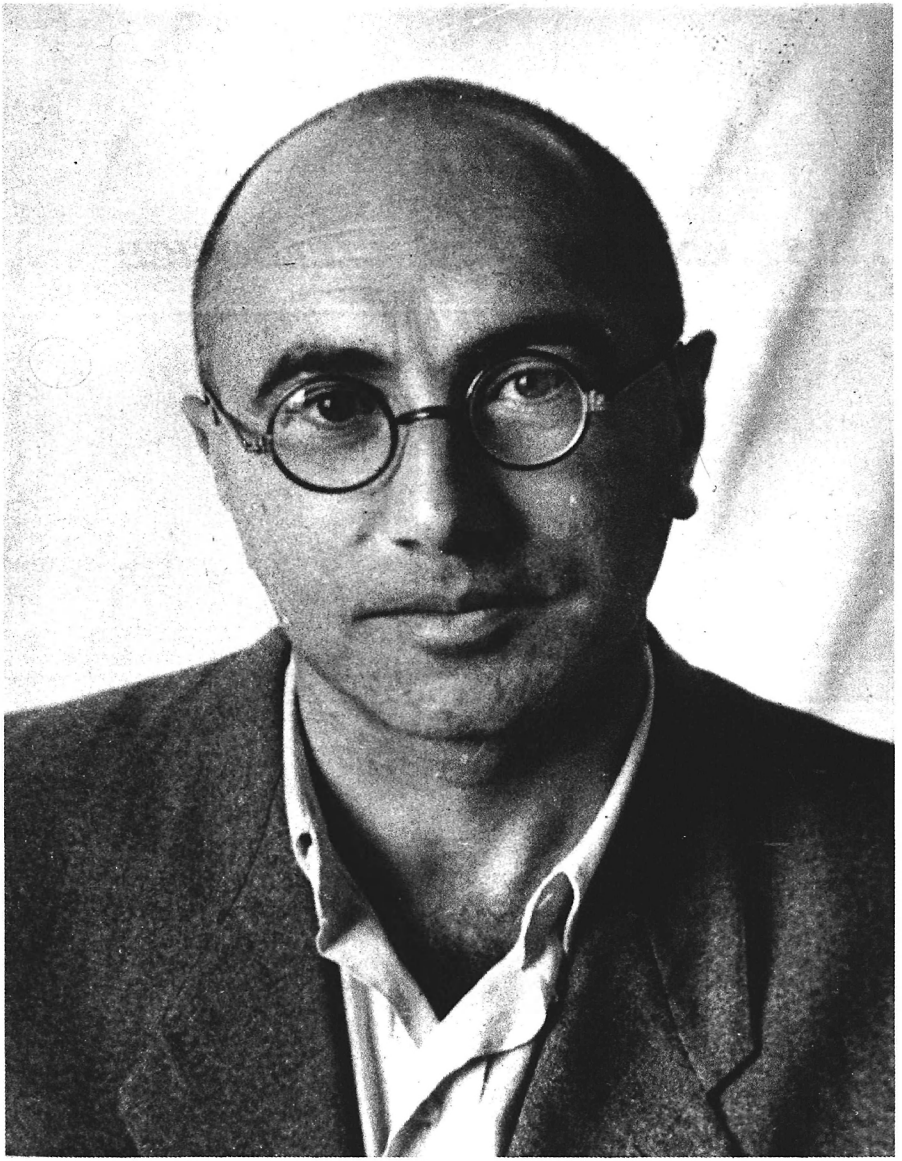
YAKOV BORISOVICH ZELDOVICH

# Selected Works of Yakov Borisovich Zeldovich, Volume II

*Particles, Nuclei, and the Universe*

PRINCETON LEGACY LIBRARY

# *SELECTED WORKS OF YAKOV BORISOVICH ZELDOVICH*



Ya. B. Zeldovich, ca. 1956

# *SELECTED WORKS OF YAKOV BORISOVICH ZELDOVICH*

Editor of English Edition
*J. P. Ostriker*

Coeditors of English Edition
*G. I. Barenblatt*
*R. A. Sunyaev*

Technical Supervisor of English Edition
*E. Jackson*

Translators of English Edition
*A. Granik*
*E. Jackson*

VOLUME II
*PARTICLES, NUCLEI, AND THE UNIVERSE*

PRINCETON UNIVERSITY PRESS
PRINCETON, NEW JERSEY

Published by Princeton University Press, 41 William Street,
Princeton, New Jersey 08540
In the United Kingdom Princeton University Press, Chichester, West Sussex



**Library of Congress Cataloging-in-Publication Data**

Zel'dovich, I͡A. B. (I͡Akov Borisovich)
[Chastit͡sy, i͡adra, vselennai͡a. English]
Particles, nuclei, and the universe / editor of English edition J. P. Ostriker; coeditors of English edition G. I. Barenblatt, R. A. Sunyaev; technical supervisor of English edition E. Jackson; translators of English edition A. Granik, E. Jackson.
p. cm. -- (Selected works of Yakov Borisovich Zeldovich; v. 2)
Translation of: Chastit͡sy, i͡adra, vselennai͡a.
Includes bibliographical references.
ISBN 0-691-08742-3
1. Nuclear physics. 2. Particles (Nuclear physics). 3 Nuclear astrophysics. 4. Cosmology. I. Ostriker, J. P. II. Barenblatt, G. I. III. Sunyaev, R. A. IV. Title. V. Series: Zel'dovich, I͡A. B. (I͡Akov Borisovich). Selections. English. 1992 ; v. 2
QC3.Z44213 1993 vol. 2
[QC776]
530 s--dc20
[539.7'2] 91-14814

*Selected Works of Yakov Borisovich Zeldovich*, Volume I: *Chemical Physics and Hydrodynamics*, was published in 1992

Both Volumes I and II were composed in Computer Modern Roman using the $\TeX$ typesetting system

Princeton University Press books are printed on acid-free paper, and meet the guidelines for permanence and durability of the Committee on Production Guidelines for Book Longevity of the Council on Library Resources

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

Princeton Legacy Library edition 2017
Paperback ISBN: 978-0-691-60047-5
Hardcover ISBN: 978-0-691-62994-0

# Contents

## *III. Atomic Physics and Radiation*



## Part Two
## Astrophysics and Cosmology

## *IV. Elementary Particles and Cosmology*



## *V. General Theory of Relativity and Astrophysics*

## *VI. Neutron Stars and Black Holes. Accretion.*



## *VII. Interaction of Matter and Radiation in the Universe*



## *VIII. Formation of the Large-Scale Structure of the Universe*

## *IX. Observational Effects in Cosmology*



## Part Three
## The History of Physics. Personalia

# Preface to the English Edition of the Selected Works of Ya. B. Zeldovich

There has been no physical scientist in the second half of the twentieth century whose work shows the scope and depth of the late Yakov Borisovich Zeldovich. Born in Minsk in 1914, he was the author of over 20 books and over 500 scientific articles on subjects ranging from chemical catalysis to large-scale cosmic structure, with major contributions to the theory of combustion and hydrodynamics of explosive phenomena. His passing in Moscow in December of 1987 was mourned by scientists everywhere. To quote Professor John Bahcall of the Institute for Advanced Study: "We were enriched in Princeton as in the rest of the world by his insightful mastery of physical phenomena on all scales. All of us were his students, even those of us who never met him." In his range and productivity, Zeldovich was the modern equivalent of the English physicist Raleigh (1842-1919) whose name is associated with phenomena ranging from optics to engineering.

The breadth of Zeldovich's genius (characterized as "probably unique" by the Soviet physicist Andrei Sakharov) was alternately intimidating or enthralling to other scientists. A letter sent to Zeldovich by the Cambridge physicist Steven Hawking, after a first meeting in Moscow, compares Zeldovich to a famous school of pre-war mathematicians who wrote under a single fictitious pseudonym: "Now I know that you are a real person, and not a group of scientists like Bourbaki."

No selection from an opus of such scope can capture its full range and vigor. While basing ourselves primarily on the Russian edition, published by the Soviet Academy of Sciences in 1984–1985, we were delayed repeatedly as important and hitherto untranslated (but frequently cited) papers were brought to our attention as clearly warranting inclusion. Zeldovich played a major role in re-editing the Russian edition before translation and in choosing additional material for the present work. All told, this edition is approximately 15% longer than the Russian edition and the second volume contains one largely new section: *The History of Physics. Personalia*, including impressions of Einstein and Landau, and ending with *An Autobiographical Afterword.*

Because he wrote in Russian during a period when relations between that culture and the western world were at an historically low ebb, international recognition for Zeldovich's achievements were slower to arrive than merited. Within the Soviet Union his accomplishments were very well recognized, in part, due to his major contributions to secret wartime work. As the world's

leading expert on combustion and detonation, he had naturally been drafted early into the effort for national survival. He had written entirely prescient papers in 1939 and 1940 (included in this volume) on the theoretical possibility of chain reactions among certain isotopes of uranium. The physicist Andrei Sakharov wrote that "from the very beginning of Soviet work on the atomic (and later the thermonuclear) problem, Zeldovich was at the very epicenter of events. His role there was completely exceptional." Zeldovich was intensely proud of his contributions to the wartime Soviet scientific effort and was the most decorated Soviet scientist. His awards include the Lenin Prize, four State Prizes, and three Gold Stars.

As a corollary to internal recognition, of course, Zeldovich's scientific work was burdened by the enormous handicaps of isolation, secrecy and bureaucracy in a closed society, made more extreme for him by restrictions due to defense work. He was not permitted to attend conferences outside of the Soviet Block until August 1982 at age 68, when he delivered an invited discourse "Remarks on the Structure of the Universe" to the International Astronomical Union in Patras, Greece. When asked then by this Editor when he was last out of the Soviet Union, he answered without hesitation "sixty eight years ago," i.e., only in a prior life. Previous to that meeting, his access to preprints, normal correspondence, all of the human interchange of normal scientific life, were severely circumscribed with contacts increasing as he moved out of defense work. Then, as international relations improved, international acclaim followed. Elected in 1979 as a foreign associate of the U. S. National Academy of Sciences, he had already been made a member of the Royal Society of London and other national scientific academies. Despite having turned relatively late in his scientific career to astrophysics, his accumulated achievements in that area, rewarded with the Robertson Prize of the U. S. National Academy of Sciences for advances in cosmology, put him among the world's leading theoretical astrophysicists.

The science is of course more interesting than the honors it wins for the scientist. Let me note just two items from astrophysics, my own specialty, where Zeldovich showed extraordinary vision and imagination. He argued shortly after their discovery that quasars were accreting black holes, and that the universe was likely to have a large-scale porous structure, anticipating in both cases the standard paradigms for interpreting these cosmic phenomena. In addition, he was among the first to realize that the early universe could be used as our laboratory for very high energy physics, leaving as fossils strange particles and cosmic microwave background fluctuations.

If the matter is more important than the recognition, it was also true, for Zeldovich in particular, that the manner was as significant as the matter. He always proceeded by a direct intuitive *physical* approach to problems. Even in areas where his ultimate accomplishment was a mathematical formulation adopted by others such as the "Zeldovich number" in combustion theory

or the "Zeldovich spectrum" and the "Zeldovich approximation" to linear perturbations in cosmology, the reasoning and approach are initially and ultimately physical and intuitive. His view was that if you cannot explain an idea to a bright high school student, then you do not understand it. He backed up this conviction, and his interest in the education of young scientists, with the book *Higher Mathematics for Beginners*, which presented in a clear and intuitive way the elementary mathematical tools needed for modern science. Here again Zeldovich was in good company; from Einstein to Feynman, the greatest physicists have felt that they could and *should* make clear to anyone who cared to listen, the excitement of modern science.

The value of Zeldovich's papers, unlike those of most scientists, has outlived the novelty of his results. But, inevitably, one must question the logic of republishing scientific papers. Is not all valid scientific work included in and superseded by later work. Of course there is a value in collecting, for the record, in one place the major works of a truly great scientist. The fact that we include with each paper, commentaries (often revised from the Soviet edition) by the author on the significance of these papers will further enhance their value to historians and philosophers of science. But Zeldovich was almost above all else the teacher, the founder of a school of today's world famous scientists and author of widely read texts at all levels. He had strong views on *how* science should be done and how it should be taught. To him, the "how" of the scientific method, of his own scientific method, was central; it was what he most wanted to communicate in making his work available to a broader audience.

We are happy to be able to provide a complete enlarged edition of the works of this great scientist for the English-speaking world. We would like to thank the Academy of Sciences of the USSR for permission to utilize (a) *Izbrannye Trudy: I. Khimicheskaĭa Fizika i Gidrodinamika* and (b) *II. Chastitsy, Iadra, Vselennaĭa*, but especially offer our thanks to Professors G. I. Barenblatt and R. A. Sunyaev for their dedication and expertise in closely reading (Volumes One and Two, respectively) the entire manuscript in its English edition.

J. P. Ostriker
19 January 1990

# Acknowledgements

The editors acknowledge with thanks permission from the following owners of copyrights, both of translations, some of which have been modified at the request of the scientific editors of this edition, and of original English-language publications.

*Acta Physicochimica*

*Akusticheskiĭ journal*

American Institute of Physics

*Annual Review of Fluid Mechanics*

*Astronomy and Astrophysics*

*Astrophysics and Space Science*

Blackwell Scientific Publications, Ltd., *Royal Astronomical Society of London Monthly Notices*

Cambridge University Press

*Combustion and Flame*

*Communications in Mathematical Physics*

Editrice Compositori

Elsevier Science Publishing Co., Inc.

*Fizika goreniĭa i vzryva*

W. H. Freeman and Co.

Gordon and Breach Science Publishers, Inc.

*Journal of Fluid Mechanics*

National Academy of Sciences, *Proceedings of the National Academy of Sciences*

*National Advisory Comm. Aeronautics, Techn. Mem.*

North-Holland Physics Publishing

Professor E. S. Phinney

*Physikalische Zeitschrift der Sowjetunion*

Reidel Publishing Company, "The Theory of Large-Scale Structure of the Universe" by Ya. B. Zeldovich, in *Large-Scale Structure of the Universe*, D. Reidel, Dordrecht, pp. 409–421. Copyright 1978 by International Astronomical Union. Reproduced by permission of Kluwer Academic Publishers.

*Second international colloquium on explosion and reacting systems gasdynamics*

Società Italiana di Fisica

Springer-Verlag

*Supplemento al Nuevo Cimento*

# Я. Б. ЗЕЛЬДОВИЧ

## Избранные труды

# ЧАСТИЦЫ, ЯДРА, ВСЕЛЕННАЯ

Под редакцией
академика Ю. Б. ХАРИТОНА

МОСКВА
«НАУКА»
1985

# PART ONE

# PARTICLES AND NUCLEI

I. NUCLEAR PHYSICS

II. THEORY OF ELEMENTARY PARTICLES

III. ATOMIC PHYSICS AND RADIATION

# I

# Nuclear Physics

## 1

## On the Problem of the Chain Decay of the Main Uranium Isotope*

*With Yu. B. Khariton*

In this work we consider the problem of the moderation of neutrons which form in uranium decay and of the conditions necessary for the chain decay of uranium.

For the chain decay of the main uranium isotope to be possible it is essential that the neutrons which form in the fission of the uranium atom manage with sufficient probability to induce the next decay event, not only before they leave the mass of the uranium involved in the decay [1], but also before they are slowed to an energy below which they are no longer able to induce decay of the main isotope.

In the present note we consider precisely this last problem. If we find a probability $\gamma$ that neutrons forming with energy $E_0$ without absorption accompanying the act of decay are slowed to the energy $E_k$, below which the decay of the main isotope can no longer be induced, then under optimal conditions of a maximal mass of uranium the probability of a chain reaction will be determined by the inequality

$$\nu(1-\gamma) > 1, \tag{1}$$

where $\nu$ is the (average) number of neutrons arising for one neutron captured in the energy interval $E_0 - E_k$, and $\gamma$ is the probability that a neutron is slowed without being absorbed (equivalent to breaking of the chain).

*Zhurnal eksperimentalnoĭ i teoreticheskoĭ fiziki **9** 12, 1425–1427 (1939).

Below we carry out a calculation which accounts only for elastic neutron scattering.

The conditions found under this assumption are necessary for an explosion to occur, but may not be sufficient due to the presence of inelastic collisions.

Let us find the quantity $\gamma$. To do this we simultaneously consider the equation determining the variation in the mean energy of the particles in scattering and the equation for the variation in the number of particles due to their absorption.

Restricting ourselves, as we have said, to elastic collisions, we write

$$\frac{dE}{dt} = -Eu \sum \sigma_{s_i} c_i \lambda_i, \tag{2a}$$

$$\frac{dN}{dt} = -Nu \sum \sigma_{c_i} c_i, \tag{2b}$$

where $E$ and $N$ are the mean energy and the number of particles, respectively, $u$ is the velocity of the neutrons, $\sigma_{s_i}$ and $\sigma_{c_i}$ are the scattering and capture cross-sections by the nuclei of atoms of the $i$-th sort, $c_i$ is the concentration, $\lambda_i = 2m_i/(m_i + 1)^2$, where $m_i$ is the mass of the $i$-th nucleus, expressed in neutron masses. Hence

$$\frac{dN}{dt} = \frac{N \sum \sigma_{c_i} c_i}{E \sum \sigma_{s_i} c_i \lambda_i} = \frac{N\psi}{E}. \tag{3}$$

If $\psi$ is independent of the energy

$$\gamma = \frac{N_k}{N_0} = \left(\frac{E_k}{E_0}\right)^{\psi} \tag{4}$$

and in the general case

$$\gamma = \frac{N_k}{N_0} = \exp\left(\int_{E_0}^{E_k} \psi(E)\, d\ln E\right). \tag{4a}$$

In constructing the criterion of feasibility of chain reaction (1) it should be kept in mind that the number of neutrons $\nu$ entering into it is taken with respect to one neutron captured in the energy interval $E_0 - E_k$. Thus, if the number of neutrons produced in a single decay event is $\nu_f$, then we obtain the number $\nu$ entering into (1) by the formula

$$\nu = \frac{\nu_f \sigma_f c_U}{\sum \sigma_{c_i} c_i}, \tag{5}$$

where the summation in the denominator includes, as in formulas (2b) and (3), the term $\sigma_f c_U$ as well.

Let us do a concrete calculation applied to the proposed use of uranium oxide (see, for example, [1]).

We take the following values: $\nu = 1.5, 2, 3$ [2–4]; $E_0$ in the two versions of the calculation is equal to 3 and 2 MeV [5]; $E_k = 1.5$ MeV [6, 7]; $\sigma_{SO} = 2 \cdot 10^{-24}$ cm$^2$; $\sigma_{SU} = 6 \cdot 10^{-24}$ cm$^2$; $\sigma_{CO} = 0$, $\sigma_{CU} = 0.5 \cdot 10^{-24}$ cm$^2$ [7]; $c_O : c_U = 8 : 3$ for the composition of $U_3O_8$.

Substituting formula (1) into (4) we obtain the final table for the quantities $\nu(1-\gamma)$.

| $E_0$, MeV | $\nu$ | | |
|---|---|---|---|
| | 1.5 | 2 | 3 |
| 3 | 0.63 | 0.84 | 1.26 |
| 2 | 0.3 | 0.4 | 0.6 |

As we see from the table, the presence of oxygen, if not completely eliminates, then strongly inhibits the chain decay of uranium.

The situation for pure uranium is completely different: in this case, in all versions, we obtain negligible chain breaking, not exceeding $5 \cdot 10^{-3}$ (in the absence of inelastic scattering) and the feasibility of explosion.

It is clear that these considerations may also be applied to the question of the feasibility of decreasing the critical mass of uranium by surrounding it with material which slows the diffusion of neutrons to the outside [1]. As a result of neutron moderation for a large number of collisions (note that the subsequent fate of the slowed neutrons is unimportant for the chain reaction), the effectively working thickness of the neutron-isolation layer is of order $\lambda\sqrt{M/\sigma}$, where $\lambda$ is the free path and $M$ is the mass of a nucleus of the isolating material.

This last remark is related to the problem of the controllability of decay by the effect of fast neutrons: in the immediate vicinity of the explosion limit (critical conditions for development of the chain) a change by even a very small additional number of neutrons arising from the fission of the 235 isotope under the action of slow neutrons can affect the behavior of such a very sensitive system. Thus, in principle, it is possible to regulate the decay of the main isotope using the decay of the 235 isotope under the action of slow neutrons in conditions when this latter decay cannot possibly lead to explosion.

All of the above calculations were carried out under the assumption that the system is of unlimited extent, i.e., that there are no additional losses of neutrons carried from the system by diffusion.

Comparison with Perrin's calculations allows us to conclude that with the decrease of $\nu(1-\gamma)$ approaching unity, the critical size of the system grows as $[\nu(1-\gamma)-1]^{-1/2}$ and the volume as $[\nu(1-\gamma)-1]^{-3/2}$.

In contrast, for $\nu(1-\gamma) < 1$ critical conditions for chain branching cannot be achieved for any size of the system.

We note, finally, that in light of the above individual experiments which observe an increase in the number of source neutrons by 10–20% [8] in the

presence of uranium cannot yet be considered proof of the realizability of uranium chain decay. Such proof can be provided only by an increase of 5–10 times, corresponding to multiple chain branching, which requires using uranium mass of the same order as the critical mass.

*Institute of Chemical Physics*
*Leningrad*

*Received*
*October 7, 1939*

*Note added in proof.* On the basis of the theory of N. Bohr and J. Wheeler [*Bohr N., Wheeler J.*—Phys. Rev. **56**, 299 (1939)], published while the present article was in press, we carried out a calculation of chain breaking related to inelastic neutron scattering. Due to the absence of data on the energy levels of uranium 238, the calculation was done for Th C, which would appear not to introduce significant errors. The results of the calculation lead to the conclusion that even in the case of pure metallic uranium, no chain reaction apparently takes place.

## *REFERENCES*

1. *Perrin F.*—C. r. Acad. Sci. **208**, 1394 (1939).
2. *Anderson H. L., Fermi E., Szillard L.*—Phys. Rev. **56**, 284 (1939).
3. *Anderson H. L., Fermi E., Hanstein H. B.*—Phys. Rev. **55**, 797 (1939).
4. *Halban H., Joliot F., Kowarski L.*—Nature **143**, 680 (1939).
5. *Halban H., Joliot F., Kowarski L.*—Nature **143**, 939 (1939).
6. *Roberts R. B., Mayer R. C., Hafstad L. R.*—Phys. Rev. **55**, 416 (1939).
7. *Ladenburg R., Kanner M. N., Barshall H., Van Voorhis C. C.*—Phys. Rev. **56**, 168 (1939).
8. *Haenny C., Rosenberg A.*—C. r. Acad. Sci. **208**, 898 (1939).

# 2

# On the Chain Decay of Uranium Under the Action of Slow Neutrons*

*With Yu. B. Khariton*

In realizing the chain decay of uranium under the action of slow neutrons [1], it is necessary for continuation of the chain to slow neutrons created in the act of decay with an energy of several million volts to a velocity close to the thermal velocity at which they are sufficiently likely to cause the next act of decay of the isotope with atomic weight 235. In the interval between the energy of the neutrons formed and the region in which they cause decay (continue the chain) there is a region near 25 eV of resonant absorption of neutrons[1] by the basic isotope 238; this absorption does not lead to the appearance of new neutrons and is, consequently, a break in the chain just as is the absorption of neutrons by any admixtures to uranium present in the system.

However, quantitatively there is a significant difference between these two types of chain breaking. The capture cross-sections of slow neutrons by various atoms, including the cross-section for capture by the uranium nucleus which leads to its decay, vary identically with the energy of the neutrons (inversely proportional to the velocity, i.e., as $E^{-1/2}$). The distribution of neutrons among the various possible processes—absorption by various nuclei, absorption by uranium with its subsequent decay—does not depend on the energy, and therefore on the velocity, of the moderated neutrons. The probabilities of the different processes are in a constant ratio; specifically, they are proportional to the products of the number density of the nuclei participating in the process with the capture cross-section, measured (bearing in mind its dependence on the energy, i.e., as $E^{-1/2}$) for all processes at a single energy, for example at room temperature. Thus, the number of neutrons captured in one or another particular way is easily found by a formula of the form

$$N_i = \frac{N\sigma_i c_i}{\sum_l \sigma_l c_l}. \tag{1}$$

*Zhurnal eksperimentalnoĭ i teoreticheskoĭ fiziki **10** 1, 29–36 (1940).

[1]The action of slow neutrons causes decay of isotope 235 whose content in uranium is 0.7%.

The situation is quite different with resonant capture. Qualitatively it is clear that, due to the sharply different dependence of the cross-section on the energy of the neutrons (compared with the capture of slow neutrons according to the law $E^{-1/2}$), the simple formula (1), which contains neither the dependence on the initial neutron energy nor their rate of moderation, which determines the time that they spend in the dangerous zone of resonant capture, cannot correctly describe the dependence of the amount of resonance-absorbed neutrons on the number density.[2] By analogy, while all processes with cross-section $\sim E^{-1/2}$ are included in parallel, resonant capture in the region of 25 eV, where the above processes in practical terms have not yet begun, is included in series before them during the course of slowing of the neutrons.[3]

In the present note we shall attempt to establish the patterns of resonant capture and their consequences relating to chain decay on slow neutrons.

Preliminary calculations using the methods of our previous note showed that for uranium oxide or, even more so for pure uranium, the resonant absorption is extremely large and completely eliminates the possibility of a chain reaction on slow neutrons.

In order to accomplish this reaction strong slowing of the neutrons is necessary, which may be practically accomplished by the addition of a significant amount of hydrogen. Thus the conditions for the reactions on slow and fast neutrons turn out to be significantly different so that their simultaneous calculation is necessary in the closest vicinity to the critical conditions for one of the reactions, where the system is so sensitive that even minimal participation of the second reaction can change its behavior (see our note [2]).

Let us note the peculiarities of the forthcoming calculations.

1. The quite significant radius of scattering of neutrons by protons and the favorable ratio of the masses make calculation of the slowing of neutrons in scattering by the other nuclei unnecessary.

2. In each collision of a neutron with a proton the energy of the neutron after scattering varies throughout the interval from 0 to the initial energy before the collision. A neutron with an energy of 35–50 eV has a greater probability of being immediately slowed to an energy less than 25 eV, after which it cannot be resonance-absorbed. For such a strong exchange of energy the method of the previous article, in which the change in energy after collisions was considered to be a continuous process, is inapplicable.

3. Absorption (even very little) of thermal neutrons by protons restricts the possible dilution of uranium with hydrogen with the aim of accelerating

[2]Under constant conditions (above all, composition) resonant absorption can of course be approximately described by the introduction of an equivalent capture cross-section of neutrons at room temperature (Joliot [5]). However, this fictive cross-section is an unknown function of all the parameters of the system.

[3]Near 25 eV cross-sections proportional to $E^{-1/2}$ are negligibly small.

the energy exchange and decreasing resonant absorption near 25 eV. We shall have to find the optimal proportions of uranium and hydrogen taking both effects into account.

4. Finally, for concrete calculations the form of the energy distribution function of neutrons with given initial energy after one collision with a proton is extremely important.

As may be shown in a general form, the wave function of scattered particles possesses spherical symmetry;** the other wave functions vanish in the center and therefore enter with coefficients $\sim r/\lambda$, where $r$ is the radius of the nucleus and $\lambda$ is the wavelength of the neutron, so that at the energies much less than $10^6$ eV of interest to us they may be disregarded. The equal probability of all directions, independent of the angle of scattering, just as in the collision of two elastic spheres in classical mechanics, leads in the calculation of the conservation laws to a very simple energy distribution; specifically, it leads to equal probabilities of all values of the energy less than the initial value:

$$\begin{aligned} a(E)\,dE &= \frac{dE}{E_0}, \qquad 0 < E < E_0, \\ a(E)\,dE &= 0, \qquad\quad E > E_0. \end{aligned} \tag{2}$$

In the derivation of the formula we disregard the thermal energy of the scattering protons since it is quite small compared to the energy in the resonance region.

The cross-section of resonant capture in the presence of one level obeys the Breit-Wigner formula:

$$\sigma_E = \sigma_r \sqrt{\frac{E_r}{E}} \frac{(\Gamma/2)^2}{(E - E_r)^2 + (\Gamma/2)^2}. \tag{3}$$

In order to close this part of the task, we have extracted from expression (3) the term $\sigma_2$, which behaves at small $E$ as $E^{-1/2}$, so that the corresponding cross-section may be accounted for with the others in the form (1):

$$\left.\begin{aligned} &\sigma = \sigma_1 + \sigma_2;\ \sigma_1 = \sigma_r \sqrt{\frac{E_r}{E}} \left[ \frac{(\Gamma/2)^2}{(E - E_r)^2 + (\Gamma/2)^2} - \frac{(\Gamma/2)^2}{E_r^2 + (\Gamma/2)^2} \right]; \\ &\sigma_2 = \sigma_r \sqrt{\frac{E_r}{E}} \frac{(\Gamma/2)^2}{E_r^2 + (\Gamma/2)^2}. \end{aligned}\right\} \tag{3a}$$

The remaining function vanishes at $E = 0$, and from now one we will consider resonant absorption to mean only this part of the capture function.

The advantage of such a definition lies in the fact that we may now rigorously pose the question of the probability that neutrons will be slowed without suffering absorption at the resonant level, and this quantity is no longer tied to the question of the fate of thermal neutrons, as it would have

**This refers to the symmetry in the system of the center of inertia of a proton at rest and a moving neutron—*Editor's note.*

been had we not excluded from the resonance curve the capture at small energies.

The probabilities that a neutron will be resonantly captured in the next collision by uranium or scattered by hydrogen are in the ratio $\sigma_{cU}c_U : \sigma_{sH}c_H$ (we do not consider all the other processes). Let us introduce the normalized probability of capture in the next collision:

$$W = W(E) = \frac{\sigma_{cU}(E)c_U}{\sigma_{cU}(E)c_U + \sigma_{sH}c_H}. \tag{4}$$

We construct the equation for the joint probability $\varphi(E)$ that we seek for a neutron having energy $E$ to be slowed without being captured in the resonant region: in the first collision the neutron has a probability $W$ of being captured, and with probability $(1 - W)$ may be scattered with a uniform distribution of energy. Hence we obtain the integral equation

$$\varphi(E) = \frac{1 - W(E)}{E} \int_0^E \varphi(\varepsilon)\, d\varepsilon. \tag{5}$$

The equation is integrated in quadratures; to do this we take the derivative $\varphi'$ and express the integral entering into it again in terms of the quantity $\varphi$. Thus we arrive at the differential equation

$$\varphi' = \frac{-W\varphi}{E} - \frac{W'\varphi}{1 - W}, \tag{6}$$

and finally find

$$\varphi = (1 - W) \exp\left( - \int_0^E W\, d \ln E \right). \tag{7}$$

At an energy $E$ which is greater than the resonant energy, beginning from the value at which $W$ may be disregarded, $\varphi$ no longer depends on $E$ so that the desired limiting value is

$$\varphi = \exp\left( - \int_0^\infty W\, d \ln E \right). \tag{8}$$

For the simplest form of the dependence of the cross-section of resonant capture on the energy

$$\sigma_{cU}(E) = \infty, \quad E_1 < E < E_2; \qquad \sigma_{cU} = 0, \quad E < E_1 \text{ and } E > E_2, \tag{9}$$

formula (8) will yield

$$\varphi = \frac{E_1}{E_2}. \tag{10}$$

We again emphasize that our arguments and the results (7), (8) and (10) refer exclusively to slowing by protons for which each scattering collision leads to a uniform distribution in the energy interval from the initial value to zero.

For calculations with a resonance curve of the form (3) we note that for very large values of the capture cross-section $W$ approaches unity and then changes very little; therefore, (3) may be replaced by the expression

$$\sigma_{cU} = \text{const} \cdot (E - E_r)^{-2}. \tag{11}$$

Now it is easy to write the concrete form of the function, although we do not need it:

$$W = W_0[c_U(E - E_r)^{-2}/c_H], \tag{12}$$

$$\int W \, d\ln E = \text{const} \cdot \sqrt{c_U/c_H}, \tag{13}$$

or, denoting $c_H/c_U = \eta$, we obtain

$$\varphi = e^{-\alpha\sqrt{c_U/c_H}} = e^{-\alpha\eta^{-1/2}}. \tag{14}$$

Let us turn to the consideration of uranium chain decay itself. We denote by $N$ the total number of fast neutrons appearing in the system in unit time both from the source ($N_0$) and from the fission of uranium nuclei by slow neutrons ($N_1$) so that $N = N_0 + N_1$. The number of neutrons which arise for each slow neutron captured (in any way) by uranium we denote by $\nu$, and the probability that a neutron (already slowed to an energy much less than $E_r = 25$ eV) will be captured by uranium rather than hydrogen calculated from a formula of the form (1), $-\theta$:

$$\theta = \frac{\sigma_{cU}c_U}{\sigma_{cU}c_H + \sigma_{cH}c_H} = \frac{1}{1 + \beta c_H/c_U} = \frac{1}{1 + \beta\eta}. \tag{15}$$

We note that if we had wanted to use the number of neutrons $\nu_f$ arising for each decay of a uranium atom, then, bearing in mind the possibility of capture of neutron by uranium without decay, we would have had to introduce in place of the general probability of capture by uranium $\theta$ the probability of capture with decay:

$$\theta_f = \frac{\sigma_f c_U}{\sigma_{cU}c_U + \sigma_{cH}c_H} = \frac{\gamma}{1 + \beta\eta}, \tag{16}$$

where

$$\gamma = \frac{\sigma_f}{\sigma_{cU}} = 1 - \frac{\sigma_1}{\sigma_{cU}},$$

so that identically

$$\nu_f\theta_f = \nu\theta = \frac{\nu}{1 + \beta\eta}. \tag{17}$$

Of the overall number of fast neutrons $N$ arising in unit time, the number which are slowed and pass successfully through the resonant level is $\varphi N$; these, causing decay, lead to $N_1 = \nu\theta\varphi N$ new neutrons in unit time. By definition

$$N = N_0 + N_1 = N_0 + \nu\theta\varphi N, \tag{18}$$

$$N = \frac{N_0}{1 - \nu\theta\varphi}, \tag{19}$$

whence the critical condition [6]

$$\nu\theta\varphi = 1. \tag{20}$$

Thus, the determination of optimal conditions for branching of the nuclear chain reduces to finding the maximum of $\theta\varphi$ as a function of $\eta = c_H/c_U$, i.e., to finding the maximum of the function

$$\frac{\exp(-\alpha\eta^{-1/2})}{1+\beta\eta}. \tag{21}$$

Turning to practical calculations, it should be noted that while the quantity $\beta$, equal to the ratio of the cross-sections, is known comparatively well, data on resonant capture in contrast have been insufficiently determined.

A numerical calculation of the quantity $\varphi$ directly from formula (7), where the capture cross-section was taken in the form (3) with the constants $\sigma_2 = 3000 \cdot 10^{-24}$, $\Gamma = 0.2$ and a scattering cross-section by hydrogen as $\sigma_{sH} = 20 \cdot 10^{-24}$ leads for $\eta = 1$ to the quantity $\varphi = 0.844$, which corresponds in the interpolation formula (14) to $\alpha = 0.168$.

A direct experiment by Halban, Kowarski and Savitch [3] gives practically the same quantity 0.84 at $\eta = 62$ (see note below), which corresponds to significantly greater capture under equal conditions and a corresponding $\alpha = 1.36$.

There have been indications [4] that the simple formula (3) with a single level is not applicable at all.[4]

Until the problem is clarified experimentally we have nothing but to perform a dual calculation with the two values:

$$\alpha = 0.168 \quad \text{and} \quad \alpha = 1.36.$$

In the following we take $\sigma_{cH} = 0.27 \cdot 10^{-24}$, $\sigma_{fU} = 2 \cdot 10^{-24}$, and the cross-section of idle capture by uranium as $1.2 \cdot 10^{-24}$.

From the numerical calculations we obtain respectively the position and height of the maximum of the quantity $\theta\varphi$ and the minimum value of $\nu$ for which the critical inequality (20) holds. We show the detailed calculation.

From the relations between the cross-sections we find in formulae (17), (16), (15):

$$\frac{\nu}{\nu_f} = \gamma = \frac{\sigma_f}{\sigma_{cU}} = 0.625; \qquad \beta = \frac{\sigma_H}{\sigma_{cU}} = 0.0845.$$

The function whose maximum we seek in two variants has the following forms:

$$\theta\varphi_1 = \frac{10^{-0.0745\eta^{-1/2}}}{1+0.0845\eta}; \qquad \theta\varphi_2 = \frac{10^{-0.603\eta^{-1/2}}}{1+0.0845\eta}. \tag{22}$$

[4] It should be noted that when a nucleus is strongly excited due to capture of a neutron, it is natural to expect the presence of a series of resonant levels which differ from one another by several dozen volts. This, however, will have little effect on the form of formula (14). Thus recalculation of the experimental data using (14) will be completely legitimate even when a series of levels is present.

We compile a table of both functions:

| $\eta$ | 62 | 17 | 8 | 4 | 2 | 1 | 1/2 | 1/4 |
|---|---|---|---|---|---|---|---|---|
| $\theta$ | 0.160 | 0.410 | 0.597 | 0.748 | 0.855 | 0.922 | 0.960 | 0.980 |
| $\varphi_1$ | 0.980 | 0.960 | 0.942 | 0.918 | 0.885 | 0.840 | 0.785 | 0.710 |
| $\varphi_2$ | 0.840 | 0.716 | 0.613 | 0.501 | 0.377 | 0.521 | — | — |
| $\theta\varphi_1$ | 0.157 | 0.384 | 0.562 | 0.686 | 0.757 | 0.775 | 0.752 | 0.696 |
| $\theta\varphi_2$ | 0.134 | 0.284 | 0.366 | 0.374 | 0.331 | 0.231 | — | — |

At $\alpha = 0.168$, $\eta_{\max} = 1$, $\theta\varphi_{\max} = 0.775$, $\nu_{\min} = 1.29$.
At $\alpha = 1.36$, $\eta_{\max} = 4$, $\theta\varphi_{\max} = 0.374$, $\nu_{\min} = 2.64$.

The difference in the results of the calculations for the two variants decreases if we take into account the fact that the calculation of $\nu$ itself from experiments like those of Joliot and Fermi must also be consistently carried out in two variants. Here the greater value of $\alpha$, which is less favorable for chain decay (yielding a smaller $\nu$), obviously leads in processing the experimental data to an increase in the neutron output $\nu$ calculated from the observed experimental data. Thus, the quantity $\nu\theta\varphi$ in which we are ultimately interested varies much less with the choice of one or another $\alpha$.

Let us do a detailed calculation from Joliot's experiment [5]. In order to avoid introducing the new concept of neutron lifetime, we will now consider the number of neutrons $N_H$ which are absorbed by hydrogen of the solution in both the presence and absence of uranium salt in the solution. Since the neutron detector used by Joliot absorbs neutrons also with a probability proportional to $E^{-1/2}$, the absorption of neutrons by hydrogen is exactly proportional to the product of the detector indication and the hydrogen concentration.

Integrating over the entire volume we obtain for the spherically symmetric problem

$$N_H \sim \int c_H I r^2 \, dr, \tag{23}$$

where $I$ is the detector reading at the given point. In a solution with a constant concentration of the dissolved substance and, consequently, of hydrogen

$$N_H \sim c_H \int I r^2 \, dr. \tag{24}$$

We find the quantity $c_H$ in the two solutions used by Joliot, taking as 1 the concentration of hydrogen in pure water. For this we supplement the data on the relative density of solutions of ammonium nitrate and uranyl nitrate as a function of percentage of dissolved substance $z$ taken from the physical-chemical tables of Landolt, Bernstein and Roth with the following quantities: the molarity of the solution according to the formula $\mu = 1000\,dz/100M$, where $M$ is the molecular weight of the compound; the water content in a unit volume of the solution $\varepsilon_1 = d(1-z/100)$; and the hydrogen content (with respect to pure water) in the solution $\varepsilon$, to which in the case of ammonium nitrate is added the hydrogen content of the salt itself. Finally we find the

desired result by graphic interpolation in the coordinates $\mu - \varepsilon$. At $\mu = 1.6$ we find the hydrogen content in a 1.6-molar solution of ammonium nitrate to be 0.982, and in the same solution of uranyl nitrate—0.893.

We have discussed these elementary calculations in such detail because in Joliot's note one finds the assertion that the concentrations of hydrogen in the two solutions used by him differ by not more than 2%. These results, which contradict our own, can be obtained either by forgetting about the decrease in the hydrogen concentration in dissolving uranyl nitrate in water and comparing the $\epsilon$ of the solution $NH_4NO_3$ with $\epsilon = 1$ for pure water, or by forgetting the hydrogen content in the ammonium nitrate itself and having equal water contents (or hydrogen only in the form of water) of the two solutions. Both assumptions are obviously unfounded.

The integral on the right side of (24) is nothing other than the area under the curve $Ir^2$, whose variation was determined by Joliot. According to his data, this area increases in the ratio 1 : 1.05 when ammonium nitrate is replaced by uranyl nitrate.

Thus, the total number of neutrons absorbed by hydrogen varies in the ratio

$$\frac{0.893}{0.982} \cdot 1.05 = 0.955.$$

Consequently, when uranium is introduced, as our calculation shows, the number of neutrons absorbed by hydrogen in fact falls.

This still does not preclude the formation of more than one neutron for each thermal neutron absorbed by a uranium nucleus since when uranium is introduced an immediate consequence is the quite noticeable absorption of fast neutrons of the source as they are slowed to thermal velocity on the resonant capture level at 25 eV.

We shall show this numerically. In the notations introduced earlier, of the total number $N$ of fast neutrons arising in unit time, $\varphi N$ neutrons will be slowed without absorption on the resonant level; the slowed neutrons are distributed between uranium and hydrogen as $\theta : (1 - \theta)$ so that, finally, from (19) the number of neutrons absorbed by hydrogen is

$$N_H = N_0(1 - \theta)\varphi/(1 - \nu\theta\varphi), \tag{25}$$

rather than $N_0$ without uranium.

Equating

$$(1 - \theta)\varphi/(1 - \nu\theta\varphi) = 0.955, \tag{26}$$

we find $\varphi$ at $\eta = 62$, $\theta = 0.160$ in two variants: $\varphi_1 = 0.98$ (extrapolated by calculations based on data on the capture curve) and $\varphi_2 = 0.840$ (directly measured by Savitch, Halban, and Kowarski). Finally we obtain

$$\nu_1 = 0.88, \qquad \nu_2 = 1.95. \tag{27}$$

The corresponding quantities taken with respect to a single event of uranium decay, under the relations taken between capture cross-sections with

decay and idle capture of slow neutrons:

$$\nu_f = 3.2\nu/2.0, \qquad \nu_{1f} = 1.41, \quad \nu_{2f} = 3.12. \tag{28}$$

The last number, 3.12, was calculated under the same assumptions as Joliot used to obtain the output $\nu_f = 3.5$. Thus, refining the hydrogen content in the solution and refining the calculation (Joliot considered all effects related to the introduction of uranium as small and systematically discarded terms of second order) have changed the final value relatively little.

Returning to the question of interest, we find the magnitude of the criterion for explosion $\nu\theta\varphi$; in the two consistently performed versions of the calculation it turns out to be equal:

$$(\nu\theta\varphi)_{1\,\max} = 0.88 \cdot 0.775 = 0.68 \quad \text{at } \eta = 1,$$
$$(\nu\theta\varphi)_{2\,\max} = 1.95 \cdot 0.374 = 0.73 \quad \text{at } \eta = 4,$$

This corresponds to a maximum intensity of the source due to an increase in neutrons from uranium decay by not more than 3–4 times under the optimal choice of uranium–water ratio.

Thus Joliot's experimental data give a value for the product $\nu\theta\varphi$ which is almost independent of the choice of $\alpha$ and is insufficient for chain decay to occur.

A calculation from Fermi's experiment is difficult due to the separate distribution of uranium and water in his instrument. In any case, it would not provide any more consoling result, and our conclusion based on experimental data about the impossibility of powerful chain decay in a uranium-water system turns out to be related in the final analysis only to the law chosen for variation of capture on the resonant level as a function of the ratio of hydrogen to uranium in the form (14); over a broad range it is independent of the value of the coefficient $\alpha$, as is clear from comparison of the final results of the two variants of the calculation with two widely different values of $\alpha$.

From this it follows that in order to realize conditions for the chain explosion of uranium it is necessary to use for neutron moderation heavy hydrogen or, perhaps, heavy water, or some other substance which ensures a sufficiently small capture cross-section. The significantly smaller scattering cross-section compared to hydrogen and the somewhat lower effectiveness of energy exchange may be compensated by the negligibly small capture cross-section of neutrons and the related possibility of extreme dilution of uranium (large $\eta$).

Another possibility lies in the enrichment of uranium with the isotope 235.

If water (hydrogen) is used as the solvent, the quantity $\nu\theta\varphi$ becomes equal to one when the uranium 235 content is increased by 1.9 times (from 0.7 to 1.3%) at an optimal value $\eta \sim 8$.

All that has been said above refers to a solution of uranium and hydrogen (or water solution of uranium salt, since the effects of other nuclei may be neglected) of infinite extent. Taking account of the finite size of the volume

occupied by the mixture (solution) leads to lowering of the effective value of $\theta\varphi$ due to diffusion of neutrons to the outside.

Near the critical conditions of explosion it may be shown that the following relation holds:

$$(\theta\varphi)_{\text{eff}} = \theta\varphi(1 - A/d^2), \tag{29}$$

where $d$ is the characteristic size of the system, and $A$ is a quantity which depends on the free path.

It is clear that the achievement of critical conditions here is made more difficult. Conversely, the greater is the quantity $\nu\theta\varphi - 1$, the smaller the critical size of the system may be.

We would like to take this opportunity to express our gratitude to I. I. Gurevich, I. V. Kurchatov and I. Ya. Pomeranchuk for a number of valuable comments in discussions of this work.

*Institute of Chemical Physics*
*Leningrad*

*Received*
*October 22, 1939*

## *REFERENCES*

1. *Perrin F.*—C. r. Acad. Sci. **208**, 1573 (1939).
2. *Zeldovich Ya. B., Khariton Yu. B.*—ZhETF **9**, 1425 (1939).
3. *Halban H., Kowarski L., Savitch P.*—C. r. Acad. Sci. **208**, 1396 (1939).
4. *Anderson H. L., Fermi E., Szillard L.*—Phys. Rev. **56**, 284 (1939).
5. *Halban H., Joliot F., Kowarski L.*—Nature **143**, 470, 680 (1939).
6. *Semenov N. N.* Tsepnye reaktsii [Chain Reactions]. Leningrad: Goskhimtekhizdat, 555 p. (1934).

# 3

# Kinetics of the Chain Decay of Uranium*

*With Yu. B. Khariton*

We consider the development of a chain nuclear reaction in a mass of uranium in the transition across the critical mass. It is shown that thermal expansion is a powerful regulating factor which makes the transition across the limit—if such exists—completely safe. For a critical mass of 1 ton heating to 1000° can be accomplished by adding only $\sim$ 50 kg above the critical mass. A gradual increase in mass above the critical value leads to an oscillatory reaction regime whose period is inversely proportional to the square root of the rate of uranium supply. Delayed neutrons significantly increase the oscillation period of the reaction rate.

## *1. Introduction*

In our previous papers [1, 2] we considered the question of the possibility, in principle, of realizing a chain decay reaction of uranium on fast and slow neutrons, without taking into account diffusional evacuation of neutrons, i.e., in essence, the calculations related to an infinitely extended mass of uranium or solution of a uranium compound in water.

It would appear (the lack of experimental data precludes any categorical assertions) that by applying some technique, creating a large mass of metallic uranium either by mixing uranium with substances possessing a small capture cross-section (e.g., with heavy water) or by enriching the uranium with the $U^{235}$ isotope, which is thought to decay under the action of slow neutrons—it will be possible to establish conditions for the chain decay of uranium by branching chains in which an arbitrarily weak radiation by neutrons will lead to powerful development of a nuclear reaction and macroscopic effects. Such a process would be of much interest since the molar heat of the nuclear fission reaction of uranium exceeds by $5 \cdot 10^7$ times the heating capacity of coal. The abundance and cost of uranium would certainly allow the realization of some applications of uranium.

Therefore, despite the difficulties and unreliability of the directions indicated, we may expect in the near future attempts to realize the process.

In this paper we investigate the details of the behavior of a system in which conditions for branching of chains of nuclear reaction have somehow

*Zhurnal eksperimentalnoĭ i teoreticheskoĭ fiziki **10** 5, 477–482 (1940).

been achieved. In an infinitely extended system the neutron number density and reaction intensity then grow exponentially (see, for example, Flügge [3]):

$$\frac{dn}{dt} = bn \tag{1}$$

until a significant portion of the substance has already reacted. If the probability of branching is 0.1, i.e., the quantity $\alpha = \nu_f(1-\gamma) - 1 = 0.1$ (see our paper [1]), or $\alpha = \nu\theta\varphi - 1 = 0.1$ (see our paper [2]), then the inverse time of relaxation $b$ of equation (1) for fast neutrons turns out to be of order $10^7\,\mathrm{s}^{-1}$, for slow neutrons it is around $10^4\,\mathrm{s}^{-1}$. In general form [4]

$$b = \alpha u \sum n_i \sigma_{c_i}, \tag{2}$$

where $\alpha$ is defined above, $u$ is the neutron velocity, $n_i$ is the number of particles of the $i$-th sort in a unit volume, $\sigma_{c_i}$ is the capture cross-section of particles of the $i$-th sort.

In the case of a system of finite size, the evacuation of neutrons into the surrounding space is of course equivalent to breaking of chains. From these considerations, analyzing the diffusion of neutrons, Perrin [5] found the critical dimensions beginning from which a branching chain of reactions is possible. Perrin's calculation was generalized by Peierls [6]; he confirmed the critical condition for the existence of a steady regime, found by Perrin, and also analyzed the practically unimportant case of high probability of branching and large decay cross-section in which the critical dimensions of the system are small compared with the free path and the diffusion equation cannot be written.

Restricting ourselves to the only interesting case when the critical dimensions are significantly larger than the free path, we construct the equation of the variation in the number density of neutrons in the absence of an outside source:

$$\frac{\partial n}{\partial t} = bn + D\Delta n, \tag{3}$$

where the coefficient of diffusion is

$$D = \frac{1}{3}\lambda u = \frac{u}{3\sum n_i \sigma_{s_i}}. \tag{4}$$

The general solution of (3) may be found in the form of a sum:

$$n = n(x, y, z, t) = \sum \psi_i(x, y, z) e^{(b-g_i)t}, \tag{5}$$

where $\psi_i$ and $g_i$ are the eigenfunctions and eigenvalues, respectively, of the equation

$$D\Delta\psi + g\psi = 0 \tag{6}$$

with boundary condition $\psi = 0$ on the surface of the body.

From dimensionality considerations it is clear that

$$g_i = \frac{k_i D}{d^2}, \tag{7}$$

where $k_i$ is a dimensionless coefficient depending only on the form of the body, $d$ is the linear dimension of the body. In the case of a spherical form, equating $d$ with the diameter of the sphere we find

$$k_i = 4\pi^2 i^2, \qquad k_1 = 4\pi^2 \cong 40. \tag{8}$$

The critical condition is

$$b - g_1 = 0; \quad b = \frac{k_1 D}{d^2}, \quad d_{\text{cr}} = 2\pi\sqrt{\frac{D}{b}}, \tag{9}$$

which coincides with Perrin's result [5].

Substituting (2) and (4), we find at the limit

$$d_{\text{cr}}\sqrt{\sum n_i \sigma_{c_i} \sum n_i \sigma_{s_i}} = \frac{2\pi}{\sqrt{3\alpha}}. \tag{10}$$

In a mixture of constant composition in which all $n_i$ are in a constant ratio, at the limit

$$d_{\text{cr}} n_i = \text{const}, \tag{11}$$

and the critical mass,

$$M_{\text{cr}} = d_{\text{cr}}^3 n_i \sim n_i^{-2}, \tag{12}$$

decreases as the density increases.

## 2. The Kinetics of the Decay

This calculation obviously is not sufficient to give a macroscopic description of the process under realistic conditions.

As is clear from equation (1), far from the critical conditions, when diffusional evacuation is small, the neutron number density grows exponentially at a huge rate, increasing by $e$ times in a time of order $10^{-7}$ s for decay on fast neutrons, $10^{-3}$ s for decay on slow neutrons. Given such rapid development of the chain decay we can no longer put off consideration of the creation of the supercritical conditions which are uniquely necessary for chain decay to occur.

The time of occurrence of processes which bring about the transition to critical conditions, e.g., the time of approach of two uranium masses, each of which separately is in the subcritical region with respect to chain decay, is hardly likely to be even comparable with the time required for the reaction to get going. We may expect, therefore, that in reality in all cases we shall have to deal with conditions which are quite close to critical. On the one hand, it is necessary to consider the start-up and acceleration of the reaction not under given conditions (of unknown origin), but in the gradual transition of critical conditions corresponding to some concrete setup of the experiment, the approach of two uranium masses, addition of uranium powder, etc. On the other hand, in the immediate vicinity of the critical conditions the behavior of the system is extremely sensitive to factors whose

effect could be neglected far from the limit. As examples of such factors which need investigation we may note uranium consumption and the appearance of new nuclei capable of capturing neutrons in decay, the thermal expansion of the uranium mass being used as a result of the release of decay energy; the release of some small ($\sim 10^{-2}$) amount of all neutrons with a delay of about 10 s after decay. The effect of all these factors on the critical conditions, in themselves insignificant, turns out to be decisive in the case when the system is so close to the critical conditions that the effect, for example, of thermal expansion or the release of neutrons which have been delayed by a half-period of 10 s, can carry the system from the supercritical to the subcritical region or *vice versa.*

The kinetics of the development of chain decay are decisive in judging one or another path for practical energy or explosive use of uranium decay. Hasty conclusions made without regard for the considerations above [3], for example on the extreme danger of experiments with large masses of uranium and the catastrophic consequences of such experiments (counting on complete decay of all the uranium nuclei) do not correspond to reality. Also unnecessary, it appears, are special additives such as cadmium to control the process [7, 8]. In all the works cited the specifics of the reaction, its extreme sensitivity near the limit, were ignored.

Let us turn to setting up the equations. It is important for us to note that in the general formula (5) directly below the limit and in the supercritical region the coefficient of the first eigenfunction (with the smallest characteristic number) is incomparably larger than all the other coefficients. Disregarding the latter, we come to the conclusion that, in practical terms, throughout the region of interest to us the spatial distribution of neutrons remains self-similar and is described by the first eigenfunction of the Poisson equation of our problem (6). Because of this in what follows we do not need to consider the dependence of the neutron number density on both coordinates and time, which would at best lead to an equation in partial derivatives. Instead, in our investigation of the kinetics of the reaction we will limit ourselves to consideration of the dependence on one variable—time—of the coefficient of the first eigenfunction or of the total number of neutrons in the system.

For constant external conditions the exponential growth (or decrease) of the total number of neutrons (proportional to the coefficient of the first eigenfunction) with time in the absence of an external source,

$$N = \int n\,dv, \quad n = c_0 e^{pt}\psi(x, y, z), \quad N = N_0 e^{pt} \tag{13}$$

corresponds to the differential equations

$$\frac{\partial n}{\partial t} = pn \quad \text{or} \quad \frac{dN}{dt} = pN, \tag{14}$$

$$p = b - \frac{k_i D}{d^2} = \alpha u \sum n_i \sigma_{c_i} - \frac{k_1 u}{3d^2 \sum n_i \sigma_{s_i}}. \tag{15}$$

The supply of neutrons by an external source is introduced with a coefficient $\beta$ which depends on the position of the source; this last, however, even in the least favorable case is not much smaller than the ratio of the free path to the dimensions of the system, i.e., in any case is not less than several one hundredths:

$$\frac{dN}{dt} = pN + \beta m, \tag{16}$$

where $m$ is the intensity of the source—the number of neutrons per second, $\beta$ is the already-mentioned coefficient. Together with this we introduce into the analysis the rate of change of the quantity $p$ itself, which characterizes the distance from the limit: $p < 0$ in the subcritical region, $p > 0$ in the supercritical region:

$$\frac{dp}{dt} = c - \alpha N, \tag{17}$$

here $c$ characterizes the rate at which uranium is added, the approach of two masses of uranium, or another process by means of which we carry the system through the critical conditions. In contrast, the coefficient $\alpha$ describes the self-regulation of the system, its departure from the limit as a result of the consequences of uranium decay due to consumption of the material, thermal expansion of the system in connection with the release of energy in decay. The numerical values of $c$ and $\alpha$ under given experimental conditions are easily found from the definition of $p$ in formula (15) which reveals the dependence of $p$ on the dimensions and form of the system, the uranium concentration, and so on.

We introduce, finally—for the first time in our paper—consideration of delayed neutrons.

The observed half-period $\sim 10$ s is, apparently, the half-period of the process of $\beta$-transformation of one of the fragments which form in the decay; the evaporation of a neutron from a nucleus which has gained sufficient energy as a result of the $\beta$-transformation occurs, according to existing conceptions, in a time not exceeding $10^{-13}$ s. Denoting by $l$ the number of nuclei capable after $\beta$-transformation of discarding one neutron, we write the equation

$$\frac{dl}{dt} = \zeta g N - fl, \tag{18}$$

where $f$ is the probability of the $\beta$-decay of interest, $10^{-1}\ \mathrm{s}^{-1}$, according to what we have said, $gN$ is the number of decay events occurring in unit time, $\zeta$ is the probability (dimensionless) of formation in the decay event of the neutron-active nucleus of interest.

In equation (16) an additional term appears

$$\frac{dN}{dt} = pN + \beta m + \xi\zeta\eta fl, \tag{19}$$

which results precisely from the "delayed" neutrons.

The number of delayed neutrons arising in a single $\beta$-decay event is denoted by $\eta$. The product $\zeta\eta$ is the experimentally determined ratio of the delayed neutron output to the number of decays which have occurred ($\sim 10^{-2}$).

The factor $\xi$ has been introduced to account for the fact that delayed neutrons have a different energy distribution and therefore are equivalent with respect to causing further decay by the primary neutrons which formed in the decay process with a delay of $10^{-13}$ s. The magnitude of $\xi$ does not differ from 1 in working on slow neutrons, and is not smaller than $10^{-2}$ in working on fast neutrons due to the presence of a concentration $10^{-2}$ of isotope $U^{235}$.

Let us first consider the system (18) and (19) assuming constant $p$. We find the solution in the form

$$N = Ae^{\gamma t} + B, \qquad l = Ce^{\gamma t} + E.$$

For the quantity $\gamma$ we obtain the quadratic equation

$$(\gamma - p)(\gamma + f) - \zeta\eta\xi fg = 0. \tag{20}$$

The critical condition $\gamma = 0$ in which we are interested will be reached at

$$p = -\xi\eta\zeta g, \tag{21}$$

i.e., earlier than the limit is reached in the absence of delayed neutrons, $p = 0$.

Substituting the expression for (15) and recalling that $b = \nu g$ we obtain

$$p = \nu g - \frac{H_0 D}{r^2} = -\xi\eta\zeta g. \tag{22}$$

It is clear that the critical radius changes by no more than 1% in accordance with the small output of delayed neutrons.

We further write

$$\left(\frac{d\gamma}{dp}\right)_{\gamma=0} = \frac{1}{1 + \zeta\eta\xi g/f}. \tag{23}$$

This quantity turns out to be significantly smaller than unity in decay on both fast and slow neutrons. Physically this means that in the region where delayed neutrons are necessary for the realization of a branching chain, i.e., $\zeta\eta\xi f < p < 0$, a process at an equal distance from the limit develops more slowly—in the absence of delayed neutrons, obviously, it would be

$$\gamma = p, \qquad \frac{d\gamma}{dp} = 1.$$

Calculations of particular cases in which integration of the equations is relatively simple (for example, a steady regime, small oscillations, etc.) have convinced us of the feasibility of the following approximate interpretation of the influence of delayed neutrons: equation (19) is replaced by

$$\frac{dN}{dt} = \gamma N + \beta m,$$

with

$$\frac{d\gamma}{dt} = \left(\frac{d\gamma}{dt}\right)_{\gamma=0}, \quad \frac{dp}{dt} = \frac{c}{1+\zeta\eta\xi g/f} - \frac{\alpha N}{1+\zeta\eta\xi g/f}$$

in accord with (17) and (23).

Let us consider several particular solutions which illustrate the properties of the system.

Disregarding the release of neutrons by the external source, which is quite small for any macroscopic process, we find the stationary state

$$\gamma = 0, \quad c = \alpha N, \quad N_{\text{stationary}} = \frac{c}{\alpha}$$

The stationary number of neutrons is such that $\gamma = 0$ is maintained despite the supply of uranium (the term $c$). At constant density and form in the stationary state, the amount of uranium decaying in unit time, accurate to within a numerical factor close to one, is equal to the amount of uranium supplied.

It turns out, however, that, for example, for a solid mass of uranium weighing 1 ton the decay of $10^{-3}$ g of uranium will heat the uranium to a temperature of about 1000°, which corresponds to an expansion of about 1%. This expansion resulting from the decay of $10^{-3}$ g of uranium compensates for the effect on the limit of the addition of 50 kg of uranium so that in this temperature interval natural regulation through the density leads to burning of a $1.5 \cdot 10^{-8}$ part of the amounts supplied. Conversely, when heat is removed from a mass of uranium which has heated to 1000°, its temperature falls to room temperature only after $10^{12}$ kcal has been removed, i.e., after $\sim 50$ kg of the uranium has burned (in fact it occurs earlier due to the effect of other regulating factors).

An analysis of small oscillations about the stationary state gives us the period of these oscillations:

$$\tau = 2\pi\sqrt{\frac{1+\zeta\eta\xi f/g}{c}},$$

which characterizes the relaxation time of the system. In the absence of a supply of neutrons by an external source these oscillations turn out not to decay. The equation is integrated in the variables $\gamma - N$ by separation of variables even at large amplitudes:

$$N_{\max} = N_{\text{stationary}} \ln N_{\text{stationary}} \,/\, N_{\min}.$$

The period here of the oscillations varies only logarithmically. The order of magnitude of the period of oscillation and of the relaxation time of the system, e.g., at a critical mass of $10^6$ g and supply of 10 g/s, are around 0.1 s (for fast neutrons). It is not difficult to estimate the initial amount of neutrons from which oscillations will begin (cycles in the $\gamma - N$ plane) when

the limit is reached: in order of magnitude this amount is

$$N_{\min} = \beta m\tau = \beta m\sqrt{\frac{1+\zeta\eta\xi f/g}{c}}$$

equal to the product of the supply rate (source intensity) and the relaxation time. Accordingly

$$N_{\max} = \frac{c}{2a}\ln\frac{\beta^2a^2m^2}{c^3}\cdot\left(1+\frac{\zeta\eta\xi f}{g}\right).$$

It turns out here that taking account of the neutrons of the source leads to gradual decay of the oscillations.

When the uranium supply is abruptly cut off, or two uranium masses suddenly stop their approach, the amount of uranium which burns "by inertia" as the neutron number density falls as a result of the departure from the limit—this amount is equal to the average amount which burns over the relaxation time in the stationary supply regime.

Let us summarize the results of this last part of the work.

A chain disintegration** of uranium, unlike the combustion of explosives and other similar processes, practically instantaneously stops when the system moves back from the super- to the sub-critical region without affecting the remaining amount of uranium, which is quite close to the critical value.

When the process runs isothermically the amount of uranium which decays in unit time is equal to the amount supplied.

In the adiabatic process as a result of thermal expansion the amount burned is $\sim 10^8$ times less than that supplied.

The relaxation time of the process, inversely proportional to the square root of the rate of uranium supply, of order $10^2$ s for a supply $\sim 50$ kg/hr and at a critical mass of about 1 ton, is approximately $10^3$ times larger than that which would result in the absence of delayed neutrons. These numbers refer to chain decay on fast neutrons. The formulas obtained are of course applicable to decay on slow neutrons as well. Such properties of the system (above all the regulation *via* thermal expansion) make experimental investigation and energy production use of uranium decay safe. Explosive use of chain decay requires special devices for a very fast and deep transition to the supercritical region and decrease in the natural thermal regulation.

*Institute of Chemical Physics*
*Leningrad*

*Received*
*March 7, 1940*

## *REFERENCES*

1. *Zeldovich Ya. B., Khariton Yu. B.*—ZhETF **9**, 1425 (1939).
2. *Zeldovich Ya. B., Khariton Yu. B.*—ZhETF **10**, 29 (1940).
3. *Flügge S.*—Naturwissenschaften **27**, 402 (1939).

**Today the term "disintegration" is not used; it has been replaced by the term "uranium fission."—*Editor's note*

4. *Semenov N. N.* Tsepnye reaktsii [Chain Reactions]. Leningrad: Goskhimtekhizdat, 555 p. (1934).
5. *Perrin F.*—C. r. Acad. Sci. **208**, 1934 (1939).
6. *Peierls R.*—Proc. Cambridge Philos. Soc. **35**, 610 (1939).
7. *Perrin F.*—C. r. Acad. Sci. **208**, 1537 (1939).
8. *Halban H.*—Nature **143**, 793 (1939).

## *Commentary*

The discovery in 1938–1939 of the fission of uranium nuclei, which led eventually to the development of nuclear energy, heralded a new, extraordinarily fruitful stage in Ya. B.'s scientific activity. His interests became concentrated on the study of the mechanism of fission of heavy nuclei and, what proved especially important, on the development of a theory of the chain reaction of uranium fission. On these subjects during two years (1939–1940) Ya. B., in collaboration with Yu. B. Khariton, performed three basic studies which are of enormous, fundamental value. The papers of this cycle are the foundation of the modern physics of reactors and nuclear power, they are widely known and do not require special commentary—just a short summary of the results is eloquent enough.

Ya. B.'s interest in problems of nuclear physics, and then in the physics of elementary particles, was also stimulated by the discovery of the phenomenon of fission of heavy nuclei and by practical work in nuclear power.

We shall comment on each paper in turn.

*Paper 1.* The conditions for the appearance of a chain fission reaction of the main isotope of uranium are considered, taking into account the slowing of neutrons below the threshold of $U^{238}$. The basic text of the article considers only elastic moderation of neutrons. At the same time the authors, taking plausible values for the number of secondary neutrons from fission, conclude that chain reaction, practically speaking, is impossible in uranium oxide, and possible in pure uranium. However, in a note added in proof the authors report that a calculation carried out by them on the basis of the Bohr-Wheeler theory of inelastic scattering (moderation) of neutrons shows that even in the case of metallic uranium a chain reaction is impossible. Thus the impossibility of a chain nuclear reaction on fast neutrons in natural uranium was shown.

In this same paper, for the first time, an estimate is given for the thickness of the reflector for fast neutrons in the case of threshold fission.

*Paper 2* is pioneering and classical in the highest sense of the word. The basic content of the work may be summarized as follows.

a) The clear introduction of resonant absorption of $U^{238}$ as one of the determining factors in the coefficient of multiplication in systems on slow neutrons (uranium + moderator).

b) The formulation of the history of one generation of a neutron and the derivation of the famous expression for the multiplication coefficient in an infinite medium: $K_\infty = \nu\varphi\theta$, where $\nu$ is the number of secondary neutrons per event of capture of a thermal neutron by uranium, $\varphi$ is the probability of avoiding resonant capture in the process of moderation, and $\theta$ is the coefficient of consumption of thermal neutrons.

c) The derivation of an equation for $\varphi$ which relates it with the resonant cross-sections of $U^{238}$ for a hydrogen moderator. The equation for $\varphi$ is easily generalized for a non-hydrogen moderator ($A > 1$), but the solution is more complicated since it leads to equations in finite differences. The paper does not do this.

d) The introduction of the effect of self-shielding of uranium atoms. A square root law is obtained for the dependence of the resonance absorption on the concentration. The later-understood possibility of reducing resonance absorption by heterogeneous (block-wise) placement of uranium in the moderator is based on the existence of two effects, one of which is the self-shielding of uranium atoms discovered in the present paper (internal regions of a block do not participate in the absorption). The second effect, not considered in this paper, is that only resonance neutrons formed directly near the block are absorbed. The remaining neutrons in the process of moderation will exit the dangerous resonant zone before they reach the block.

e. A detailed recalculation of Joliot's experiments is performed, with the conclusion that it is not possible to obtain a self-supporting chain fission reaction in a system of natural uranium + light water at any concentration. A homogeneous mixture is of course assumed. On this basis the paper makes the very important assertion, later fully justified, that new moderators must be used and proposed the use of heavy water and graphite due to the smallness of the absorption cross-sections of thermal neutrons by deuterium, carbon and oxygen (see below).

Regarding the points d) and e), more detailed explanations should be given.

1. In a number of isotopes, including $Pu^{239}$, there is a deviation from the $1/v$ law in the thermal region, however at the time this was unknown. For $U^{235}$ which is considered in the paper the $1/v$ law holds.

2. Accounting for reaction on fast neutrons, which is mentioned in the paper but not carried out, becomes more important in a heterogeneous (block) system. The corresponding coefficient $\mu$ attains $1.03 - 1.04$ and is quite significant for reactors with graphite and water cooling moderation.

3. Formula (11) is valid only for large uranium concentrations (omission of the factor $\Gamma^2/4$ in the denominator of the Breit-Wigner formula). An exact solution of equation (8) gives

$$-\ln\varphi = \frac{\pi}{2}\sum_i \frac{\Gamma_i}{E_{0i}}\left[\frac{\eta\sigma_s(1+\eta\sigma_s/\sigma_{0i})}{\sigma_{0i}}\right]^{-1/2}$$

$$\cong \frac{\pi}{2}\sum_i \frac{\Gamma_i}{E_{0i}}\sqrt{\frac{\sigma_{0i}}{\eta\sigma_s}} - \sigma_{0i} \gg \eta\sigma_s \qquad \text{(I)}$$

in the case of strong levels (self-shielding), and

$$\ln\varphi \simeq \frac{\pi}{2}\sum_i \frac{\Gamma_i}{E_{0i}}\frac{\sigma_{0i}}{\eta\sigma_s} - \sigma_{0i} \ll \eta\sigma_s \qquad \text{(II)}$$

in the case of weak levels.

The context of the paper corresponds to taking only strong (self-shielding) levels of resonance absorption into account.

For hydrogen at concentrations $c_H/c_v = 62$ and even 17 levels in the region 100–200 eV should be considered using the exact formula.

4. Subsequent measurements significantly changed the values of the constants. For current values of the constants $\varphi \simeq 0.6$ at $c_H/c_v = 1$. However, the estimate taken in the article is much more realistic than the completely incorrect estimate of A. Khalban and L. Kovarskiĭ at which even a heavy-water reactor could not work on natural uranium.

The ratio of the cross-sections of fission and capture in uranium on thermal neutrons is taken close to the current value. But $\sigma_a$ itself for natural U is 2.4 times smaller than the true value (3.2 barn instead of 7.5 barn), while for hydrogen it is only 20% lower. Therefore sharply lowered values of $\theta$ are obtained. Now the optimum is close to $c_H/c_v = 3$ and $\{\varphi\theta\}_{max} = 0.69$, i.e., the minimum enrichment for a homogeneous system at $\mu = 1.02$ corresponds to $\nu_{ef} = 1.43$ for $K_\infty = 1$, i.e., enrichment of order 1%:

$$\theta = \frac{8.96}{8.96 + 0.96} = 0.9; \qquad \varphi = 0.77;$$
$$\mu\varphi\theta = 0.707, \qquad K_\infty = 1.01.$$

Finally, we note that self-shielding of resonance levels of capture leads to the advantageousness of using uranium in the form of bodies (blocks) several centimeters in size. The block-effect was discovered in the USSR by I. I. Gurevich and I. Yu. Pomeranchuk in 1945, after the paper by Ya. B. and Yu. B. Khariton. The block-effect noticeably increases $K_\infty$ and is extremely important for work with reactors which use natural unenriched uranium. At the same time, the introduction of the block-effect has not changed the very important qualitative conclusions of the paper here. Even when the size of the blocks is made optimal, natural (unenriched) uranium with ordinary (light) water in an infinite system does not attain criticality.

We note that in a review article in *UFN*[1] Ya. B. and Yu. B. Khariton indicated substances which should be investigated as moderators, including helium, heavy water ($D_2O$) and carbon. As we know, heavy-water and graphite reactors are in practical use.

*Paper 3*, like paper 2, is classic and pioneering. For the first time the kinetics of the chain decay of uranium were considered in detail in the transition to the supercritical state. We note here the most important results.

a) Most important in the paper is the consideration of the role of delayed neutrons in the kinetics of the chain reaction. In the interval of effective multiplication coefficients (given by $K = K_\infty P$, where $P$ is the probability that a neutron will be absorbed rather than leave the system),

$$1 < K < 1 + \beta, \qquad (III)$$

where $\beta$ is the fraction of delayed neutrons, the kinetics become very soft and are primarily determined by the periods of the delayed neutrons. The authors produce a complete system of kinetic equations with delayed neutrons from which, in particular, follows the equation of the so-called "inverse clock" which characterizes the rate of acceleration of the reactor; this last was not directly obtained in the paper. This brilliant idea explains the fact that a nuclear reactor proved to be an easily regulated system, which in turn was one of the basic factors ensuring the success of atomic energy (all reactors work in the interval (III)).

[1] *Zeldovich Ya. B., Khariton Yu. B.*—UFN **23**, 329–357 (1940).

b) Also prophetic in the paper is the statement of the possibility of the appearance of new, strongly absorbing nuclei. The well-known phenomenon of the "iodine well" in reactors is related to the accumulation of an isotope of xenon which absorbs thermal neutrons at a record rate ($Xe^{135}$: $\sigma_c = \pi\lambda^2/10 = 3 \cdot 10^6$ barn).

c) The effect of heating of the uranium on the kinetics is considered in detail and it is shown that thermal expansion is an effective regulating factor.

d) Fluctuations near the equilibrium position are studied which, in the absence of external sources, prove to be non-decaying.

e) Finally, the conditions for generation of a strong explosion follow directly from the paper—significant supercriticality in the initial state and multiplication on fast neutrons. These conclusions, not explicitly formulated, were fully used by the authors in subsequent work.

Let us note that a correct estimate of the critical mass of $U^{235}$ was given by the authors together with I. I. Gurevich as early as 1941. It was also noted then that in the distant past the content of $U^{235}$ was greater than now, which ensured the appearance of the chain reaction. This is contained in the second part of the review,[2] submitted in 1941 but published only in 1983 in connection with the eightieth birthday of I. V. Kurchatov. As is known, signs of a chain reaction which occurred 2 billion years ago were discovered in a uranium deposit in Oklo (Gabon, Africa).

Overall, the papers 1–3 are unique in world literature. Similar papers in other countries were not published until the Geneva conference in 1955.

An introduction to the subsequent development of the ideas presented in these articles may be found in the books *The Physical Theory of Neutron Chain Reactors*[3] and *Theoretical Foundations and Calculation Methods for Nuclear Power Reactors.*[4]

[2] *Zeldovich Ya. B., Khariton Yu. B.*—UFN, Part 1 **25**, 381–405 (1941); Part 2 **139**, 501–527 (1983).

[3] *Weinberg A., Wigner E. The Physical Theory of Neutron Chain Reactors.* Chicago Univ. of Chicago Press (1958).

[4] *Vartolomeĭ G. G., Bat' G. A., Baibakov V. D., Altukhov M. S. Osnovy teorii i metody rascheta ĭadernykh energeticheskikh reaktorov [Theoretical Foundations and Computational Methods for Nuclear Power Reactors].* Moscow: Energiĭa, 511 p. (1982).

# 4

# On the Theory of Disintegration of Nuclei*

*With Yu. A. Zysin*

The possible state of a nucleus at the moment of its disintegration into two approximately equal nuclei is considered. A calculation of the energy of two ellipsoids of rotation in contact refutes Ya. I. Frenkel's arguments in favor of the existence of significantly non-spherical nuclei. The order of magnitude of the energy of ellipsoids found allows us to satisfactorily describe the observed formation of several fast neutrons for each disintegration event as evaporation of these neutrons by fragments excited in the process of fission.

Bohr's theory describes a nucleus as a drop of liquid with uniform charge density which gives rise to the electrostatic energy. The short-range attractive forces of nuclear particles specify their particular evaporation heat and also the surface tension of the drop.

In 1939 a very important success of the theory was the description of the fission discovered by L. Meitner and O. Frisch [1] of heavy nuclei under neutron bombardment into two approximately equal fragments with the release of huge—even for radioactive processes—amounts of energy (100–200 MeV) and the formation of several neutrons ("neutron dust") for each disintegration event. This last peculiarity is of particular interest since it opens the possibility in principle of the chain decay of macroscopic amounts of uranium [2]. The theory of the decay, which has been especially thoroughly developed by three physicists—N. Bohr (Denmark), J. Wheeler (USA) and Ya. I. Frenkel (USSR) [3], considers the stability of a spherical uniformly charged drop of incompressible fluid possessing a specific surface tension.

As it turns out, the spherical form becomes instable with respect to small deformations when the ratio of the electrostatic energy $E$ to the surface energy $O$ is

$$E/O \geq 2. \tag{1}$$

All three of the above authors then consider two contacting spherical nuclei which have resulted from the division of the original nuclei. It is not difficult to find that the energy of two contacting spheres of half the volume is equal to or less than the energy of the original sphere if for the

*Zhurnal eksperimentalnoĭ i teoreticheskoĭ fiziki **10** (8), 831–834 (1940).

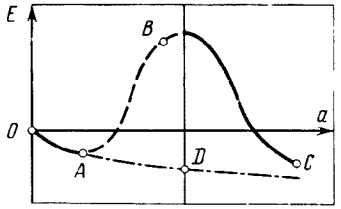



Fig. 1

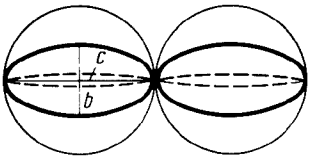



Fig. 2

latter[1]

$$E/O \geq 2.42. \tag{2}$$

Let us introduce some parameter $a$ which describes the process of division; for convenience in the graphical representation we choose $a$ such that for a spherical nucleus $a = 0$. At the moment when the two nuclei which have formed are in contact at a point (which is a necessary stage of the division), $a = 1$; finally when the two nuclei have separated by an infinite distance, $a = 2$.

For the value

$$2 < E/O < 2.42 \tag{3}$$

the above calculations lead to the form of the curve of the energy variation during decay shown in Fig. 1 with solid lines.

The left segment $OA$ is the result of the calculation of small deformations of the drop at $E/O > 2$, the right, $BC$—of a calculation of the energy of the two spheres as a function of the distance. By comparing the solid lines shown in Fig. 1 two substantially different conclusions may be made.

1. Connecting the two solid segments with a smooth curve (the dashed line $OABC$), we necessarily obtain the minimum energy at $a < 1$. Physically this means that heavy nuclei have a stable non-spherical form (Frenkel).

2. The other possibility is decay through a form which differs from two equal contacting spheres—"a tidally perturbed form" (Bohr and Wheeler). If this form corresponds to a sufficiently small energy (point $D$ in Fig. 1), then the basis for conceptions of stable nonspherical forms disappears.

The calculations of the present paper relate precisely to the energy at the moment of decay, i.e., to finding the ordinate of the point $D$, $a = 1$.

It is easy to see that pear-shaped forms possess the minimum energy for a given charge and volume at the moment of contact.

[1]The number 2.17 cited by Frenkel is the result of an arithmetic error since the corresponding formula in his article

$$\Delta W = E(1 - 2^{1/3}/2 - 5 \cdot 2^{1/3}/24) - u(2^{1/3} - 1)$$

is written correctly. Cf. also the graph of the function $f^*(x)$ (the above-cited work by Bohr and Wheeler, Fig. 4): $f^*(x) = 0$ at $x = 1.2$, where $x = E/2O$ is Bohr's parameter.

However, even a calculation for ellipsoids of revolution elongated along the line of centers (Fig. 2) gave sufficiently definite qualitative results.

A cross-section of the ellipsoids is depicted by the solid lines in Fig. 2. Let us denote the length of the large semiaxis by $c$, and the length of the smaller one by $b$. For a given ratio $c/b$, we easily find each of the semiaxes from the condition of conservation of volume:

$$2cb^2 = r_0^3, \tag{4}$$

where $r_0$ is the radius of the original nuclei.

We find the surface energy from the well-known formula

$$O = 2\pi ab\left(\frac{b}{c} + \frac{1}{\varepsilon}\arcsin\varepsilon\right), \tag{5}$$

where $\varepsilon = \sqrt{1 - b^2/c^2}$.

We determine the electrostatic energy of an individual ellipsoid by the formula

$$F = \frac{3}{10}\frac{(e/2)^2}{c\varepsilon}\ln\frac{(1+\varepsilon)}{(1-\varepsilon)}. \tag{6}$$

It is somewhat more complicated to find the mutual energy of the ellipsoids. An exact analytical calculation for the mutual energy of two ellipsoids, carried out on the basis of the method proposed by Laguerre for the particular case of two uniform elongated ellipsoids of rotation, led to the formula

$$E_{1,2} = \frac{9}{16}\frac{(e/2)^2}{\pi}\int_{-1}^{+1}\int_{-1}^{+1}\int_{0}^{\pi}\frac{dt\,dt_0\,d\varphi\,(1-t^2)(1-t_0^2)}{2c - t\sqrt{c^2-b^2} - t_0\sqrt{c^2-b^2}}. \tag{7}$$

The evaluation of this formula led to a quite cumbersome expression containing more than 100 terms. The calculation, which was carried out for $c/b = 2$, agreed within 3% with the corresponding data of the approximate method given below, which was used for all the calculations.

If one also takes into account that the mutual energy comprises only a part of the total energy, then the possible error in the calculation of the total energy will be of the order of a fraction of a percent. For given charge $(e/2)$ of each ellipsoid and given distance between the centers $2c$, the electrostatic energy of interaction is easily found from the two limiting cases:

at $b = c$ spheres, the thin lines in Fig. 2,

$$E_{1,2} = \frac{(e/2)^2}{2c}; \tag{8}$$

at $b \to 0$, "sticks", the dashed lines in Fig. 2,

$$E_{1,2} = \frac{(e/2)^2}{1.74c}. \tag{9}$$

In the interval of interest,

$$0 < b < c \tag{10}$$

| $E/O$ | $c/b$ | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| 2.60 | 3.581 | 3.414 | 3.318 | 3.312 | 3.493 |
| 2.40 | 3.402 | 3.254 | 3.118 | 3.180 | 3.293 |
| 2.20 | 3.231 | 3.095 | 3.038 | 3.050 | 3.05 |
| 2.00 | 3.047 | 2.939 | 2.898 | 2.918 | 2.929 |
| 1.80 | 2.869 | 2.779 | 2.755 | 2.789 | 2.807 |
| 1.70 | 2.778 | 2.70 | 2.686 | 2.723 | 2.747 |
| 1.65 | 2.734 | 2.659 | 2.650 | 2.69 | 2.718 |
| 1.60 | 2.689 | 2.620 | 2.614 | 2.658 | 2.687 |
| 1.40 | 2.511 | 2.462 | 2.473 | 2.527 | 2.565 |

we interpolate according to the formula

$$E_{1,2} = \frac{(e/2)^2}{\sqrt{3.04c^2 + 0.96b^2}}. \tag{11}$$

The form of formula (11) reflects the very simple conceptions of the dependence of the energy $E_{1,2}$ on $b$; the coefficients in (11) are chosen such that both limiting expressions (8) and (9) are satisfied.

The results of the calculations are summarized in the table. For various ratios $E/O$ of the original nucleus (Column 1) we show the energies with respect to the surface energy of the original nucleus of two contacting ellipsoids resulting from it for various values of $c/b = 1, 2, 3, 4, 5$.

As is clear from the table, the minimum energy is attained in the interval of $E/O$ under consideration at

$$3 < \frac{c}{b} < 4. \tag{12}$$

This energy is less than the energy of the original nucleus at $E/O > 1.65$.

A nucleus for which relation (3) holds cannot be in the form of a sphere. It also cannot decay through the form of two contacting spheres. However, our calculation shows that decay through two elongated contacting ellipsoids is not prohibited.

For the interval

$$1.65 < E/O < 2 \tag{13}$$

our data on the change in energy in the process of decay are shown in Fig. 3 where all notations are taken from Fig. 1.

The segment $OMA$ (where $M$ is the maximum of the energy) is borrowed from Bohr. The position of the point $D$, which corresponds to two ellipsoids, is taken from our data. Since in the interval (13) $D$ is located below $B$, there is no basis for assuming the presence of any additional maxima in the interval. If we restricted our attention to the spheres (segment $BC$),

we would reach completely different conclusions having no connection with reality.

If we consider non-symmetric pear-shaped forms rather than ellipsoid we will undoubtedly decrease the lower boundary (13).

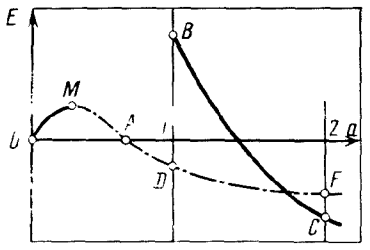



*Fig. 3*

Finally, comparing the energy at the point $D$ with the energy at the point $M$, but not at the point $O$ (Fig. 3) we must obtain $E_D < E_M$ for any value of $E/O$.

It is interesting that after separation by a large distance the energy of two ellipsoids, naturally, proves larger than the energy of two separated spheres (see the positions of the points $F$ and $C$ at $a = 2$ in Figs. 1 and 3).

The energy difference at $c/b \sim 3.5$, $E/O \sim 1.8$ attains about $0.078\,O$ ($O$ is the surface energy of the original nucleus), i.e., around 42 MeV for each nucleus that forms.

The excitation energy of the nucleus will first of all be directed to evaporation of neutrons. For a comparatively small binding energy of neutrons in the nuclei of the fragments with an anomalous ratio of the charge to the mass one could thus explain the release of a large number of neutrons per fission event, as well as the observed, sometimes very large energies of the neutrons [4].

In fact, even in the case when the fission occurs through the form of two contacting ellipsoids, their form changes as they move away from one another. Calculations of the part of the energy of deformation of the fragments which goes to kinetic energy and the part which in the form of excitation energy may be used for the evaporation of neutrons, is the task of nuclear fluid dynamics, an area which is completely undeveloped.

Our elementary calculations have one meaning—they indicate the order of magnitude of the possible energy of excitation.

In any case, evaporation of neutrons by excited fragments seems more likely to us than the mechanism proposed by N. Bohr and J. Wheeler. They note that in the division of one droplet into two, there usually form several small droplets at the point where the connection breaks and they identify neutrons precisely with these small droplets.

*Institute of Chemical Physics*
*Leningrad*

*Received*
*June 22, 1940*

## REFERENCES

1. *Bohr N.*—Phys. Rev. **56**, 426 (1939) [see Bibliography therein].
2. *Zeldovich Ya. B., Khariton Yu. B.*—ZhETF **9**, 1425 (1939); **10**, 475 (1940) [and earlier references therein].
3. *Frenkel J.*—Phys., USSR **1** (1939).
4. *Joliot F.*—C. r. Acad. Sci. **208**, 341 (1939).

## *Commentary*

A calculation is performed of the energy of two fragments in contact at the moment of decay of a nucleus undergoing fission. The form of the splitting nucleus is modeled using two ellipsoids in contact. An estimate is obtained of the difference in the energy between nonspherical contacting fragments and spherical fragments flying apart which is transformed into excitation energy (thermal energy) of the fragments ($\sim$ 40 MeV). The magnitudes of this energy are sufficient to explain the release of fission neutrons by evaporation from the excited fragments. Thus, the authors propose a new mechanism for the emission of secondary fission neutrons which differs from that considered by N. Bohr and J. Wheeler, who assumed that the neutrons are emitted by the "neck" at the moment of fission.

Even the first experiments by J. Fraser and J. Milton[1] on the angular distribution of neutrons (the authors studied the fission of $U^{233}$ by thermal neutrons) showed a clear correlation between the direction of neutron emission and the direction of motion of the fragments and in this way completely confirmed the validity of the Zeldovich-Zysin mechanism.

Subsequent experiments and theoretical calculations continued and deepened the picture of emission of fission neutrons. Thus J. Terrell,[2] recalculating the distribution curves of the energy and masses of the fragments for $U^{233}$, $U^{235}$ and $Cf^{252}$, estimated the number of secondary neutrons $\nu$ as a function of the fragment mass and showed that the $\nu$ of a light fragment is larger than that of a heavy fragment. An investigation by H. Bowman, S. Thompson, J. Milton and W. Swiatecki[3] was devoted to the study of the dependence of the number of secondary neutrons on the mass of the fragment, its critical energy, and the angle of emission for the spontaneous fission of $Cf^{252}$. Using the measurement of the flight time of the neutrons and fragments, they found a strong correlation between the directions of motion of the fragment and neutron and measured the ratios of the numbers of neutrons emitted from a light fragment, a heavy fragment and at an angle of 90° to the direction of motion of the fragments. These ratios turned out to be equal to 9 : 5 : 1. In a paper by R. Vanden Bosch[4] it was shown that the experimentally observed kinetic energy and number of fission neutrons require for their explanation the introduction of a shell-like dependence of the rigidity of the fragments with respect to quadrupole deformation (rigidity is defined to be the co-

[1] *Fraser J. S. A., Milton J. C. D.*—Phys. Rev. **93**, 818–824 (1954).

[2] *Terrell J.*—Phys. Rev. **127**, 880–904 (1962).

[3] *Bowman H., Thompson S., Milton J., Swiatecki W.*—Phys. Rev. **129**, 2120–2147 (1963).

[4] *Vanden Bosch R.*—Nucl. Phys. **46**, 129 (1963).

efficient $c$ in the expression for the quadrupole energy of deformation: $E = c\alpha^2/2$). Subsequent calculations have usually used the method of Strutinskiĭ to account for shell corrections.

Thus, this paper by Ya. B. and Yu. A. Zysin gave the correct mechanism for the emission of secondary fission neutrons and stimulated a large number of experimental and theoretical studies on the physics of fission (see the collection *Achievements of the Physics of Nuclear Fission*).[5]

We should remark separately on the statement at the beginning of the article about "refuting Ya. I. Frenkel's arguments in favor of the existence of significantly non-spherical nuclei." Today it is well known that nonspherical nuclei exist and that, in fact, Ya. I. Frenkel was right. It is also true, however, that this is related to shell rather than electrostatic effects, and also that nuclear fission is not a proof of nonsphericality.

[5] *Uspekhi fiziki deleniia iader [Achievements of the Physics of Nuclear Fission].* Collection of Papers. Moscow: Atomizdat, 307 p. (1965).

# 5

# Storage of Cold Neutrons*

The idea of retaining slow neutrons has been mentioned many times, but the corresponding experiments have not yet been performed, and the literature does not contain even rough estimates pertaining to this problem.

It is known that slow neutrons experience total internal reflection in glancing incidence on the surface of most substances. At sufficiently low velocities, the neutrons cannot penetrate in such a substance even under normal incidence. Thus, for carbon with a density $\sim 2$ g/cm$^3$ the critical neutron velocity is close to 5 m/s, for beryllium it is approximately 7 m/s. Let us place neutrons in a cavity surrounded on all sides by graphite. The neutrons of speed higher than critical will rapidly leave the cavity, but neutrons of less than critical speed are blocked in the cavity and vanish only as they decay, with a half-life of approximately 12 minutes. Such slow neutrons will penetrate into the wall only a depth on the order of their wavelength; taking into account dimensionless factors, the depth is $\sim 10^{-6}$ cm. Therefore if the cavity has a considerable volume, the fraction of the time that the neutrons stay in the material of the shell is quite small; for a one-cubic-meter cavity this fraction is $\sim 10^{-7}$.

The capture cross section of carbon ($4.5\times10^{-27}$ cm$^2$ at $v = 2.2\times10^5$ cm/s) obeys the $1/v$ law and corresponds to a neutron lifetime in carbon of $\sim 0.01$ s regardless of its velocity. For neutrons in a cavity we obtain an absorption time of $0.01/10^{-7}\text{s} = 10^5\ s = 1$ day. Slow neutrons will also be lost, as they acquire energy by collision; obviously, however, this process is greatly suppressed, because the neutrons are for the most time in the cavity and not in the material of the shell.

The most difficult feat is to obtain a sufficient number of such neutrons. For a Maxwellian distribution at room temperature, the fraction of such neutrons is on the order of $10^{-8}$.

It is advisable first to cool the neutrons in a volume filled with liquid helium, and then the fraction of the necessary neutrons increases to $10^{-5}$. As a result of the long life of the slow neutrons in the cavity, their number density after a few seconds becomes equal to the Maxwellian equilibrium density. The principal difficulty is connected with the need for having a large volume of liquid helium, because of the long range of the neutrons in helium (50 cm).

*Zhurnal eksperimentalnoĭ i teoreticheskoĭ fiziki **36** (6), 1952–1953 (1959).

With a fully moderated neutron flux of $10^{12}$ $cm^{-2}s^{-1}$ from a reactor, the flux of neutrons emitted with a temperature of 3°K can amount to $10^{11}$ $cm^{-2}s^{-1}$, which corresponds at an average velocity on the order of $2 \times 10^4$ cm/s to a density of $5 \times 10^6$ $cm^{-3}$ of thermal neutrons, including 50 $cm^{-3}$ slow ones (with velocity less than 500 cm/s). Thus, under the most favorable assumptions, it is possible to accumulate up to $5 \times 10^7$ slow neutrons in a cavity 1 $m^3$ in volume.

By placing a graphite partition over the opening that joins the cavity with the liquid helium it is possible to remove the cavity with slow neutrons from the reactor and make the measurements at a small background.

It may prove advantageous to use a palliative variant without helium, by cooling neutrons, say, to 70°K and accumulating up to $10^5$ neutrons. We note that the index of refraction of the moderator should be less than the index of refraction of the cavity material, or else the moderator will not admit necessary neutrons in from the vacuum, and consequently will not let any out.

An experiment of this type is quite difficult, but it seems that it can give experimenters a valuable method of investigating the interaction of slow neutrons with substances introduced into the cavity. By introducing an $(n, \gamma)$ absorber of neutrons into the cavity, it is easy to measure the number of neutrons left intact at the instant of observation.

We note that the neutrons in the cavity can be effectively heated to a speed above critical by mechanical displacement of the graphite surfaces at a speed of several meters per second.

The theory of the coefficient of refraction and the total internal reflection of neutrons is well known; we note only that it remains valid also at those small energies, at which the absorption cross section, following the $1/v$ law, becomes equal to or greater than the scattering cross section. It is easy to verify that the imaginary part of the pseudo-potential, the part describing the absorption, is small compared with the real part, which describes the scattering. Their ratio is equal to $\sqrt{\pi\sigma_s}/\lambda_1$, where $\lambda_1$ is the wavelength of the neutron for which $\sigma_s = \sigma_a$. Consequently, in the case of total internal reflection, absorption does not change the exponential law of damping of the wave function of the neutron in the medium.

*Received*
*April 3, 1959*

## *REFERENCES*

1. *Hughes D. J. Pile Neutron Research.* Cambridge, Mass.: Addison-Wesley (1953).
2. *Vlasov N. V. Neĭtrony [Neutrons].* Gostekhizdat, 427 p. (1955).

## Commentary

A method is proposed for containment of very slow neutrons in a cavity, based on the fact that, beginning from some critical energy (velocity) the refraction index of the neutrons attains a zero value and complete internal reflection of neutrons occurs at any angle of incidence. "Zeldovich's nuclear bottle" opened an area of neutron physics which is now undergoing rapid development—the physics of ultra-cold neutrons (UCN).[1] Using UCN measurements are made of the electrical dipole moment of a neutron (to date only an estimate of its upper bound has been achieved) in connection with the violation of symmetry in time reversal, the lifetime of the neutron, related states of the neutron in matter and much more. All of this has its roots in this ground-laying paper by Ya. B. We note also another method—the magnetic method of neutron containment,[2] proposed by V. V. Vladimirskiĭ shortly after the paper by Ya. B. Vladimirskiĭ's method may prove more convenient for the exact determination of the probability of beta-decay of a neutron. However, for certain important physical experiments a magnetic field, which interacts with a neutron, is inadmissible. This relates to the measurement of the electrical dipole moment of a neutron and to the detection of neutron-antineutron oscillations.

[1] *Shapiro F. L.*—In: *Nuclear Structure Study with Neutrons.* Plenum Press, 259 p. (1974).

[2] *Vladimirskiĭ V. V.*—ZhETF **39**, 1062–1070 (1960).

# 6

# Quasistable States with Large Isotopic Spin in Light Nuclei*

We consider an odd nucleus $A$ with one excess neutron, with a minimum value of isotopic spin $T = 1/2$ in the ground state, and with a neutron binding energy $Q$. The excited states of the nucleus $A^*$ with excitation energy $E > Q$ have as a rule a rather large probability of neutron emission, i.e., a large width $\Gamma_n$ of the process $A^* \to B + n$, where $B$ is an even nucleus.

Let the ground state of the nucleus $B$ have $T = 0$, and let the state $B^*$ with $T = 1$ have an excitation energy $\Delta$. We assume that the nucleus $A$ has an excited state $A_3^*$ with $T = 3/2$ and excitation energy $E_3$ such that $Q < E_3 < Q + \Delta$. The decay of $A_3^*$ to $B^* + n$ is energetically impossible, while the decay of $A_3^*$ into $B + n$ proceeds via a change in isotopic spin and should therefore have an anomalously small width $\Gamma_n$. The state $A_3^*$ is quasistable and should appear in a unique manner in the scattering of neutrons by nuclei $B$, and also in the photoeffect $A + \gamma = B + n$.

When $n$ is scattered by $B$ the isotopic spin of the system in the initial state is $T = 1/2$, and it is usually assumed that states with $T = 3/2$ should make only a small contribution to the scattering cross section. However, if a quasistate exists, then sharp scattering resonance takes place at a neutron energy $E_n = E_3 - Q$, with a maximum cross section

$$4\pi\lambda^2 \frac{2J+1}{2S+1}.$$

The low probability of the process, connected with the disturbance of the isotopic spin, manifests itself not in a reduction in the scattering cross section, but in a reduction of the width of the resonance scattering. Therefore observation of resonance is quite possible if the neutrons are sufficiently monochromatic.

At resonance the increase in the scattering cross section will be accompanied by an increased probability of the process $B(n,\gamma)A$, since $\sigma_{n,\gamma}/\sigma_{sc} =$

*Zhurnal eksperimentalnoĭ i teoreticheskoĭ fiziki **38** (1), 278–280 (1960).

$\Gamma_\gamma/\Gamma_n$ and an anomalously small $\Gamma_n$ should give[1] an anomalously large $\Gamma_\gamma/\Gamma_n$. Incidentally, the inequality $\Gamma_\gamma/\Gamma_n \ll 1$ remains in force, since $\Gamma_n \sim e^2$, when the isotopic spin is disturbed by the Coulomb interaction, like $\Gamma_\gamma$, which contains, however, other small factors, $(v/c)^u$, $(R/\lambda\!\!\!^{-})^v$, and $\hbar/Mc\lambda\!\!\!^{-})^w$ in degrees that depend on the type of transition (for $E1$: $u = 2$, $v = 2$, and $w = 0$; for $M1$: $u = v = 0$, and $w = 2$, etc.).

The existence of a quasistable $A_3^*$ should lead to a narrow resonance in the reverse process[2] $A(\gamma, n)B$ and also to resonant scattering of $\gamma$ by $A$. Incidentally, owing to the inequality $\Gamma_\gamma/\Gamma_n \ll 1$, the latter process can apparently not be observed.

The state $A_3^*$ forms an isotopic multiplet with the ground state of the nucleus with three excess neutrons, and, by introducing a known Coulomb correction, it is possible to determine the expected position of the quasistable level. Thus, knowing the masses [1] of the boron isotopes $B^{12}$ and $B^{13}$, it is possible to determine the energies of the corresponding states of $C_2^{12*}$ ($T = 1$) and $C_3^{13*}$ ($T = 3/2$). The result (in our notation) is $E_3 = 11.2$ MeV at $Q = 4.95$ MeV and $\Delta = 11.54$ MeV. Consequently, the level $C_3^{13*}$ should be quasistable, since its energy is insufficient for decay into $C_2^{12*} + n$.

One should expect a narrow resonance in the scattering of $n$ by $C^{12}$ at $E_n = 11.2 - 4.95 = 6.25$ MeV, corresponding to a neutron energy of 7.20 MeV in the laboratory system.

From the similarity between $C^{12}$ in the state with $T = 3/2$ and the ground state of $B^{13}$ one expects $C_3^{13*}$ to be in the state $3/2^-$, which leads to a scatter of neutrons in the state $P_{3/2}$ on $C^{12}$, with a cross section

$$4\pi\lambda\!\!\!^{-2}\frac{2J+1}{2S+1} = 0.8 \text{ barn}$$

A relatively narrow resonance was observed experimentally [2] at $E_n = 6.30$ MeV, along with a superposition of two resonances at $E_n = 7.4$ and 8.7 MeV.

The state of interest to us can be investigated by studying the angular distribution and polarization of the scattered neutrons. On the other hand, at least in principle, there is a possibility of ascertaining the existence of the unknown isobars by resonance in the scattering of neutrons by stable nuclei. Thus, narrow resonance in neutron scattering on $Be^{10}$ or $C^{14}$ could denote the existence of stable (with respect to emission of neutrons) nuclei $Li^{11}$ or $B^{15}$.

[1] $A_3^* \to A + \gamma$ is allowed, $\Gamma_\gamma$ has a normal value.

[2] It is possible that the best method of observing the quasistable level is to let the reaction proceed against the continuous spectrum of bremsstrahlung and to determine the maxima in the spectrum of the emitted neutrons from the time of flight, using a pulsed $\gamma$ source.

I take this opportunity to express my gratitude to V. I. Goldanskiĭ for discussions.

*Institute of Theoretical*
*and Experimental Physics*
*USSR Academy of Sciences. Moscow*

*Received*
*September 24, 1959*

## *REFERENCES*

1. *Ajzenberg-Selove F., Lauritsen T.*—Nucl. Phys. **11**, 5 (1959).
2. *Bondelid R., Dunning K. L., Talbott F. L.*—Phys. Rev. **105**, 193 (1957).

## *Commentary*

This paper was the first to indicate that since the isotopic spin in light nuclei is a sufficiently good quantum number, the decay of a level with $T = 3/2$ per nucleus to a state with $T = 0$ and a neutron should be forbidden.

Generalizing the idea of the paper, one may say that it was the first to point out the possibility of the existence of peculiar "isospin isomers"—narrow nuclear levels which are higher than the bonding energy of a nucleon, but which decay not through strong, but through electromagnetic interaction since the release of a nucleon from these levels is possible only through a change in the isotopic spin ($\Delta T = 1$).

Unfortunately, the paper proposes far from the best means of populating such narrow levels—through neutron scattering or the photo-effect. This means has still not been realized.

However, as was first shown by V. I. Goldanskiĭ, "isospin isomers" should be clearly evident in processes of release of $\beta^+$-delayed protons[1] and proton pairs[2] in the form of so-called analog states, populated at superallowed ($\Delta T = 0$) $\beta^+$-decay of the mother-nuclei, for example, $^{33}\mathrm{Ar} \longrightarrow {}^{33}\mathrm{Cl}^* \longrightarrow p + {}^{32}\mathrm{S}$ or $^{22}\underset{T=2}{\mathrm{Al}} \xrightarrow{\beta^+} {}^{22}\underset{T=2}{\mathrm{Mg}}{}^* \longrightarrow 2p + {}^{20}\underset{T=0}{\mathrm{Ne}}$. Many such examples have indeed been observed in experiments. The paper also predicts the existence of nuclei $^{11}\mathrm{Li}$ and $^{15}\mathrm{B}$ which are stable with respect to neutron decay; these were later discovered in experiments.

[1] *Goldanskiĭ V. I.*—Dokl. AN SSSR **146**, 1309–1311 (1962).
[2] *Goldanskiĭ V. I.*—Pisma v ZhETF **32**, 572–574 (1980).

# 7

# New Isotopes of Light Nuclei and the Equation of State of Neutrons*

The limits of stability (relative to nucleon emission) of light nuclei are considered. The existence (in the sense of stability against decay with emission of a nucleon) of the following nuclei is predicted: $He^{8}$, $Be^{12}$, $O^{13}$, $B^{15,17,19}$, $C^{16-20}$, $N^{18-21}$, $Mg^{20}$. The problem of the possibility of existence of heavy nuclei composed of neutrons only is considered. The problem is reduced to that of a Fermi gas with a resonance interaction between the particles. The energy of such a gas is proportional to $\omega^{2/3}$, where $\omega$ is its density. The accuracy of the calculations is not sufficient to determine the sign of the energy and answer the question as to the existence of neutron nuclei.

The problem of the possible isotopes has been treated by Nemirovskiĭ [1, 2] for $8 \leq Z \leq 84$, and by Baz [3] for the region $17 \leq A \leq 40$. The former uses the one-particle approximation, with an attempt to find the dependence of the parameters of the well on the numbers of neutrons and protons. For nuclei with an excess of protons Baz bases his discussion on the experimental data on the mirror nuclei (with excess of neutrons) and on the well-known expression for the Coulomb energy. For nuclei with an excess of neutrons he extrapolates the binding energy in a series of nuclei with constant isotopic spin.

These papers predict the existence of many as yet unknown $\beta$-active isotopes. In the table given below the isotopes so predicted are enclosed in dashed-line squares. One of them ($O^{20}$) has very recently been observed experimentally [4].

In the present paper (Sec. 1) we make additional predictions in the region of the lightest nuclei; the isotopes so predicted are enclosed in solid-line squares in the table. We point out particularly the conclusion that there is a large probability that $He^{8}$ exists. For nuclei with an excess of neutrons the writer has tried to take the effect of shells and the pair interaction of neutrons into account as accurately as possible.

In Sec. 2 the question is raised of the existence of nuclei composed solely of neutrons. In the limiting case of a large number of neutrons, by using the data on resonance in the $^{1}S$ scattering, one can find the general form of the dependence of the energy on the density of the nuclear matter, but the

*Zhurnal eksperimentalnoĭ i teoreticheskoĭ fiziki **38** (4), 1123–1131 (1960).

accuracy of the first approximation obtained in this paper is insufficient to give a definite answer to the question of the existence of such nuclei.

## 1. Light Nuclei

Following the method of Baz [3], one easily convinces oneself that there should exist a nucleus $O^{13}$ with a proton binding energy not smaller than 1.2 MeV and with $\beta^+$-decay energy 16 to 17 MeV. Using the data [4] on the mass of $O^{20}$, we conclude that the mirror nucleus $Mg^{20}$ should exist with proton binding energy not less than 2.7 MeV and $\beta^+$-decay energy about 7 MeV. The existence of $O^{12}$, $Ne^{16}$, and $Mg^{19}$ is not excluded (empty spaces in the table);[1] the corresponding mirror isotopes $Be^{12}$, $C^{16}$, and $N^{19}$ are predicted in this paper (see later argument), but their energies cannot be predicted with enough accuracy to give a definite conclusion about $O^{12}$, $Ne^{16}$, and $Mg^{19}$. The isotopes $Ne^{17}$, $Na^{19}$, $Mg^{21}$, and $Mg^{22}$ are predicted by Baz.

Regarding all the other nuclei in the upper right-hand part of the table we can assert with assurance that they are unstable against emission of a proton, i.e., they do not exist, which is shown in the table by the minus signs in all the upper cells.

Let us turn to the nuclei with an excess of neutrons. A nucleus with an excess of neutrons does not exist in the case in which all the discrete levels are already filled up with neutrons. An important point here is that the nuclear forces fall off rapidly with distance, and therefore the number of levels in the field of the nuclear forces is limited (in contradistinction, for example, to the case of the Coulomb field). With the spin taken into account the number of levels is always even; therefore if a nucleus exists containing an odd number of neutrons $(2n + 1)$, then there is also a place for a subsequent $(2n + 2)$-nd neutron. On account of the mutual attraction of a pair of neutrons the binding energy of the $(2n+2)$-nd neutron is always larger than that of the preceding $(2n + 1)$-st neutron.

In each cell of the table that corresponds to an experimentally known isotope there is written the binding energy of the last neutron. It is easily verified that in all cases $E_{2n+2} > E_{2n+1}$. Therefore the existence of the nuclei $Be^{12}$ and $C^{16}$ definitely follows from the existence of $Be^{11}$ and $C^{15}$. As a rough estimate, the binding energy of a neutron in $Be^{12}$ is about 2–3 MeV, and the $\beta$-decay energy is 12–13 MeV; for $C^{16}$ these values are 3–4 MeV and 8–9 MeV, respectively.

It is much harder to settle the existence of other isotopes. Extrapolation

[1] These nuclei may be unstable with respect to the emission of two protons at once. On the other hand, at the limit of stability the expression for the Coulomb energy of the last proton, $1.2(Z-1)A^{-1/3}$, gives too large a result; for example, in the pair $Li^8$–$B^8$ we have for $Li^8$ the binding energy $Q_n = 2$ MeV and for $B^8$ the value $Q_p = 0.2$ MeV, so that the difference is 1.8 MeV, whereas by the formula we would get $1.2 \times 4 \times 7^{-1/3} = 2.5$ MeV.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $p^1$ | — | — | — | — | — | — | — | — | — | — | — |
| $N^1$ | $D^2$ 2,23 | $He^3$ | — | — | — | — | — | — | — | — | — | — |
| — | $T^3$ 6,26 | $He^4$ 20,58 | — | — | — | — | — | — | — | — | — | — |
| — | — | — | $Li^6$ | $Be^7$ | $B^8$ | $C^9$ | — | — | — | — | — | — |
| | $?H^5$ | $He^6$ 0,94 | $Li^7$ 7,25 | $Be^8$ 18,10 | $B^9$ 18,58 | $C^{10}$ 21,9 | — | | — | — | — | — |
| | | — | $Li^8$ 2,04 | $Be^9$ 1,67 | $B^{10}$ 8,44 | $C^{11}$ 13,3 | $N^{12}$ | $O^{13}$ | — | — | — | — |
| | | $He^8$ | $Li^9$ 3,58 | $Be^{10}$ 6,81 | $B^{11}$ 11,50 | $C^{12}$ 18,73 | $N^{13}$ 20,9 | $O^{14}$ | — | | — | — |
| | | — | — | $Be^{11}$ 0,55 | $B^{12}$ 3,36 | $C^{13}$ 4,95 | $N^{14}$ 10,55 | $O^{15}$ 13,23 | — | $Ne^{17}$ | — | |
| | | — | | $Be^{12}$ | $B^{13}$ 4,89 | $C^{14}$ 8,17 | $N^{15}$ 10,84 | $O^{16}$ 15,66 | $F^{17}$ 17,01 | $Ne^{18}$ | $Na^{19}$ | $Mg^{20}$ |
| | | | | — | | $C^{15}$ 1,23 | $N^{16}$ 2,50 | $O^{17}$ 4,15 | $F^{18}$ 9,77 | $Ne^{19}$ 11,39 | $Na^{20}$ | $Mg^{21}$ |
| | | | | | $B^{15}$ | $C^{16}$ | $N^{17}$ 5,84 | $O^{18}$ 8,07 | $F^{19}$ 10,42 | $Ne^{20}$ 16,92 | $Na^{21}$ 18,57 | $Mg^{22}$ |
| | | | | — | | $C^{17}$ | $N^{18}$ | $O^{19}$ 3,96 | $F^{20}$ 6,61 | $Ne^{21}$ 6,76 | $Na^{22}$ 11,05 | $Mg^{23}$ |
| | | | | | $B^{17}$ | $C^{18}$ | $N^{19}$ | $O^{20}$ 7,65 | $F^{21}$ 8,18 | $Ne^{22}$ 10,36 | $Na^{23}$ 12,42 | $Mg^{24}$ 16,4 |
| | | | | — | | $C^{19}$ | $N^{20}$ | $O^{21}$ | $F^{22}$ | $Ne^{23}$ 5,20 | $Na^{24}$ 6,97 | $Mg^{25}$ 7,34 |
| | | | | | $B^{19}$ | $C^{20}$ | $N^{21}$ | $O^{22}$ | $F^{23}$ | $Ne^{24}$ 8,90 | $Na^{25}$ 9,15 | $Mg^{26}$ 11,12 |
| | | | | | | | | $O^{23}$ | $F^{24}$ | $Ne^{25}$ | $Na^{26}$ | $Mg^{27}$ 6,44 |
| | | | | | | | | $O^{24}$ | $F^{25}$ | $Ne^{26}$ | $Na^{27}$ | $Mg^{28}$ 8,56 |
| | | | | | | | | $O^{25}$ | $F^{26}$ | $Ne^{27}$ | $Na^{28}$ | $Mg^{29}$ |
| | | | | | | | | $O^{26}$ | $F^{27}$ | $Ne^{28}$ | $Na^{29}$ | $Mg^{30}$ |
| | | | | | | | | $O^{27}$ | $F^{28}$ | $Ne^{29}$ | $Na^{30}$ | $Mg^{31}$ |
| | | | | | | | | $O^{28}$ | $F^{29}$ | $Ne^{30}$ | $Na^{31}$ | $Mg^{32}$ |

for fixed isospin [3] $T$ is not reliable, since it involves comparison of neutrons that are in different shells.

For the lightest nuclei the idea of a smooth dependence of the parameters of the well on $N$ and $Z$ [1, 2] does not take sufficient account of the individual peculiarities of the shells. We shall try to make maximum use of the experimental data. It is known from the scattering of neutrons by $He^4$ that for the partial wave $P_{3/2}$ there is a resonance at the energy +1.0 MeV (i.e., in the continuous spectrum) with width 0.55 MeV (which corresponds to an $He^5$ lifetime of $10^{-21}$ s). The nucleus $He^5$ does not exist, and consequently there is no discrete bound state of a neutron in the field of $He^4$.

In the same sense, the dineutron does not exist, since from experiments on the scattering of neutrons by protons it is known that in the $^1S$ state, which is allowed for two neutrons by the Pauli principle, the attraction is not sufficient for the formation of a bound state. Therefore the $He^6$ nucleus is a remarkable system of three particles ($n+n+He^4$), which are not bound together in pairs, but all three together form a bound system. Quite crudely we can imagine that $He^6$ consists of two neutrons in the state $(P_{3/2})^2$ in the field of $He^4$. The energy of interaction between the two neutrons (about −3 MeV) is more than enough to compensate for the positive energy of each neutron in the state $P_{3/2}$ (+1 MeV) in the field of $He^4$.

The $P_{3/2}$ shell has four places in all. Therefore we can raise the question of the possibility of $He^7$ and $He^8$. According to Kurath [5], in the limit of small range of the forces and weakly bound nucleons, and for large radius of the orbits of the shell ($r_0 \ll r_1$, $L = 3K$, in his notation), one gets a simple result: if the energy of interaction of two neutrons is $B$, then the energy of the interaction of three neutrons is also $B$, and the energy of the interaction of four neutrons is $2B$, i.e., the neutrons combine in pairs, as it were. From this there follows the conclusion that $He^7$ does not exist, but $He^8$ exists; the expected binding energy of a neutron is 0.5–0.8 MeV, and the $\beta$=decay energy is about 12 MeV. It would be extremely desirable to verify the existence of $He^8$ experimentally and determine its binding energy.

How accurately the rule of the combining of neutrons in pairs in a single shell around a doubly magic (closed) core holds experimentally can be seen from two examples.[2]

1) The filling up of the $d_{5/2}$ shell on the closed $O^{16}$ (see table). We quote the binding energies (in megavolts). The subscript on $E$ is the number of neutrons in the $d_{5/2}$ shell (the upper index is the atomic weight):

$$E_1^{17} = 4.15, \quad E_2^{18} = 8.07, \quad E_3^{19} = 3.96, \quad E_4^{20} = 7.65.$$

There are no data on $E_5$ and $E_6$, which finish the filling of the shell; the nuclei $O^{21}$ and $O^{22}$ have not yet been observed.

2) The filling up of the $f_{7/2}$ shell on $Ca^{40}$, which has closed shells (this

[2]The mass data are taken from review articles [6–8].

example has been treated partially by Nemirovskiĭ [2]). The binding energies are:

| $E_1^{41}$ | $E_2^{42}$ | $E_3^{43}$ | $E_4^{44}$ | $E_5^{45}$ | $E_6^{46}$ | $E_7^{47}$ | $E_8^{48}$ |
|---|---|---|---|---|---|---|---|
| 8.3 | 11.4 | 8.0 | 11.4 | 7.4 | 11.0(?)[3] | 6.8?[3] | 10.8 |

At the end of the filling-up of the $f_{7/2}$ shell the binding energy $E$ falls sharply: $E_9^{49} = 5.1$. Since $He^4$ is a closed doubly magic nucleus (and an even more stable one than $O^{16}$ and $Ca^{40}$), these examples speak convincingly for the existence of $He^8$.

If the proton shell is not filled, then $E$ drops off extremely sharply within the range of the given neutron shell; we may imagine that the first neutrons unite in pairs with the "free" protons (those outside the closed shells), and later neutrons can no longer do this. As an example let us consider the $d_{5/2}$ shell of $Ne^{18}$—a nucleus with two protons beyond $O^{16}$. We have:

| $E_1^{19}$ | $E_2^{20}$ | $E_3^{21}$ | $E_4^{22}$ | $E_5^{23}$ | $E_6^{24}$ |
|---|---|---|---|---|---|
| 11.4 | 16.9 | 6.8 | 10.4 | 5.2 | 8.9 |

If the proton shell falls short of being closed by one, two, or three protons, the binding energy of the neutrons is decreased as compared with the binding to a closed shell (cf. $C^{15}$, $N^{16}$, and $O^{17}$ in the table). But within the limits of a given neutron shell (on a core with holes in the proton shell) $E$ varies little, in contradistinction to the case in which excess protons are present.

We give examples of the filling of the $f_{7/2}$ shell with neutrons in nuclei with unfilled proton shells:

| | | | | |
|---|---|---|---|---|
| Nucleus $K_{19}^{39}$: | $E_1^{40} = 7.9$ | $E_2^{41} = 10.0$ | $E_3^{42} = 7.4$ | $E_4^{43} = 10.8$ |
| Nucleus $Ar_{18}^{38}$: | $E_1^{39} = 6.7$ | $E_2^{40} = 9.7$ | $E_3^{41} = 6.1$ | |

Thus we can formulate the rule that on nuclei with closed proton shells and with holes in the proton shell (but not on nuclei with excess protons), the binding energies of the odd neutrons are practically constant within the limits of a given neutron shell. The binding energies of the even neutrons are also constant within the limits of a given shell, but are larger by the amount of the pairing energy. Carrying this rule over to the $d_{5/2}$ shell, we come to the conclusion that the experimental fact of the existence of bound $d_{5/2}$ states in the nuclei $C^{15}$ and $N^{16}$, $N^{17}$ guarantees the possibility of filling up the entire $d_{5/2}$ shell, to $C^{20}$ and $N^{21}$, respectively.

An examination of the binding energies of neutrons in the table reveals a regular increase of $E$ in each row, with increase of the number of protons (the single exception is the pair $Li^8$–$Be^9$, which is due to the special structure of $B^8$). Extrapolation of $E$ to the left along the rows makes probable the

[3]The nucleus $Ca^{46}$ has not been studied, so that one knows experimentally only the sum $E_6^{46} + E_7^{47} = 17.8$; the separate terms in the table are obtained by interpolation.

existence of $B^{15}$, and from this—by the principle of the constancy of the binding energy in a shell—of $B^{17}$ and $B^{19}$. The existence of the nuclei with odd numbers of neutrons, $B^{14}$, $B^{16}$, $B^{18}$, remains questionable. With considerable assurance we can assert that the odd (in $n$) nuclei $Be^{13}$, $Be^{15}$, $Be^{17}$, $Li^{10}$ do not exist.

On the whole, however, the assertions that can be made reliably about nuclei with excess neutrons not known to exist are extremely weak. From studies of scattering only the nonexistence of $n^2$ and $He^5$ is quite accurately proved. From principles of the pair interaction of neutrons it is obvious that $n^3$ and $He^7$ do not exist. There is no longer such certainty regarding $H^5$ ($H^5$ is entered in the table with a question mark), and the hypothesis that it exists has been suggested [9]. We note that if $n^4$ and $H^5$ existed, then there would be isotopically similar quasi-stable systems $H^4$ with $T = 2$ and $He^5$ with $T = 3/2$, which would manifest themselves in the scattering of $n$ by $T$ and of $n$ by $He^4$; this situation has been examined in detail in a separate note [10]. At present there are no experimental data in the required range of neutron energies.

Unlike the upper right-hand part of the table, which is almost solidly filled with minus signs ("does not exist"), in most of the cells of the lower left-hand part we can put neither a minus nor the symbol of a nucleus ("exists"). The obscurity of the problem of the limits of existence of isotopes with excess neutrons is a consequence of the fact that the limiting case is not clear; it is not known whether a heavy nucleus composed solely of neutrons could exist.

## 2. *The Neutron Liquid*

The problem of the limiting number of neutrons that can adhere to a heavy nucleus has been considered by Wheeler [11]; he came to the conclusion that for $Z \sim 90$–$100$ the maximum mass number is $A_{\max} \sim 500$–$600$. Wheeler used the Weizsäcker formula; Nemirovskĭ [2] correctly criticizes this formula near the limits of existence, and therefore, Wheeler's conclusions are not reliable.

Let us consider the extreme case of a very large nucleus consisting of neutrons alone. If it does exist, it surely does so only with a density much smaller than that of ordinary nuclei. Let us first examine the properties of a neutron liquid of small density; these properties are determined by the pair interactions of the neutrons at small energies (up to a few MeV). In this region only the interaction of pairs of neutrons in the $^1S$ state is of importance, and here this interaction is completely determined by the

scattering length[4] (cf., e.g., [12]):

$$a = -\left(\frac{d\ln\varphi}{dr}\right)^{-1} = -19\cdot 10^{-13}\,\text{cm};$$

the sign corresponds to the absence of a bound state, and the quantity $a$ corresponds to the so-called energy of a virtual level ($\mu$ is the reduced mass, equal to $M/2$):

$$E_v = \frac{\hbar^2}{2\mu a^2} = 0.11\ \text{MeV}.$$

We cite here the well-known calculation [13, 14] of the energy of interaction of particles in the continuous spectrum, confining ourselves at once to the $S$ wave. As usual, we consider first a spherical box for $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the coordinates of the two particles, i.e., we set $\psi(r) = 0$ at $|\mathbf{r}| = R$. Without interaction the normalized $S$-wave function in such a box is

$$\psi = \frac{\sin(n\pi r)}{r\sqrt{2\pi R}}.$$

With an interaction corresponding to scattering with the phase shift $\alpha$ we have

$$\psi = \frac{\sin[\alpha + R^{-1}(n - \alpha/\pi)\pi r]}{r\sqrt{2\pi R}},$$

which corresponds to a change of the energy of the $n$-th state given by

$$\Delta E_n = -\frac{\hbar^2 n\pi\alpha}{\mu R^2}.$$

Let us eliminate the auxiliary quantities $R$ and $n$ from the expression for $\Delta E_n$. The state under consideration is characterized by the momentum of the relative motion

$$p_n = \frac{\hbar n\pi}{R}$$

and the density at the coordinate origin in the unperturbed motion

$$\rho_n(0) = \psi^2(0) = \frac{\pi n^2}{2R^3}.$$

Let us express $\Delta E_n$ in terms of $p$ and $\rho(0)$; after this we can set $R \to \infty$, $n \to \infty$, and forget about $n$. We get

$$\Delta E = -\frac{2\pi\hbar^3\alpha\rho(0)}{\mu p}. \tag{1}$$

We express the phase in terms of the scattering length:

$$\alpha = -\tan^{-1}\left(\frac{ap}{\hbar}\right).$$

[4]For $pp$-scattering $a = -17.2$, and for $np$-scattering $a = -23.7$; we assume that $a$ depends linearly on the product of the magnetic moments.

For $E \ll E_v$, $ap \ll \hbar$ we have

$$\alpha = \frac{ap}{\hbar}, \qquad \Delta E = -\frac{2\pi\hbar^2 a\rho(0)}{\mu}; \tag{2}$$

for $E \gg E_v$, $ap \gg \hbar$ we get

$$\alpha = \frac{\pi}{2}, \qquad \Delta E = -\frac{\pi^2\hbar^3\rho(0)}{p\mu}. \tag{3}$$

Let us apply the expressions (2) and (3) to a Fermi gas consisting of neutrons only with mean density $\omega$. We single out one neutron with a definite spin direction. At the point where this neutron is located, the density of other neutrons with the same spin direction is zero by the Pauli principle; if there were no interaction, the density of the other neutrons with antiparallel spins would not differ from that of those with parallel spins on the average over all space; that is, $\omega(0) = \omega/2$. We recall that $\omega$ is the total density of neutrons with both spin directions and that the formula for $\Delta E$ contains just the density in the state without interaction.[5]

We still have to take into account the fact that the change of energy $\Delta E$ relates to a system of two particles; in order not to include the interaction of each pair twice, we recall that the decrease of the energy of one particle is $\Delta E/2$. We finally find that if for a pair of particles in the $^1S$ state $\Delta E = k\rho(0)$, where $k$ is a coefficient that depends on the momentum, then the change of the energy of all the gas in unit volume on account of the interaction is

$$U = \frac{\omega^2\overline{k}}{4}; \tag{4}$$

here $k$ is averaged over the Fermi distribution.

The Fermi distribution is characterized by the boundary momentum $p_f$, the boundary energy $E_f$, and the total kinetic energy $\mathcal{E}$ of all the gas in unit volume; as is well known,

$$\mathcal{E} = \omega\overline{E} = \frac{3}{5}\omega E_f, \qquad E_f = \frac{p_f^2}{2M},$$

$$\omega = \frac{p_f^3}{3\pi^2\hbar^3}, \qquad \mathcal{E} = \frac{p_f^5}{10\pi^2\hbar^3 M}. \tag{5}$$

When we average $k$ we get a result which depends on the ratio of $E_f$ to the energy $E_v$ of the virtual level. For $E_f < E_v$ the quantity $k$ is constant

[5] Another possible approach is based on the fact that the statistical weights of the triplet and singlet are in the ratio 3 : 1; a given neutron interacts with only 1/4 of the others. But in the singlet state without scattering the density at the coordinate origin is twice as large as the average density throughout the volume, since in the singlet state only even angular momenta $l$ are possible, and therefore the $S$ state, the only one that contributes to $\rho(0)$, makes up twice as large a fraction of all singlet states as in the case of different particles. We finally find (1 is the index for the singlet) $\omega(0) = 2\overline{\omega}_1 = 2(\omega/4) = \omega/2$, which agrees with the result obtained in the text.

and ($\mu = M/2$)

$$U = -\frac{\pi\hbar^2\alpha\omega^2}{2\mu}. \tag{6}$$

In the limiting case $E_f \gg E_v$ we must average over the Fermi distribution $p^{-1}$, where $p$ is the momentum of the relative motion of two particles. We have

$$\mathbf{p} = \mu(\mathbf{v}_1 - \mathbf{v}_2) = \frac{1}{2}M(\mathbf{v}_1 - \mathbf{v}_2) = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2). \tag{7}$$

Using the electrostatic analogy[6] we easily find

$$\overline{|\mathbf{p}_1 - \mathbf{p}_2|^{-1}} = \frac{6}{5p_f}, \qquad \overline{p^{-1}} = \frac{12}{5p_f}, \tag{8}$$

and finally,

$$U = -\frac{3\pi^2\hbar^3\omega^2}{5\mu p_f} = -\frac{2p^5}{15\pi^2\hbar^3 M} = -\frac{4\mathcal{E}}{3}. \tag{9}$$

This is a remarkable result: the interaction energy is a constant multiple of the kinetic energy.

If we take these results literally, we get the following physical conclusions about the dependence on the density of the average energy of a neutron, $E_1(\omega) = (\mathcal{E} + U)/\omega$: at small density, in the limit

$$E_1 = \frac{3}{5}E_m > 0, \qquad E_1 \sim \omega^{2/3}, \tag{10}$$

the interaction is proportional to a higher power of $\omega$ (first) at the density $\omega_0$ that corresponds to $E_f = 5E_v$, the energy $E_1$ goes to zero, and then changes sign and at larger densities

$$E_1 = -\frac{1}{3}E_m < 0, \qquad E_1 \sim \omega^{2/3}. \tag{11}$$

This expression holds for[7] $\omega > \omega_0 \approx a^{-3}$. From this it follows that a nucleus can exist that consists of neutrons only, with a binding energy given by $-E_1$.

This treatment does not give the equilibrium density, since according to (11) as the density increases $E_1$ continues to decrease ($E_1$ is negative and its absolute value increases). To find the equilibrium density and the binding energy at this density we must bring in the effective range of nuclear forces and the interaction in states with $l \neq 0$. Qualitatively, however, the fact of the existence of neutron nuclei itself follows just from the change of sign of $E_1$, which is obtained from a calculation at the density $\omega_0 = a^{-3}$. Since $a$ is extremely large, we have $\omega_0 \approx 0.001\omega_n$, where $\omega_n$ is the density of ordinary nuclei. In a state corresponding to the density $\omega_0$ for which $E_1 = 0$

[6]For any body, $\overline{r_{12}^{-1}} = \int\int r_{12}^{-1}\, dv_1\, dv_2 = \int \varphi_1\, dv_1 = \overline{\varphi}$, where $\varphi$ is the potential for unit charge density, which satisfies the equation $\Delta\varphi = -4\pi$ inside the body and $\Delta\varphi = 0$ outside the body.

[7]A consistent calculation on the assumptions made above gives a value of the coefficient very close to unity.

the boundary kinetic energy $E_f$ is about 0.5 MeV, so that the contribution from $l \neq 0$ and the influence of the effective range are negligible; thus the assumptions about the interaction of the neutrons that were the basis for the calculation are very well satisfied at $\omega = \omega_0$. We note that if the existence of a range of values of $\omega$ in which $E_1 < 0$ is confirmed, then the surface tension of the neutron liquid will give a definite critical size of the neutron droplet, i.e., a minimum number of neutrons for which the existence of a neutron nucleus is possible. Therefore if it is proved that bound states $n^4$, $n^6$ or $n^8$ do not exist, this does not by itself exclude the existence of the heavier neutron nuclei.

Nevertheless the main result—the change of sign of $E_1$—is by no means to be regarded as established, since only the pair interaction of the neutrons has been considered and no account has been taken of the influence of the other neutrons on the wave functions of the interacting pair. The result is doubly unreliable because for $\omega > \omega_0$ the desired quantity $E_1$ is the small difference of two nearly equal quantities:

$$E_1 = \overline{E} + U_1, \quad \overline{E} = \frac{3}{5}E_f, \quad U_1 = -\frac{4}{5}E_f = -\frac{4}{3}\overline{E}. \tag{12}$$

For $\omega > \omega_0$, $E_f \gg E_v$, the scattering does not depend on the length $a$, and we can set $a = \infty$, $a^{-1} = 0$, i.e., consider resonance scattering. Then the problem contains no dimensionless parameters. From dimensional considerations it follows that in this region

$$E_1 \sim U_1 \sim \overline{E} \sim E_f \sim \omega^{2/3}. \tag{13}$$

The formula (11) for $E_1$ is in agreement with this requirement. But then the correction to $E_1$ because of the influence of a third neutron on the wave functions of a given pair is also proportional to $E_f$, i.e., depends on the same power of the density and can differ from $E_f$ and $E_1$ only by a numerical coefficient. This case is not like the usual one; in the Fermi gas at absolute zero with resonance scattering one cannot expand in a series of powers of the density.

We have not found the corrections for the interactions of three and more particles; it is quite possible that they will change the sign of $E_1$ in the region $\omega < \omega_0$. We know that $E_1 > 0$ for $\omega < \omega_0$. On the other hand, for values of $\omega$ approaching the density of ordinary nuclei it is to be expected that the energy will lie above that calculated from the resonance $S$ scattering.[8] Therefore, if from an exact solution of the problem of the Fermi gas with resonance interaction it is found that $E_1 > 0$, this will mean that the existence of nuclei composed of neutrons only is impossible.

We note that the expression (11) for $E_1$ found by using the pair interaction is not the mathematical expectation of the energy, calculated with

[8]By the method described above we would get for nuclear matter consisting of equal numbers of neutrons and protons, with the Coulomb interaction neglected, the result $U_1 = -4\overline{E}$; for the ordinary nuclear density this would give a binding energy $\sim 60$ MeV, many times the experimental value.