regular g -cage for $g \geq 3$ is a CUBIC GRAPH of GIRTH g with the minimum possible number of points. More generally, an (v, g) -cage graph is a smallest v -regular graph with GIRTH g . Cubic cages were first discussed by Tutte (1947), but the intensive study of cage graphs did not begin until publication of an article by Erdos and Sachs (1963). There exists a $(3, g)$ -cage

for all $g \geq 3$, and the $(3, g)$ -cages are unique for $g = 3$ to 8. The number of nonisomorphic $(3, g)$ cages for $g = 1, 2, \dots$ are given by 0, 0, 1, 1, 1, 1, 1, 18, 3, ... (Sloane's A052453; Gould 1988, Royle). The number of vertices in the $(3, g)$ cages for $g = 3, 4, \dots$ are 4, 6, 10, 14, 22, 30, 46, 62, 94, ... (Sloane's A052454). A selection of known $(3, g)$ -cages are illustrated above. There are a number of special cases (Wong 1982). The $(2, g)$ -cage is the CYCLE GRAPH C_g , the $(v, 2)$ -cage is the MULTIGRAPH of v edges on two vertices, the $(v, 3)$ -cage is the COMPLETE GRAPH K_{v+1} , and the $(v, 4)$ -cage is the BIPARTITE GRAPH $K_{v, v}$.

Computing the number of vertices in a (v, g) -cage is very difficult for $g \geq 5$ and $n \geq 3$ (Wong 1982). The following table summarizes known cages. A lower bound for the number of vertices $f(v, g)$ in a (v, g) -cage is given by

$$f_l(v, g) = \begin{cases} \frac{v(v-1)^r - 2}{v-2} & \text{for } g = 2r + 1 \\ \frac{2(v-1)^r - 2}{v-2} & \text{for } g = 2r \end{cases}$$

(Tutte 1967, p. 70; Bollobás 1978, p. 105; Wong 1982). Sauer (1967ab) has obtained the best known upper bounds

$$f_u(3, g) = \begin{cases} \frac{4}{3} + \frac{29}{12} 2^{g-2} & \text{for } g \text{ odd} \\ \frac{2}{3} + \frac{29}{12} 2^{g-2} & \text{for } g \text{ even} \end{cases} \quad (1)$$

$$f_u(n, g) = \begin{cases} 2(n-1)^{g-2} & \text{for } g \text{ odd} \\ 2(n-1)^{g-3} & \text{for } g \text{ even,} \end{cases} \quad (2)$$

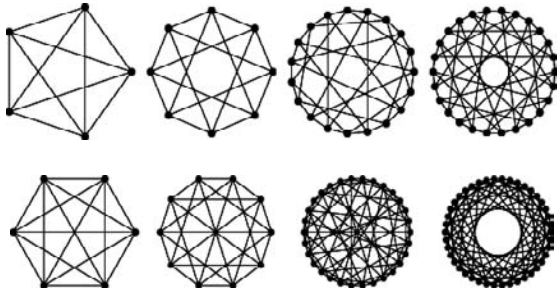
with $v \geq 4$ (Wong 1982).

In the table, K_n denotes a COMPLETE GRAPH, and $K_{m, n}$ a complete bipartite graph.

g	$(3, g)$	$(4, g)$	$(5, g)$	$(6, g)$	$(7, g)$ -cage
3	K_4	K_5	K_6	K_7	K_8
4	$K_{3, 3}$	$K_{4, 4}$	$K_{5, 5}$	$K_{6, 6}$	$K_{7, 7}$
5	PETERSEN GRAPH	ROBERTSON GRAPH	ROBERTSON-WEGNER GRAPH		HOFFMAN-SINGLETON GRAPH
6	HEAWOOD GRAPH				
7	MC GEE GRAPH				
8	LEVI GRAPH				

g	$f(3, g)$	$f(4, g)$	$f(5, g)$	$f(6, g)$	$f(7, g)$
3	4	5	6	7	8
4	6	8	10	12	14
5	10	19	30	40	50
6	14	26	42	62	90
7	24				
8	30				
9	[54, 58]				
10	70				
11	< 112				

The first (3, 9)-cage was found by Biggs and Hoare (1980), and Brinkmann *et al.* (1995) completed an exhaustive search yielding all 18 (3, 9)-cages (Royle). The three (3, 10)-cages were found by O'Keefe and Wong (1980). Computations by McKay and W. Myrvold have demonstrated that a (3, 11)-cage must have 112 vertices (Royle). The single known example was found by Balaban (1973).



The known (4, g)- and (5, g)-cages are shown above (Wong 1982).

See also CAYLEY GRAPH, CUBIC GRAPH, EXCESS, HOFFMAN-SINGLETON GRAPH, MOORE GRAPH, REGULAR GRAPH, ROBERTSON GRAPH, ROBERTSON-WEGNER GRAPH, UNITRANSITIVE GRAPH

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Cahn-Hilliard Equation

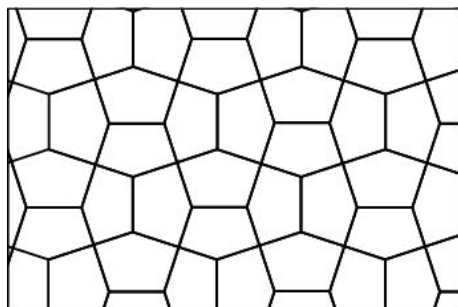
The PARTIAL DIFFERENTIAL EQUATION

$$u_t = \nabla \cdot \left[M(u) \nabla \left(\frac{\partial f}{\partial u} - K \nabla^2 u \right) \right].$$

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Cairo Tessellation



A TESSELLATION appearing in the streets of Cairo and in many Islamic decorations. Its tiles are obtained by projection of a DODECAHEDRON, and it is the DUAL TESSELLATION of the semiregular tessellation of squares and equilateral triangles.

See also DODECAHEDRON, TESSELLATION

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Cake Cutting

It is always possible to "fairly" divide a cake among n people using only vertical cuts. Furthermore, it is possible to cut and divide a cake such that each person believes that *everyone* has received $1/n$ of the cake according to his own measure (Steinhaus 1983, pp. 65–71). Finally, if there is some piece on which two people disagree, then there is a way of partitioning and dividing a cake such that each participant believes that he has obtained more than $1/n$ of the cake according to his own measure.

There are also similar methods of dividing collections of individually indivisible objects among two or more people when cash payments are used to even up the final division (Steinhaus 1983, pp. 67–68).

Ignoring the height of the cake, the cake-cutting problem is really a question of fairly dividing a CIRCLE into n equal AREA pieces using cuts in its plane. One method of proving fair cake cutting to always be possible relies on the FROBENIUS-KÖNIG THEOREM.

See also CIRCLE DIVISION BY CHORDS, CIRCLE DIVISION BY LINES, CYLINDER CUTTING, ENVYFREE, FROBENIUS-KÖNIG THEOREM, HAM SANDWICH THEOREM, PANCAKE THEOREM, PIZZA THEOREM, SQUARE DIVISION BY LINES, TORUS CUTTING, VOTING

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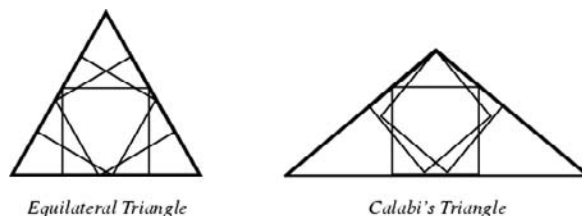
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Cal

WALSH FUNCTION

Calabi's Triangle



The one TRIANGLE, in addition to the EQUILATERAL TRIANGLE, for which the largest inscribed SQUARE can be inscribed in three different ways. The ratio of the sides to that of the base is given by $x = 1.55138752454 \dots$ (Sloane's A046095), where

$$x = \frac{1}{3} + \frac{(-23 + 3i\sqrt{237})^{1/3}}{3 \cdot 2^{2/3}} + \frac{11}{3[2(-23 + 3i\sqrt{237})]^{1/3}}$$

is the largest POSITIVE ROOT of

$$2x^3 - 2x^2 - 3x + 2 = 0,$$

which has CONTINUED FRACTION $[1, 1, 1, 4, 2, 1, 2, 1, 5, 2, 1, 3, 1, 1, 390, \dots]$ (Sloane's A046096).

See also GRAHAM'S BIGGEST LITTLE HEXAGON, TRIANGLE

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Calabi-Yau Manifold

CALABI-YAU SPACE

Calabi-Yau Space

Calabi-Yau spaces are important in string theory, where one model posits the geometry of the universe to consist of a ten-dimensional space OF THE FORM $M \times V$, where M is a four dimensional manifold (space-time) and V is a six dimensional COMPACT Calabi-Yau space. They are related to KUMMER SURFACES. Although the main application of Calabi-Yau spaces is in theoretical physics, they are also interesting from a purely mathematical standpoint. Consequently, they go by slightly different names, depending mostly on context, such as Calabi-Yau manifolds or Calabi-Yau varieties.

Although the definition can be generalized to any dimension, they are usually considered to have three complex dimensions. Since their COMPLEX STRUCTURE may vary, it is convenient to think of them as having six real dimensions and a fixed SMOOTH STRUCTURE.

A Calabi-Yau space is characterized by the existence of a NONVANISHING HARMONIC SPINOR ϕ . This condition implies that its CANONICAL BUNDLE is TRIVIAL.

Consider the local situation using coordinates. In $\mathbb{R}^6$, pick coordinates x_1, x_2, x_3 and y_1, y_2, y_3 so that

$$z_j = x_j + iy_j \quad (1)$$

gives it the structure of $\mathbb{C}^3$. Then

$$\phi_z = dz_1 \wedge dz_2 \wedge dz_3 \quad (2)$$

is a local section of the canonical bundle. A unitary change of coordinates $w = Az$, where A is a UNITARY MATRIX, transforms ϕ by $\det A$, i.e.

$$\phi_w = \det A \phi_z. \quad (3)$$

If the linear transformation A has DETERMINANT 1, that is, it is a special unitary transformation, then ϕ is consistently defined as ϕ_z or as ϕ_w .

On a Calabi-Yau manifold V , such a ϕ can be defined globally, and the LIE GROUP $SU(3)$ is very important in the theory. In fact, one of the many equivalent definitions, coming from RIEMANNIAN GEOMETRY, says that a Calabi-Yau manifold is a $2n$ -dimensional manifold whose HOLONOMY GROUP reduces to $SU(n)$. Another is that it is a CALIBRATED MANIFOLD with a CALIBRATION FORM ψ , which is algebraically the same

as the REAL PART of

$$dz_1 \wedge \dots \wedge dz_n. \quad (4)$$

Often, the extra assumptions that V is SIMPLY CONNECTED and/or COMPACT are made.

Whatever definition is used, Calabi-Yau manifolds, as well as their MODULI SPACES, have interesting properties. One is the symmetries in the numbers forming the HODGE DIAMOND of a compact Calabi-Yau manifold. It is surprising that these symmetries, called MIRROR SYMMETRY, can be realized by another Calabi-Yau manifold, the so-called mirror of the original Calabi-Yau manifold. The two manifolds together form a MIRROR PAIR. Some of the symmetries of the geometry of mirror pairs have been the object of recent research.

See also CALIBRATED MANIFOLD, CANONICAL BUNDLE, COMPLEX MANIFOLD, DOLBEAULT COHOMOLOGY, HARMONIC, HODGE DIAMOND, KÄHLER FORM, LIE GROUP, MIRROR PAIR, MODULI SPACE, SPINOR, VARIETY

Calabi-Yau Variety

CALABI-YAU SPACE

Calculus

In general, "a" calculus is an abstract theory developed in a purely formal way.

"The" calculus, more properly called ANALYSIS (or REAL ANALYSIS or, in older literature, INFINITESIMAL ANALYSIS) is the branch of mathematics studying the rate of change of quantities (which can be interpreted as SLOPES of curves) and the length, AREA, and VOLUME of objects. The calculus is sometimes divided into DIFFERENTIAL and INTEGRAL CALCULUS, concerned with DERIVATIVES

$$\frac{d}{dx} f(x)$$

and INTEGRALS

$$\int f(x) dx,$$

respectively.

While ideas related to calculus had been known for some time (Archimedes' EXHAUSTION METHOD was a form of calculus), it was not until the independent work of Newton and Leibniz that the modern elegant tools and ideas of calculus were developed. Even so, many years elapsed until the subject was put on a mathematically rigorous footing by mathematicians such as Weierstrass.

See also ARC LENGTH, AREA, CALCULUS OF VARIATIONS, CHANGE OF VARIABLES THEOREM, DERIVATIVE, DIFFERENTIAL CALCULUS, ELLIPSOIDAL CALCULUS, EXTENSIONS CALCULUS, FLUENT, FLUXION, FRAC-

TIONAL CALCULUS, FUNCTIONAL CALCULUS, FUNDAMENTAL THEOREMS OF CALCULUS, HEAVISIDE CALCULUS, INTEGRAL, INTEGRAL CALCULUS, JACOBIAN, LAMBDA CALCULUS, KIRBY CALCULUS, MALLIAVIN CALCULUS, PREDICATE CALCULUS, PROPOSITIONAL CALCULUS, SLOPE, STOCHASTIC CALCULUS, TENSOR CALCULUS, UMBRAL CALCULUS, VOLUME

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Calculus of Variations

A branch of mathematics which is a sort of generalization of CALCULUS. Calculus of variations seeks to find the path, curve, surface, etc., for which a given FUNCTION has a STATIONARY VALUE (which, in physical problems, is usually a MINIMUM or MAXIMUM). Mathematically, this involves finding STATIONARY VALUES of integrals of the form

$$i = \int_b^a f(y, \dot{y}, x) dx. \quad (1)$$

i has an extremum only if the EULER-LAGRANGE

DIFFERENTIAL EQUATION is satisfied, i.e., if

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) = 0. \quad (2)$$

the FUNDAMENTAL LEMMA OF CALCULUS OF VARIATIONS states that, if

$$\int_a^b M(x)h(x) dx = 0 \quad (3)$$

for all $h(x)$ with CONTINUOUS second PARTIAL DERIVATIVES, then

$$M(x) = 0 \quad (4)$$

on (a, b) .

A generalization of calculus of variations known as MORSE THEORY (and sometimes called "calculus of variations in the large" uses nonlinear techniques to address variational problems.

See also BELTRAMI IDENTITY, BOLZA PROBLEM, BRACHISTOCURVE PROBLEM, CATENARY, ENVELOPE THEOREM, EULER-LAGRANGE DIFFERENTIAL EQUATION, ISOPERIMETRIC PROBLEM, ISOVOLUME PROBLEM, LINDELOF'S THEOREM, MORSE THEORY, PLATEAU'S PROBLEM, POINT-POINT DISTANCE-2-D, POINT-POINT DISTANCE-3-D, ROULETTE, SKEW QUADRILATERAL, SPHERE WITH TUNNEL, SURFACE OF REVOLUTION, UNDULOID, WEIERSTRASS-ERDMAN CORNER CONDITION

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