

*PRINCETON LANDMARKS*

IN MATHEMATICS

Abraham Robinson

**Non-standard  
Analysis**

# NON-STANDARD ANALYSIS

PRINCETON LANDMARKS  
IN MATHEMATICS AND PHYSICS

Non-standard Analysis, *by Abraham Robinson*

General Theory of Relativity, *by P.A.M. Dirac*

Angular Momentum in Quantum Mechanics,  
*by A. R. Edmonds*

Mathematical Foundations of Quantum Mechanics,  
*by John von Neumann*

Introduction to Mathematical Logic, *by Alonzo Church*

Convex Analysis, *by R. Tyrrell Rockafellar*

Riemannian Geometry, *by Luther Pfahler Eisenhart*

The Classical Groups, *by Hermann Weyl*

Topology from the Differentiable Viewpoint, *by John W. Milnor*

Algebraic Theory of Numbers, *by Hermann Weyl*

Continuous Geometry, *by John von Neumann*

Linear Programming and Extensions, *by George B. Dantzig*

Operator Techniques in Atomic Spectroscopy, *by Brian R. Judd*

# NON-STANDARD ANALYSIS

*Revised Edition*

Abraham Robinson

PRINCETON UNIVERSITY PRESS  
PRINCETON, NEW JERSEY

Copyright © 1996 by Princeton University Press  
Published by Princeton University Press, 41 William Street,  
Princeton, New Jersey 08540  
In the United Kingdom: Princeton University Press,  
Chichester, West Sussex

All Rights Reserved

**Library of Congress Cataloging-in-Publication Data**

Robinson, Abraham, 1918–1974.

Non-standard analysis / Abraham Robinson. — Rev. ed.

p. cm. — (Princeton landmarks in mathematics and physics)

Originally published: Rev. ed. Amsterdam : North-Holland Pub. Co.;

New York : American Elsevier Pub. Co., 1974, in series: Studies in logic and  
the foundations of mathematics.

Includes bibliographical references and index.

ISBN 0-691-04490-2 (pb : alk. paper)

1. Nonstandard mathematical analysis. 2. Logic, Symbolic and mathematical.

I. Title. II. Series.

QA299.82.R6 1995

515'.1—dc20 95-43750

Princeton University Press books are printed on acid-free paper and meet the  
guidelines for permanence and durability of the Committee on Production  
Guidelines for Book Longevity of the Council on Library Resources

Printed in the United States of America

10 9 8 7 6 5 4 3

*To Renée*



Engraving reproduced from Leonard Euler's 'Introductio in Analysin Infinitorum', first ed., Lausanne, 1748. By courtesy of the Amsterdam University Library.

## FOREWORD (1996)

The second edition of Abraham Robinson's monumental work, *Non-standard Analysis*, has been out of print for some time. It gives me great pleasure, as one of the earlier followers of Robinson, to have the opportunity to welcome the reissuing of this unique mathematical work by Princeton University Press.

During this century the mathematical community has witnessed the solution of a number of long-standing problems and conjectures. One of these problems, attributed to W. G. Leibniz, concerns the foundation of the calculus. In the early history of the calculus, arguments involving infinitesimals played a fundamental role in the derivation of the basic rules of Newton's method of fluxions. The notion of an infinitesimal, however, lacked a precise mathematical definition, and soon their widespread use came under severe attack, notably by Bishop Berkeley in England, who disdainfully referred to infinitesimals as the "ghosts of departed quantities." To counteract the critics of the theory of infinitesimals, Leibniz proposed a program to conceive of a system of numbers that would include infinitesimally small as well as infinitely large numbers. The process to arrive at such a system of numbers was to adjoin ideal objects to the existing finite quantities and to define arithmetic rules of combining them in such a way that the new ideal numbers and their reciprocals could be viewed as being either infinitesimally small or infinitely large; yet, such that the entire system would have, in some well-defined sense, the same properties as the system of real numbers. This process was inspired by the manner in which the so-called imaginary numbers were introduced as ideal objects to solve equations. When Leibniz and his followers failed in their attempts to create such a system, infinitesimal reasoning gradually lost ground and survived only as a figure of speech. In fact, infinitesimals were gradually replaced by the D'Alembert-Cauchy concept of a limit to provide a firm foundation for the calculus. The creation of a mathematical theory of infinitesimals, as envisaged by Leibniz, on which to base the calculus remained an open problem.

Despite the discovery of non-archimedean ordered field extensions of the reals around the turn of the century, Leibniz' problem remained unsolved and lay dormant until the end of the 1950s. At that time the situation changed dramatically. It then became clear that a rigorous mathematical theory of infinitesimals as proposed by Leibniz was within the grasp of mathematics.

In 1958, the aerodynamicist C. Schmieden, in collaboration with D. Laugwitz, constructed a partially ordered ring extension of the reals that included elements that could be viewed to play the role of infinitesimals for the purpose of representing the Dirac delta function by a function defined point-wise on the ring which is infinitely small except at the origin where it is infinitely large. Since their ring extension contained divisors of zero and elements of indeterminate size that could not be classified, with respect to the reals, as either infinitely small or infinitely large or finite, it did not lend itself in a straightforward manner as a number system that could be used to provide a satisfactory solution to Leibniz' problem. Nevertheless their approach was the first step in the right direction.

At the end of the 1950s, however, unaware of the Schmieden-Laugwitz approach, Abraham Robinson showed that the ordered fields that are non-standard models of the theory of real numbers could be viewed in the meta-mathematical or external sense as non-archimedean ordered field extensions of the reals that externally contain numbers that behave like infinitesimals with respect to the reals and whose reciprocals are infinitely large; that is, larger in absolute value than any positive real number. As models of the reals, these ordered fields had, in a precise formal manner, the same properties of the reals. As a consequence, these new number systems obtained by using model theoretic methods, provided a complete, satisfactory solution of Leibniz' three-centuries-old problem.

Since the models employed are known as nonstandard models, Robinson announced the creation of his theory of infinitesimals in a paper that appeared in 1961 in the *Proceedings of the Royal Academy of Sciences of Amsterdam* under the title *Non-standard Analysis*. Robinson introduced his history-making paper by giving credit to the ideas of the world-renowned logician T. Skolem. In a paper published in the *Fundamenta Mathematicae* in 1934, Skolem showed that the system of natural numbers could not be characterized by any set of its arithmetical properties that were formulated in the lower predicate calculus. In view of the categorical nature of the Peano

system of axioms of arithmetic, Skolem's result came as a great surprise. The proof given by Skolem consisted of a construction of a proper extension of the system of natural numbers  $\mathbb{N}$  that, under a precisely defined interpretation of its symbols, satisfies all the properties of  $\mathbb{N}$  that are formulated in the lower predicate calculus in terms of a given set of arithmetic operations such as addition and multiplication. We may add here that Skolem's proper extension of  $\mathbb{N}$  was the first explicit example of what are now referred to as nonstandard models of arithmetic.

After recalling Skolem's result, Robinson continued by applying the same idea to the set  $K_0$  of all sentences of the lower predicate calculus in terms of the operations of the reals expressing properties of the real number system. The new number systems were then defined as nonstandard models of  $K_0$ . It followed that, in the meta-mathematical sense, the new number systems were non-archimedean ordered field extensions of the reals and thus, in the meta-mathematical sense, contained numbers that could serve the role of infinitesimals and their reciprocals that could serve as infinitely large numbers. After making these observations and using his new and rigorous theory of infinitesimals, Robinson derived the fundamental properties of the theory of limits, and more generally of the calculus, and in doing so, presented a completely satisfactory solution of Leibniz' problem.

Robinson did not stop there, however. He continued to test—successfully—his new methods in such fields as classical differential geometry, the classical theory of Lie groups, the theory of differential equations, and the theory of distributions that includes an improved version of the Schmieden-Laugwitz representation of the Dirac delta distribution. From the area of classical applied mathematics, Robinson discussed briefly the way in which the theory of infinitesimals could be used in the theory of boundary layers and concluded by stating that the probability that the nonstandard reals may provide a better explanation of certain observable phenomena than the standard reals should be borne in mind.

Abraham Robinson is considered to be one of the prominent mathematicians of this century for his ground-breaking work in logic and various fields of mathematics and applied mathematics, but above all for his discovery of nonstandard analysis, which he used to unite ideas of mathematical logic with mathematics proper. He was born in 1918 in Waldenburg, Lower Silesia and spent the larger part of his childhood in Breslau. In 1933, his family fled Nazi Germany and settled in Palestine. He studied mathematics

at the Hebrew University under the wing of the famous mathematician Abraham Fraenkel. In June of 1940, after a brief stay in Paris studying at the Sorbonne, he barely escaped to England. During World War II, Robinson worked at the Royal Aircraft Establishment in Farnborough analyzing the design of supersonic airfoils. In a short amount of time, this work made him a specialist in aerodynamics. After the war, he continued his work in mathematical logic begun with Fraenkel and in 1949 he received his Ph.D. at the University of London. His thesis appeared later in book form under the title *The Meta-Mathematics of Algebra*. During those years he also continued his aeronautical research as a staff member of the Cranfield College of Aeronautics. In 1951, he joined the Department of Applied Mathematics of the University of Toronto, where I met him for the first time in the fall of 1956. Robinson succeeded his mentor, Abraham Fraenkel, in 1957 as Professor of Pure Mathematics at the Hebrew University. The creation of the theory of infinitesimals happened during the academic year 1959–1960, while he was visiting the Institute for Advanced Study in Princeton. During his years as Professor of Mathematics and Philosophy at the University of California, Los Angeles (from 1962 through 1967), Robinson worked full force at developing nonstandard techniques for applications to various other branches of mathematics. He also wrote this book, the first edition of which appeared in 1966, and the second in 1973. From 1967 until his untimely death on April 11, 1974, Robinson was Professor of Mathematics at Yale. During those years he had started to collaborate with P. Roquette from the University of Heidelberg on a program analyzing problems of algebraic number theory with nonstandard methods.

This most welcome reissuing of Robinson's *Non-standard Analysis* follows on the heels of the recent publication of Robinson's biography by the renowned historian of mathematics, Joseph Warren Dauben, entitled *Abraham Robinson, The creation of nonstandard analysis, a personal and mathematical odyssey*, with a forward by Benoit B. Mandelbrojt (Princeton University Press, 1995). These two books present a most valuable documentation of the history and the meaning of the creation of nonstandard analysis.

Since Robinson's death in 1974, nonstandard analysis has continued to attract the attention of the mathematical community. Recently, these methods have been applied, with great success, in probability theory, notably in the theory of stochastic processes. At regular intervals conferences and

workshops devoted to nonstandard analysis have been organized. Proceedings of some of them have appeared in print.

For the interested reader we mention here the following publications: *Nonstandard Analysis, Recent Developments*, Proceedings of the Second Victoria Symposium on Non-standard Analysis, edited by A. E. Hurd; *Nonstandard Analysis and Its Applications, Lecture Notes in Mathematics* 983 (Springer Verlag, 1983); Proceedings of a Conference and Workshop held at the University of Hull in 1986, edited by Nigel Cutland, London Mathematical Society Student Texts 10 (Cambridge University Press, 1988); and *Advances in Analysis, Probability and Mathematical Physics; Contributions of Nonstandard Analysis*, Proceedings of a 1992 International Conference honoring D. Laugwitz on the Occasion of his Sixtieth Birthday, edited by S. A. Albeverio, W.A.J. Luxemburg, and M.P.H. Wolff, Mathematics and its Application Series 314 (Kluwer Academic Publishers, 1995).

In addition, a number of textbooks on nonstandard analysis at various levels have appeared, each with its own flavor. Nevertheless, paraphrasing the advice Lagrange is alleged to have given his students about reading Euler, I conclude by stating, "Read Robinson; he is the master of us all."

WILHELMUS A. J. LUXEMBURG



## PREFACE

*Je vois plus que jamais qu'il ne faut  
juger de rien sur sa grandeur apparente.  
O Dieu! qui avez donné une intelligence  
à des substances qui paraissent si  
méprisables, l'infiniment petit vous  
coûte autant que l'infiniment grand.*

VOLTAIRE, Micromégas

In the fall of 1960 it occurred to me that the concepts and methods of contemporary Mathematical Logic are capable of providing a suitable framework for the development of the Differential and Integral Calculus by means of infinitely small and infinitely large numbers. I first reported my ideas in a seminar talk at Princeton University (November 1960) and, later, in an address at the annual meeting of the Association for Symbolic Logic (January 1961) and in a paper published in the Proceedings of the Royal Academy of Sciences of Amsterdam (ROBINSON [1961]). The resulting subject was called by me Non-standard Analysis since it involves and was, in part, inspired by the so-called Non-standard models of Arithmetic whose existence was first pointed out by T. Skolem.

In the intervening years, Non-standard Analysis has developed considerably in several directions. Since many of the results have been reported so far only in courses or lectures, and in mimeographed reports, it was thought that a book dedicated entirely to this subject would be in order.

Over the years, my thinking in this area has been enlivened by discussions with several colleagues, among whom I venture to mention here R. Arens, C. C. Chang, A. Erdelyi, A. Horn, G. Kreisel, I. Lakatos, and J. B. Rosser. Special thanks are due to W. A. J. Luxemburg, whose lectures and lecture notes on Non-standard Analysis have done much to make the subject known among mathematicians.

I acknowledge with thanks the support received from the National Science Foundation (Grants GP-1812 and GP-4038) in connection with much of the research incorporated in the present book. I also wish to mention that Chapter vi is based on research sponsored in part by the Air Force Office of Scientific Research and published previously in report form (ROBINSON [1962]). Finally, I wish to put on record my indebtedness to M. Machover who read the book at the proof stage and suggested various improvements.

ABRAHAM ROBINSON

*University of California, Los Angeles*  
April, 1965

## PREFACE TO THE SECOND EDITION

Seven years have passed since the publication of the first edition of this book and almost twice as long since the inception of its subject. The fact that a new edition has become necessary indicates that the ideas described here attracted a measure of attention. The least that can be said is that the intrinsic interest of the subject and its historical relevance are, by now, widely appreciated. Beyond this, numerous articles in the recent mathematical literature contain applications of non-standard analysis to contemporary problems in areas as widely apart as algebraic number theory and mathematical economics. In particular, the interested reader may wish to consult the proceedings of several symposia in this area which were held in recent years. They are contained in: *Applications of Model Theory to Algebra, Analysis and Probability*, edited by W. A. J. Luxemburg (Holt, Rinehart and Winston, Toronto 1969); *Contributions to Non-standard Analysis*, edited by W. A. J. Luxemburg and A. Robinson, in: *Studies in Logic and the Foundations of Mathematics*, Vol. 69 (North-Holland, Amsterdam 1972); and the proceedings of a conference on Non-standard Analysis which was held in the spring of 1972 at the University of Victoria, B.C., Canada, to be published in the Lecture Note series of the Springer-Verlag.

Although the situation may change some day, the non-standard methods that have been proposed to date are *conservative* relative to the commonly accepted principles of mathematics (e.g., the axioms of Zermelo–Fraenkel, including the axiom of choice). This signifies that a non-standard proof can always be replaced by a standard one, even though the latter may be more complicated and less intuitive. Thus, the present writer holds to the view that the application of non-standard analysis to a particular mathematical discipline is a matter of choice and that it is natural for the actual decision of an individual to depend on his early training.

A more definite opinion has been expressed in a statement which was

made by Kurt Gödel after a talk that I gave in March 1973 at the Institute for Advanced Study, Princeton. The statement is reproduced here with Professor Gödel's kind permission.

'I would like to point out a fact that was not explicitly mentioned by Professor Robinson, but seems quite important to me; namely that non-standard analysis frequently simplifies substantially the proofs, not only of elementary theorems, but also of deep results. This is true, e.g., also for the proof of the existence of invariant subspaces for compact operators, disregarding the improvement of the result; and it is true in an even higher degree in other cases. This state of affairs should prevent a rather common misinterpretation of non-standard analysis, namely the idea that it is some kind of extravagance or fad of mathematical logicians. Nothing could be farther from the truth. Rather there are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future.

One reason is the just mentioned simplification of proofs, since simplification facilitates discovery. Another, even more convincing reason, is the following: Arithmetic starts with the integers and proceeds by successively enlarging the number system by rational and negative numbers, irrational numbers, etc. But the next quite natural step after the reals, namely the introduction of infinitesimals, has simply been omitted. I think, in coming centuries it will be considered a great oddity in the history of mathematics that the first exact theory of infinitesimals was developed 300 years after the invention of the differential calculus. I am inclined to believe that this oddity has something to do with another oddity relating to the same span of time, namely the fact that such problems as Fermat's, which can be written down in ten symbols of elementary arithmetic, are still unsolved 300 years after they have been posed. Perhaps the omission mentioned is largely responsible for the fact that, compared to the enormous development of abstract mathematics, the solution of concrete numerical problems was left far behind.'

I am greatly indebted to Peter Winkler for correcting a considerable number of printing errors which appeared in the first edition of this book and for closing a gap in the original statement of Theorem 8.1.12, page 206.

*Yale University, New Haven*  
October 1973

ABRAHAM ROBINSON

# LIST OF CONTENTS

## CHAPTER I

### GENERAL INTRODUCTION

1.1. Purpose of this book . . . . .	1
1.2. Summary of contents . . . . .	2

## CHAPTER II

### TOOLS FROM LOGIC

2.1. The Lower Predicate Calculus . . . . .	6
2.2. Interpretation . . . . .	8
2.3. Ultraproducts . . . . .	10
2.4. Prenex normal form. . . . .	10
2.5. The finiteness principle . . . . .	13
2.6. Higher order structures and corresponding languages . . . . .	19
2.7. Type symbols . . . . .	23
2.8. Finiteness principle for higher order theories . . . . .	26
2.9. Enlargements . . . . .	30
2.10. Examples of enlargements . . . . .	34
2.11. General properties of enlargements . . . . .	41
2.12. Remarks and references . . . . .	48

## CHAPTER III

### DIFFERENTIAL AND INTEGRAL CALCULUS

3.1. Non-standard Arithmetic . . . . .	49
3.2. Non-standard Analysis . . . . .	55
3.3. Convergence . . . . .	58
3.4. Continuity and differentiation . . . . .	65
3.5. Integration . . . . .	71
3.6. Differentials . . . . .	79
3.7. Total differentials . . . . .	81

3.8. Elementary Differential Geometry . . . . .	83
3.9. Remarks and references . . . . .	88

## CHAPTER IV

## GENERAL TOPOLOGY

4.1. Topological spaces . . . . .	89
4.2. Sequences, nets, mappings . . . . .	95
4.3. Metric spaces . . . . .	100
4.4. Topologies in $*T$ . . . . .	106
4.5. Functions, limits, continuity, in metric spaces . . . . .	110
4.6. Sequences of functions. Compact mappings . . . . .	116
4.7. Euclidean space . . . . .	120
4.8. Remarks and references . . . . .	122

## CHAPTER V

## FUNCTIONS OF A REAL VARIABLE

5.1. Measure and integration . . . . .	123
5.2. Sequences of functions . . . . .	130
5.3. Distributions . . . . .	133
5.4. Remarks and references . . . . .	145

## CHAPTER VI

## FUNCTIONS OF A COMPLEX VARIABLE

6.1. Analytic theory of polynomials . . . . .	147
6.2. Analytic functions . . . . .	155
6.3. Picard's theorems and Julia's directions . . . . .	160
6.4. Compactness arguments in classical Function Theory . . . . .	173
6.5. Remarks and references . . . . .	176

## CHAPTER VII

## LINEAR SPACES

7.1. Normed spaces . . . . .	177
7.2. Hilbert space. . . . .	181
7.3. Spectral theory of compact operators. . . . .	185
7.4. An invariant subspace problem . . . . .	195
7.5. Remarks and references . . . . .	200

## CHAPTER VIII

## TOPOLOGICAL GROUPS AND LIE GROUPS

8.1. Topological groups . . . . .	202
8.2. Metric groups . . . . .	207
8.3. One-parametric subgroups . . . . .	215
8.4. The Lie algebra of a group. . . . .	225
8.5. Remarks and references . . . . .	229

## CHAPTER IX

## SELECTED TOPICS

9.1. Variations . . . . .	230
9.2. Riemann's mapping theorem . . . . .	232
9.3. Dirichlet's principle. . . . .	233
9.4. Sources and doublets . . . . .	238
9.5. Local perturbations. . . . .	241
9.6. Boundary layer theory . . . . .	247
9.7. Saint-Venant's principle . . . . .	253
9.8. Remarks and references . . . . .	258

## CHAPTER X

## CONCERNING THE HISTORY OF THE CALCULUS

10.1. Introduction . . . . .	260
10.2. Leibniz . . . . .	261
10.3. De l'Hospital . . . . .	264
10.4. Lagrange and d'Alembert . . . . .	267
10.5. Cauchy . . . . .	269
10.6. Bolzano, Weierstrass, and after . . . . .	276
10.7. Infinitely small and large numbers and the infinite . . . . .	279

BIBLIOGRAPHY . . . . .	283
------------------------	-----

INDEX OF AUTHORS . . . . .	290
----------------------------	-----

SUBJECT INDEX . . . . .	292
-------------------------	-----



## CHAPTER I

### GENERAL INTRODUCTION

**1.1 Purpose of this book.** Underlying the fundamental notions of the branch of mathematics known as Analysis is the concept of a limit. Derivatives and integrals, the sum of an infinite series and the continuity of a function all are defined in terms of limits. For example, let  $f(x)$  be a real-valued function which is defined for all  $x$  in the open interval  $(0,1)$  and let  $x_0$  be a number which belongs to that interval. Then the real number  $a$  is the *derivative of  $f(x)$  at  $x_0$* , in symbols

$$1.1.1 \quad f'(x_0) = \left( \frac{df}{dx} \right)_{x=x_0} = a$$

if

$$1.1.2 \quad \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = a.$$

Suppose we ask a well-trained mathematician for the meaning of 1.1.2. Then we may rely on it that, except for inessential variations and terminological differences (such as the use of certain topological notions), his explanation will be thus:

For any positive number  $\varepsilon$  there exists a positive number  $\delta$  such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - a \right| < \varepsilon$$

for all  $x$  in  $(0,1)$  for which  $0 < |x - x_0| < \delta$ .

Let us now ask our mathematician whether he would not accept the following more direct interpretation of 1.1.1 and 1.1.2.

For any  $x$  in the interval of definition of  $f(x)$  such that  $dx = x - x_0$  is *infinitely close to 0* but not equal to 0, the ratio  $df/dx$ , where

$$df = f(x) - f(x_0),$$

is *infinitely close to  $a$* .

To this question we may expect the answer that our definition may be simpler in appearance but unfortunately it is also meaningless. If we then try to explain that two numbers are infinitely close to one another if their distance (the modulus of their difference) is *infinitely small*, i.e., smaller than any positive number, we shall probably be faced with the rejoinder that this is possible only if the numbers coincide. And, so we may be told charitably, this obviously is not what we meant since it would make our explanation trivially wrong.

However, in spite of this shattering rebuttal, the idea of infinitely small or *infinitesimal* quantities seems to appeal naturally to our intuition. At any rate, the use of infinitesimals was widespread during the formative stages of the Differential and Integral Calculus. As for the objection quoted above, that the distance between two distinct real numbers cannot be infinitely small, G. W. Leibniz argued that the theory of infinitesimals implies the introduction of ideal numbers which might be infinitely small or infinitely large compared with the real numbers but which were *to possess the same properties as the latter*. However, neither he nor his disciples and successors were able to give a rational development leading up to a system of this sort. As a result, the theory of infinitesimals gradually fell into disrepute and was replaced eventually by the classical theory of limits.

It is shown in this book that Leibniz' ideas can be fully vindicated and that they lead to a novel and fruitful approach to classical Analysis and to many other branches of mathematics. The key to our method is provided by the detailed analysis of the relation between mathematical languages and mathematical structures which lies at the bottom of contemporary model theory.

**1.2 Summary of contents.** The plan of this book is as follows. In Chapter II we describe the formal tools from mathematical logic which are required for subsequent developments. Our discussion deals with first and higher order theories and includes a proof of the finiteness principle (compactness theorem), which is of central importance for our purpose.

In Chapter III we detail the basic properties of the mathematical structures which serve as frameworks for Non-standard Arithmetic and Non-standard Analysis. We show that these structures provide an adequate foundation for a theory of infinitesimals, and we develop the ele-

ments of the Differential and Integral Calculus by means of that theory. Next we introduce first and higher order differentials and give applications to some simple problems of classical Differential Geometry. Naturally, these differentials are infinitely small, as they were taken to be, naively, in the earliest texts on the Calculus published in continental Europe, such as the *Traité des infiniment petits* of de l'Hospital. It turns out that after the removal of some glaring and frequently attacked inconsistencies, the method used in these texts can be put on a firm foundation.

In Chapter VI we show that the theory of infinitesimals possesses a generalization which is applicable to (non-metric) topological spaces. Within this theory, we reformulate various basic notions of Topology. In particular, we obtain a striking characterization of compact spaces, which has several applications.

Chapter V deals with functions of a real variable. The Lebesgue measure is defined in terms of Non-standard Analysis and several standard theorems are proved within this framework. Next, we discuss the theory of Schwartz distributions. Our approach supplies a concrete realization of these generalized functions and, in particular, provides effective means for the discussion of the notion of the local value of a distribution.

In Chapter VI, we develop the non-standard theory of functions of a complex variable. Fields of application considered in detail are (i) the analytic theory of polynomials, which deals with the location of zeros of polynomials in the complex domain, and (ii) the theory of exceptional values of entire functions, including Picard's theorems and Julia's directions. However, it is even more significant that the theory of normal families is replaced by certain generalized functions provided in a natural way by our approach.

Chapter VII brings us to the theory of normed linear spaces. The Non-standard theory of compact operators is developed in several directions. In particular, it is shown that a linear operator in Hilbert space which has a compact square, possesses a non-trivial invariant subspace. It is noteworthy that the analysis given here provided the first proof of this result, which settles a problem raised by K. T. Smith and P. R. Halmos (see BERNSTEIN and ROBINSON [1966]). Another application of the theory deals with spectral analysis.

In Chapter VIII we consider topological groups and, more particularly, Lie groups. In our theory, the infinitesimal neighborhood of the unit

element of a group actually exists and constitutes a group. This group realizes the intuitive notion of the infinitesimal group for a given topological group or Lie group. It can be related to the standard concepts of the theory.

Chapter IX contains applications of variational principles to several mathematical problems within the framework of Non-standard Analysis. In particular, we give adaptations of a classical proof of Riemann's mapping theorem and of the Dirichlet integral method in Potential Theory. Next we consider several topics from Hydrodynamics. We expound the basic notions of boundary layer theory by means of infinitesimals, and we give a counterpart of de Saint-Venant's principle in the Theory of Elasticity. Finally de Saint-Venant's principle itself is interpreted rationally within the framework of Non-standard Analysis.

The last chapter contains a review of certain stages in the history of the Differential and Integral Calculus that had to do with the theory of infinitesimals. The fact that the more recent writers in this field were convinced that no such theory can be developed effectively, colored their historical judgment. Thus, a revision has now become necessary.

It is natural to ask whether a non-standard method (in the technical sense in which the term non-standard is used here, i.e., a method of Non-standard Analysis) can always be replaced by a standard mathematical proof. This question presumes that the methods of Mathematical Logic are set apart from ordinary Mathematics, and we may agree for our present purposes that the distinction is meaningful in practice. The answer to the question then is that the method of ultrapowers provides a ready means for translating a non-standard proof into a standard mathematical proof in each particular case. However, in the course of doing so we may complicate the proof considerably, so that frequently the resulting procedure will be less desirable from a heuristic point of view. At the same time there may well exist a shorter mathematical proof which can be obtained independently.

The development of the theory of infinitesimals has very nearly stood still during the last one hundred and fifty years. During the same period an enormous amount of effort and ingenuity has been poured into the classical methods. Nevertheless, we believe to have shown that even at this late stage non-standard methods can supplement standard methods effectively, both in throwing new light on old theories and in order to find

new results. We hope that some of the experts, in the subjects touched upon in this book or in subjects not touched upon here, will consider the application of non-standard methods to their field. They may find that in some cases the appropriate non-standard interpretation of a classical theory is not discovered without effort. But once it has been found the subsequent reformulation and development of the theory can be a rewarding experience.

One might express the hope that some branches of modern Theoretical Physics, in particular those that are afflicted with divergence problems, might be treated with profit by Non-standard Analysis. The fact that the present book contains only applications to classical Applied Mathematics most probably is a testimony to the limitations of the author and not of the method.

## CHAPTER II

### TOOLS FROM LOGIC

**2.1 The Lower Predicate Calculus.** In this chapter we introduce the reader to some formalisms whose appreciation is essential for the sequel. The text is written in such a way that it can be understood by anyone with a rudimentary knowledge of Mathematical Logic and of the elements of Abstract Set Theory.

We proceed to describe the language of the Lower, or First Order, Predicate Calculus. Disregarding variations of minor importance, this may be said to be the basic formalism of Mathematical Logic.

A first order language  $L$  is defined in the following way.

The atomic symbols of  $L$  are:

(i) Individual object symbols, or individual constants, usually small italics, with or without subscripts or primes, such as  $a, b', c_n^m$ , or occasionally, to conform with common usage, other symbols such as the numerals  $0, 1, 2, \dots$ . The set of individual constants is arbitrary but fixed (and is usually transfinite).

(ii) (Individual) variables,  $x, y', z_{kl}$  (Italics from the end of the alphabet, with or without subscripts, etc.). Their number is supposed to be infinite but countable.

(iii) Relation symbols of order  $n, n \geq 1$ ,  $S( \quad ), R( \quad , \quad )$ , where  $n$  denotes the number of places in the parentheses. The set of relation symbols is arbitrary but fixed and may be transfinite. (Relation symbols of order  $n=0$  are not required and, for ease of understanding, will be excluded here.)

(iv) The connectives,  $\neg$  (negation),  $\vee$  (disjunction),  $\wedge$  (conjunction),  $\supset$  (implication),  $\equiv$  (equivalence).

(v) The quantifiers,  $(\exists)$  – existential, and  $(\forall)$  – universal.

(vi) The brackets,  $[ \quad ]$  and  $\{ \quad \}$ .

Atomic formulae are obtained from the atomic symbols by filling the empty places of relative symbols with either individual constants or variables,

e.g.,  $S(a)$ ,  $S(x_n)$ ,  $R(x,a,y)$ . From these, well-formed formulae (wff) are obtained in successive stages, beginning with the atomic formulae, by applying connectives and quantifiers in the appropriate manner to well-formed formulae obtained already. At the same time brackets are introduced in such a way that the mode of construction of a well-formed formula can be determined unambiguously from its form. In more detail –

If  $X$  is an atomic formula,  $[X]$  is a well-formed formula; if  $X$  and  $Y$  are wff, then  $[\neg X]$ ,  $[X \vee Y]$ ,  $[X \wedge Y]$ ,  $[X \supset Y]$ ,  $[X \equiv Y]$  are wff; and if  $X$  is a wff then  $[(\exists y)X]$  and  $[(\forall y)X]$  are wff (where  $y$  stands for an arbitrary variable) provided  $y$  does not already appear in  $X$  under the sign of a quantifier, i.e., provided  $X$  does not already contain  $(\exists y)$  or  $(\forall y)$ .  $X$  is said to be the scope of the quantifier under consideration in  $[(\exists y)X]$  or  $[(\forall y)X]$  and in any wff which is obtained from these by the further (repeated) application of connectives or quantifiers.

An occurrence of a variable, e.g.  $y$ , in a wff  $X$  is *free* if  $y$  is not in  $(\exists y)$  or  $(\forall y)$  or in their scope in  $X$ . The wff  $X$  is called a *sentence* if it does not contain a free occurrence of a variable, otherwise it is called a *predicate*.

It is usual to omit brackets if this does not lead to any ambiguity, or if any ambiguity which is introduced thereby is irrelevant. Thus, we write  $[X_1 \vee X_2 \vee X_3]$  indiscriminately for  $[[X_1 \vee X_2] \vee X_3]$  and for  $[X_1 \vee [X_2 \vee X_3]]$ , since (as we shall see) it is irrelevant which of the two latter expressions is meant.

We say that the first order language  $L'$  is an *extension* of the first order language  $L$  if the set of atomic symbols of  $L$  is a subset of the set of atomic symbols of  $L'$ .

A *first order structure*,  $M$ , consists of a non-empty set of *individuals*,  $A$ , in which a set  $P$  of  $n$ -ary relations e.g.,  $S( , , \dots)$  are *defined* in the following sense. If  $S( , , \dots)$  is an  $n$ -ary relation which belongs to  $P$  and  $(a_1, \dots, a_n)$  is an ordered  $n$ -tuple of elements of  $A$  then it is determinate whether or not  $S(a_1, \dots, a_n)$  *holds* in  $M$ . (In set theoretic versions of the notion of a structure, a relation is actually identified with the set of  $n$ -tuples which hold for it.) For example, we may consider that an algebraic field is given by its set of individuals and by the three relations  $E( , )$ ,  $S( , , )$ ,  $P( , , )$ , where  $E$  is binary and  $S$  and  $P$  are ternary and where  $E(a,b)$  stands for  $a=b$  (equality), and  $S(a,b,c)$  and

$P(a,b,c)$  stand for  $a+b=c$  and  $ab=c$  (addition and multiplication) respectively.

**2.2 Interpretation.** Now let  $M$  be a first order structure and let  $L$  be a first order language. Suppose that we are given a one-to-one mapping  $C$  between the set of individuals of  $M$  and a subset of the set of individual constants of  $L$  and between the relations of  $M$  and a subset of the set of relation symbols of  $L$  in such a way that  $n$ -ary relations correspond to  $n$ -ary relation symbols. We shall say that a well-formed formula  $X$  of  $L$  is *defined* in  $M$  (with respect to the mapping  $C$ ) if all the individual constants and relation symbols which occur in  $X$  correspond to individuals and relations of  $M$  under  $C$ .

For any sentence  $X$  which is defined in  $M$  we shall now define what it signifies that  $X$  *holds in  $M$*  or *is true in  $M$* , or that  $M$  *is satisfied by  $X$* , by means of the following conditions. If  $X$  is defined, but does not hold in  $M$  we shall say also that  $X$  *is false in  $M$* .

(i) Let  $X=[Y]$  be defined in  $M$  where  $Y$  is an atomic formula. Thus  $Y=R(a_1, \dots, a_n)$  where  $R$  is an  $n$ -ary relation symbol and  $a_1, \dots, a_n$  are individual constants such that  $Y$ , under  $C$ ,  $R$  corresponds to a relation  $R'$  in  $M$  and  $a_1, \dots, a_n$  correspond to individuals  $a'_1, \dots, a'_n$  respectively. Then  $X$  *holds in  $M$*  (under  $C$ ) by definition, if and only if  $R'(a'_1, \dots, a'_n)$  holds in  $M$ .

(ii) Suppose that the sentence  $X=[\neg Y]$  is defined in  $M$ . Thus,  $Y$  also is defined in  $M$ . Then  $X$  *holds in  $M$*  if and only if  $Y$  does not hold in  $M$ . Again, suppose that the sentence  $X$  which is defined in  $M$  is given by  $X=[Y \vee Z]$ . Then  $X$  *holds in  $M$*  if and only if at least one of the two sentences  $Y$  or  $Z$  holds in  $M$ . For the remaining connectives, supposing the sentence  $X$  to be defined in  $M$ , the rules which decide whether or not  $X$  holds in  $M$  are as follows.

If  $X=[Y \wedge Z]$ ,  $X$  *holds in  $M$*  if and only if both  $Y$  and  $Z$  hold in  $M$ . If  $X=[Y \supset Z]$ ,  $X$  *holds in  $M$*  provided  $Z$  holds in  $M$ , and also if neither  $Y$  nor  $Z$  hold in  $M$ ;  $X$  does *not hold in  $M$*  if  $Y$  holds in  $M$  but  $Z$  does not hold in  $M$ . If  $X=[Y \equiv Z]$ ,  $X$  *holds in  $M$*  if and only if both  $Y$  and  $Z$  hold in  $M$  or neither  $Y$  nor  $Z$  holds in  $M$ .

(iii) Suppose now that  $X$  has been obtained by existential quantification. Thus,  $X$  is of the form  $X=[(\exists y)Z(y)]$ . Then  $X$  *holds in  $M$*  if and only if there exists an individual constant  $a$  such that  $Z(a)$  holds in  $M$ . (This implies that  $a$  corresponds to some individual of  $M$  under  $C$ ). We

have used the notation  $Z(y), Z(a)$  in order to indicate that  $Z(a)$  is obtained from the scope of  $(\exists y)$  by substituting  $a$  for (every occurrence of)  $y$ .

If  $y$  does not occur in  $Z$  the correct interpretation of our rule is that  $X$  holds in  $M$  if and only if  $Z$  holds in  $M$ .

Finally, suppose that  $X$  is of the form  $X = [(\forall y)Z(y)]$ . Then  $X$  holds in  $M$  if for every individual constant  $a$  of  $L$  which corresponds to an individual of  $M$  under  $C$ , the sentence  $Z(a)$  holds in  $M$ .

One shows by induction following the construction of a well-formed formula (or, more concretely, following the number of square brackets in a formula) that conditions (i), (ii), and (iii) jointly decide unambiguously for every  $X$  which is defined in  $M$  whether or not  $X$  holds in  $M$ , for the given  $C$ . The question whether or not a sentence  $X$  holds in a structure  $M$  depends only on the mapping  $C_X$  between the individual constants and relation symbols which actually occur in  $X$  and the appropriate individuals and relations of  $M$ , provided there are enough individual constants and relation symbols available in  $L$  to extend  $C_X$  to a mapping  $C$  in which all individuals and relations of  $M$  are mapped on corresponding entities in  $L$ . For a given  $M$ , we may at any rate extend  $L$  to a language  $L'$  in which this condition is satisfied. In order to avoid trivial but tiresome exceptions, we shall suppose from now on that such an extension of the given language has been carried out tacitly where necessary. That is to say, when stating that  $X$  holds in a structure  $M$  for a one-to-one mapping  $C$  between the individual constants and relation symbols which occur in  $X$  and certain individuals and relations in  $M$ , we shall imply that this is the case, according to our original definition, in an extension  $L'$  of  $L$  in which  $C_X$  can be extended to a mapping  $C$  as above.

Similarly, let  $K$  be a set of sentences in a first order language  $L$  and let there be given a one-to-one mapping  $C_K$  between the individual constants and relation symbols that occur in  $K$  and individuals and relations of a (first order) structure  $M$ , so that individual constants correspond to individuals and, for each  $n \geq 1$ ,  $n$ -ary relation symbols correspond to  $n$ -ary relations. Then we shall say that  $K$  holds in  $M$  (is true in  $M$ , is satisfied by  $M$  or, that  $M$  is a *model* of  $K$ ) for the given mapping  $C_K$  if in some extension  $L'$  of  $L$ ,  $C_K$  can be extended to a mapping  $C$  as above, which ranges over all individuals and relations of  $M$ , such that all sentences of  $K$  hold in  $M$  under this mapping, in the sense explained previously. Here again, the answer to the question whether or not  $K$  holds in  $M$  depends

only on  $C_K$  in the sense that if  $C = C_1$  and  $C = C_2$  are two extensions of  $C_K$  which range over all of  $M$  then the sentences of  $K$  hold in  $M$  under  $C_1$  if and only if they hold in  $M$  under  $C_2$ .

**2.3 Ultraproducts.** Let  $Q = \{M_v\}$  be a set of first order structures, where  $v$  varies over a non-empty index set  $I$ . Suppose that the  $M_v$  are similar, that is to say that the same relations are defined in all of them. (Notice that this concept of similarity is very wide. For example, all algebraic fields are similar if we take them to be given by the three relations of equality, addition, and multiplication, as introduced above. We do not mean that the relations need be the same extensionally, i.e., we identify, e.g., the relations of addition in different fields although they will in general be different in the set theoretic sense.) Let  $D$  be an ultrafilter in  $I$  (or, which is the same, a maximal dual ideal in the Boolean algebra of subsets of  $I$ ). That is to say,  $D$  is a set of subsets of  $I$  which satisfies the following conditions.

- (i) The empty set does not belong to  $D$ ,  $\emptyset \notin D$ .
- (ii) If  $A \in D$  and  $B \in D$  then  $A \cap B \in D$ .
- (iii) If  $A \in D$  and  $A \subset B \subset I$  then  $B \in D$ .
- (iv) For any  $A \subset I$ , either  $A \in D$  or  $I - A \in D$  (although, by (i) and (ii) the sets  $A$  and  $I - A$  cannot both belong to  $D$ ).

By means of  $Q$  and  $D$  we now define a new first order structure  $Q_D$ , called an *ultraproduct* as follows. The set of individuals of  $Q_D$  is the set of functions  $f(x)$  defined on  $I$  and satisfying the condition that  $f(v) \in M_v$  for every  $v \in I$ . Now let  $R(x_1, x_2, \dots, x_n)$  be a relation which is given for the structures  $M_v$  (where we have inserted the variables  $x_1, \dots, x_n$  into the empty places of  $R$  for easier reading). Let  $f_1, \dots, f_n$  be individuals of  $Q_D$ . Thus,  $f_i = f_i(x)$ ,  $i = 1, \dots, n$  where  $f_i(v) \in M_v$ , for  $v \in I$ , as detailed. We determine the relation  $R$  in  $Q_D$  by defining that  $R(f_1, \dots, f_n)$  shall hold in  $Q_D$  if and only if the set  $\{v | R(f_1(v), \dots, f_n(v))\}$  holds in  $M_v$  — i.e., the set of elements  $v$  of  $I$  for which  $R(f_1(v), \dots, f_n(v))$  holds in  $M$  — is an element of the ultrafilter  $D$ .

This completes the definition of  $Q_D$ .

**2.4 Prenex normal form.** A wff  $X$  in  $L$  is said to be in *prenex normal form* if, in the construction of  $X$  from atomic formulae, the quantifiers (if any) are applied after the connectives (if any) or, which is the same, if the connectives are in the scope of all quantifiers.

For the purposes of this and the following section, two sentences  $X$  and  $Y$  will be called *equivalent* if they contain the same individual constants and relation symbols, and if  $X \equiv Y$  holds in all structures in which this sentence is defined. This will be the case if and only if for any structure  $M$  in which  $X$  and  $Y$  are defined, and by the same correspondence, either both  $X$  and  $Y$  hold in  $M$  or neither  $X$  nor  $Y$  holds in  $M$ .

**2.4.1 THEOREM.** For every sentence  $X$ , there exists an equivalent sentence  $X_0$  which is in prenex normal form.

In order to establish this result, we observe that if  $X$  is equivalent to  $Y$  and  $Y$  is equivalent to  $Z$  according to the above definition, then  $X$  is equivalent to  $Z$ , i.e., the (meta-mathematical) relation of equivalence between sentences is transitive. For a given  $X$ , we may therefore find an equivalent  $Y$  by a *succession* of modifications each leading from a given sentence to an equivalent sentence. The modifications required here are based on the following facts which can be verified without difficulty by means of the definitions of section 2.2.

(i) If  $X$  contains a wff of the form  $[Y \equiv Z]$  and we replace it by  $[[Y \supset Z] \wedge [Z \supset Y]]$  then we obtain a sentence  $X'$  which is equivalent to  $X$ , by 2.2(ii).

(ii) If  $X$  contains a wff of the form  $[Y \supset Z]$ , and we replace it by  $[[\neg Y] \vee Z]$  then we obtain a sentence  $X'$  which is equivalent to  $X$ , again by 2.2(ii).

(iii) The sentence  $Y = [\neg[Z \vee W]]$  is equivalent to  $Y' = [[\neg Z] \wedge [\neg W]]$ , by 2.2(ii), and the sentence  $Y = [\neg[Z \wedge W]]$  is equivalent to  $Y' = [[\neg Z] \vee [\neg W]]$ , by 2.2(ii) (these are 'de Morgan's laws'). Hence, if a sentence  $X$  contains a *well-formed formula*  $Y = [\neg[Z \vee W]]$ , and we replace  $Y$  by  $Y' = [[\neg Z] \wedge [\neg W]]$  then we obtain a sentence  $X'$  which is equivalent to  $X$ . And if the sentence  $X$  contains a wff  $Y = [\neg[Z \wedge W]]$  and we replace the latter by  $Y' = [[\neg Z] \vee [\neg W]]$  then we obtain a sentence  $X'$  which is equivalent to  $X$ .

(iv) If the sentence  $Y$  is given by  $Y = [\neg[(\exists y)Z]]$  then  $Y$  is equivalent to  $Y' = [(\forall y)[\neg Z]]$ , by 2.2(ii) and 2.2(iii), and if  $Y = [\neg[(\forall y)Z]]$  then  $Y$  is equivalent to  $Y' = [(\exists y)[\neg Z]]$ , also by 2.2(ii) and 2.2(iii). Accordingly, if in any sentence we replace a *well-formed formula*  $[\neg[(\exists y)Z]]$  or  $[\neg[(\forall y)Z]]$  by  $[(\exists y)[\neg Z]]$  or  $[(\forall y)[\neg Z]]$  respectively, then we obtain an equivalent sentence.