PRINCETON LANDMARKS IN PHYSICS

Raymond F. Streater and Arthur S. Wightman

PCT, Spin and Statistics, and All That

PCT, Spin and Statistics, and All That

PCT, Spin and Statistics, and All That

R.F. STREATER
Kings College London

A.S. WIGHTMAN
Princeton University

Princeton University Press
Princeton and Oxford

Published by Princeton University Press, 41 William Street,
Princeton, New Jersey 08540
In the United Kingdom: Princeton University Press,
3 Market Place, Woodstock, Oxfordshire OX20 1SY

First paperback printing, with revised Preface and corrections, 2000

The publisher is pleased to acknowledge the assistance of Cecelia Duray-Bito, who produced the illustrations.

Originally published in 1964 as part of the Mathematical Physics Monograph series, by W.A. Benjamin, Inc.

PCT, Spin and Statistics, and All That
First printing, 1964
Second printing, with additions and corrections, 1978
Third printing, 1980

Library of Congress Cataloging-in-Publication Data

Streater, R. F.
PCT, spin and statistics, and all that / R.F. Streater, A.S. Wightman.
p. cm.
"Originally published in 1964 as part of the Mathematical physics monograph series, by W.A. Benjamin"—T.p.
Includes bibliographical references and index.
ISBN 0-691-07062-8
1. Quantum field theory. I. Wightman, A. S. II. Title.

QC174.45 .S87 2000
530.14′3—dc21 00-061116

The paper used in this publication meets the minimum requirements of ANSI/NISO Z39.48-1992 (R1997) (*Permanence of Paper*)

ABCDEFGHIJ-AL-89

www.pup.princeton.edu

10 9 8 7 6 5 4 3 2

ISBN-13: 978-0-691-07062-9 (pbk.)

ISBN-10: 0-691-07062-8 (pbk.)

Contents

Preface

The idea of this book arose in a conversation with H.A. Bethe, who remarked that a little book about modern field theory which contained only Memorable Results would be a Good Thing. In the field of historical research this approach led to the publication of a treatise† which has become a standard text for serious students. Although it is often dangerous to use the tried and true methods of one subject in another field of research, the application to physics of the principles of that book has led to at least one good result: we have eliminated all theorems whose proofs are non-existent.

R.F. Streater
A.S. Wightman
November 1963

In the 1978 edition of this book, we added Appendix A, which outlined three significant developments in the general theory of quantized fields that had occurred since the appearance of the first edition: constructive quantum field theory, the theory of local algebras, and the theory of superselection rules.

Neither of these first two editions contained an account of the Haag-Ruelle collision theory. For that the reader of the first edition was referred to the then forthcoming excellent book by R. Jost (*The General Theory of Quantized Fields,* American Mathematical Society, 1965). For the 1978 edition this reference was supplemented by a reference to the treatise *Introduction to Axiomatic Quantum Field Theory* by N. Bogolubov, A. Logunov and I. Todorov, W.A. Benjamin (1975).

For the present Princeton University Press reissue (2000), we want to add a

†W.C. Sellar and R.J. Yeatman, *1066 and All That*, Dutton, New York, 1931.

third recommendation to this list: H. Araki's *Mathematical Theory of Quantum Fields,* Oxford University Press, 2000. In other respects the Princeton University Press edition differs from the 1978 edition mainly in the correction of a few misprints.

R.F. Streater
A.S. Wightman

PCT, Spin and Statistics, and All That

INTRODUCTION

In the beginning, when Dirac, Jordan, Heisenberg, and Pauli created the quantum theory of fields, it was not expected that it would provide a consistent description of Nature. After all, it was only a quantized version of the classical theory of Maxwell and Lorentz, a theory which was well known to be afflicted with diseases arising from the infinite electromagnetic inertia of point particles. Many physicists were of the opinion that any project to make the theory's mathematical foundation more rigorous was probably ill-advised; first the classical foundation should be set right. Such alterations might so change the basis of the theory that a mathematically rigorous discussion of any preceding version would be entirely irrelevant. More recently, it has been suggested that the trouble is that the theory is too modest; it is not designed to predict the masses of the elementary particles or the values of the coupling constants, and should be fundamentally changed with this in view.

However, attempts to go beyond the theory foundered again and again. What successes were achieved were either phenomenological, or were due to systematic developments of the original formalism. But the quantum theory of fields never reached a stage where one could say with confidence that it was free from internal contradictions—nor the converse. In fact, the Main Problem of quantum field theory turned out to be to kill it or cure it: either to show that the idealizations involved in the fundamental notions of the theory (relativistic invariance, quantum mechanics, local fields, etc.) are incompatible in some physical sense, or to recast the theory in such a form that it provides a practical language for the description of elementary particle dynamics.

The last ten years have seen a number of attempts to meet the situation head on. (The physicists who have engaged in this kind of work are sometimes dubbed the Feldverein. Cynical observers have compared them to the Shakers, a religious sect of New England who built solid barns and led celibate lives, a non-scientific equivalent of proving rigorous theorems and calculating no cross sections.) These efforts have not yet led to a solution of the Main Problem, but they have yielded a number of by-products, very general insights into the structure of a field theory. The present book is devoted to an exposition of some of these general results, the physical ideas they embody, and the mathematics necessary for their proofs.

We have included only results which have a certain definitive character. In particular, this has resulted in the omission of a description of attempts to establish a connection with the important work of Lehmann, Symanzik, and Zimmermann and others on time-ordered and retarded functions and their connection with collision theory. A great deal more work will be necessary before these subjects can be properly understood and put on a rigorous basis. Although the connection with Lehmann, Symanzik, and Zimmermann is not yet firmly established, a rigorous collision theory (based on the axioms of Chapter 3 of the present book) has been set up by D. Ruelle along lines laid down by R. Haag. The omission of this theory from this book will be mitigated by the availability of the excellent book by R. Jost, which will contain a full account (*The General Theory of Quantized Fields*, American Mathematical Society, 1965).

The first chapter contains a summary of the transformation properties of physical states in relativistic quantum mechanics. It is assumed that the reader has had an introduction to Hilbert space and its application to the description of states in quantum mechanics. It is probably worthwhile to point out to younger readers that the simplicity of the transformation laws of the physical vacuum and one-particle states under Lorentz transformations, well known today, was buried in the quantum field theory of fifteen years ago under a kitchen midden of difficult and ambiguous formalism. The task of the first chapter is to provide a language in which physical states with simple transformation properties have a simple description. For example, the concepts of bare mass and bare vacuum need not be and are not introduced.

The second chapter is an exposition of the mathematical tools used in the following. Technical details of some proofs have been omitted but an attempt has been made to get across the main mathematical ideas. The theorems are stated precisely. The presentation presupposes nothing that an undergraduate physics major has not met.

The third chapter defines the notion of field as used in this book. It is shown that a field theory is defined by the vacuum expectation values of products of field operators. While this chapter is essentially self-contained, a brush with elementary quantum field theory at the level of, say, Part II of Schweber's book† might be a help.

In Chapter 4 the three preparatory chapters are applied to get some general theorems of quantum field theory, of which the *PCT* theorem and the theorem on the connection of spin with statistics are the best known.

† S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Harper and Row, New York, 1961.

The reader who wants to get on with the systematic discussion of quantum field theory could well begin with Chapter 3 and only go back to Chapters 1 and 2 when he finds it necessary to fill in details.

Each chapter has been equipped with a bibliography to guide the reader to relevant literature. No attempt at completeness has been made. The notation is standard: Theorem 3–1 refers to the first theorem of the third chapter, and similarly for equations. Halmos notation ▮ has been used to signify the end of a proof.

CHAPTER 1

RELATIVISTIC TRANSFORMATION LAWS

You point out that care is needed in the analysis of the representations of the Lorentz group; I promise you that I will be careful.

E. WIGNER

Throughout this book, states will be described in the Heisenberg picture of quantum mechanics. The Schrödinger picture is much less convenient for the description of a relativistic theory because it treats the time coordinate on a very different footing from the space coordinates; as will be proved in Chapter 4, the other commonly used picture, the interaction picture, in general does not exist. In the Heisenberg picture, to each state of the system under consideration there corresponds a unit vector, say Φ, in a Hilbert space, $\mathscr{H}$. The vector does not change with time, whereas the observables, represented by hermitian linear operators acting on $\mathscr{H}$, in general do. The scalar product of two vectors Φ and Ψ in $\mathscr{H}$ is denoted by (Φ, Ψ), called the *transition amplitude* of the corresponding states.

Two vectors that differ only by multiplication by a complex number of modulus one describe the same state, because the results of all experiments on a state described by Ψ may be expressed in terms of the quantities

$$|(\Phi, \Psi)|^2$$

which gives the probability of finding Φ if Ψ is what you have. The set $\mathbf{\Phi}$ of vectors $e^{i\alpha}\Phi$, where α varies over all real numbers, and the norm of Φ (written $\|\Phi\|$ and defined as $[(\Phi, \Phi)]^{1/2}$) is unity, is called a *unit ray*. For brevity, we shall speak of the state $\mathbf{\Phi}$. The condition $\|\Phi\| = 1$ is obviously equivalent to the convention of normalizing the probability to unity. The preceding remarks can be summarized: *states*† *of a physical system are represented by unit rays.*

† By "state" we shall always mean pure state. A "mixed state" can always be formed from several states by a *classical* superposition, each entering with a certain known probability which describes our ignorance of the system.

1–1. SUPER-SELECTION RULES

Suppose that the rays which describe the states of a physical system lie in a Hilbert space $\mathscr{H}$. Does every unit ray in $\mathscr{H}$ describe a possible state of the system? The answer is, in general, no. For example, no one has ever succeeded in producing a state which is a superposition of states with different charges, Q, and it is believed that they do not occur in Nature. It also seems that every *physically realizable* state must be an eigenstate of B, the baryon number, and $(-1)^F$, where F is an even integer for states of integer spin and an odd integer for states of half odd integer spin.

The operators Q,B and $(-1)^F$ are conserved in time, but these conservation laws should be distinguished from ordinary conservation laws such as, say, that for the x-component of the angular momentum, J_x. Physically realizable states do exist which are not eigenstates of J_x, for example, states with a definite value of the z-component of the angular momentum J_z.

The operator $(-1)^F$ arises because of the invariance of the results of experiment under rotation through an angle of 2π around any axis. If ψ_1 is a state of half odd integer spin, and ψ_2 one of integer spin, then a rotation through an angle 2π takes $\alpha\psi_1 + \beta\psi_2$ into $-\alpha\psi_1 + \beta\psi_2$. These two, which are physically indistinguishable, must belong to the same ray, which is possible only if $\alpha = 0$ or $\beta = 0$.

Any statement that singles out certain unit rays as not physically realizable is called a *super-selection rule*. If there are super-selection rules in a theory, then not all hermitian operators are observables, and the superposition principle does not hold in $\mathscr{H}$. However, if Q,B and $(-1)^F$ define the only super-selection rules, we can form a linear combination of any two states with the same values of Q,B and $(-1)^F$ and get a physical state. The superposition principle then holds unrestrictedly in any subspace of $\mathscr{H}$ consisting of states belonging to given eigenvalues of Q,B and $(-1)^F$.

The super-selection rules associated with Q,B and $(-1)^F$ are known as the *charge*, *baryon*, and *univalence* super-selection rules, respectively.

To study super-selection rules of a general theory systematically, one considers the set θ of all observables of the system under consideration. Each observable determines a hermitian operator in $\mathscr{H}$, not necessarily bounded (an operator A is bounded if $\|A\Phi\| \leqslant C\|\Phi\|$ for some constant C and all $\Phi \in \mathscr{H}$). In this general case a ray is said to be *physically realizable* if the projection operator onto it is an observable.† Consider the set of all bounded operators which commute with all the

† The projection operator, E_Φ, onto a vector Φ is given by the formula

$$E_\Phi \Psi = (\Phi, \Psi)[\|\Phi\|^2]^{-1}\Phi.$$

observables; this is a set θ', called the *commutant* of θ. The limitation to bounded operators in the definition of θ' is purely a matter of convenience. In fact the operators Q and B, being unbounded, do not lie in θ', but the associated projection operators, which project onto the states of various possible values of Q and B, do.

The set θ' partly characterizes the super-selection rules present in the theory. For example, if every hermitian operator is observable, every state is physically realizable since any projection operator is hermitian. Therefore, in this case there are no super-selection rules; the set θ' consists only of multiples of the identity operator.

If we now make the hypothesis that all operators of θ' commute with each other (that is sometimes called *the hypothesis of commutative super-selection rules*), the structure of the set of physically realizable states simplifies considerably. The super-selection rules in θ' can be diagonalized simultaneously, and $\mathscr{H}$ splits up into orthogonal subspaces in which each of the operators defining a super-selection rule takes a definite value. These are called *coherent subspaces.* The observables map coherent subspaces into themselves, and the only operators which are defined on a single coherent subspace, map it into itself, and commute with all observables are constant multiples of the identity; i.e., the observables, when restricted to a single coherent subspace, form an irreducible set of operators.

There is one important case in which one can prove that the hypothesis of commutative super-selection rules holds: when there exists a complete commuting set of observables.† Any operator that commutes with all operators of such a set is a function of the operators of the set. In particular, any operator that is in θ' is a function of the observables of the set. Therefore, in this case all operators in θ' commute.

Although the observables of a particular coherent subspace are irreducible, this by no means implies that they include among them every hermitian operator. For example, in a particular coherent subspace there are normalizable states with infinite energy, and states of this kind ought not to be classified as physically realizable. Thus, the projection operator onto such a state, although hermitian, is not an observable. Nevertheless, in the following we shall assume that θ' is commutative and that every ray of a coherent subspace is physically realizable. This hypothesis is made purely for mathematical convenience. Actually, the analysis could be carried out under much more general assumptions at a cost of more effort.

† Complete commuting set is the standard Dirac terminology; it is also called a *maximal Abelian set.*

It should be emphasized that the above super-selection rules for Q, B, and $(-1)^F$, like all laws of physics, depend upon experiment. It is quite unclear at the moment whether further super-selection rules exist. For example, it may be that there are laws of lepton conservation which define super-selection rules.

1–2. SYMMETRY OPERATIONS

A *symmetry operation* (sometimes called an *invariance principle*, or simply a *symmetry*) of a physical system is a correspondence which yields for each physically realizable state $\mathbf{\Phi}$, another, $\mathbf{\Phi}'$, such that all transition probabilities are preserved:

$$|(\Phi', \Psi')|^2 = |(\Phi, \Psi)|^2. \tag{1-1}$$

It is assumed that the mapping $\mathbf{\Phi} \to \mathbf{\Phi}'$ is one to one. This means that as $\mathbf{\Phi}$ runs over all physically realizable states, so does $\mathbf{\Phi}'$, and if $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are distinct, so are $\mathbf{\Phi}'$ and $\mathbf{\Psi}'$. An example of a symmetry is the operator translating the system by the four-vector a. This is represented by an operator $V(a)$ which is unitary [i.e., $(V\Phi, V\Psi) = (\Phi, \Psi)$]. Another example is the CPT operator Θ, which is anti-unitary [i.e., $(\Theta\Phi, \Theta\Psi) = \overline{(\Phi, \Psi)}$]. Incidentally, the operator Θ interchanges the coherent sub-spaces with opposite charge and baryon number. Clearly, both unitary and anti-unitary operators satisfy (1–1). In fact, all mappings $\mathbf{\Phi} \to \mathbf{\Phi}'$ lead essentially to a unique transformation $\Phi \to \Phi'$ satisfying (1–1), and this transformation is either unitary or anti-unitary (Ref. 1).

Theorem 1–1

Let $\mathbf{\Phi} \to \mathbf{\Phi}'$ be a symmetry of a physical theory satisfying the hypothesis of commutative super-selection rules.

If the symmetry leaves coherent subspaces invariant, then there exists in each coherent subspace a unitary or anti-unitary operator V such that for all physically realizable states of that subspace

$$\mathbf{\Phi}' = V\mathbf{\Phi}. \tag{1-2}$$

The operator V is uniquely determined up to a phase.

If the symmetry does not leave coherent subspaces invariant, then restricted to a coherent subspace it is a one-to-one mapping onto another coherent subspace, unitary or anti-unitary and unique up to a phase.

We shall not prove the theorem here; its essence is the following. The one-to-one ray correspondence $\mathbf{\Phi} \rightarrow \mathbf{\Phi}'$ can be induced by one of many different vector correspondences, $\Phi \rightarrow \Phi'$ in the underlying Hilbert space, but in general such a correspondence is neither linear nor anti-linear; i.e., neither

$$\alpha\Phi' + \beta\Psi' = (\alpha\Phi + \beta\Psi)'$$

nor

$$\bar{\alpha}\Phi' + \bar{\beta}\Psi' = (\alpha\Phi + \beta\Psi)'$$

holds. What the theorem says is that when the vectors of a single coherent subspace are considered there is a linear or anti-linear transformation unique up to a phase (one or the other, not both!) which yields the given correspondence between rays.

A final remark about symmetry operations. Our definition clearly makes every unitary and anti-unitary operator a symmetry operator and Theorem 1–1 shows that these are essentially the only such. How then can one system be more symmetrical than another? The answer lies in the physical interpretation of the operations. Consider for example a theory of two spinless particles with coordinate and momentum operators $\mathbf{q}_1(t)$, $\mathbf{q}_2(t)$, $\mathbf{p}_1(t)$, $\mathbf{p}_2(t)$. Then the mapping $\mathbf{q}_i(t) \rightarrow -\mathbf{q}_i(t)$, $\mathbf{p}_i(t) \rightarrow -\mathbf{p}_i(t)$ of the observables onto themselves can always be defined, but only when the system possesses symmetry under space inversion will the mapping be induced by a unitary operator V according to

$$V\mathbf{q}_j(t)V^{-1} = -\mathbf{q}_j(t), \qquad V\mathbf{p}_j(t)V^{-1} = -\mathbf{p}_j(t), \qquad j = 1,2 \quad (1\text{–}3)$$

with V independent of t.

On the other hand, it may be possible to define a unitary operator, V, which has the effect of a parity operator for the center of mass motion and total momentum, even if the theory is *not* invariant under space-inversion, i.e., even if there is no V satisfying (1–3). For example, V might satisfy

$$V\mathbf{q}_1(t)V^{-1} = -\mathbf{q}_2(t), \qquad V\mathbf{q}_2(t)V^{-1} = -\mathbf{q}_1(t)$$
$$V\mathbf{p}_1(t)V^{-1} = -\mathbf{p}_2(t), \qquad V\mathbf{p}_2(t)V^{-1} = -\mathbf{p}_1(t).$$

Then

$$V(\mathbf{q}_1(t) + \mathbf{q}_2(t))V^{-1} = -(\mathbf{q}_1(t) + \mathbf{q}_2(t))$$
$$V(\mathbf{p}_1(t) + \mathbf{p}_2(t))V^{-1} = -(\mathbf{p}_1(t) + \mathbf{p}_2(t)).$$

This example makes clear that the statement that a physical system possesses space-inversion symmetry only makes sense when one has specified what space inversion is supposed to do to the observables of the system. In the remaining sections of this chapter it will be assumed that some such specification has been made for relativity transformations. In Chapter 3 the specification will be made explicit in terms of fields.

1–3. THE LORENTZ AND POINCARÉ GROUPS

Among the most important symmetries of relativistic quantum theory are those which arise from the Lorentz transformations themselves. The next few paragraphs are devoted to establishing our notation and summarizing the main facts about these symmetries.

The Lorentz-invariant scalar product of two four-vectors $x = (x^0, x^1, x^2, x^3)$ and $y = (y^0, y^1, y^2, y^3)$ will be written

$$x\cdot y = x^0y^0 - \mathbf{x}\cdot\mathbf{y} \equiv x^\mu g_{\mu\nu} y^\nu \equiv x^\mu y_\mu \tag{1–4}$$

with the usual summation convention on repeated indices.

Here

$$x_\mu = g_{\mu\nu}x^\nu$$

and $g^{\mu\nu} = g_{\mu\nu}$ is the μ,ν component of the matrix G,

$$G = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Bmatrix}.$$

A Lorentz transformation Λ is a linear transformation mapping space-time onto space-time which preserves the scalar product (1–4): $(\Lambda x)\cdot(\Lambda y) = x\cdot y$. If $(\Lambda x)^\mu = \Lambda^\mu{}_\nu x^\nu$ the (real) matrix, $\Lambda^\mu{}_\nu$, of the transformation must satisfy

$$\Lambda^\kappa{}_\mu \Lambda_{\kappa\nu} = g_{\mu\nu} \qquad \text{or} \qquad \Lambda^T G \Lambda = G, \tag{1–5}$$

where the transpose Λ^T of Λ is defined by $(\Lambda^T)^\mu{}_\nu = \Lambda^\nu{}_\mu$ and indices on Λ are lowered according to

$$\begin{aligned} \Lambda_{\kappa\nu} &= g_{\kappa\sigma}\Lambda^\sigma{}_\nu \\ &= (G\Lambda)_{\kappa\nu}. \end{aligned}$$

If Λ and M satisfy (1–5), so do ΛM and Λ^{-1}. Here

$$(\Lambda M)^\mu{}_\nu = \Lambda^\mu{}_\kappa M^\kappa{}_\nu$$

$$(\Lambda^{-1})^\mu{}_\kappa \Lambda^\kappa{}_\nu = g^\mu{}_\nu = \begin{cases} 0 & \mu \neq \nu \\ 1 & \mu = \nu, \end{cases} \tag{1–6}$$

so the Lorentz transformations form a group, the Lorentz group, L. Two Lorentz transformations Λ and M are defined to be close to one another if the numbers $\Lambda^\mu{}_\nu$ and $M^\mu{}_\nu$ are close for all $\mu,\nu = 0,1,2,3$. Clearly, with this definition, Λ^{-1} and ΛM are continuous functions of Λ and M, respectively. Furthermore, it makes sense to say that two Lorentz transformations can be connected to one another by a continuous curve of Lorentz transformations.

L has four components, each of which is connected in the sense that any one point can be connected to any other, but no Lorentz transformation in one component can be connected to another in another component. To see this, note that $\det \Lambda$ and $\operatorname{sgn} \Lambda^0{}_0$ are both continuous functions of the matrix elements $\Lambda^\mu{}_\nu$. Furthermore, $\det \Lambda = \pm 1$ and $\Lambda^0{}_0 \geqslant 1$ or $\leqslant -1$. [The first follows if one takes the determinant of (1–5); the second becomes evident if one looks at the 00 element of (1–5). It reads

$$(\Lambda^0{}_0)^2 - \sum_{j=1}^{3} (\Lambda^j{}_0)^2 = 1.$$

Therefore, $|\Lambda^0{}_0| \geqslant 1$.] Thus, $\det \Lambda$ and $\operatorname{sgn} \Lambda^0{}_0$ must be constant on any one component. The four possibilities are

$$\begin{aligned} L_+^\uparrow&: \det \Lambda = +1, \operatorname{sgn} \Lambda^0{}_0 = +1 \text{ which contains } 1 \\ L_-^\uparrow&: \det \Lambda = -1, \operatorname{sgn} \Lambda^0{}_0 = +1 \text{ which contains } I_s \\ L_+^\downarrow&: \det \Lambda = +1, \operatorname{sgn} \Lambda^0{}_0 = -1 \text{ which contains } I_{st} \\ L_-^\downarrow&: \det \Lambda = -1, \operatorname{sgn} \Lambda^0{}_0 = -1 \text{ which contains } I_t\,. \end{aligned} \tag{1–7}$$

Here, the Lorentz transformations I_s (*space inversion*), I_t (*time inversion*), and I_{st} (*space-time inversion*) are defined by

$$\begin{aligned} &(I_s x)^0 = x^0 && (I_s x)^j = -x^j,\ j = 1,2,3 \\ &(I_t x)^0 = -x^0 && (I_t x)^j = x^j,\ j = 1,2,3 \\ &(I_{st} x) = -x = (I_s I_t x). \end{aligned} \tag{1–8}$$

Clearly I_s maps $L_-^\uparrow$ one to one onto $L_+^\uparrow$, I_t maps $L_-^\downarrow$ one to one onto $L_+^\uparrow$, and I_{st} maps $L_+^\downarrow$ one to one onto $L_+^\uparrow$. All Λ for which $\Lambda^0{}_0 \geqslant +1$ are called *orthochronous*, Λ for which $\det \Lambda = +1$ *proper*, and Λ for which $\operatorname{sgn} \Lambda^0{}_0 \det \Lambda = +1$ *orthochorous*. To complete the proof of our assertion it has to be shown that $L_+^\uparrow$ is connected. This is customarily done by proving that any $\Lambda \in L_+^\uparrow$ has a decomposition,

$$\Lambda = \Lambda_1 \Lambda_2 \Lambda_3, \tag{1–9}$$

where Λ_1 and Λ_3 are rotations and Λ_2 is a pure Lorentz transformation along the three-axis, defined by

$$x \to \hat{x} = \Lambda_2 x,$$

where

$$\begin{aligned} \hat{x}^0 &= x^0 \cosh \chi + x^3 \sinh \chi \\ \hat{x}^3 &= x^0 \sinh \chi + x^3 \cosh \chi \qquad \tanh \chi = \frac{v}{c} \\ \hat{x}^1 &= x^1, \qquad \hat{x}^2 = x^2. \end{aligned} \tag{1–10}$$

We can then get from any one transformation of the form (1–9) to any other by varying the axes and angles of rotation of Λ_1 and Λ_3 and the parameter χ of Λ_2 continuously. We shall not prove (1–9) here but refer the reader to Ref. 7 of the bibliography of this chapter. This completes the proof that L has just four components. They are displayed in Figure 1–1.

In Figure 1–1 we have also indicated three important subgroups of L:

$L^{\uparrow} = L_{+}^{\uparrow} \cup L_{-}^{\uparrow}$ the *orthochronous Lorentz group*
$L_{+} = L_{+}^{\uparrow} \cup L_{+}^{\downarrow}$ the *proper Lorentz group*
$L_0 = L_{+}^{\uparrow} \cup L_{-}^{\downarrow}$ the *orthochorous Lorentz group.*

Associated with the restricted Lorentz group, $L_{+}^{\uparrow}$, is the group of 2×2 complex matrices of determinant one, which we shall denote $SL(2, C)$. (S stands for special meaning determinant one, L for linear, 2 is the dimension, and C stands for complex.) It is important in describing the transformation properties of spinors. The connection between $L_{+}^{\uparrow}$ and $SL(2, C)$ is obtained in the following standard fashion.

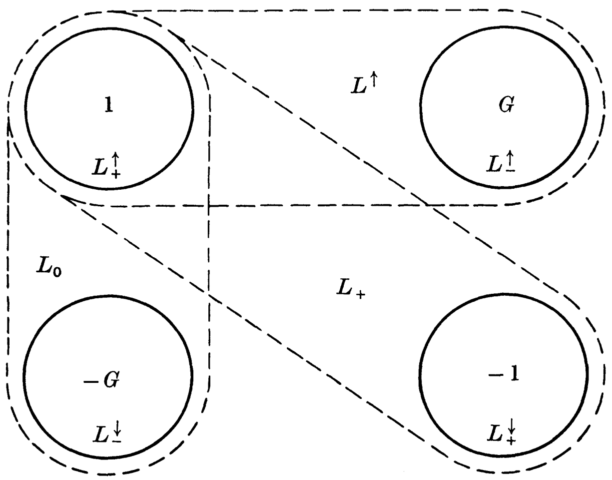

FIGURE 1–1. Connectivity properties of the Lorentz group, L, and its subgroups: the proper Lorentz group, L_{+}; the orthochronous Lorentz, $L^{\uparrow}$; the orthochorous Lorentz group, L_0; and the restricted Lorentz group, $L_{+}^{\uparrow}$.