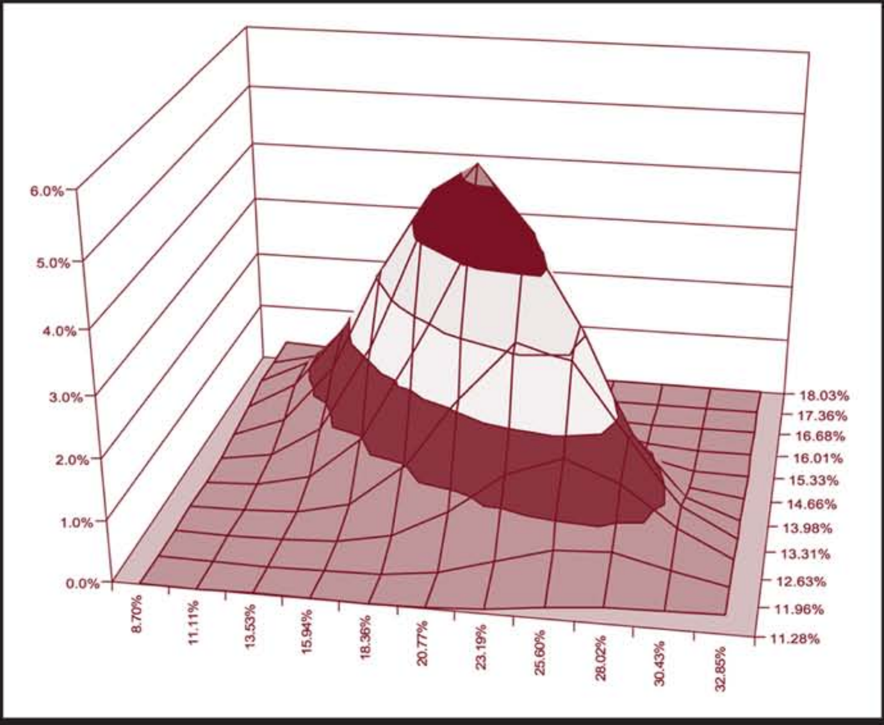
Edward E. Qian, Ronald H. Hua,
and Eric H. Sorensen

Quantitative Equity Portfolio Management

Modern Techniques and Applications



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Preface

Over the last 40 years, academic researchers have made major breakthroughs in advancing modern practice in finance. These include portfolio theory, corporate finance, financial engineering of derivative instruments, and many other applications pertaining to financial markets overall. Formal portfolio theory research saw major advances in the context of normative choice modeling, including how to form an optimal portfolio, beginning with Harry Markowitz. Parallel with this, we saw new advances in capital market theory in the context of descriptive equilibrium propositions in terms of the risk/return tradeoff, beginning with Bill Sharpe and the Capital Asset Pricing Model (CAPM). Many related academic developments provided rich portfolio management insight, including Arbitrage Pricing Theory (APT), market efficiency proposition, market anomalies, and behavioral finance.

Against this backdrop, it is therefore not surprising, over the past two decades, that modernizing portfolio management has been the ambition of hundreds of professional investment management practitioners as well as fiduciaries. Driven by market demand and the search of higher returns, a new breed of investment professionals has emerged — quants, i.e., quantitative professions with advanced degrees in science and economic/finance, seeking to exploit market anomalies with increasing success.

As a result, quantitative equity investment strategies have been gaining acceptance and popularity in the investment community. They are deployed in many forms, from enhanced products that aim to beat market indices while limiting the amount of risk, to absolute return strategies (long-short hedge funds) that strive to produce positive return regardless of the overall market condition.

Quantitative equity portfolio management combines theories and advanced techniques from several disciplines, including financial economics, accounting, mathematics, and operational research. Although many books are devoted to these disciplines, few deal with quantitative equity investing in a systematic and mathematical framework that is suitable for quantitative investment professionals and students with interests in quantitative equity investing.

The motivation for this book is to provide a self-contained overview and detailed mathematical treatment of various topics that serve collectively as the foundation of quantitative equity portfolio management. In many cases, we frame related problems in this field in mathematical terms and solve these problems with mathematical rigor while establishing an analytical framework. We also illustrate the mathematical concepts and solutions with numerical and empirical examples. In the process, we provide a review of quantitative investment strategies or factors accompanied by their academic origins.

This book serves as a guide for practitioners in the field who are frustrated with certain naïve treatments of many common modeling issues and wish to gain in-depth insights from mathematical analysis. We hope that the book will also serve as a text and reference for students in computational and quantitative finance programs interested in quantitative equity investing out of pure curiosity or in search of employment opportunities. As practitioners, we feel strongly that current curriculum of many such programs is often light on portfolio theory and portfolio management, and long on option pricing theory and various microscopic views of market efficiency (or lack thereof).

As practitioners and active researchers in the field, we have selected topics essential to quantitative equity portfolio management, from theoretical foundation to recently developed techniques. Due to our variety of topics, we adopt a flexible style: we employ theoretical, numerical, and empirical approaches, when appropriate, for specific subjects within the book.

Many people have helped us in making this book possible. We are grateful to Joe Joseph of Putnam Investments who is responsible for many ideas developed in Chapter 6. We thank Dan diBartolomeo of Northfield and participants of Northfield research conferences for feedbacks to several research presentations that have made their way into the book. Frank Fabozzi and Gifford Fong also deserve credit in recognizing the value of our research and publishing it in the *Journal of Portfolio Management* and the *Journal of Investment Management*, respectively. We also thank our colleagues at PanAgora and Putnam for helpful comments. Betty Anne Case, Craig Nolder, and Alec Kercheval of Florida State University provided encouragement and academic perspective for our effort. Others who provided feedback to us include Artemiza Woodgate and Fred Copper. Last, but not least, we are very grateful to Jennifer Crotty for editorial assistance. Any errors, however, remain entirely ours.

Abstract

This book provides a self-contained overview, empirical examination, and detailed mathematical treatment of various topics from financial economics/accounting, mathematics, and operational research that serve collectively as the foundation of quantitative equity portfolio management. In the process, we review quantitative investment strategies or factors that are commonly used in practice, including value, momentum, and quality, accompanied by their academic origins. We present advanced techniques and applications in return forecasting models, risk management, portfolio construction, and portfolio implementation. Examples include optimal multifactor models, contextual and nonlinear models, factor timing techniques, portfolio turnover control, Monte Carlo valuation of firm values, and optimal trading.

We frame and solve related problems in mathematical terms and also illustrate the mathematical concepts and solutions with numerical and empirical examples. This book serves as a guide for practitioners in the field who wish to gain in-depth insights from mathematical analysis. We hope that the book will also serve as a text and reference for students in finance/economics, computational, and quantitative finance programs, interested in quantitative equity investing, out of pure curiosity, or in search of employment opportunities.

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Eric H. Sorensen, Ph.D., is the president and CEO of PanAgora Asset Management in Boston. PanAgora specializes in quantitative money management solutions implementing modern portfolio strategies totaling approximately $24 billion in institutional assets. He and his team are industry leaders in the field of quantitative investment strategies and focus on a variety of innovative bottom-up equity as well as multialpha macro strategies. Between 2000 and 2004, Dr. Sorensen was the global head of Quantitative Research and a chief investment officer at Putnam Investments. Prior to joining Putnam Investments, Dr. Sorensen was global head of quantitative research at Salomon Brothers (now Citigroup). During his 14 years on Wall Street, he published extensively, and consulted with institutional investor clients around the world and developed a global reputation as a leader in the field of modern investment strategies. Prior to Wall Street, he was a professor of finance and department head at the University of Arizona. He has published over 50 academic journal articles and has served on the editorial boards of several academic finance journals. His background also includes a tour as a U.S. Air Force officer and jet pilot from 1969–1974. Dr. Sorensen received his Ph.D. in finance from University of Oregon in 1976 and M.B.A. from Oklahoma City University in 1974.

CHAPTER 1

Introduction: Beliefs, Risk, and Process

THIS BOOK IS ABOUT QUANTITATIVE EQUITY INVESTMENT STRATEGIES, focusing on modern techniques and applications. Three fundamental activities form the basis of a modern investment practice: in order to be successful, the investment team must have (1) a strong philosophy based on commitment to a set of beliefs, (2) a clear approach in translating uncertainty into an appropriate risk/return trade-off, and (3) a comprehensive investment process from beginning to end.

1.1 BELIEFS

What do markets give us, and how do we believe we can go after it? This two-part question is essential to a portfolio manager's belief system. In the premodern 1950s world of fundamental stock picking, the analysis focused exclusively on the second part of the question — go for the "best" stocks and enjoy the results. Inherent in this belief is that one has sufficient skill and is significantly blessed above others who compete in the same game. Across a diverse spectrum of stock-picking techniques, there certainly have been (and are) some that win more than others. However, over the years, formal academic research and practitioner experience converge on the conclusion that it is difficult to win consistently if we account for the proper risks. With consideration of the risks, we should think of the game as well worth winning but not necessarily worth playing.

As for the first part of the question, there has been a common evolution of beliefs. What does the opportunity set look like? How do the distributions of relative stock returns behave? Are these return differences exploitable? In the 1960s, there began a tension surrounding the true value of past price and volume information in security returns — "technical analysis." A well-accepted investment approach was to study the pattern of past price returns in order to forecast future returns. As we will see in later chapters, the same underlying price data may be also relevant today, though in the context of a modern, comprehensive process.

As academics began to formally study return distributions, they gravitated to a concept of "random walk." They increasingly came to the conclusion that "price has no memory" (Lorie and Hamilton 1973). If the investor's technique is conditioned on some *ad hoc* price configuration, there will be little value added because a random walk stock will give us no profitable clues about future prices.

It was Fama (1970) who artfully formed and expanded the notion of random walk into what he popularized as the efficient market hypothesis (EMH). In summary, it is hard (if not impossible) to beat the market depending on the investors' information set. Past price data does not cut it. Taken to an extreme, a very strong EMH belief is that all information, both public and private, is not sufficient to beat the market, after consideration of appropriate costs and proper risk specifications.

By the 1970s, variations of efficient markets beliefs were firmly implanted in the brains of many financial economists. In fact, it was quite difficult for a bright assistant professor of finance to publish any empirical findings that disproved the EMH. However, by the early 1980s, the ambitious and persistent academic empiricists found a way — just call it something else! In the 1980s, there came a volume of formal literature that discovered inefficiencies that could lead to abnormal returns if rigorously applied. The list includes size effect, January effect, value irregularities, momentum effect, etc. We called them anomalies[1] and reverently acknowledged in the conclusion that these discoveries (1) were likely not repeatable in the future (now that we know them), (2) may be inconclusive because of potential "risk misspecification," or (3) were lacking the proper allocation of costs in the strategy. In a modern quantitative process we call these anomalies "factors," which are an in-depth topic of later chapters.

What are our beliefs? What are the principles underlying our book? We choose rather safe ones that are explained in many of the subsequent chapters. First, skill and return dispersion are the key drivers of opportunity.

Second, the market is not efficient, which, in many cases, is attributable to investors' irrational behavior described by "behavioral finance." Third, the variables or factors we use to predict return must be grounded in financial theory and reflect logical cause and effect. (Sunspots do not cut it.) Fourth, true alpha-generation is available to practitioners who creatively combine modern tools — econometrics, mathematics, investment theory, financial accounting, psychology, operations research, and computer science. Fifth, objective discipline is essential in the implementation of strategies. This is not to say subjective judgment is lacking in the world of quantitative management — but it lies in perfecting the comprehensive portfolio system, rather than in comprehending the perfect stock selection.

This comprehensive system is the core of quantitative investment process. Active investment is about the processing of information. One must have the best information as well as the best way to process and implement them in a portfolio. With the advent of the information age, advance of financial markets, and increasing computing power, quantitative investment process provides a way of unifying all these together to deliver consistent returns. In a way, this is analogous to combining the best machinery with the best operators. In the late 1960s, there was a common belief in the U.S. Air Force that advances in aeronautical engineering would obviate any role for the human pilot. On the contrary, air superiority today resides with the force that combines the best equipment with the best-trained pilots. The best equipment is not knowable without design inputs from the best pilots.

1.2 RISK

The quantification of uncertainty is also one of the evolutionary breakthroughs in the theory of investment during the last century. Frank Knight (1921) laid the groundwork with a quite intuitive definitional distinction between uncertainty and risk: (1) decision makers crudely operate in a world of random uncertainty, and (2) risk is a condition in which the decision maker assigns formal mathematical probabilities to specify the uncertainty. Later, Von Neumann and Morgenstern (1944) formalized the specification of risk into microeconomic theory, laying a foundation for rational decision making under uncertainty with the concept of expected utility.[2]

It was Markowitz (1952) who inaugurated the vast body of literature we know as modern portfolio theory (MPT). Markowitz combined the notion that when a rational investor is faced with a set of security choices that follow a normal distribution, he or she will seek to maximize expected utility by formally trading off expected return with risk measured by variance.

In a world characterized by diminishing marginal utility for wealth, the optimal portfolio is specified and the security weights are solved using the mean and variance of the portfolio return distribution (see Chapter 2 for a complete treatment).

Bill Sharpe's article in 1964 took the normative mean–variance portfolio concept to the next level by developing an equilibrium pricing model to describe the first formal capital market pricing of risk framework — the capital asset pricing model (CAPM).[3] For this, he later received the Nobel Prize, as did Harry Markowitz. Assuming frictionless markets and homogeneous expectations of investors, the pricing relationship is depicted in terms of expected returns. The expected return of a security (or a portfolio) consists of two parts: (1) market price of time — the risk-free rate and (2) market price of risk — beta times the market excess return.

For investors, CAPM concludes that the market provides a fair risk premium — take systematic or market (beta) risk and be rewarded. As such, prudent investments should be combinations of two passively managed portfolios — the market portfolio and the risk-free portfolio; the precise combination is governed by the risk tolerance of a particular investor.

In theoretical equilibrium, beta is the elasticity of the portfolio return with the market and presents a linear trade-off between risk and return in the long run, i.e., capital market line (CML). However, can't we do better in practice? Isn't what this book and myriads of writings before are about? How can we generate alpha — the return above the CML that is in excess of the risk? It takes positive skill!

1.2.1 Beta, Benchmarks, and Risk

Risk-adjusted positive skill is the true goal of the game. The development of risk and capital market theory from the 1950s, and for 30 years thereafter, ushered in a host of phenomena and participants to the game. Three stand out. First, beginning in the 1980s, the attraction of indexing to a benchmark — index such as the S&P 500 — exploded. Entrepreneurs at Wells Fargo (BGI today), Mellon, and later, Vanguard and State Street, offered passive zero alpha index funds with an efficient beta of 1 and low fees. It was as if the new risk tools combined with the now acceptable belief in market efficiency to produce a powerful antidote to those that had been stung by underdelivered promises of traditional active return managers.

Second, a new player category entered the fray in the 1980s. Managers who promised active strategies (positive alpha) found themselves increasingly exposed to benchmark comparisons by a new labor force

— the influential pension plan consultants. Within the consulting firms emerged armies of analysts equipped with MPT devices to conduct manager research, evaluating them against designated benchmarks (growth/value, large/small, domestic/international, developed/emerging, etc.). Their objective was to provide service to institutional investors and the ability to "separate alpha from beta" by performing scientific attribution of active managers, as well as to pronounce an active strategy dead or alive. The game was still worth "winning" but now had more talented officials evaluating the "playing."

Third, enter hedge fund managers who got away with no benchmarks. Hedge fund is not a new phenomenon — combining subjective long and short positions (asset classes of securities) goes back to the 1960s. For example, equity hedge funds are long-short — buy securities as well as sell borrowed ones — but they are not necessarily market beta neutral. It is often hard, if not impossible, to disentangle what is alpha and what is beta. For a long time, nobody cared because most of the investors in the hedge funds were high-net-worth individuals who had their eyes on the absolute returns, not abstract geeks. Today, the situation has changed dramatically. Equity market neutral managers (mostly quants) manage zero-beta funds with refined risk management systems, and often deliver pure alpha. Institutional investors are increasingly pursuing and paying handsomly for alpha, but are unwilling to pay excessively for beta management. Hence, we have the rise of market-neutral hedge funds with a new benchmark — cash.

1.3 QUANTITATIVE INVESTMENT PROCESS

What steps characterize a quantitative investment process? What are the instruments in the toolbox of quantitative investment professionals? There are at least five essential components.

Alpha model: First and foremost is an alpha model that forecasts excess return of stocks. If return distribution is characterized by the expected return and the standard deviation, it is often the expected return that determines whether we buy or sell, overweight or underweight, and the standard deviation that determines the size of the portfolio allocations. It is easier to find random factors that represent non-compensated market risks than to find alpha factors that represent incremental rewards. The alpha model is often proprietary and highly guarded, reflecting creativity as well as superior systems. It is the most important differentiator within the investment firm.

Risk models: Good quantitative investment processes require sophisticated risk tools that embody many "drivers' of risk beyond the one-factor CAPM — plain vanilla beta. Today, commercial risk models such as BARRA serve to isolate and control stock specific factors that measure unwanted risk, such as size, value and the like. However, some BARRA factors, first estimated in the mid-1980's, overlap with potential stock-specific alpha factors. Ross and Roll (1976, 1977) introduced the arbitrage pricing model (APT), and estimated it with a set of four purely macroeconomic time-series factors, such as the cycle of long-term interest rates. Later others developed more complete specifications of macro models using such phenomenon as economic growth, term structure of rates, inflation, oil and so on. Salomon Brothers quantitative team first estimated a set of macroeconomic risk systems for local and global equity markets in the late 1980's Similarly, the Northfield Company delivered a portfolio optimization package using a macro risk model in the 1990's.

Portfolio optimization: The normative machinery that calculates the tradeoff between alpha factors (wanted risk) with risk factors (unwanted risk) formally is the optimization tool. Effectively, portfolio optimization formally combines both proprietary alpha with exogenous risk to create the ex ante optimum set of portfolio weights, subject to the risk appetite of the manager. Managers can optimize active portfolios versus a benchmark such as S&P 500 index, or against cash for market-neutral long/short portfolios. These tools allow managers to dissect the ex ante risks, and place their exposures with their alphas. However, there is a tendency to be overconfident in risk model outputs. As we will see later, there is alpha model risk also, and it must be modeled to achieve the best portfolio results.

Portfolio implementation: Risks and alphas change. The complete process requires trading — turnover. Relatively high-turnover active portfolios demand close attention to transaction costs. Since the 1970's, market maker competition and computer networking technology influenced and drove down the costs of trading — both commissions as well as market pricing impact proportional to volume. Nevertheless, trading costs are positive and less subject to randomness than are security prices (and alphas). The modern implementation process, therefore, includes a risk/return framework to address the portfolio implementation. Asset management firms and brokerage firms are increasingly relying on proprietary or commercial models to implement trades with the goal of minimizing implementation shortfall under uncertainty.

Performance attribution: Well, in the end does this all work? If so, how much is *working* and how much is *random*? Modern managers perform attributions regularly to ascribe ex post returns to ex ante factor exposures. It is increasingly imperative for active managers to identify their skill vis-a-vis ex ante alpha efficacy, and to attribute ex post results to maintaining exposure of these alpha sources. Here quantitative mangers possess a clear advantage over pure fundamental managers.

Successful investment firms would find a way to integrate these five components together and constantly search for improvements in all of them to stay ahead of the market and the competitors.

1.3.1 Quantitative vs. Fundamental

It is inaccurate to say that fundamental managers dig deep at the solo stock level, but have no models or disciplines. It is also unfair to say that quantitative managers apply skills to so broad a set of stocks that the process is superficial at the fundamental level, and often labeled black-box, data-mining nerds. This is a misrepresentation. Many quantitative investment strategies rely on factors that are based on not only solid economic principles, but also on sound fundamental intuition (more on this in Chapters 5 and 6). At the same time, fundamental managers all use models. These may be rules-of-thumb or heuristics, and not subject to rigorous testing, but the deep implementation of the *model* into the security makes up for the lack of breadth. To repeat, quantitative management — lies in broadly perfecting the comprehensive portfolio system, whereas, fundamental management lies in deeply comprehending the perfect stock selection.

In many instances, the underlying principles of quantitative investment are no different from traditional fundamental research. At a basic level, all investment strategies seek to buy low and sell high — requiring a measured valuation methodology. John Burr Williams [1938] developed the first modern expression for the fundamental valuation of intrinsic value — that a company's stock should achieve a market price that quantifies the present value of all future potentially profitable operations of the firm that accrue to shareholders. This is the forerunner of the now common dividend discount model (DDM) and a variety of related cash flow valuation expressions. This valuation framework is indispensable to fundamental analysis. Who can say it is not quantitative analysis — do we value bonds, even those with embedded options, similarly?

Notably, Benjamin Graham (1934, 1949) laid the foundation of fundamental investing, which deemphasizes movements of market prices and focus on a firm's intrinsic value and fundamental analysis. Warren Buffet is perhaps the best-known disciple of Graham and offers at least an implicit process firmly founded on the original valuation principals. Can quantitative investing have a much closer affinity and be kindred spirit to the Ben Graham principles? We provide some answers to this question in the book.

Perhaps, some of the misperception about quantitative investing is self-inflicted. After all, we are quants — as some would assume all it takes is a brainy nerd and a fast computer, right? Many become easily get excited about mean–variance optimization and Monte Carlo simulation but are bored with balance sheet and cash-flow analysis. This is the wrong attitude, perhaps. Some of the most valuable information, quantitative or fundamental, is only garnered through painstaking analysis of financial statements. We hope readers would agree with this after reading the book.

1.4 INFORMATION CAPTURE

Investing without true information is just speculation. How do we know we have true information that can predict security returns? On one level, predicting a market crash is not enough, even if you are correct once. In the same vein, neither is finding the correct target prices for a couple of stocks a proof of skill. The key to investment success is consistency in forecasting (skill) applied repeatedly (breadth).

We have Grinold and Kahn (2000) to thank for introducing the fundamental law of active management (FLAM). It has become an important framework for evaluating skills in active management. In their framework, the skill is measured by the information coefficient (IC) — the cross-sectional correlation coefficient between forecasts and subsequent returns. Consistency is measured by the information ratio (IR) — the ratio of average excess return to the standard deviation of excess return. Under a host of assumptions, FLAM combines skill and opportunity set together into a convenient expression for IR:

$$\mathrm{IR} = \mathrm{IC}\sqrt{N}$$

where N is the number of independent securities.

Although FLAM represents a milestone in active portfolio management theory, important practical extensions have gone in two directions. First, we can reexamine FLAM and modify for portfolios with real world constraints. For instance, Grinold and Kahn (2000) compare the IR of long-only portfolios with long-short portfolios. Clarke et al. (2002) generalize FLAM introducing the concept of transfer coefficient to approximate the loss of information due to constraints. These studies highlight the dampening effect of overly stringent constraints on investment performance. This awareness across the investment community has created increased receptivity to long-short portfolios, either "pure" or constrained, in the search of more consistent alpha (see Chapter 11).

The second extension, more subtle but arguably more significant, is a multiperiod version of IR. Unknown to many, FLAM is a result for a single period — the expected excess return to the targeted tracking error. Qian and Hua (2004) first pointed out that, in a multiperiod framework, the standard deviation of IC plays an important role in determining the *ex post* tracking error, which is not necessarily the same as the *ex ante* tracking error. This insight is further extended in Sorensen et al. (2004), using an alternative expression for IR to combine multiple alpha factors with optimal factor weights that achieves maximum IR (Chapter 4 and Chapter 7).

Multiperiod portfolio management is dynamic in nature. This dynamic link is amplified by portfolio turnover constraints (Sneddon 2005; Grinold 2006). The turnover constraint, while controlling transaction costs, inhibits information transfer to the portfolio. However, its impact varies across alpha factors with differing information horizon (Chapter 8 and Chapter 12). Such recent research raises the awareness of important normative implications of the fundamental law and proposed various methods to modify it for practical use.

Quality information is the most precious substance in the investment business. Simple yet naïve models that are unconditional and one-size-fits-all do not capture all the information available. These simple models fall short in two ways. First, stocks are idiosyncratic in nature. A one-size-fits-all model assumes that all stocks respond to the factor exposure in the same way all the time. Practitioners know this is not true, and are beginning to analyze factor significance within this context. How do we systemize this approach? Second, the market is inherently dynamic due to influences from macroeconomic factors and the changing behavior of players — firms, investors, etc. As a result, the efficacy of alpha factors does not necessarily remain stable as the market environment changes. There is a growing list of academic literatures covering conditional CAPM. For practical purposes, how do we build a forecasting model that is adaptive to allow its factor combination to change over time? We cover this topic in the book.

Much of this book goes deep into the elements of FLAM. Our purpose is to enrich this framework to highlight key elements of a modern process. It will be apparent that our approach is part art, part science, part quantitative, and part fundamental. These steps may not be the ultimate way to capture all the information, but they represent considerable improvement in our journey to build the perfect comprehensive portfolio system.

1.4.1 Alpha

True risk-adjusted alpha has always been scarce. Some refer to the search for alpha as a zero-sum game. To win the game — using a baseball analogy — a team must play well by having a high batting average, similar to a high average IC. Skill combined with many times at bat is tantamount to a high average IC. Great batters can't win if the game is rained out. Poor batters can't win no matter how many times they get to the plate. To win more games than its opponents, a team must play consistently throughout the year by not having prolonged slumps, analogous to a low standard deviation of IC. In order to do this, the players must complement each other: when some are not playing well, others are there to pick up the slack, similar to a diversifying set of alpha factors. To win a division title, a team must play a lot of games, and players' time at the plate is high. The best team is expected to always win the division, but the play-off could be a toss-up in a seven-game series.

Alpha can also be allusive, and today's alpha could be gone tomorrow or reclassified as beta in the future. However, one thing is constant: investors such as institutional fiduciaries, pension funds, endowments, and the like, will continue to pursue risk-adjusted alpha through active equity management. It might be that the latest surge of formal quantitative investing has, in part, ushered in better metrics for "separating alpha from beta" and therefore led to a higher level of general understanding of the difference. It is our hope that this book can contribute to that pursuit by presenting investors and researchers the best practice of quantitative equity investing and what it takes to be successful in the search for alpha.

1.5 THE CHAPTERS

The rest of the book consists of 3 parts with 11 chapters. Part I lays the basics of MPT framework. We present the modern portfolio theory from Markowitz through the CAPM and introduce some applications in Chapter 2. In Chapter 3, we develop modern risk models to include APT, fundamental factor models, and macroeconomic risk models, with emphasis on how these are used in quantitative portfolio management.

In Part II, we have 4 chapters devoted to the development and implementation of quantitative factors that form the bases for security selection. Chapter 4 introduces the typical objective functions of IR and Sharpe ratio, with a focus on cross-sectional estimation of the predictive power of factors, represented by average information coefficient, and the inherent risks of alpha strategies, represented by the standard deviation of IC. Chapter 5 focuses on the broad set of factors that academics and practitioners have researched over the last decade. We outline their economic and behavior intuition and analyze their efficacy through the framework developed in Chapter 4. Chapter 6 devotes attention to firm valuation based on the discount cash flow method. It extends the one-path-one-value approach to a multipath approach, which gives rise to measures of confidence around the fair-value estimation. Lastly, Chapter 7 presents mathematical frameworks for constructing multifactor models, with a focus on exploiting the diversification benefit among factors and maximizing information ratio.

Part III, the final section, puts it all together with a series of advanced implementation issues. These include Chapter 8, portfolio turnover and alpha integration; Chapter 9, advanced alpha modeling techniques to account for security context and nonlinear patterns; Chapter 10, dynamic factor timing; Chapter 11, dealing with real-world portfolio constraints optimally; and lastly, Chapter 12, incorporating transactions costs in the comprehensive optimal strategy.

Although we have tried to blend theoretical analyses and empirical examinations throughout the book, each chapter tends to have either a theoretical or empirical focus. Chapters with more analytical focus are 2, 3, 4, 7, 8, 11, and 12. Chapters with more empirical emphasis are 5, 6, 9, and 10.

APPENDIX: PSYCHOLOGY AND BEHAVIOR FINANCE

The literature on behavior finance has exploded in recent years, much of it goes beyond the scope of the book. However, it is important for readers to have some basic understanding of its tenets, which will provide some insight into materials in the later chapters.

A1.1 ADVANCES IN PSYCHOLOGY

In the 1960s, cognitive psychology began to describe the brain as an information processing device, as opposed to a stimulus–response machine. Psychologists such as Ward Edwards, Duncan Luce, Amos Tversky, and Daniel Kahneman began to explore cognitive models of decision-making under uncertainty and to benchmark their models against neoclassical economic models of rational behavior. Their works had far-reaching impact on finance as well as many other fields, such as economics, political science, and consumer behavior. Kahneman and Tversky (1979) wrote the seminal paper, "Prospect theory: Decision making under risk," which detailed an alternative model of choice under uncertainty — prospect theory — in contrast to the expected utility theory from Von Neumann and Morgenstern (1944). Prospect theory provided explanations for a number of documented anomalies beyond the capabilities of the expected utility theory. They also articulated the difference between a normative model, such as the expected utility theory, and a descriptive model such as their prospect theory. Kahneman and Tversky (1984) noted, "The normative analysis is concerned with the nature of rationality and the logic of decision making. The descriptive analysis, in contrast, is concerned with people's beliefs and preferences as they are, not as they should be." Their later work regarded the framing of decisions. Kahneman and Tversky (1986) articulated four normative rules underlying the expected utility theory: cancellation, transitivity, dominance, and invariance. They noted, "Because these rules are normatively essential but descriptively invalid, no theory of choice can be both normatively adequate and descriptively accurate."

A1.2 BEHAVIORAL FINANCE

Behavioral finance flourished in the 1990s. Its research integrates insights from psychology with neoclassical economic theory, with a foundation rooted in alternative views that question the assumption of rational agents (homo-economicus) and the notion of riskless arbitrage. Historically, fundamental equity investing came into vogue in the last half century. Demand for fundamental research attracted interests in three research areas within the accounting discipline, including fundamental analysis, accounting-based valuation, and value relevance of financial reporting.

After years of unsatisfactory efforts to explain market anomalies by efficient market theorists, behavioral economists took an alternative approach to challenge two key tenets of equilibrium pricing models: (1) arbitrage activity eliminates pricing discrepancies completely and (2) investors behave rationally. A series of papers, known as "Limits to Arbitrage," showed that irrationality can have a substantial and long-lived impact on prices, and they provided a differing view from Friedman's (1953) classical arbitrage argument. In essence, this literature argued that the arbitrage strategy designed to correct mispricing can be both risky and costly, rendering it unattractive. On an intuitive level, risk simply comes from the imperfection of the substitution, thus exposing the arbitrageur to fundamental risk. On a more sophisticated level, the arbitrageur also faces the noise trader risk. Shleifer (2000) argued that irrationality is to some extent unpredictable, and it is plausible for today's mispricing to become even more extreme tomorrow. In other words, convergence of price dislocation is not a certainty. Hirshleifer (2001) argued that pricing equilibrium reflects the beliefs of both rational and irrational traders. Because each group has a risk-bearing capacity, both influence security prices. The years of 1999 and 2000 are salient reminders, as many value shops went out of business when the market became more and more irrational. Experimental psychology documented a long list of behavioral biases of investors when making decisions under risk. Hirshleifer (2001) argued that heuristic simplification, self-deception, and emotional loss of control provide a unified explanation for most biases.

Heuristic simplification: Kahneman and Riepe (1998) dubbed heuristic simplification as biases of preference. The premise of this bias lies in the fact that humans have limited time, attention, memory, and processing capacity in tackling information and making decisions. As such, problem solving is simplified to a rules-of-thumb or heuristic approach. Commonly cited behavioral anomalies include narrow framing, mental accounting, loss aversion, and representativeness heuristic.

Self-deception: Kahneman and Riepe (1998) referred to it as biases of judgment. Overconfidence, optimism, and biased self-attribution are the three major cognitive illusions, wherein perceptions deviate, sometimes significantly, from reality. Overconfidence relates to the observation that humans are poor judges of probability and that their predictions tend to fail more often than they expect.

Optimism means that people display unrealistically rosy views of their own abilities and underestimate the likelihood of bad outcomes over which they have no control. Biased self-attribution is that phenomenon in which people attribute success to skill and failure to bad luck. Kahneman and Riepe (1998) noted, "The combination of overconfidence and optimism is a potent brew, which causes people to overestimate their knowledge, underestimate risks, and exaggerate their ability to control events."

Emotions and self-control: Hirshleifer (2001) posited that emotion could overpower reason. For example, people who are in good moods are more optimistic in their choices.

A1.3 BEHAVIORAL MODELS

Three behavioral models, shown in Table 1.1, provide an integrated explanation of several cross-sectional pricing anomalies, including short-term price momentum (Jegadeesh 1993), long-term reversal of price momentum (DeBondt and Thaler 1985), excess volatility (Shiller 1981), earnings announcement drift (Ball and Brown 1968), earnings revision (Givoly and Lakonishok 1979), analyst recommendations (Womack 1996), and the value premium.

1. Daniel, Hirshleifer, and Subrahmanyam (DHS) (1998) assume that investors are overconfident about their private information, and their overconfidence increases gradually with the arrival of public information with biased self-attribution. The pattern of increased confidence leads to a prediction of the return pattern, manifested in short-run positive autocorrelation and long-run negative autocorrelation. Specifically, overconfidence induces overreaction, which pushes prices beyond the underlying fundamentals when information is positive, and below the fundamentals when negative. Such over- or underpricing is eventually eliminated as price reverts back to fundamental, thus resulting in long-term return reversal. Short-term return continuation is traced to the progressive nature of the increased overconfidence, largely due to biased self-attribution. As an investor becomes more and more overconfident, he pushes the stock price further and further away from its fair value, thus giving rise to short-term momentum continuation.

TABLE 1.1 Summary of Behavioral Models

Models	Departure from EMH Assumptions	Short-Term Momentum Continuation	Long-Run Momentum Reversal	Representative Agents
HS	1. Investors are boundely rational with limited computational capacity 2. Information diffuses slowly across the population	Underreaction	Overreaction	1. News-watchers 2. Momentum traders
DHS	1. Informed investors are overconfident about their private information 2. Their overconfidence increase progressively due to biased self-attribution	Overreaction	More overreaction	1. The informed and the risk-neutral price setter 2. The uninformed and the risk-averse price taker
BSV	Investors exhibit two biases in updating their prior beliefs: conservatism and representativeness	Underreaction	Overreaction	A risk-averse investor who shifts his or her belief between two regimes: trending or reverting

2. Hong and Stein (HS) (1999) make two assumptions: (1) investors are bounded rational, meaning that they have limited intellectual capacity and that they are rational in processing only a small subset of the available information; and (2) information diffuses slowly across the population. They specify two bounded rational agents — news-watchers and momentum traders. Both are risk-averse, and their interactions set security prices. On the one hand, news-watchers exhibit similar behavior to a typical fundamental manager in practice, observe some private information, and ignore information in past and current prices. On the other hand, momentum traders condition their forecasts *only* on past price changes, and their forecast method is simple. The slow diffusion of information among news-watchers induces underreactions in the short-horizon. Underreaction leads to

positively autocorrelated returns — momentum continuation. Upon observing this predictable return pattern, momentum traders condition their forecast only on past price changes and arbitrage the profit opportunity. Arbitrage activity eventually leads to overreaction in the long-horizon, creating dislocation between price and fundamentals. The reversion of price back to fundamental is the source of long-term momentum reversal.

3. Barberis, Shleifer, and Vishny (BVS) (1998) suggest that investors exhibit two biases in updating their prior beliefs with public information: conservatism and representativeness. Conservatism (Edwards 1968) states that investors are slow to change their beliefs in the face of new evidence; representativeness heuristic (Tevrsky and Kahneman 1974) involves assessing the probability of an event by finding a "similar known" event and assuming that the probabilities will be similar, i.e., "if it walks like a duck and quacks like a duck, it must be a duck." Conservatism underweights new information and causes underreaction. For example, after a positive earnings surprise, conservatism means that the investor reacts insufficiently, creating a positive postannouncement drift. In contrast, after a series of positive surprises, representativeness causes people to extrapolate and overreact, pushing price beyond the fundamental value. This eventually results in long-term momentum reversal.

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ENDNOTES

1. Anomalies: Pricing anomalies began to appear in the literature in the 1980s. An early example is firm size. Banz (1981) and Reinganum (1981) concluded that small capitalization stocks earned higher average return than the CAPM might predict. Keim (1983) showed that much of the abnormal return to small stocks occurs in January (the "January Effect"). Similarly, the abnormal returns to cheap (value) stocks also received significant attention, starting with Basu (1983), who documented that high-earnings-yield (E/P) firms delivered positive abnormal returns. Rosenberg (1985) further showed that stocks with high book-to-market ratios outperform others as a group. In the realm of technical analysis, new momentum strategies emerged. DeBondt and Thaler (1985) identified long-term reversals of returns to both winner and loser portfolios. Jegadeesh and Titman (1993) further documented a short-term reversal (1st month after portfolio formation) and an intermediate-term momentum continuation (2nd to 12th month after portfolio formation). Ball and Brown (1968) were the first to document the postearnings-announcement drift, in which the market appears to underreact to earnings news. Givoly and Lakonishok (1979) concluded that market reaction to analysts' earnings revisions was relatively slow.
2. This work ushered in a series of other important pieces: Arrow and Debreu (1954), Savage (1954), and Samuelson (1969).

3. Academic literature also examines the effect of relaxing the assumptions of the CAPM: (1) different riskless lending and borrowing rates, (2) the inclusion of personal taxes, (3) existence of nonmarketable assets such as human capital, and (4) heterogeneity of expectations. These research projects typically examine CAPM's assumptions one at a time. The intertemporal CAPM (ICAPM) was devised to extend CAPM into multiperiod to discover other sources of risk that may be priced in the equilibrium. They included aggregate consumption growth (Breeden 1979), inflation risk (Friend 1976), or other sources of risk concerning investors in general (Merton 1971, 1973) beyond the movement of the market portfolio, such as default risk or term structure risk that are generally related to business cycles.

Part I

CHAPTER 2

Portfolio Theory

THE TRADITIONAL OBJECTIVE OF ACTIVE PORTFOLIO MANAGEMENT is to consistently deliver excess return against a benchmark index with a given amount of risk. The benchmark in question could be one of the traditional market indices, such as the Standard & Poor's (S&P) 500 Index and the Russell 2000 Index, or a cash return, such as Treasury bill rate, or LIBOR, in the case of market-neutral hedge funds. To be successful, quantitative equity managers must rely on four key components to their investment process. First and foremost on the list is an alpha model, which predicts the relative returns of stocks within a specified investment. The second component is a risk model that estimates the risks of individual stocks and the return correlations among different stocks. The third piece is a portfolio construction methodology to combine both return forecasts and risk forecasts to form an optimal portfolio. Lastly, one must have the portfolio implementation process in place to execute the trades. We present the portfolio construction methodology in this chapter. Risk models, alpha models, and portfolio implementations are introduced in later chapters.

Ever since the seminal work by Markowitz (1959), the mean–variance optimization has served as the workhorse for many areas of quantitative finance, including asset allocation, equity, and fixed income portfolio management. It finds the appropriate portfolio weights by solving an optimization problem. There could be several versions of this optimization: one to maximize expected portfolio return for a given level of risk, and another to minimize portfolio variance for a required expected return. Yet another version is to maximize an objective function, that is, the expected portfolio return minus a multiple (risk-aversion parameter) of the portfolio variance. Despite some of its shortcomings, one of them being the sensitivity of optimal weights to the inputs (noted by practitioners over the years), and many variants of portfolio construction methods aimed to overcome these shortcomings, the mean–variance optimization remains a core tenet of modern portfolio management. A firm understanding of the method and its intuition is thus essential to the understanding and successful implementation of quantitative investment strategies.

We shall first introduce the basic assumptions in the mean–variance optimization. We then present the mathematical analysis for the procedure, deriving the optimal portfolio and analyzing its implications. We shall form the portfolio with minimal constraints in order to derive an analytic solution, allowing us to develop insights and intuitions that might otherwise be obscured in numerical simulations. We analyze two versions of the mean–variance optimization: one for total risk and total return, and the other for active risk and active return. The latter version can be used for both an active portfolio managed against a traditional benchmark and long-short hedge funds.

In this chapter, we also introduce the capital asset pricing model (CAPM) as a risk model and consider optimal portfolios with a beta-neutral constraint as well as a dollar neutral constraint. These portfolios can be obtained by solving a constrained mean–variance optimization or by finding a linear combination of characteristic portfolios.

2.1 DISTRIBUTIONS OF INVESTMENT RETURNS

Return and risk are two inherent characteristics of any investment. The limiting case being cash, which is risk free — devoid of uncertainty — in the short term. The return of an uncertain investment is best described by a probability distribution. One of the most challenging tasks in quantitative finance is to select a type of distribution function that adequately models a given investment instrument and yet is amendable to mathematical analysis. For stocks, the simplest choice is either a normal or lognormal distribution, both of which have their advantages and disadvantages.

A normal distribution, describing the return of a stock over the next time period, can be denoted by $r \sim N\left(\mu,\sigma^2\right)$, where μ is the average or expected return and σ is the standard deviation. The term σ^2 is the variance. The most attractive feature of modeling security return with normal distribution is that the return distribution of a portfolio investing in a number of stocks would also be normal. First, we denote the joint return distribution of multiple stocks as a multivariate normal distribution

$\mathbf{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\mathbf{r} = (r_1, \cdots, r_N)'$ is the return vector, $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_N)'$ is the expected return vector, and $\boldsymbol{\Sigma} = (\sigma_{ij})_{i,j=1}^{N}$ is the covariance matrix among returns of different stocks. The covariance matrix is symmetric with $\sigma_{ij} = \sigma_{ji}$ and positive definite. If we denote the portfolio weights by the weight vector $\mathbf{w} = (w_1, \cdots, w_N)'$, then the portfolio returns distribution is

$$r_p \sim N(\mathbf{w}' \cdot \boldsymbol{\mu}, \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}). \tag{2.1}$$

Therefore, the portfolio expected return is a weighted average of individual expected returns, and the portfolio return variance is a quadratic function of the weight vector.

Several features of the normal distribution are undesirable or unrealistic when it is used to model stock returns. First, a stock investor has only limited liability — he could not lose more than what he invested in. Therefore, the return of a stock over any time horizon should never be less than –100%. But a normal distribution assigns nonzero probability to losses of any size, even those exceeding –100%. Second, if we assume that a single-period return for a stock is normal, the compound return over multiple periods is no longer normal. This can be illustrated with an example for just two periods. If the return for the first period is r_1 and for the second period is r_2, the compound return over the two periods is $r = (1+r_1)(1+r_2) - 1 = r_1 + r_2 + r_1 r_2$. The compound return consists of the sum of two individual period returns and their product. Because the product of two normal variables is not normal, the compound return is not normal. However, note the following remark:

- There are other drawbacks in using a normal distribution to model stocks and returns. The normal distribution is symmetric, whereas in reality, returns exhibit skewness and often have fatter tails (higher probabilities of a large loss or gain) than a normal distribution.

Some of these issues are negated if we use a lognormal distribution for stock returns, i.e., $\ln(1+r)$ obeys a normal distribution function. The lognormal distribution not only eliminates the possibility of return being less then –100% but also assures that the compound return over multiple time periods is also lognormal. Unfortunately, we know that a linear combination of lognormal variables is not lognormal. Therefore, portfolio returns will not be lognormal even if individual stock returns are. This makes it difficult for us to use lognormal distributions in portfolio analysis. Therefore, although we are aware of some of its limitations, we will use the normal distribution function to model stock returns throughout this book.

2.1.1 Correlation Coefficient and Diversification

The concept of diversification refers to the fact that the total risk of a portfolio is often less than the sum of all its parts. Diversification arises when the returns among different stocks are not perfectly correlated.

The correlation coefficient between two stocks relates to their covariance and standard deviations by

$$\rho_{1,2} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} . \tag{2.2}$$

It is known that $\left|\rho_{1,2}\right| \leq 1$. When given the covariance matrix $\boldsymbol{\Sigma} = \left(\sigma_{ij}\right)_{i,j=1}^{N}$, the standard deviations $\left(\sigma_1, \cdots, \sigma_N\right)$ are the square roots of its diagonal elements. The equivalent of (2.2) in the matrix form gives the correlation matrix of N assets:

$$\mathbf{C} = \operatorname{diag}\left(\sigma_1^{-1}, \cdots, \sigma_N^{-1}\right) \boldsymbol{\Sigma} \operatorname{diag}\left(\sigma_1^{-1}, \cdots, \sigma_N^{-1}\right) . \tag{2.3}$$

In Equation 2.3, $\operatorname{diag}\left(\sigma_1^{-1}, \cdots, \sigma_N^{-1}\right)$ denotes a diagonal matrix with $\left(\sigma_1^{-1}, \cdots, \sigma_N^{-1}\right)$ as diagonal elements and zero elsewhere.

Example 2.1

Before we delve into any mathematical analysis, we first consider a simple hypothetical example to illustrate the benefit of diversification. Imagine two stocks A and B, both priced at \$1. Stock A goes up 100% to \$2 in the first month, and then goes down 50% and back to \$1 again in the second month. Stock B does the opposite, down 50% in the first month and then up 100% in the second month. In this hypothetical case, the two stocks have a correlation of −1. Now, if we have invested in either stock, we would have gone nowhere with our investments after two turbulent months. However, if we had invested in both stocks with a 50/50 split and *rebalanced* the mix back to 50/50 after the first month, we would have grown our investment by 56.25% after the 2 months.

It is informative to analyze the diversification benefit of a portfolio of just two stocks. The total portfolio variance is then

$$\sigma_p^2 = w_1^2\sigma_1^2 + 2\rho_{1,2}w_1w_2\sigma_1\sigma_2 + w_2^2\sigma_2^2. \tag{2.4}$$

It is easy to see that when both weights are nonnegative,

$$\sigma_p = \begin{cases} w_1\sigma_1 + w_2\sigma_2 & \text{if } \rho_{1,2} = 1 \\ \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2} & \text{if } \rho_{1,2} = 0 \\ \left|w_1\sigma_1 - w_2\sigma_2\right| & \text{if } \rho_{1,2} = -1 \end{cases}. \tag{2.5}$$

At one extreme, when the correlation is 1, the portfolio volatility is the weighted sum of two stock volatilities, and there is no diversification benefit. At the other extreme, when the correlation is –1, the portfolio volatility is the absolute difference of the two, and the diversification is at the maximum. When the correlation is 0, the portfolio volatility is between the two extremes. In this case, the variances are additive instead.

Example 2.2

For a portfolio of N stocks, assume each has the same return standard deviation denoted by σ. Further assume the returns are uncorrelated, and the portfolio return standard deviation is then

$$\sigma_p = \sqrt{\sum_{i=1}^{N} w_i^2\sigma^2} = \sigma\sqrt{\sum_{i=1}^{N} w_i^2}. \tag{2.6}$$

For an equally weighted portfolio, $\sigma_p = \sigma/\sqrt{N}$, the risk declines as the square root of *N*.

We have just seen how the portfolio variance changes with the correlation. It is also instructive to see how it changes when the underlying security weights change. Still using the stock example, we require $w_1 + w_2 = 1$. In other words, the portfolio is fully invested in the two risky securities under consideration. Figure 2.1 displays the variance as a function of w_1 with $\sigma_1 = 40\%$, $\sigma_2 = 30\%$, and $\rho_{1,2} = 0.3$. In the plot, we let the weight to be both negative and greater than 100% to allow shorting of both stocks.

The portfolio variance (2.4) is a quadratic function of the weight, and it attains the minimum when

$$w_1 = \frac{\sigma_2^2 - \rho_{1,2}\sigma_1\sigma_2}{\sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2},\ w_2 = 1 - w_1. \tag{2.7}$$

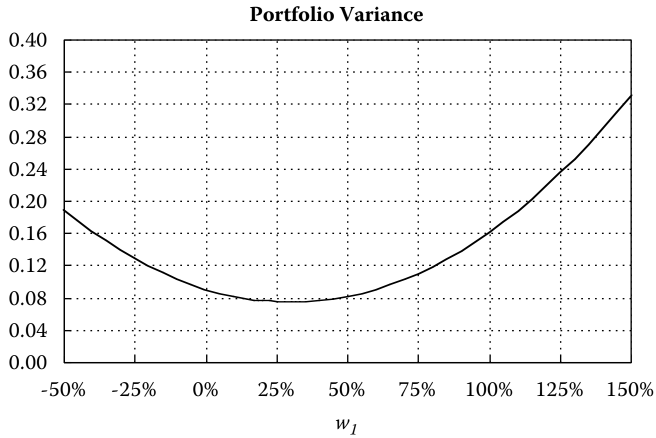

FIGURE 2.1. Portfolio variance as a function of stock weight w_1.

This is the minimum variance portfolio that has the least risk. For parameters used in Figure 2.1, the minimum occurs when w_1 = 30%, and in this case the minimum portfolio volatility is 27%, smaller than either of the individual volatilities.

2.2 OPTIMAL PORTFOLIOS

In this section, we shall derive various optimal portfolios with different objective functions.

2.2.1 Minimum Variance Portfolio

Suppose there are N stocks in the investmentable universe and we have a fully invested portfolio investing 100% of the capital. The covariance matrix is denoted as $\mathbf{\Sigma}$. We are interested in finding the portfolio with minimum variance. An investor choosing this portfolio is only concerned about the risk of the portfolio. Denoting a vector of ones by $\mathbf{i}=\left(1,\cdots,1\right)'$, we have the following optimization problem:

$$\begin{gathered}\text{Minimize } \frac{1}{2}\mathbf{w}'\mathbf{\Sigma}\mathbf{w} \\ \text{subject to: } \mathbf{w}'\cdot\mathbf{i}=w_1+w_2+\cdots+w_N=1.\end{gathered} \tag{2.8}$$

The constraint in (2.8) is often referred to as a *budget constraint*. The fraction one half is merely a scaling constant, and the reason for including it will soon be apparent. The problem can be solved by the method of Lagrangian multipliers. We form a new objective function

$$Q(\mathbf{w},l)=\frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}-l(\mathbf{w}'\cdot\mathbf{i}-1). \tag{2.9}$$

The additional term in (2.9) is the Lagrangian multiplier times a constraint-related term. Taking the partial derivative of the new function with respect to the weight vector and equating it to zero yields the condition for the optimal weight

$$\boldsymbol{\Sigma}\mathbf{w}-l\mathbf{i}=0 \tag{2.10}$$

and solving for the weight vector gives

$$\mathbf{w}=l\boldsymbol{\Sigma}^{-1}\mathbf{i}, \tag{2.11}$$

where $\boldsymbol{\Sigma}^{-1}$ is the inverse matrix of $\boldsymbol{\Sigma}$. To determine the Lagrangian multiplier l, we substitute the weight vector into the constraint in Equation 2.8 to obtain

$$l=\frac{1}{\left(\mathbf{i}'\boldsymbol{\Sigma}^{-1}\mathbf{i}\right)}. \tag{2.12}$$

Finally, substituting Equation 2.12 into Equation 2.11 yields the minimum variance portfolio weight vector

$$\mathbf{w}^{*}_{\min}=\frac{\boldsymbol{\Sigma}^{-1}\mathbf{i}}{\mathbf{i}'\boldsymbol{\Sigma}^{-1}\mathbf{i}}. \tag{2.13}$$

It is easy to verify that the optimal weight (2.13) satisfies the budget constraint. Finally, the minimum variance is

$$\sigma^2_{\min}=\left(\mathbf{w}^{*}_{\min}\right)'\boldsymbol{\Sigma}\mathbf{w}^{*}_{\min}=\frac{1}{\mathbf{i}'\boldsymbol{\Sigma}^{-1}\mathbf{i}}, \tag{2.14}$$

equal to the Lagrangian multiplier (2.12).

2.2.2 Mean–Variance Optimal Portfolio with Cash

The minimum variance portfolio focuses solely on the risk and ignores the expected return of the portfolio. Most investors prefer a balance between the two, provided they have return expectation for stocks. The mean–variance optimization serves as the main tool for finding the optimal portfolio with the maximum expected return for a given level of risk. We first consider portfolios that include cash and denote its return by r_f and its weight by w_0. We denote the expected return vector of N stocks by $\mathbf{f} = \left(f_1, \cdots, f_N\right)'$, which is a collection of forecasts generated by investors through investment research. For the time being, we take these forecasted returns as given inputs. In Part II of this book, we will identify some quantitative factors for forecasting stock returns. The mean–variance optimal portfolio with a risk-aversion parameter λ is

$$\begin{aligned} &\text{Maximize } w_0 r_f + \mathbf{w}' \cdot \mathbf{f} - \frac{1}{2}\lambda\left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\right) \\ &\text{subject to: } w_0 + \mathbf{w}' \cdot \mathbf{i} = 1 \end{aligned} \tag{2.15}$$

Note that cash is risk free — it only contributes to return but has no risk, at least for a single-period optimization. The risk-aversion parameter $\lambda > 0$ determines the degree of influence that risk has on the portfolio. If $\lambda = 0$, then the risk term drops out and the problem reduces to maximizing expected return under the assumed budget constraint. The solution is generally unbounded because one can borrow unlimited amount from the low-return asset and invest that sum in the higher return asset. On the other hand, if $\lambda \to \infty$, (meaning the investor is extremely risk averse and), then the optimal portfolio would have 100% in cash and have no risk at all.

The problem (2.15) can be converted into an unconstrained optimization problem for the stock weights by using the constraint in the objective function. Writing the constraint as $w_0 = 1 - \mathbf{w}' \cdot \mathbf{i}$ and substituting it into the objective function yields

$$\text{Maximize } \mathbf{w}' \cdot \mathbf{f}_e - \frac{1}{2}\lambda\left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\right), \quad \text{with } \mathbf{f}_e = \mathbf{f} - r_f \mathbf{i}. \tag{2.16}$$

The vector $\mathbf{f}_e$ represents the stocks' excess returns above cash. The optimal weights are found by equating partial derivatives of the objective function (2.16) to zero. We have