A PRINCETON UNIVERSITY PRESS **E-BOOK**

Spin Geometry

H. Blaine Lawson, jr.

and

Marie-Louise Michelsohn

PRINCETON

Spin Geometry

PRINCETON MATHEMATICAL SERIES

Editors: WILLIAM BROWDER, ROBERT P. LANGLANDS, JOHN D. MILNOR, and ELIAS M. STEIN

1. The Classical Groups, By HERMANN WEYL
3. An Introduction to Differential Geometry, By LUTHER PFAHLER EISENHART
4. Dimension Theory, By W. HUREWICZ and H. WALLMAN
6. The Laplace Transform, By D. V. WIDDER
7. Integration, By EDWARD J. MCSHANE
8. Theory of Lie Groups: I, By C. CHEVALLEY
9. Mathematical Methods of Statistics, By HARALD CRAMÉR
10. Several Complex Variables, By S. BOCHNER and W. T. MARTIN
11. Introduction to Topology, By S. LEFSCHETZ
12. Algebraic Geometry and Topology, Edited by R. H. FOX, D. C. SPENCER, and A. W. TUCKER
14. The Topology of Fibre Bundles, By NORMAN STEENROD
15. Foundations of Algebraic Topology, By SAMUEL EILENBERG and NORMAN STEENROD
16. Functionals of Finite Riemann Surfaces, By MENAHEM SCHIFFER and DONALD C. SPENCER.
17. Introduction to Mathematical Logic, Vol. I, By ALONZO CHURCH
19. Homological Algebra, By H. CARTAN and S. EILENBERG
20. The Convolution Transform, By I. I. HIRSCHMAN and D. V. WIDDER
21. Geometric Integration Theory, By H. WHITNEY
22. Qualitative Theory of Differential Equations, By V. V. NEMYTSKII and V. V. STEPANOV
23. Topological Analysis, By GORDON T. WHYBURN (revised 1964)
24. Analytic Functions, By AHLFORS, BEHNKE and GRAUERT, BERS, et al.
25. Continuous Geometry, By JOHN VON NEUMANN
26. Riemann Surfaces, By L. AHLFORS and L. SARIO
27. Differential and Combinatorial Topology, Edited by S. S. CAIRNS
28. Convex Analysis, By R. T. ROCKAFELLAR
29. Global Analysis, Edited By D. C. SPENCER and S. IYANAGA
30. Singular Integrals and Differentiability Properties of Functions, By E. M. STEIN
31. Problems in Analysis, Edited by R. C. GUNNING
32. Introduction to Fourier Analysis on Euclidean Spaces, By E. M. STEIN and G. WEISS
33. Étale Cohomology, By J. S. MILNE
34. Pseudodifferential Operators, By MICHAEL E. TAYLOR
36. Representation Theory of Semisimple Groups: An Overview Based on Examples, By ANTHONY W. KNAPP
37. Foundations of Algebraic Analysis, By MASAKI KASHIWARA, TAKAHIRO KAWAI, and TATSUO KIMURA. Translated by GORO KATO
38. Spin Geometry, By H. BLAINE LAWSON and MARIE-LOUISE MICHELSOHN

Spin Geometry

H. BLAINE LAWSON, JR.
and
MARIE-LOUISE MICHELSOHN

PRINCETON UNIVERSITY PRESS
PRINCETON, NEW JERSEY
1989

Published by Princeton University Press, 41 William Street,
Princeton, New Jersey 08540
In the United Kingdom: Princeton University Press,
Chichester, West Sussex

Library of Congress Cataloging-in-Publication Data

Lawson, H. Blaine.
Spin geometry/H. B. Lawson and M.-L. Michelsohn.
p. cm.—(Princeton mathematical series: 38)
Includes index.
ISBN 0-691-08542-0 (alk. paper):
1. Nuclear spin—Mathematics. 2. Geometry. 3. Topology.
4. Clifford algebras. 5. Mathematical physics. I. Michelsohn,
M.-L. (Marie-Louise), 1941– II. Title. III. Series.
QC793.3.S6L39 1989
539.7′25—dc20 89-32544

Princeton University Press books are printed on acid-free paper
and meet the guidelines for permanence and durability of the
Committee on Production Guidelines for Book Longevity
of the Council on Library Resources

Printed in the United States of America

Second Printing, with
errata sheet, 1994

5 7 9 10 8 6

http://pup.princeton.edu

ISBN-13: 978-0-691-08542-5
ISBN-10: 0-691-08542-0

For Christie, Didi,
Michelle, and Heather

Contents

Preface

In the late 1920's the relentless march of ideas and discoveries had carried physics to a generally accepted relativistic theory of the electron. The physicist P.A.M. Dirac, however, was dissatisfied with the prevailing ideas and, somewhat in isolation, sought for a better formulation. By 1928 he succeeded in finding a theory which accorded with his own ideas and also fit most of the established principles of the time. Ultimately this theory proved to be one of the great intellectual achievements of the period. It was particularly remarkable for the internal beauty of its mathematical structure which not only clarified much previously mysterious phenomena but also predicted in a compelling way the existence of an electron-like particle of negative energy. Indeed such particles were subsequently found to exist and our understanding of nature was transformed.

Because of its compelling beauty and physical significance it is perhaps not surprising that the ideas at the heart of Dirac's theory have also been discovered to play a role of great importance in modern mathematics, particularly in the interrelations between topology, geometry and analysis. A great part of this new understanding comes from the work of M. Atiyah and I. Singer. It is their work and its implications which form the focus of this book.

It seems appropriate to sketch some of the fundamental ideas here. In searching for his theory, Dirac was faced, roughly speaking, with the problem of finding a Lorentz-invariant wave equation $D\psi = \lambda\psi$ compatible with the Klein-Gordon equation $\Box\psi = \lambda\psi$ where $\Box = (\partial/\partial x_0)^2 - (\partial/\partial x_1)^2 - (\partial/\partial x_2)^2 - (\partial/\partial x_3)^2$. Causality required that D be first order in the "time" coordinate x_0. Of course by Lorentz invariance there could be no preferred time coordinate, and so D was required to be first-order in all variables. Thus, in essence Dirac was looking for a first-order differential operator whose square was the laplacian. His solution was to replace the complex-valued wave function ψ with an n-tuple $\Psi = (\psi_1, \ldots, \psi_n)$ of such functions. The operator D then became a first-order system of the form

$$D = \sum_{\mu=0}^{3} \gamma_\mu \frac{\partial}{\partial x_\mu}$$

where $\gamma_0, \ldots, \gamma_3$ were $n \times n$-matrices. The requirement that

$$D^2 = \begin{pmatrix} \square & & \\ & \ddots & \\ & & \square \end{pmatrix}$$

led to the equations

$$\gamma_\nu \gamma_\mu + \gamma_\mu \gamma_\nu = \pm 2\delta_{\nu\mu}.$$

These were easily and explicitly solved for small values of n, and the analysis was underway.

This construction of Dirac has a curious and fundamental property. Lorentz transformations of the space-time variables $(x_0, \ldots, x_3)$ induce linear transformations of the n-tuples Ψ which are determined only up to a sign. Making a consistent choice of sign amounts to passing to a non-trivial 2-fold covering $\tilde{L}$ of the Lorentz group L. That is, in transforming the Ψ's one falls upon a representation of $\tilde{L}$ which does not descend to L.

The theory of Dirac had another interesting feature. In the presence of an electromagnetic field the Dirac Hamiltonian contained an additional term added on to what one might expect from the classical case. There were strong formal analogies with the additional term one obtains by introducing internal spin into the mechanical equations of an orbiting particle. This "spin" or internal magnetic moment had observable quantum effects. The n-tuples Ψ were thereby called *spinors* and this family of transformations was called the *spin representation.*

This physical theory touches upon an important and general fact concerning the orthogonal groups. (We shall restrict ourselves for the moment to the positive definite case.) In the theory of Cartan and Weyl the representations of the Lie algebra of SO_n are essentially generated by two basic ones. The first is the standard n-dimensional representation (and its exterior powers). The second is constructed from the representations of the algebra generated by the γ_μ's as above (the *Clifford algebra* associated to the quadratic form defining the orthogonal group). This second representation is called the spin representation. It does not come from a representation of the orthogonal group, but only of its universal covering group, called Spin_n. It plays a key role in an astounding variety of questions in geometry and topology: questions involving vector fields on spheres, immersions of manifolds, the integrality of certain characteristic numbers, triality in dimension eight, the existence of complex structures, the existence of metrics of positive scalar curvature, and perhaps most basically, the index of elliptic operators.

In the early 1960s general developments had led mathematicians to consider the problem of finding a topological formula for the index of any elliptic operator defined on a compact manifold. This formula was to generalize the important Hirzebruch-Riemann-Roch Theorem already established in the complex algebraic case. In considering the problem, Atiyah and Singer noted that among all manifolds, those whose SO_n-structure could be simplified to a $Spin_n$-structure had particularly suggestive properties. Realizing that over such spaces one could carry out the Dirac construction, they produced a globally defined elliptic operator canonically associated to the underlying riemannian metric. The index of this operator was a basic topological invariant called the $\hat{A}$-genus, which was known always to be an integer in this special class of spin manifolds. (It is not an integer in general.) Twisting the Dirac-type operator with arbitrary coefficient bundles led, with some sophistication, to a general formula for the index of any elliptic operator.

Atiyah and Singer went on to understand the index in the more proper setting of K-theory. This led in particular to the formulation of certain KO-invariants which have profound applications in geometry and topology. These invariants touch questions unapproachable by other means. Their study and elucidation was a principal motivation for the writing of this tract.

It is interesting to note in more recent years there has been another profound and beautiful physical theory whose ideas have come to the core of topology, geometry and analysis. This is the non-abelian gauge field theory of C. N. Yang and R. L. Mills which through the work of S. Donaldson and M. Freedman has led to astonishing results in dimension four. Yang-Mills theory can be plausibly considered a highly non-trivial generalization of Dirac's theory which encompasses three fundamental forces: the weak, strong, and electromagnetic interactions. This theory involves modern differential geometry in an essential way. The theory of connections, Dirac-type operators, and index theory all play an important role. We hope this book can serve as a modest introduction to some of these concepts.

H. B. LAWSON AND M.-L. MICHELSOHN
Stony Brook

Acknowledgments

This book owes much to the fundamental work of Michael Atiyah and Iz Singer. Part of our initial motivation in writing the book was to give a leisurely and rounded presentation of their results.

The authors would like to express particular gratitude to Peter Landweber, Jean-Pierre Bourguignon, Misha Katz, Haiwan Chen and Peter Woit, each of whom read large parts of the original manuscript and made a number of important suggestions. We are also grateful to the National Science Foundation and the Brazilian C. N. Pq. for their support during the writing of this book.

H. B. LAWSON AND M.-L. MICHELSOHN

This author would like to take this opportunity to express her deep appreciation to Mark Mahowald who held out a hand when one was so dearly needed.

M.-L. MICHELSOHN

Spin Geometry

Introduction

Over the past two decades the geometry of spin manifolds and Dirac operators, and the various associated index theorems have come to play an increasingly important role both in mathematics and in mathematical physics. In the area of differential geometry and topology they have become fundamental. Topics like spin cobordism, previously considered exotic even by topologists, are now known to play an essential role in such classical questions as the existence or non-existence of metrics of positive curvature. Indeed, the profound methods introduced into geometry by Atiyah, Bott, Singer and others are now indispensible to mathematicians working in the field. It is the intent of this book to set out the fundamental concepts and to present these methods and results in a unified way.

A principal theme of the exposition here is the consistent use of Clifford algebras and their representations. This reflects the observed fact that these algebras emerge repeatedly at the very core of an astonishing variety of problems in geometry and topology.

Even in discussing riemannian geometry, the formalism of Clifford multiplication will be used in place of the more conventional exterior tensor calculus. There is a philosophical justification for this bias. Recall that to any vector space V there is naturally associated the exterior algebra $\Lambda^* V$, and this association carries over directly to vector bundles. Applied to the tangent bundle of a smooth manifold, it gives the de Rham bundle of exterior differential forms. In a similar way, to any vector space V equipped with a quadratic form q, there is associated the Clifford algebra $C\ell(V, q)$, and this association carries over directly to vector bundles equipped with fibre metrics. In particular, applied to the tangent bundle of a smooth riemannian manifold, it gives a canonically associated bundle of algebras, called the Clifford bundle. As a vector bundle it is isomorphic to the bundle of exterior forms. However, the Clifford multiplication is strictly richer than exterior multiplication; it reflects the inner symmetries and basic identities of the riemannian structure. In fact fundamental curvature identities will be derived here in the formalism of Clifford multiplication and applied to some basic problems.

Another justification for our approach is that the Clifford formalism gives a transparent unification of all the fundamental elliptic complexes in differential geometry. It also renders many of the technical arguments involved in applying the Index Theorem quite natural and simple.

This point of view concerning Clifford bundles and Clifford multiplication is an implicit, but rarely an explicit theme in the writing of Atiyah and Singer. The authors feel that for anyone working in topology or geometry it is worthwhile to develop a friendly, if not intimate relationship with spin groups and Clifford modules. For this reason we have used them explicitly and systematically in our exposition.

The book is organized into four chapters whose successive themes are algebra, geometry, analysis, and applications. The first chapter offers a detailed introduction to Clifford algebras, spin groups and their representations. The concepts are illuminated by giving some direct applications to the elementary geometry of spheres, projective spaces, and low-dimensional Lie groups. K-theory and KR-theory are then introduced, and the fundamental relationship between Clifford algebras and Bott periodicity is established.

In the second chapter of this book, the algebraic concepts are carried over to define structures on differentiable manifolds. Here one enters properly into the subject of spin geometry. Spin manifolds, spin cobordism, and spinor bundles with their canonical connections are all discussed in detail, and a general formalism of Dirac bundles and Dirac operators is developed. Hodge-de Rham Theory is reviewed in this formalism, and each of the fundamental elliptic operators of riemannian geometry is derived and examined in detail.

Special emphasis is given here to introducing the notion of a $C\ell_k$-linear elliptic operator and discussing its index. This index lives in a certain quotient of the Grothendieck group of Clifford modules. For the fundamental operators (which are discussed in detail here) it is one of the deepest and most subtle invariants of global riemannian geometry. The systematic discussion of $C\ell_k$-linear differential operators is one of the important features of this book.

In the last section of Chapter II a universal identity of Bochner type is established for any Dirac bundle, and the classical vanishing theorems of Bochner and Lichnerowicz are derived from it.

This seems an appropriate time to make some general observations about spin geometry. To begin it should be emphasized that spin geometry is really a special topic in riemannian geometry. The central concept of a spin manifold is often considered to be a topological one. It is just a manifold with a simply-connected structure group. This is understood systematically as follows. On a general differentiable n-manifold ($n \geqq 3$), the tangent bundle has structure group GL_n. The manifold is said to be oriented if the structure group is reduced to GL_n^+ (the connected component of the identity). The manifold is said to be spin if the structure group GL_n^+ can be "lifted" to the universal covering group $\widetilde{\mathrm{GL}}_n \to \mathrm{GL}_n^+$. This approach is perfectly correct, but there is a hidden obstruction to the viability of the concept: namely, *the group* $\widetilde{\mathrm{GL}}_n$ (*for* $n \geqq 3$) *has no finite dimensional representations that do not come from* GL_n^+. This means that in terms of standard tensor calculus, nothing has been gained by this refinement of the structure.

However, if one passes from GL_n to the maximal compact subgroup O_n, that is, if one introduces a riemannian metric on the manifold, the story is quite different. An orientation corresponds to reducing the structure group to SO_n, and a spin structure corresponds to then lifting the structure group to the universal covering group $\mathrm{Spin}_n \to \mathrm{SO}_n$. Maximal compact subgroups are homotopy equivalent to the Lie groups which contain them, and there is essentially no topological difference in viewing spin structures this way. However, *there do exist finite dimensional representations of* Spin_n *which are not lifts of representations of* SO_n. Over a spin manifold one can thereby construct certain new vector bundles, called bundles of spinors, which do not exist over general manifolds. Their existence allows the introduction of certain important analytic tools which are not generally available, and these tools play a central role in the study of the global geometry of the space. It is, by the way, an important fact that this construction is metric-dependent; *the bundle of spinors itself depends in an essential way on the choice of riemannian structure on the manifold.*

These observations lead one to suspect that there must exist a local spinor calculus, like the tensor calculus, which should be an important component of local riemannian geometry. A satisfactory formalism of this type has not yet been developed. However, the spinors bundles have yielded profound relations between local riemannian geometry and global topology.

The main tools by which we access the global structure of spin manifolds are the various index theorems of Atiyah and Singer. These are presented and proved in Chapter III of the book. They include not just the standard G-Index Theorem but also the Index Theorem for Families and the $C\ell_k$-Index Theorem (for $C\ell_k$-linear elliptic operators). There are in existence today many elegant proofs of index theorems which use the methods of the heat equation. These do not apply to the $C\ell_k$-Index Theorem however, because of the non-local nature of this index. For this reason our exposition follows the "softer," or more topological, arguments given in the original proofs.

Chapter IV of the book is concerned with applications of the theory. There is no attempt to be exhaustive; such an attempt would be pointless and nearly impossible. We have tried however to demonstrate the broad range of problems in which the considerations of spin geometry can be effectively implemented.

It is of some historical interest to note that while Dirac did essentially use Clifford modules in the construction of his wave operator, he was not really responsible for what is commonly called the "Dirac operator" in riemannian geometry. The construction of this operator is due to Atiyah and Singer and is, in our estimation, one of their great achievements. It required for its discovery an understanding of the subtle geometry of spin manifolds and a recognition of the central role it would play in the general theory of elliptic operators. Even the formidable Élie Cartan, who sensed the importance of the question and, of course, authored the general theory of spinors and who was not unaware of the fundamentals of global analysis, never reached the point of defining this operator in the proper context of spin manifolds. In keeping with historical developments we shall call the general construction of operators from modules over the Clifford bundle, the Dirac construction, and we shall call the specific operator so defined on the spinor bundle, the Atiyah-Singer operator.

It is this operator which in a very specific sense generates all elliptic operators over a spin manifold. It introduces a direct relationship between curvature and topology which exists only under the spin hypothesis. The $C\ell_k$-linear version of this operator carries an index in KO-theory. In fact its index gives a basic ring homomorphism $\Omega_*^{\mathrm{Spin}} \to KO^{-*}(\mathrm{pt})$ which generalizes to KO-theory the classical $\hat{A}$-genus. The applications of this to geometry include the fact that half the exotic spheres in dimensions one and two (mod 8) do not carry metrics of positive scalar curvature.

The presentation in this book is aimed at readers with a knowledge of elementary geometry and topology. Important things, such as the concept of spin manifolds and the theory of connections, are developed from basic definitions. The Atiyah-Singer index theorems are formulated and proved assuming little more than a knowledge of the Fourier inversion formula. There are several appendices in which principal bundles, classifying spaces, Thom isomorphisms, and spin manifolds are discussed in detail.

The references to theorems and equations within each chapter are made without reference to the chapter itself (e.g., 2.7 or (5.9)). References to other chapters are prefaced by the chapter number (e.g., III.2.7 or (IV.5.9)).

CHAPTER I

Clifford Algebras, Spin Groups and Their Representations

The object of this chapter is to present the algebraic ideas which lie at the heart of spin geometry. The central concept is that of a Clifford algebra. This is an algebra naturally associated to a vector space which is equipped with a quadratic form. Within the group of units of the algebra there is a distinguished subgroup, called the spin group, which, in the case of the positive definite form on $\mathbb{R}^n$ ($n > 2$), is the universal covering group of SO_n.

It is a striking (and not commonplace) fact that Clifford algebras and their representations play an important role in many fundamental aspects of differential geometry. These include such diverse topics as Hodge-de Rham Theory, Bott periodicity, immersions of manifolds into spheres, families of vector fields on spheres, curvature identities in riemannian geometry, and Thom isomorphisms in K-theory. The effort invested in becoming comfortable with this algebraic formalism is well worthwhile.

Our discussion begins in a very general algebraic context but soon moves to the real case in order to keep matters simple and in the domain of most interest. In §§7 and 8 we present some applications of the purely algebraic theory to topology and to the appearence of exceptional phenomena in the theory of Lie groups.

The last part of the chapter is devoted to K-theory. Basic definitions are given and fundamental results are reviewed. The discussion culminates with the Atiyah-Bott-Shapiro isomorphisms which directly relate the periodicity phenomena in Clifford algebras to the classical Bott Periodicity Theorems. In particular, explicit isomorphisms are given between $K^{-*}(\mathrm{pt}) = \bigoplus_n K(S^n)$ (and $KO^{-*}(\mathrm{pt}) = \bigoplus_n KO(S^n)$) and a certain quotient of the ring of Clifford modules. Section 10 is concerned with KR-theory which later plays a role in the index theorem for families of real elliptic operators. This is a bigraded theory and the corresponding Atiyah-Bott-Shapiro isomorphism entails representations of Clifford algebras $C\ell_{r,s}$ for quadratic forms of indefinite signature.

§1. Clifford Algebras

Let V be a vector space over the commutative field k and suppose q is a quadratic form on V. The **Clifford algebra** $C\ell(V,q)$ associated to V and

q is an associative algebra with unit defined as follows. Let

$$\mathscr{T}(V) = \sum_{r=0}^{\infty} \bigotimes{}^r V$$

denote the tensor algebra of V, and define $\mathscr{I}_q(V)$ to be the ideal in $\mathscr{T}(V)$ generated by all elements of the form $v \otimes v + q(V)1$ for $v \in V$. Then the Clifford algebra is defined to be the quotient

$$C\ell(V,q) \equiv \mathscr{T}(V)/\mathscr{I}_q(V).$$

There is a natural embedding

$$V \hookrightarrow C\ell(V,q) \tag{1.1}$$

which is the image of $V = \bigotimes^1 V$ under the canonical projection

$$\pi_q : \mathscr{T}(V) \longrightarrow C\ell(V,q). \tag{1.2}$$

We prove that $\pi_q|_V$ is injective as follows. We say that an element $\varphi \in \mathscr{T}(V)$ is of **pure degree** s if $\varphi \in \bigotimes^s V$. (Every element of $\mathscr{T}(V)$ is a finite sum of elements of pure degree.) We want to show that any element $\varphi \in \mathscr{I}_g(V) \cap V$ is zero. Any such element can be written as a finite sum $\varphi = \sum a_i \otimes (v_i \otimes v_i + q(v_i)) \otimes b_i$ where we may assume that the a_i's and b_i's are of pure degree. Since $\varphi \in V = \bigotimes^1 V$, we conclude that $\sum a_{i'} \otimes (v_{i'} \otimes v_{i'}) \otimes b_{i'} = 0$, where this sum is taken over those indices with $\deg a_i + \deg b_i$ maximal. This equation implies, by contraction with q, that $\sum a_{i'} q(v_{i'}) \cdot b_{i'} = 0$. Proceeding inductively, we prove that $\varphi = 0$.

The algebra $C\ell(V,q)$ is generated by the vector space $V \subset C\ell(V,q)$ (and the identity 1) subject to the relations:

$$v \cdot v = -q(v)1 \tag{1.3}$$

for $v \in V$. If the characteristic of k is not 2, then for all $v,w \in V$,

$$v \cdot w + w \cdot v = -2q(v,w) \tag{1.4}$$

where $2q(v,w) \equiv q(v + w) - q(v) - q(w)$ is the polarization of q. The relations (1.3) can be used to give the following universal characterization of the algebra.

Proposition 1.1. *Let $f: V \to \mathscr{A}$ be a linear map into an associative k-algebra with unit, such that*

$$f(v) \cdot f(v) = -q(v)1 \tag{1.5}$$

for all $v \in V$. Then f extends uniquely to a k-algebra homomorphism $\tilde{f}: C\ell(V,q) \to \mathscr{A}$. Furthermore, $C\ell(V,q)$ is the unique associative k-algebra with this property.

Proof. Any linear map $f: V \to \mathscr{A}$ extends to a unique algebra homomorphism $\tilde{f}: \mathscr{T}(V) \to \mathscr{A}$. Property (1.5) implies that $\tilde{f} = 0$ on $\mathscr{I}_q(V)$, and so $\tilde{f}$ descends to $C\ell(V,q)$. Suppose now that $\mathscr{C}$ is an associative k-algebra with unit and that $i: V \hookrightarrow \mathscr{C}$ is an embedding with the property that any linear map $f: V \to \mathscr{A}$ ($\mathscr{A}$ as above) with property (1.5) extends uniquely to an algebra homomorphism $\tilde{f}: \mathscr{C} \to \mathscr{A}$. Then the isomorphism from $V \subset C\ell(V,q)$ to $i(V) \subset \mathscr{C}$ clearly induces an algebra isomorphism $C\ell(V,q) \xrightarrow{\approx} \mathscr{C}$. ■

This characterization of Clifford algebras is extremely useful. It shows, for example, that they are functorial in the following sense. Given a morphism $f: (V,q) \to (V',q')$, i.e., a k-linear map $f: V \to V'$ between vector spaces which preserves the quadratic forms ($f^*q' = q$), there is, by Proposition 1.1, an induced homomorphism $\tilde{f}: C\ell(V,q) \to C\ell(V',q')$. Given another morphism $g:(V',q') \to (V'',q'')$, we see from the uniqueness in Proposition 1.1, that $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$.

A particular consequence of this is that the orthogonal group $O(V,q) \equiv \{f \in GL(V): f^*q = q\}$ extends canonically to a group of automorphisms of $C\ell(V,q)$. We shall see later that this embedding

$$O(V,q) \subset \operatorname{Aut}(C\ell(V,q)) \tag{1.6}$$

actually lies in the subgroup of inner automorphisms.

An element here of particular importance is the automorphism

$$\alpha: C\ell(V,q) \longrightarrow C\ell(V,q) \tag{1.7}$$

which extends the map $\alpha(v) = -v$ on V. Since $\alpha^2 = Id$, there is a decomposition

$$C\ell(V,q) = C\ell^0(V,q) \oplus C\ell^1(V,q) \tag{1.8}$$

where $C\ell^i(V,q) = \{\varphi \in C\ell(V,q): \alpha(\varphi) = (-1)^i\varphi\}$ are the eigenspaces of α. Clearly, since $\alpha(\varphi_1\varphi_2) = \alpha(\varphi_1) \cdot \alpha(\varphi_2)$, we have that

$$C\ell^i(V,q) \cdot C\ell^j(V,q) \subseteq C\ell^{i+j}(V,q) \tag{1.9}$$

where the indices are taken modulo 2. An algebra with a decomposition (1.8) satisfying (1.9) is called a **$\mathbb{Z}_2$-graded algebra.** Note that $C\ell^0(V,q)$ is a subalgebra of $C\ell(V,q)$. It is called the **even part** of $C\ell(V,q)$. The subspace $C\ell^1(V,q)$ is called the **odd part.** It is an observation of Atiyah, Bott and Shapiro that this $\mathbb{Z}_2$-grading plays an important role in the analysis and application of Clifford algebras.

There exist some elementary and important relationships between the Clifford algebra $C\ell(V,q)$ of a space and its exterior algebra $\Lambda^* V$ (whose definition is, of course, independent of the quadratic form q). There is a natural filtration $\tilde{\mathscr{F}}^0 \subset \tilde{\mathscr{F}}^1 \subset \tilde{\mathscr{F}}^2 \subset \ldots \subset \mathscr{T}(V)$ of the tensor algebra, which is defined by

$$\tilde{\mathscr{F}}^r \equiv \sum_{s \leqq r} \bigotimes^s V,$$

and has the property that $\tilde{\mathscr{F}}^r \otimes \tilde{\mathscr{F}}^{r'} \subseteqq \tilde{\mathscr{F}}^{r+r'}$. If we set $\mathscr{F}^i = \pi_q(\tilde{\mathscr{F}}^i)$ we obtain a filtration $\mathscr{F}^0 \subset \mathscr{F}^1 \subset \mathscr{F}^2 \subset \ldots \subset C\ell(V,q)$ of the Clifford algebra, which also has the property that

$$\mathscr{F}^r \cdot \mathscr{F}^{r'} \subseteqq \mathscr{F}^{r+r'} \tag{1.10}$$

for all r,r'. This makes $C\ell(V,q)$ into a **filtered algebra.** It follows from (1.10) that the multiplication map descends to a map $(\mathscr{F}^r/\mathscr{F}^{r-1}) \times (\mathscr{F}^s/\mathscr{F}^{s-1}) \to (\mathscr{F}^{r+s}/\mathscr{F}^{r+s-1})$ for all r,s. Setting $\mathscr{G}^* \equiv \bigoplus_{r \geqq 0} \mathscr{G}^r$ where $\mathscr{G}^r \equiv \mathscr{F}^r/\mathscr{F}^{r-1}$, we obtain the **associated graded algebra.**

Proposition 1.2. *For any quadratic form q, the associated graded algebra of $C\ell(V,q)$ is naturally isomorphic to the exterior algebra $\Lambda^* V$.*

Proof. The map $\bigotimes^r V \xrightarrow{\pi_r} \mathscr{F}^r \to \mathscr{F}^r/\mathscr{F}^{r-1}$, which is given by $v_{i_1} \otimes \cdots \otimes v_{i_r} \mapsto [v_{i_1} \cdots v_{i_r}]$ clearly descends to a map $\Lambda^r V \to \mathscr{F}^r/\mathscr{F}^{r-1}$ by property (1.4). (When the characteristic of k is 2, we use the fact that $v \cdot w + w \cdot v = 0$.) This map is evidently surjective and is easily seen to give a homomorphism of graded algebras $\Lambda^* V \to \mathscr{G}^*$.

To see that this map is injective we proceed as follows. The kernel of $\bigotimes^r V \to \mathscr{G}^r$ consists of the r-homogeneous pieces of elements $\varphi \in \mathscr{I}_q(V)$ of degree $\leqq r$. Any such φ can be written as a finite sum $\varphi = \sum a_i \otimes (v_i \otimes v_i + q(v_i)) \otimes b_i$ where $v_i \in V$ and where we may assume that the a_i and b_i are of pure degree with $\deg a_i + \deg b_i \leqq r - 2$. The r-homogeneous part of φ is then of the form $\varphi_r = \sum a_i \otimes v_i \otimes v_i \otimes b_i$ (where $\deg a_i + \deg b_i = r - 2$ for each i). Since $v_i \wedge v_i = 0$ for all i, we see that the image of φ in the exterior algebra is zero. Hence the map $\Lambda^r V \to \mathscr{G}^r$ is injective. ■

Proposition 1.2 says that Clifford multiplication is an enhancement of exterior multiplication which is determined by the form q. Note that $C\ell(V,0) \cong \Lambda^* V$.

Proposition 1.3. *There is a canonical vector space isomorphism*

$$\Lambda^* V \xrightarrow{\approx} C\ell(V,q) \tag{1.11}$$

compatible with the filtrations.

REMARK 1.4. The map (1.11) is, of course, *not* an isomorphism of algebras unless $q = 0$. The point here is that the map is canonical. Thus we may speak of the embeddings

$$\Lambda^r V \subset C\ell(V,q) \qquad \text{for all } r \geqq 0. \tag{1.12}$$

Proof. We define a map of the r-fold direct product $f: V \times \cdots \times V \to \mathrm{C}\ell(V,q)$ by setting

$$f(v_1, \ldots, v_r) = \frac{1}{r!} \sum_\sigma \operatorname{sign}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(r)} \tag{1.13}$$

where the sum is taken over the symmetric group on r elements. (If the characteristic of k is not zero, one must drop the factor $1/r!$.) Clearly f determines a linear map $\tilde{f}: \Lambda^r V \to \mathrm{C}\ell(V,q)$ whose image lies in $\mathscr{F}^r$. The composition of $\tilde{f}$ with the projection $\mathscr{F}^r \to \mathscr{F}^r/\mathscr{F}^{r-1}$ is easily seen to be the map discussed in the proof of Proposition 1.2. Hence $\tilde{f}$ is injective, and the direct sum of these maps (1.11) is an isomorphism.

We now take up the question of tensor products. Recall that if $\mathscr{A}$ and $\mathscr{B}$ are algebras with unit over k, then the tensor product of the algebras $\mathscr{A} \otimes \mathscr{B}$ is the algebra whose underlying vector space is the tensor product of $\mathscr{A}$ and $\mathscr{B}$ and whose multiplication is given (on simple elements) by the rule $(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb')$. If, however,

$$\mathscr{A} = \mathscr{A}^0 \oplus \mathscr{A}^1 \qquad \text{and} \qquad \mathscr{B} = \mathscr{B}^0 \oplus \mathscr{B}^1$$

are $\mathbb{Z}_2$-graded algebras, then we can introduce a second "$\mathbb{Z}_2$-graded" multiplication, determined by the rule

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{\deg(b)\deg(a')}(aa') \otimes (bb') \tag{1.14}$$

whenever b and a' are of pure degree (even or odd). The resulting algebra is called the **$\mathbb{Z}_2$-graded tensor product** and is denoted $\mathscr{A} \hat{\otimes} \mathscr{B}$.

The $\mathbb{Z}_2$-graded tensor product is again $\mathbb{Z}_2$-graded with

$$(\mathscr{A} \hat{\otimes} \mathscr{B})^0 = \mathscr{A}^0 \otimes \mathscr{B}^0 + \mathscr{A}^1 \otimes \mathscr{B}^1$$
$$(\mathscr{A} \hat{\otimes} \mathscr{B})^1 = \mathscr{A}^1 \otimes \mathscr{B}^0 + \mathscr{A}^0 \otimes \mathscr{B}^1.$$

It also carries a filtration $\mathscr{F}^0 \subset \mathscr{F}^1 \subset \mathscr{F}^2 \subset \ldots \subset \mathscr{A} \hat{\otimes} \mathscr{B}$, where

$$\mathscr{F}^r \equiv \sum_{i+j=r} \mathscr{F}^i(\mathscr{A}) \hat{\otimes} \mathscr{F}^j(\mathscr{B}).$$

The importance of the $\mathbb{Z}_2$-graded tensor product for Clifford algebras is evident from the following proposition.

Proposition 1.5. *Let $V = V_1 \oplus V_2$ be a q-orthogonal decomposition of the vector space V (i.e., $q(v_1 + v_2) = q(v_1) + q(v_2)$ for all $v_1 \in V_1$ and $v_2 \in V_2$). Then there is a natural isomorphism of Clifford algebras*

$$\mathrm{C}\ell(V,q) \longrightarrow \mathrm{C}\ell(V_1,q_1) \hat{\otimes} \mathrm{C}\ell(V_2,q_2)$$

where q_i denotes the restriction of q to V_i and where $\hat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product.

Proof. Consider the map $f: V \to C\ell(V_1,q_1) \hat{\otimes} C\ell(V_2,q_2)$ given by $f(v) = v_1 \otimes 1 + 1 \otimes v_2$ where $v = v_1 + v_2$ is the decomposition of v with respect to the splitting $V = V_1 \oplus V_2$. From (1.14) and the q-orthogonality of this splitting we see that $f(v) \cdot f(v) = (v_1 \otimes 1 + 1 \otimes v_2)^2 = v_1^2 \otimes 1 + 1 \otimes v_2^2 = -(q_1(v_1) + q_2(v_2))1 \otimes 1 = -q(v)1 \otimes 1$. Hence, by Proposition 1.1, f extends to an algebra homomorphism $\tilde{f}: C\ell(V,q) \to C\ell(V_1,q_1) \hat{\otimes} C\ell(V_2,q_2)$. The image of $\tilde{f}$ is a subalgebra which contains $C\ell(V_1,q_1) \otimes 1$ and $1 \otimes C\ell(V_2,q_2)$. Therefore, $\tilde{f}$ is surjective. Injectivity follows easily by considering a basis for $C\ell(V,q)$ generated by a basis of V which is compatible with the splitting. ∎

We finish this section by introducing a second fundamental involution on the algebra. The tensor algebra $\mathscr{T}(V)$ has an involution, given on simple elements by the reversal of order, i.e., $v_1 \otimes \cdots \otimes v_r \mapsto v_r \otimes \cdots \otimes v_1$. This map clearly preserves the ideal $\mathscr{I}(V,q)$ and so descends to a map

$$(\quad)^t : C\ell(V,q) \longrightarrow C\ell(V,q) \tag{1.15}$$

called the **transpose.** Note that $(\quad)^t$ is an antiautomorphism, i.e., $(\varphi\psi)^t = \psi^t\varphi^t$.

§2. The Groups Pin and Spin

We now consider the **multiplicative group of units** in the Clifford algebra, which is defined to be the subset

$$C\ell^\times(V,q) \equiv \{\varphi \in C\ell(V,q) : \exists \varphi^{-1} \text{ with } \varphi^{-1}\varphi = \varphi\varphi^{-1} = 1\} \tag{2.1}$$

This group contains all elements $v \in V$ with $q(v) \neq 0$. When $\dim V = n < \infty$, and k is either $\mathbb{R}$ or $\mathbb{C}$, this is a Lie group of dimension 2^n. In general, there is an associated Lie algebra $\mathfrak{cl}^\times(V,q) = C\ell(V,q)$ with Lie bracket given by

$$[x,y] = xy - yx. \tag{2.2}$$

The group of units always acts naturally as automorphisms of the algebra. That is, there is a homomorphism

$$\mathrm{Ad} : C\ell^\times(V,q) \longrightarrow \mathrm{Aut}(C\ell(V,q)) \tag{2.3}$$

called the **adjoint representation**, which is given by

$$\mathrm{Ad}_\varphi(x) \equiv \varphi x \varphi^{-1}. \tag{2.4}$$

Taking the "derivative" of this gives a homomorphism

$$\mathrm{ad} : \mathfrak{cl}^\times(V,q) \longrightarrow \mathrm{Der}(C\ell(V,q)) \tag{2.5}$$

into the derivations of the algebra, defined by setting

$$\mathrm{ad}_y(x) \equiv [y,x].$$

REMARK 2.1. Suppose V is finite dimensional, and defined over $\mathbb{R}$ or $\mathbb{C}$. Then there is a natural exponential mapping $\exp : \mathrm{cl}^\times(V,q) \to C\ell^\times(V,q)$, defined by setting

$$\exp(y) = \sum_{m=0}^{\infty} \frac{1}{m!} y^m. \tag{2.6}$$

Note that this series converges since for any choice of positive definite inner product on $C\ell(V,q)$, we have $\|xy\| \leq c\|x\|\,\|y\|$ for some $c > 0$. It is easy to see that

$$\frac{d}{dt}\mathrm{Ad}_{\exp(ty)}(x)\Big|_{t=0} = \mathrm{ad}_y(x). \tag{2.7}$$

From this point on we shall assume that *the characteristic of the field k is different from* 2. Under this assumption, we have the following important facts concerning the adjoint representation:

Proposition 2.2. *Let $v \in V \subset C\ell(V,q)$ be an element with $q(v) \neq 0$. Then $\mathrm{Ad}_v(V) = V$. In fact, for all $w \in V$, the following equation holds:*

$$-\mathrm{Ad}_v(w) = w - 2\frac{q(v,w)}{q(v)}v. \tag{2.8}$$

Proof. Since $v^{-1} = -v/q(v)$, we have from (1.4) that

$$\begin{aligned} -q(v)\mathrm{Ad}_v(w) &= -q(v)vwv^{-1} = vwv \\ &= -v^2w - 2q(v,w)v = q(v)w - 2q(v,w)v. \quad \blacksquare \end{aligned}$$

This leads us naturally to consider the subgroup of elements $\varphi \in C\ell^\times(V,q)$ such that $\mathrm{Ad}_\varphi(V) = V$. By Proposition 2.2, this group contains all elements $v \in V$ with $q(v) \neq 0$. Furthermore, we see from equation (2.8) that whenever $q(v) \neq 0$, the transformation Ad_v preserves the quadratic form q. That is, $(\mathrm{Ad}_v^* q)(w) \equiv q(\mathrm{Ad}_v(w)) = q(w)$ for all $w \in V$. Therefore, we define $P(V,q)$ to be the subgroup of $C\ell^\times(V,q)$ generated by the elements $v \in V$ with $q(v) \neq 0$, and observe that there is a representation

$$P(V,q) \xrightarrow{\mathrm{Ad}} \mathrm{O}(V,q) \tag{2.9}$$

where

$$\mathrm{O}(V,q) = \{\lambda \in \mathrm{GL}(V) : \lambda^* q = q\} \tag{2.10}$$

is the **orthogonal group** of the form q. The group $\mathrm{P}(V,q)$ has certain important subgroups.

DEFINITION 2.3 The **Pin group** of (V,q) is the subgroup $\mathrm{Pin}(V,q)$ of $\mathrm{P}(V,q)$ generated by the elements $v \in V$ with $q(v) = \pm 1$. The associated **Spin group** of (V,q) is defined by

$$\mathrm{Spin}(V,q) = \mathrm{Pin}(V,q) \cap C\ell^0(V,q).$$

We observe now that the right-hand side of equation (2.8) is just the map $\rho_v : V \to V$ given by reflection across the hyperplane $v^\perp = \{w \in V : q(w,v) = 0\}$. That is, the map ρ_v fixes this hyperplane and maps v to $-v$. Unfortunately, there is a minus sign on the left in equation (2.8). This means, for example, that if dim V is odd, then Ad_v is always orientation preserving. This defect can be removed by considering the **twisted adjoint representation**

$$\widetilde{\mathrm{Ad}} : C\ell^\times(V,q) \longrightarrow \mathrm{GL}(C\ell(V,q))$$

defined by setting

$$\widetilde{\mathrm{Ad}}_\varphi(y) = \alpha(\varphi) y \varphi^{-1}. \tag{2.11}$$

Clearly, $\widetilde{\mathrm{Ad}}_{\varphi_1\varphi_2} = \widetilde{\mathrm{Ad}}_{\varphi_1} \circ \widetilde{\mathrm{Ad}}_{\varphi_2}$ and $\widetilde{\mathrm{Ad}}_\varphi = \mathrm{Ad}_\varphi$ for even elements φ (i.e., for $\varphi \in C\ell^0(V,q)$). Furthermore, from (2.8) we have

$$\widetilde{\mathrm{Ad}}_v(w) = w - 2\frac{q(v,w)}{q(v)} v. \tag{2.12}$$

We then define the subgroup

$$\tilde{\mathrm{P}}(V,q) \equiv \{\varphi \in C\ell^\times(V,q) : \widetilde{\mathrm{Ad}}_\varphi(V) = V\}. \tag{2.13}$$

It is clear that $\mathrm{P}(V,q) \subset \tilde{\mathrm{P}}(V,q)$. Furthermore, we have the following.

Proposition 2.4. *Suppose that V is finite dimensional and that q is nondegenerate. Then the kernel of the homomorphism*

$$\tilde{\mathrm{P}}(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{GL}(V)$$

is exactly the group $k^\times$ of non-zero multiples of 1.

Proof. Choose a basis $\{v_1, \dots, v_n\}$ for V such that $q(v_i) \neq 0$ for all i and $q(v_i,v_j) = 0$ for all $i \neq j$. Suppose $\varphi \in C\ell^\times(V,q)$ is in the kernel of $\widetilde{\mathrm{Ad}}$, that is, suppose φ has the property that $\alpha(\varphi)v = v\varphi$ for all $v \in V$. Write $\varphi = \varphi_0 + \varphi_1$, where φ_0 is even and φ_1 is odd, and observe that

$$\begin{aligned} v\varphi_0 &= \varphi_0 v \\ -v\varphi_1 &= \varphi_1 v \end{aligned} \tag{2.14}$$

for all $v \in V$. The terms φ_0 and φ_1 can be written as polynomial expressions in $v_1, \dots, v_n$. Successive use of the fact (1.4) that $v_i v_j = -v_j v_i - 2q(v_i,v_j)$ shows that φ_0 can be expressed as $\varphi_0 = a_0 + v_1 a_1$ where a_0 and a_1 are polynomial expressions in $v_2, \dots, v_n$. Applying α shows that a_0 is even and a_1 is odd. Setting $v = v_1$ in (2.14), we see that

$$\begin{aligned} v_1 a_0 + v_1^2 a_1 &= a_0 v_1 + v_1 a_1 v_1 \\ &= v_1 a_0 - v_1^2 a_1. \end{aligned}$$

Hence, $v_1^2 a_1 = -q(v_1)a_1 = 0$, and so $a_1 = 0$. This implies that φ_0 does not involve v_1. Proceeding inductively, we see that φ_0 does not involve any of the terms $v_1, \ldots, v_n$ and so $\varphi_0 = t \cdot 1$ for $t \in k$.

The analogous argument can now be applied to φ_1. Write $\varphi_1 = a_1 + v_1 a_0$, where a_0 and a_1 do not involve v_1. Note that a_1 is odd and a_0 is even; and therefore, from (2.14), $-v_1 a_1 - v_1^2 a_0 = a_1 v_1 + v_1 a_0 v_1 = -v_1 a_1 + v_1^2 a_0$. Hence, $a_0 = 0$ and so φ_1 is independent of v_1. By induction, φ_1 is independent of $v_1, \ldots, v_n$ and so $\varphi_1 = 0$.

Now we have $\varphi = \varphi_0 + \varphi_1 = t \cdot 1 \in k$. But $\varphi \neq 0$, so $\varphi \in k^\times$. ■

Note that this proposition requires the **twisted** adjoint representation and not the adjoint representation. The minus sign in (2.14) is crucial to the proof.

Proposition 2.4 is false if we do not assume that q is non-degenerate. To see this, consider the extreme case $C\ell(V,0) = \Lambda^* V$. For all $v_1, v_2 \in V$, we have $1 + v_1 v_2 \in C\ell^\times(V,0)$. In fact, $(1 + v_1 v_2)^{-1} = 1 - v_1 v_2$. However, for any $v \in V$, we see that $\alpha(1 + v_1 v_2)v(1 + v_1 v_2)^{-1} = (1 + v_1 v_2) \cdot v(1 - v_1 v_2) = v$. Hence, the kernel of the homomorphism includes many non-scalar terms.

We now introduce the **norm** mapping $N: C\ell(V,q) \to C\ell(V,q)$ defined by setting

$$N(\varphi) \equiv \varphi \cdot \alpha(\varphi^t). \tag{2.15}$$

Here φ^t denotes the transpose of φ introduced in (1.15). It is easy to see that $\alpha(\varphi^t) = (\alpha(\varphi))^t$. Note that

$$N(v) = q(v) \qquad \text{for } v \in V. \tag{2.16}$$

The importance of the norm is evident from the following proposition.

Proposition 2.5. *Suppose that V is finite dimensional and that q is non-degenerate. Then the restriction of N to the group $\tilde{P}(V,q)$ gives a homomorphism*

$$N: \tilde{P}(V,q) \longrightarrow k^\times \tag{2.17}$$

into the multiplicative group of non-zero multiples of the identity in $C\ell(V,q)$.

Proof. To begin we observe that $N(\tilde{P}(V,q)) \subset k^\times$. Choose $\varphi \in \tilde{P}(V,q)$ and recall that by definition, $\alpha(\varphi)v\varphi^{-1} \in V$ for all $v \in V$. Applying the transpose antiautomorphism, which is the identity on V, we see that

$$(\varphi^t)^{-1} v \alpha(\varphi^t) = \alpha(\varphi) v \varphi^{-1}.$$

Hence,

$$\begin{aligned}\varphi^t\alpha(\varphi)v\varphi^{-1}(\alpha(\varphi^t))^{-1} &= \alpha[\alpha(\varphi^t)\varphi]v[\alpha(\varphi^t)\varphi]^{-1}\\ &= \widetilde{\mathrm{Ad}}_{\alpha(\varphi^t)\varphi}(v) = v\end{aligned}$$

for all $v \in V$. Hence, $\alpha(\varphi^t)\varphi$ is in the kernel of $\widetilde{\mathrm{Ad}}$. It is easy to check that $\alpha(\varphi^t)$ belongs to $\tilde{\mathrm{P}}(V,q)$, and therefore so does $\alpha(\varphi^t)\varphi$. Hence, by Proposition 2.4 we have $\alpha(\varphi^t)\varphi \in k^\times$. Applying α shows that $\varphi^t\alpha(\varphi) = N(\varphi^t) \in k^\times$. Since the transpose antiautomorphism preserves $\tilde{\mathrm{P}}(V,q)$, we conclude that $N(\varphi) \in k^\times$ for all $\varphi \in \tilde{\mathrm{P}}(V,q)$.

We now observe that if $\varphi,\psi \in \tilde{\mathrm{P}}(V,q)$, then $N(\varphi\psi) = \varphi\psi\alpha(\psi^t)\alpha(\varphi^t) = \varphi N(\psi)\alpha(\varphi^t) = \varphi\alpha(\varphi^t)N(\psi) = N(\varphi)N(\psi)$. Thus, N is a homomorphism on $\tilde{\mathrm{P}}(V,q)$. ■

Continuing to assume that $\dim V < \infty$ and q is non-degenerate, we have the following.

Corollary 2.6. *The transformations $\widetilde{\mathrm{Ad}}_\varphi : V \to V$ for $\varphi \in \tilde{\mathrm{P}}(V,q)$ preserve the quadratic form q. Hence, there is a homomorphism*

$$\widetilde{\mathrm{Ad}} : \tilde{\mathrm{P}}(V,q) \longrightarrow \mathrm{O}(V,q) \tag{2.18}$$

Proof. To begin we note that $N(\alpha\varphi) = N(\varphi)$ for $\varphi \in \tilde{\mathrm{P}}(V,q)$ since $N(\alpha\varphi) = \alpha(\varphi)\varphi^t = \alpha N(\varphi) = N(\varphi)$. Consequently, if we set

$$V^\times = \{v \in V : q(v) \neq 0\}, \tag{2.19}$$

then for each $v \in V^\times (\subset \tilde{\mathrm{P}}(V,q))$, we have $N(\widetilde{\mathrm{Ad}}_\varphi(v)) = N(\alpha(\varphi)v\varphi^{-1}) = N(\alpha\varphi)N(v)N(\varphi)^{-1} = N(\varphi)N(\varphi)^{-1}N(v) = N(v)$. Since $N(\varphi) = q(v)$ for $v \in V$ (cf. (2.16)), we see that $\widetilde{\mathrm{Ad}}_\varphi$ preserves all non-zero q-lengths. Applying $\widetilde{\mathrm{Ad}}_{\varphi^{-1}}$ now shows that $\mathrm{Ad}_\varphi(V^\times) = V^\times$ and so $\widetilde{\mathrm{Ad}}_\varphi$ leaves invariant the set of vectors of zero q-length. Thus, $\widetilde{\mathrm{Ad}}_\varphi$ is q-orthogonal. ■

We now return to the group $\mathrm{P}(V,q) \subseteq \tilde{\mathrm{P}}(V,q)$ and observe that by definition

$$\mathrm{P}(V,q) = \{v_1 \cdots v_r \in C\ell(V,q) : v_1 \ldots, v_r \text{ is a finite sequence from } V^\times\}. \tag{2.20}$$

Recall that the twisted adjoint representation gives a homomorphism $\widetilde{\mathrm{Ad}} : \mathrm{P}(V,q) \to \mathrm{O}(V,q)$ such that

$$\widetilde{\mathrm{Ad}}_{v_1 \cdots v_r} = \rho_{v_1} \circ \cdots \circ \rho_{v_r} \tag{2.21}$$

where

$$\rho_v(w) = w - 2\frac{q(w,v)}{q(v)}v \tag{2.22}$$

is **reflection across** $v^\perp$. Thus the image of $P(V,q)$ under $\widetilde{\mathrm{Ad}}$ is exactly the group generated by reflections. It is an important and classical result that this is always the entire orthogonal group.

Theorem 2.7 (Cartan-Dieudonné). *Let q be a non-degenerate quadratic form on a finite dimensional vector space V. Then every element $g \in O(V,q)$ can be written as a product of r reflections*

$$g = \rho_{v_1} \circ \cdots \circ \rho_{v_r}$$

where $r \leqq \dim(V)$.

We refer the reader to Artin's book [1] for the general proof. In the special case where $V = \mathbb{R}^n$ and $q(x) = \|x\|^2$ is the standard norm, this theorem is easily proved by putting the orthogonal matrix g in "diagonal" form:

$$\begin{pmatrix} \boxed{R_{\theta_1}} & & & & & & \\ & \boxed{R_{\theta_2}} & & & & & \\ & & \ddots & & & & \\ & & & \boxed{R_{\theta_k}} & & & \\ & & & & \pm 1 & & \\ & & & & & \ddots & \\ & & & & & & \pm 1 \end{pmatrix}$$

where each R_{θ_i} is a 2×2 rotation matrix (which can be expressed as a product of two reflections).

Theorem 2.7 says that the homomorphism $\widetilde{\mathrm{Ad}}: P(V,q) \to O(V,q)$ is surjective. Furthermore, we could consider the group $SP(V,q) = P(V,q) \cap C\ell^0(V,q)$ and, since $\dim V$ is finite, the **special orthogonal group**

$$SO(V,q) = \{\lambda \in O(V,q) : \det(\lambda) = 1\}.$$

Theorem 2.7 also says that the homomorphism $\widetilde{\mathrm{Ad}}: SP(V,q) \to SO(V,q)$ is surjective. To see this, we first show that $\det(\rho_v) = -1$ for any $v \in V$. To prove this, choose a basis $v_1, \ldots, v_n$ such that $v_1 = v$ and $q(v,v_j) = 0$ for $j \geqq 2$. It follows from the definition that $\rho_v(v_1) = -v_1$ and $\rho_v(v_j) = v_j$ for $j \geqq 2$, and so $\det(\rho_v) = -1$ as claimed. Thus from Theorem 2.7 we conclude that

$$SO(V,q) = \{\rho_{v_1} \circ \cdots \circ \rho_{v_r} : q(v_j) \neq 0 \text{ and } r \text{ is even}\}. \tag{2.23}$$

From the definition (cf. (2.20)) we see that $SP(V,q) = \{v_1 \cdots v_r \in P(V,q) : r$ is even$\}$. The surjectivity of $\widetilde{\mathrm{Ad}}: SP(V,q) \to SO(V,q)$ follows immediately (see (2.21)).

We now return to the groups Pin and Spin. Recall that these are the groups generated by the generalized unit sphere $S = \{v \in V : q(v) = \pm 1\}$ in V. That is,

$$\mathrm{Pin}(V,q) = \{v_1 \cdots v_r \in \mathrm{P}(V,q) : q(v_j) = \pm 1 \text{ for all } j\} \tag{2.24}$$

and

$$\mathrm{Spin}(V,q) = \{v_1 \cdots v_r \in \mathrm{Pin}(V,q) : r \text{ is even}\}. \tag{2.25}$$

In light of the above it is natural to ask whether the homomorphism $\widetilde{\mathrm{Ad}}$ restricted to $\mathrm{Pin}(V,q)$ and $\mathrm{Spin}(V,q)$ maps *onto* $\mathrm{O}(V,q)$ and $\mathrm{SO}(V,q)$ respectively. This seems quite likely since at a glance one can see that

$$\rho_{tv} = \rho_v \tag{2.26}$$

for any non-zero scalar $t \in k$, and so one should be able to renormalize any $v \in V^\times$ to have q-length ± 1. Of course since q is quadratic, $q(tv) = t^2 q(v)$, and the equation $t^2 q(v) = \pm 1$, i.e., the equation $t^2 = \pm a$ for a given a, may or may not be solvable in a general field k. If $k = \mathbb{R}$ or $\mathbb{C}$, of course, it is always solvable. If $k = \mathbb{Q}$ (the rational numbers) it is very often not solvable. (The group $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ is infinitely generated.) If k is a finite field of characteristic $\neq 2$, then $k^\times/(k^\times)^2 \cong \mathbb{Z}_2$ and -1 may or may not lie in $(k^\times)^2$. In the cases where $\widetilde{\mathrm{Ad}}$ is not surjective, we still have the following general fact, which is interesting because the group $\mathrm{SO}(V,q)$ is often almost a simple group (see Artin [1]). (The reader interested only in the real and complex cases can skip this proposition.)

Proposition 2.8. *Each of the images* $\widetilde{\mathrm{Ad}}(\mathrm{Pin}(V,q))$ *and* $\widetilde{\mathrm{Ad}}(\mathrm{Spin}(V,q))$ *is a normal subgroup of* $\mathrm{O}(V,q)$.

Proof. Recall (cf. (1.6)) that from the universal property of $C\ell(V,q)$, the action of $\mathrm{O}(V,q)$ on V extends to automorphisms of $C\ell(V,q)$. It is easy to see that these automorphisms commute with α. Suppose then that we have $v, w \in V$ with $q(v) \neq 0$ and choose $g \in \mathrm{O}(V,q)$. Then $\widetilde{\mathrm{Ad}}_{g(v)}(w) = \alpha(gv)w(gv)^{-1} = g(\alpha v)wg(v^{-1}) = g(\alpha(v)g^{-1}(w)v^{-1}) = g\widetilde{\mathrm{Ad}}_v(g^{-1}w)$. Consequently, we have that

$$\widetilde{\mathrm{Ad}}_{gv} = g \circ \widetilde{\mathrm{Ad}} \circ g^{-1} \tag{2.27}$$

for all $v \in V$ with $q(v) \neq 0$ and for all $g \in \mathrm{O}(V,q)$. The proposition now follows immediately from (2.24), (2.25) and (2.27). ■

We now come to the main result of this section. We are primarily interested in the real and complex cases, so we shall focus on fields k that have the property discussed above. We shall say that a field k of characteristic $\neq 2$ is **spin** if at least one of the two equations $t^2 = a$ and $t^2 = -a$ can be solved in k for each non-zero element $a \in k^\times$. That is, k is spin if $k^\times = (k^\times)^2 \cup (-(k^\times)^2)$. The fields $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_p$ for p a prime with $p \equiv 3 (\mathrm{mod}\ 4)$, are spin.

Theorem 2.9. *Let V be a finite-dimensional vector space over a spin field k, and suppose q is a non-degenerate quadratic form on V. Then there are short exact sequences*

$$0 \longrightarrow F \longrightarrow \mathrm{Spin}(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(V,q) \longrightarrow 1 \tag{2.28}$$

$$0 \longrightarrow F \longrightarrow \mathrm{Pin}(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(V,q) \longrightarrow 1 \tag{2.29}$$

where

$$F = \begin{cases} \mathbb{Z}_2 = \{1,-1\} & \text{if } \sqrt{-1} \notin k \\ \mathbb{Z}_4 = \{\pm 1, \pm\sqrt{-1}\} & \text{otherwise.} \end{cases}$$

These sequences hold for general fields provided that $\mathrm{SO}(V,q)$ *and* $\mathrm{O}(V,q)$ *are replaced by appropriate normal subgroups of* $\mathrm{O}(V,q)$.

Proof. Suppose $\varphi = v_1 \cdots v_r \in \mathrm{Pin}(V,q)$ is in the kernel of $\widetilde{\mathrm{Ad}}$. Then $\varphi \in k^\times$ by Proposition 2.4, and so $\varphi^2 = N(\varphi) = N(v_1)\cdots N(v_r) = \pm 1$. This establishes the kernel of $\widetilde{\mathrm{Ad}}$ in both cases. The surjectivity of the homomorphisms follows from Theorem 2.7, the fact that $\rho_v = \rho_{tv}$, and the fact that since k is spin, any $v \in V^\times$ can be renormalized to have q-length 1. ■

It is interesting to observe that if k is a spin field, then either $\tilde{\mathrm{P}}(V,q) = \mathrm{P}(V,q)$ or $\tilde{\mathrm{P}}(V,q)/\mathrm{P}(V,q) \cong \mathbb{Z}_2$. The proof (which the reader may skip) is as follows. Since $\mathrm{P}(V,q)$ is generated by $V^\times$, we know that $t^2 q(v) \in \mathrm{P}(V,q)$ for all $t \in k^\times$ and $v \in V^\times$. Since k is spin, this implies that $\mathrm{P}(V,q)$ contains $(k^\times)^2$ or $-(k^\times)^2$ (and possibly more). In fact, if we set $k_0^\times = \{t \in k^\times : t \cdot 1 \in \mathrm{P}(V,q)\}$, then from the above and from the definition of a spin field, we see that $k^\times = k_0^\times \cup (-k_0^\times)$. Thus, $k^\times/k_0^\times = 0$ or $\mathbb{Z}_2$. Now we have the sequence

$$k_0^\times \subseteq \mathrm{P}(V,q) \subseteq \tilde{\mathrm{P}}(V,q) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(V,q)$$

where $k^\times = \ker(\widetilde{\mathrm{Ad}})$ and where $\widetilde{\mathrm{Ad}}(P(V,q)) = \mathrm{O}(V,q)$. It follows that $\mathrm{O}(V,q) \cong \tilde{\mathrm{P}}(V,q)/k^\times \cong \mathrm{P}(V,q)/k_0^\times$. It then follows without difficulty that $\tilde{\mathrm{P}}(V,q)/\mathrm{P}(V,q) \cong k^\times/k_0^\times \cong 0$ or $\mathbb{Z}_2$ as claimed.

We now examine the real case in some detail. Let V be an n-dimensional vector space over $\mathbb{R}$, and suppose q is a non-degenerate quadratic form on V. Then we may choose a basis for $V \cong \mathbb{R}^n$ so that

$$q(x) = x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2 \tag{2.30}$$

where $r + s = n$ and $0 \le r \le n$. It is standard notation to write: $q_{r,s} \equiv q$, $\mathrm{O}_{r,s} \equiv \mathrm{O}(V,q)$ and $\mathrm{SO}_{r,s} \equiv \mathrm{SO}(V,q)$. In accordance we write

$$\mathrm{Pin}_{r,s} \equiv \mathrm{Pin}(V,q) \quad \text{and} \quad \mathrm{Spin}_{r,s} \equiv \mathrm{Spin}(V,q). \tag{2.31}$$

Similarly, it is conventional to write $\mathrm{O}_n \equiv \mathrm{O}_{n,0} \cong \mathrm{O}_{0,n}$ and $\mathrm{SO}_n \equiv \mathrm{SO}_{n,0} \cong \mathrm{SO}_{0,n}$. Thus, we set

$$\mathrm{Pin}_n \equiv \mathrm{Pin}_{n,0} \quad \text{and} \quad \mathrm{Spin}_n \equiv \mathrm{Spin}_{n,0}. \tag{2.32}$$

We also write $\mathrm{P}_{r,s} \equiv \mathrm{P}(V,q)$ and $\tilde{\mathrm{P}}_{r,s} \equiv \tilde{\mathrm{P}}(V,q)$, and note from the paragraph above that

$$\mathrm{P}_{r,s} = \tilde{\mathrm{P}}_{r,s}. \tag{2.33}$$

It is a classical fact (cf. Helgason [1]) that SO_n is connected and that $\mathrm{SO}_{r,s}$, for $r,s \geqq 1$, has exactly two connected components. It is also a classical fact that $\pi_1(\mathrm{SO}_n) \cong \mathbb{Z}_2$ for $n \geqq 3$ and $\pi_1(\mathrm{SO}^0_{r,s}) \cong \pi_1(\mathrm{SO}_r) \times \pi_1(\mathrm{SO}_s)$ for all r,s. Hence, $\pi_1(\mathrm{SO}^0_{1,r}) = \pi_1(\mathrm{SO}^0_{r,1}) = \mathbb{Z}_2$ and $\pi_1(\mathrm{SO}^0_{r,s}) = \mathbb{Z}_2 \times \mathbb{Z}_2$ for all $r,s \geqq 3$. (Here $\mathrm{SO}^0_{r,s}$ denotes the connected component of the identity.)

The main result of this section is the following.

Theorem 2.10. *There are short exact sequences*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_{r,s} \longrightarrow \mathrm{SO}_{r,s} \longrightarrow 1$$
$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}_{r,s} \longrightarrow \mathrm{O}_{r,s} \longrightarrow 1$$

for all (r,s). Furthermore, if $(r,s) \neq (1,1)$, these two-sheeted coverings are non-trivial over each component of $\mathrm{O}_{r,s}$. In particular, in the special case

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_n \xrightarrow{\xi_0} \mathrm{SO}_n \longrightarrow 1 \tag{2.34}$$

the map $\xi_0 \equiv \widetilde{\mathrm{Ad}}$ represents the universal covering homomorphism of SO_n for all $n \geqq 3$.

Proof. The exact sequences are a direct consequence of Theorem 2.9. The kernel in each case is explicitly given by $\mathbb{Z}_2 = \{1,-1\}$. To prove that the coverings are non-trivial, it suffices to join -1 to 1 by a path in $Spin_{r,s}$. Choose orthogonal vectors $e_1,e_2 \in \mathbb{R}^n$ with $q(e_1) = q(e_2) = \pm 1$. (This is possible since $(r,s) \neq (1,1)$.) Then $\gamma(t) = \pm\cos(2t) + e_1e_2\sin(2t) = (e_1\cos t + e_2\sin t)(e_2\sin t - e_1\cos t)$ does the job. ∎

The above argument also shows that restricting $\widetilde{\mathrm{Ad}}$ to the identity component $\mathrm{Spin}^0_{r,1}$ of $\mathrm{Spin}_{r,1}$ gives the universal covering homomorphism

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^0_{r,1} \xrightarrow{\xi} \mathrm{SO}^0_{r,1} \longrightarrow 1 \tag{2.35}$$

for all $r \geqq 3$.

§3. The Algebras $C\ell_n$ and $C\ell_{r,s}$

We shall now study the Clifford algebras $C\ell_{r,s} \equiv C\ell(V,q)$ where $V = \mathbb{R}^{r+s}$ and

$$q(x) = x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2. \tag{3.1}$$

Of particular interest are the cases

$$C\ell_n \equiv C\ell_{n,0} \qquad \text{and} \qquad C\ell_n^* \equiv C\ell_{0,n}. \tag{3.2}$$

One reason for studying these algebras is the following. As seen in §2, the algebra $C\ell_{r,s}$ contains the groups $\mathrm{Spin}_{r,s}$ and $\mathrm{Pin}_{r,s}$, and so any representation of the algebra $C\ell_{r,s}$ restricts to a representation of these groups which is non-trivial on the element -1. (Such representations are therefore *not* induced from representations of $O_{r,s}$ or $SO_{r,s}$.)

These algebras have a simple classical presentation:

Proposition 3.1. *Let $e_1, \ldots, e_{r+s}$ be any q-orthonormal basis of $\mathbb{R}^{r+s} \subset C\ell_{r,s}$. Then $C\ell_{r,s}$ is generated (as an algebra) by $e_1, \ldots, e_{r+s}$ subject to the relations*

$$e_ie_j + e_je_i = \begin{cases} -2\delta_{ij} & \text{if } i \leqq r \\ +2\delta_{ij} & \text{if } i > r. \end{cases} \tag{3.3}$$

Proof. This follows easily from the discussion in §1. ■

We also have a pretty decomposition in terms of the $\mathbb{Z}_2$-graded tensor product.

Proposition 3.2. *There is an isomorphism*

$$C\ell_{r,s} \cong C\ell_1 \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} C\ell_1 \mathbin{\hat{\otimes}} C\ell_1^* \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} C\ell_1^* \tag{3.4}$$

where $C\ell_1$ appears r times and $C\ell_1^*$ appears s times on the right in (3.4).

Proof. Decompose $\mathbb{R}^{r+s}$ into one-dimensional q-orthogonal subspaces and apply Proposition 1.5 inductively. ■

It is not difficult to see that as algebras over $\mathbb{R}$,

$$C\ell_1 \cong \mathbb{C} \quad \text{and} \quad C\ell_1^* \cong \mathbb{R} \oplus \mathbb{R}. \tag{3.5}$$

It follows immediately that $\dim_{\mathbb{R}}(C\ell_{r,s}) = 2^{r+s}$. Proposition 3.2 is, however, not so useful if we wish to represent $C\ell_{r,s}$ as a matrix algebra. For this it is more useful to find decompositions in terms of ungraded tensor products. We shall do this in the next section.

For the remainder of this section we shall examine some of the general properties of the algebras $C\ell_{r,s}$. We begin with a discussion of the volume element. Choose an orientation for $\mathbb{R}^{r+s}$ and let $e_1, \ldots, e_{r+s}$ be any positively-oriented, q-orthonormal basis. Then the associated **(oriented) volume element** is defined to be

$$\omega = e_1 \cdots e_{r+s} \tag{3.6}$$

If $e'_1, \ldots, e'_{r+s}$ is any other such basis, then $e'_i = \sum_j g_{ij}e_j$ for $g = ((g_{ij})) \in SO_{r,s}$. From (3.3) we easily see that $e'_1 \cdots e'_{r+s} = \det(g)e_1 \cdots e_{r+s} = e_1 \cdots e_{r+s}$. Hence the definition (3.6) is independent of the choice of the basis.

Proposition 3.3. *The volume element* (3.6) *in* $C\ell_{r,s}$ *has the following basic properties. Let* $n = r + s$. *Then*

$$\omega^2 = (-1)^{\frac{n(n+1)}{2}+s}, \tag{3.7}$$

$$v\omega = (-1)^{n-1}\omega v \qquad \text{for all } v \in \mathbb{R}^n, \tag{3.8}$$

In particular, if n is odd, then the element ω *is central in* $C\ell_{r,s}$. *If n is even, then*

$$\varphi\omega = \omega\alpha(\varphi) \tag{3.9}$$

for all $\varphi \in C\ell_{r,s}$.

Proof. Choose a q-orthonormal basis and apply the relations (3.3). ■

We note that property (3.7) can be rewritten as

$$\omega^2 = \begin{cases} (-1)^s & \text{if } n \equiv 3 \text{ or } 4 \pmod 4 \\ (-1)^{s+1} & \text{if } n \equiv 1 \text{ or } 2 \pmod 4 \end{cases} \tag{3.7}'$$

We now make the following elementary but important observation.

Lemma 3.4. *Suppose the volume element* ω *in* $C\ell_{r,s}$ *satisfies* $\omega^2 = 1$, *and set*

$$\pi^+ = \frac{1}{2}(1 + \omega) \qquad \textit{and} \qquad \pi^- = \frac{1}{2}(1 - \omega). \tag{3.10}$$

Then π^+ *and* π^- *satisfy the relations*

$$\pi^+ + \pi^- = 1 \tag{3.11}$$

$$(\pi^+)^2 = \pi^+ \qquad \textit{and} \qquad (\pi^-)^2 = \pi^- \tag{3.12}$$

$$\pi^+\pi^- = \pi^-\pi^+ = 0. \tag{3.13}$$

Proof. This is a trivial consequence of the fact that $\omega^2 = 1$. ■

This leads to two basic but important facts:

Proposition 3.5. *Suppose that the volume element* ω *in* $C\ell_{r,s}$ *satisfies* $\omega^2 = 1$, *and that* $r + s$ *is odd. Then* $C\ell_{r,s}$ *can be decomposed as a direct sum*

$$C\ell_{r,s} = C\ell_{r,s}^+ \oplus C\ell_{r,s}^- \tag{3.14}$$

of isomorphic subalgebras, where $C\ell_{r,s}^\pm = \pi^\pm \cdot C\ell_{r,s} = C\ell_{r,s} \cdot \pi^\pm$ *and where* $\alpha(C\ell_{r,s}^\pm) = C\ell_{r,s}^\mp$.

Proof. Since $r + s$ is odd, we know from Proposition 3.3 that ω is central. Hence π^+ and π^- are central and the decomposition (3.14) into ideals follows directly from (3.11), (3.12) and (3.13). Since ω is an odd element, $\alpha(\pi^{\pm}) = \pi^{\mp}$ and so $\alpha(C\ell_{r,s}^{\pm}) = C\ell_{r,s}^{\mp}$. Since α is an automorphism, we conclude that these two ideals are isomorphic. ■

Proposition 3.6. *Suppose that the volume element ω in $C\ell_{r,s}$ satisfies $\omega^2 = 1$ and that $r + s$ is even. Let V be any $C\ell_{r,s}$-module (i.e., V is a real vector space with an algebra homomorphism $C\ell_{r,s} \to \mathrm{Hom}(V, V)$). Then there is a decomposition*

$$V = V^+ \oplus V^- \tag{3.15}$$

into the $+1$ and -1 eigenspaces for multiplication by ω. In fact,

$$V^+ = \pi^+ \cdot V \quad \text{and} \quad V^- = \pi^- \cdot V,$$

and for any $e \in \mathbb{R}^{r+s}$ with $q(e) \neq 0$, module multiplication by e gives isomorphisms

$$e: V^+ \longrightarrow V^- \qquad \text{and} \qquad e: V^- \longrightarrow V^+. \tag{3.16}$$

Proof. The decomposition (3.15) is a direct consequence of (3.11), (3.12) and (3.13), together with the observation that

$$\omega \cdot \pi^{\pm} = \pm \pi^{\pm}.$$

The isomorphisms (3.16) follow directly from the facts that by (3.8),

$$\begin{cases} e\pi^+ = \frac{1}{2}e(1+\omega) = \frac{1}{2}(1-\omega)e = \pi^- e \\ e\pi^- = \pi^+ e \end{cases}$$

and $e \cdot e = -q(e) \cdot 1$. ■

REMARK. The above construction will prove useful when we are dealing with vector bundles in the next chapter.

We now come to an important and basic fact. Recall the even-odd decomposition $C\ell_{r,s} = C\ell_{r,s}^0 \oplus C\ell_{r,s}^1$ given in (1.8), where the subalgebra $C\ell_{r,s}^0$ is the fixed-point set of the automorphism α.

Theorem 3.7. *There is an algebra isomorphism*

$$C\ell_{r,s} \cong C\ell_{r+1,s}^0 \tag{3.17}$$

for all r,s. In particular,

$$C\ell_n \cong C\ell_{n+1}^0 \tag{3.18}$$

for all n.

Proof. Choose a q-orthonormal basis $e_1, \ldots, e_{r+s+1}$ of $\mathbb{R}^{r+s+1}$ so that $q(e_i) = 1$ for $1 \leqq i \leqq r+1$ and $q(e_i) = -1$ for $i > r+1$. Let $\mathbb{R}^{r+s} = \operatorname{span}\{e_i | i \neq r+1\}$ and define a map $f: \mathbb{R}^{r+s} \to C\ell^0_{r+1,s}$ by setting

$$f(e_i) = e_{r+1}e_i$$

for $i \neq r+1$, and extending linearly. For $x = \sum_{i \neq r+1} x_i e_i$, we have that:

$$\begin{aligned} f(x)^2 &= \sum_{i,j} x_i x_j e_{r+1} e_i e_{r+1} e_j \\ &= \sum_{i,j} x_i x_j e_i e_j \\ &= x \cdot x = -q(x) \cdot 1 \end{aligned}$$

since $e_{r+1} \cdot e_{r+1} = -1$ and $e_{r+1}e_i = -e_i e_{r+1}$ for $i \neq r+1$. It follows from the universal property (Proposition 1.1) that f extends to an algebra homomorphism

$$\tilde{f}: C\ell_{r,s} \longrightarrow C\ell^0_{r+1,s}.$$

Checking $\tilde{f}$ on a linear basis shows that $\tilde{f}$ is an isomorphism. ∎

We now specialize to the case of $C\ell_n$.

Proposition 3.8. *Let $L: C\ell_n \to C\ell_n$ be the linear map defined by setting*

$$L(\varphi) = -\sum_j e_j \varphi e_j \tag{3.19}$$

where $e_1, \ldots, e_n$ is any orthonormal basis of $\mathbb{R}^n$. Set $\tilde{L} = \alpha \circ L$. Then the eigenspaces of $\tilde{L}$ are the canonical images of $\Lambda^p \equiv \Lambda^p \mathbb{R}^n$ in $C\ell_n$. In fact

$$\tilde{L}|_{\Lambda^p} = (n - 2p)\mathrm{Id} \tag{3.20}$$

for $p = 0, \ldots, n$.

Proof. It suffices to consider $\varphi = e_1 \cdots e_p$. Then

$$\begin{aligned} L(\varphi) &= -\sum_{j=1}^{p} e_j e_1 \cdots e_p e_j - \sum_{j=p+1}^{n} e_j e_1 \cdots e_p e_j \\ &= -\sum_{j=1}^{p} (-1)^{p-1} e_j^2 e_1 \cdots e_p - \sum_{j=p+1}^{n} (-1)^p e_j^2 e_1 \cdots e_p \\ &= (-1)^{p-1} p e_1 \cdots e_p + (-1)^p (n-p) e_1 \cdots e_p \\ &= (-1)^p (n-2p) e_1 \cdots e_p = (n-2p)\alpha(\varphi) \quad ∎ \end{aligned}$$

Under the canonical isomorphism $C\ell_n \cong \Lambda^* \mathbb{R}^n$, Clifford multiplication has a nice interpretation. Using the inner product on $\mathbb{R}^n$ we can identify $\mathbb{R}^n$ with its dual. We can thereby talk about the **interior product** or **contraction** in $\Lambda^* \mathbb{R}^n$. For $v \in \mathbb{R}^n$, this is a linear map $(v \llcorner): \Lambda^p \mathbb{R}^n \to \Lambda^{p-1} \mathbb{R}^n$

given on simple vectors by

$$v \llcorner (v_1 \wedge \ldots \wedge v_p) \equiv \sum_{i=1}^{p} (-1)^{i+1} \langle v_i, v \rangle v_1 \wedge \ldots \wedge \hat{v}_i \wedge \ldots \wedge v_p \tag{3.21}$$

where $(\hat{\cdot})$ indicates deletion. This gives a skew-derivation of the algebra, i.e., $v \llcorner (\varphi \wedge \psi) = (v \llcorner \varphi) \wedge \psi + (-1)^p \varphi \wedge (v \llcorner \psi)$ for any $\varphi \in \Lambda^p \mathbb{R}^n$. It is not difficult to see that $v \llcorner (v \llcorner) = 0$ for any $v \in \mathbb{R}^n$. Hence, by universality the interior product extends to all elements of $\Lambda^* \mathbb{R}^n$, i.e., to a bilinear map $\Lambda^* \mathbb{R}^n \times \Lambda^* \mathbb{R}^n \to \Lambda^* \mathbb{R}^n$.

Proposition 3.9. *With respect to the canonical isomorphism $C\ell_n \cong \Lambda^* \mathbb{R}^n$, Clifford multiplication between $v \in \mathbb{R}^n$ and any $\varphi \in C\ell_n$ can be written as*

$$v \cdot \varphi \cong v \wedge \varphi - v \llcorner \varphi \tag{3.22}$$

Proof. Choose an orthonormal basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$ with $v = te_1$ for some $t \in \mathbb{R}$. Let $\varphi = e_{i_1} \cdots e_{i_p}$ for $i_1 < \cdots < i_p$. Then

$$v \cdot \varphi = \begin{cases} -te_{i_2} \cdots e_{i_p} \cong (v \wedge - v \llcorner)\varphi & \text{if } i_1 = 1 \\ te_1 e_{i_1} \cdots e_{i_p} \cong (v \wedge - v \llcorner)\varphi & \text{if } i_1 > 1. \end{cases}$$

Since (3.22) holds on an additive basis of $C\ell_n$, it holds in general. ■

§4. The Classification

In this section we shall give an explicit description of the algebras $C\ell_{r,s}$ as matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (= quaternions). With little difficulty the reader can check the first few cases:

$$\begin{array}{ll} C\ell_{1,0} = \mathbb{C} & C\ell_{0,1} = \mathbb{R} \oplus \mathbb{R} \\ C\ell_{2,0} = \mathbb{H} & C\ell_{0,2} = \mathbb{R}(2) \\ \multicolumn{2}{c}{C\ell_{1,1} = \mathbb{R}(2)} \end{array} \tag{4.0}$$

where $\mathbb{R}(2)$ denotes the algebra of 2×2 real matrices.

The key facts to the classification are the following:

Theorem 4.1. *There are isomorphisms*

$$C\ell_{n,0} \otimes C\ell_{0,2} \cong C\ell_{0,n+2} \tag{4.1}$$

$$C\ell_{0,n} \otimes C\ell_{2,0} \cong C\ell_{n+2,0} \tag{4.2}$$

$$C\ell_{r,s} \otimes C\ell_{1,1} \cong C\ell_{r+1,s+1} \tag{4.3}$$

for all $n,r,s \geqq 0$.

Note that here we are using the *ungraded* tensor product.

Proof. Let $e_1, \ldots, e_{n+2}$ be an orthonormal basis for $\mathbb{R}^{n+2}$ in the standard inner product, and let $q(x) = -\|x\|^2$. Let $e'_1, \ldots, e'_n$ denote standard generators for $C\ell_{n,0}$ and let e''_1, e''_2 denote standard generators for $C\ell_{0,2}$ (in the sense of Proposition 3.1). Define a map $f: \mathbb{R}^{n+2} \to C\ell_{n,0} \otimes C\ell_{0,2}$ by setting

$$f(e_i) = \begin{cases} e'_i \otimes e''_1 e''_2 & \text{for } 1 \leqq i \leqq n \\ 1 \otimes e''_{i-n} & \text{for } i = n+1, n+2 \end{cases}$$

and extending linearly. Note that for $1 \leq i,j \leq n$, we have $f(e_i)f(e_j) + f(e_j)f(e_i) = (e'_i e'_j + e'_j e'_i) \otimes (-1) = 2\delta_{ij} 1 \otimes 1$; and for $n + 1 \leqq \alpha, \beta \leqq n + 2$ we have $f(e_\alpha)f(e_\beta) + f(e_\beta)f(e_\alpha) = 1 \otimes (e''_{\alpha-n} e''_{\beta-n} + e''_{\beta-n} e''_{\alpha-n}) = 2\delta_{\alpha\beta} 1 \otimes 1$. Also we see that $f(e_i)f(e_\alpha) + f(e_\alpha)f(e_i) = 0$. It follows that $f(x)f(x) = \|x\|^2 1 \otimes 1$ for all $x \in \mathbb{R}^{n+2}$. Hence, by the universal property (Proposition 1.1), f extends to an algebra homomorphism $\tilde{f}: C\ell_{0,n+2} \to C\ell_{n,0} \otimes C\ell_{0,2}$. Since $\tilde{f}$ maps onto a set of generators for $C\ell_{n,0} \otimes C\ell_{0,2}$, it must be surjective. Then, since $\dim C\ell_{0,n+2} = \dim C\ell_{n,0} \otimes C\ell_{0,2}$, we conclude that $\tilde{f}$ must be an isomorphism. This proves (4.1). The proof of (4.2) is entirely analogous.

For (4.3) we proceed in a similar manner. We choose a q-orthogonal basis $e_1, \ldots, e_{r+1}, \varepsilon_1, \ldots, \varepsilon_{s+1}$ for $\mathbb{R}^{r+s+2}$ such that $q(e_i) = 1$ and $q(\varepsilon_j) = -1$ for all i,j. We then let $e'_1, \ldots, e'_r, \varepsilon'_1, \ldots, \varepsilon'_s$ and e''_1, ε''_1 be corresponding bases for $\mathbb{R}^{r+s}$ and $\mathbb{R}^2$, and we define a map $f: \mathbb{R}^{r+s+2} \to C\ell_{r,s} \otimes C\ell_{1,1}$ by setting

$$f(e_i) = \begin{cases} e'_i \otimes e''_1 \varepsilon''_1 & \text{for } 1 \leq i \leq r \\ 1 \otimes e''_1 & \text{for } i = r+1, \end{cases}$$

and

$$f(\varepsilon_j) = \begin{cases} \varepsilon'_j \otimes e''_1 \varepsilon''_1 & \text{for } 1 \leq j \leq s \\ 1 \otimes \varepsilon''_1 & \text{for } j = s+1, \end{cases}$$

and then extending linearly. We now apply Proposition 1.1 and complete the argument as in the previous cases. ∎

To apply this basic proposition we shall need the following elementary facts concerning the tensor products of algebras over $\mathbb{R}$. For $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, we denote by $K(n)$ the algebra of $n \times n$-matrices with entries in K.

Proposition 4.2.

$$\mathbb{R}(n) \otimes \mathbb{R}(m) \cong \mathbb{R}(nm) \qquad \textit{for all } n,m. \tag{4.4}$$

$$\mathbb{R}(n) \otimes_{\mathbb{R}} K \cong K(n) \qquad \textit{for } K = \mathbb{C} \text{ or } \mathbb{H} \textit{ and for all } n. \tag{4.5}$$

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \tag{4.6}$$

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C}(2) \tag{4.7}$$

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4). \tag{4.8}$$

Proof. The isomorphisms (4.4) and (4.5) are obvious. The isomorphism $\mathbb{C} \oplus \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$ is determined by sending

$$(1,0) \longrightarrow \tfrac{1}{2}(1 \otimes 1 + i \otimes i),$$

$$(0,1) \longrightarrow \tfrac{1}{2}(1 \otimes 1 - i \otimes i).$$

For the isomorphism (4.7) we consider $\mathbb{H}$ as a $\mathbb{C}$-module under left scalar multiplication, and we define an $\mathbb{R}$-bilinear map $\Phi: \mathbb{C} \times \mathbb{H} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H},\mathbb{H})$ by setting $\Phi_{z,q}(x) \equiv zx\bar{q}$. This extends (by the universal property of $\otimes$) to an $\mathbb{R}$-linear map $\tilde{\Phi}: \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H},\mathbb{H}) \cong \mathbb{C}(2)$. Since $\Phi_{z,q} \circ \Phi_{z',q'} = \Phi_{zz',qq'}$, we have that $\tilde{\Phi}$ is an algebra homomorphism. Checking $\tilde{\Phi}$ on a natural basis shows that it is injective. Hence, since $\dim_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}) = \dim_{\mathbb{R}}(\mathbb{C}(2))$, $\tilde{\Phi}$ is an isomorphism.

The isomorphism (4.8) is proved similarly. Consider the $\mathbb{R}$-bilinear map $\Psi: \mathbb{H} \times \mathbb{H} \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{H},\mathbb{H}) \cong \mathbb{R}(4)$ given by setting $\Psi_{q_1,q_2}(x) \equiv q_1 x \bar{q}_2$. The resulting $\mathbb{R}$-linear map $\Psi: \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{H},\mathbb{H})$ is an algebra homomorphism between algebras of the same dimension. The injectivity of $\tilde{\Phi}$ can be checked on a natural basis for $\mathbb{H} \otimes \mathbb{H}$. ■

We now come to the first main result of the section. Before stating the result, we make the observation that for any (r,s), the complexification of the algebra $C\ell_{r,s}$ is just the Clifford algebra (over $\mathbb{C}$) corresponding to the complexified quadratic form, i.e., $C\ell_{r,s} \otimes_{\mathbb{R}} \mathbb{C} \cong C\ell(\mathbb{C}^{r+s}, q \otimes \mathbb{C})$. (This follows easily from Proposition 1.1.) However, all non-degenerate quadratic forms on $\mathbb{C}^n$ are equivalent over $C\ell_n(\mathbb{C})$. Hence, setting

$$q_{\mathbb{C}}(z) = \sum_{j=1}^{n} z_j^2$$

and defining

$$\mathbb{C}\ell_n \equiv C\ell(\mathbb{C}^n, q_{\mathbb{C}}), \tag{4.9}$$

we have that

$$\mathbb{C}\ell_n \cong C\ell_{n,0} \otimes_{\mathbb{R}} \mathbb{C} \cong C\ell_{n-1,1} \otimes_{\mathbb{R}} \mathbb{C} \cong \cdots \cong C\ell_{0,n} \otimes_{\mathbb{R}} \mathbb{C} \tag{4.10}$$

Theorem 4.3. *For all $n \geq 0$, there are "periodicity" isomorphisms*

$$C\ell_{n+8,0} \cong C\ell_{n,0} \otimes C\ell_{8,0} \tag{4.11}$$

$$C\ell_{0,n+8} \cong C\ell_{0,n} \otimes C\ell_{0,8} \tag{4.12}$$

$$\mathbb{C}\ell_{n+2} \cong \mathbb{C}\ell_n \otimes_{\mathbb{C}} \mathbb{C}\ell_2 \tag{4.13}$$

where

$$C\ell_{8,0} = C\ell_{0,8} = \mathbb{R}(16) \tag{4.14}$$

$$\mathbb{C}\ell_2 = \mathbb{C}(2). \tag{4.15}$$

Therefore, by using the identities (4.4) *and* (4.5), *all the algebras* $C\ell_{n,0}$, $C\ell_{0,n}$ *and* $\mathbb{C}\ell_n$ *can be easily deduced from the following table.*

Table I

	1	2	3	4	5	6	7	8
$C\ell_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$C\ell_{0,n}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$\mathbb{C}\ell_n$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{C}(16)$

Proof. From (4.1) and (4.2) we see that for any n, we have $C\ell_{n+8,0} \cong C\ell_{n,0} \otimes C\ell_{0,2} \otimes C\ell_{2,0} \otimes C\ell_{0,2} \otimes C\ell_{2,0}$. Using (4.0) and Proposition 4.2 we see that $C\ell_{n+8,0} \cong C\ell_{n,0} \otimes \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \cong C\ell_{n,0} \otimes \mathbb{R}(4) \otimes \mathbb{R}(4) \cong C\ell_{n,0} \otimes \mathbb{R}(16)$. This establishes (4.11). The periodicity (4.12) is proved similarly. To prove (4.13), note from (4.10) that $\mathbb{C}\ell_{n+2} \cong C\ell_{n+2,0} \otimes \mathbb{C} \cong C\ell_{n,0} \otimes C\ell_{0,2} \otimes \mathbb{C} \cong \mathbb{C}\ell_n \otimes_{\mathbb{C}} \mathbb{C}\ell_2$.

Using the isomorphisms (4.1) and (4.2), and the facts (4.4) to (4.8), one can work out the first two rows of the table in "criss-cross" fashion (starting with the initial data (4.0)). The third row of the table now follows by taking the tensor product of corresponding terms in either of the first two rows with $\mathbb{C}$. ■

Combining Table I with the fundamental periodicity isomorphism (4.3) and the fact that $C\ell_{1,1} \cong \mathbb{R}(2)$, we achieve the complete classification in Table II.

By now the reader has probably noticed some of the intrinsic beauty of this constellation of algebras and its interrelationships. There are some observations one can make from the table that are interesting exercises to prove. For example,

$$C\ell_{r,s} \cong C\ell_{r-4,s+4} \tag{4.16}$$

$$C\ell_{r,s+1} \cong C\ell_{s,r+1} \qquad \text{(symmetry about the axis } y = x + 1). \tag{4.17}$$

REMARK. The above classification reduces the Clifford algebras to familiar matrix algebras over $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Of course it is also useful to think of this result as introducing hidden and unexpected structure in the algebras $K(2^m)$. This information can be quite interesting as we shall see when we discuss vector fields on spheres in §8.

Table II. $C\ell_{r,s}$ in the box (r,s)

8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(128)$	$\mathbb{R}(128)\oplus\mathbb{R}(128)$	$\mathbb{R}(256)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64)\oplus\mathbb{R}(64)$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
6	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32)\oplus\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	$\mathbb{H}(32)\oplus\mathbb{H}(32)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(32)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)\oplus\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**

§5. Representations

Most of the important applications of Clifford algebras come through a detailed understanding of their representations. This understanding follows rather easily from the classification given in §4.

We begin with a general definition. Let V be a vector space over a field k and let q be a quadratic form on V.

DEFINITION 5.1. Let $K \supseteq k$ be a field containing k. Then a **K-representation** of the Clifford algebra $C\ell(V,q)$ is a k-algebra homomorphism

$$\rho : C\ell(V,q) \longrightarrow \mathrm{Hom}_K(W,W)$$

into the algebra of linear transformations of a finite dimensional vector space W over K. The space W is called a **$C\ell(V,q)$-module** over K. We shall often simplify notation by writing

$$\rho(\varphi)(w) \equiv \varphi \cdot w \tag{5.1}$$

for $\varphi \in C\ell(V,q)$ and $w \in W$, when no confusion is likely to occur. The product $\varphi \cdot w$ in (5.1) is often referred to as **Clifford multiplication.**

Note. By a **k-algebra homomorphism** we mean a k-linear map ρ which satisfies the property $\rho(\varphi\psi) = \rho(\varphi) \circ \rho(\psi)$ for all $\varphi,\psi \in C\ell(V,q)$.

We shall be interested in K-representations of $C\ell_{r,s}$ where $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Note that a complex vector space is just a real vector space W together with a real linear map $J : W \to W$ such that $J^2 = -\mathrm{Id}$. A complex representation of $C\ell_{r,s}$ is just a real representation $\rho : C\ell_{r,s} \to \mathrm{Hom}_{\mathbb{R}}(W,W)$ such that

$$\rho(\varphi) \circ J = J \circ \rho(\varphi) \tag{5.2}$$

for all $\varphi \in C\ell_{r,s}$. Thus the image of ρ commutes with the subalgebra $\mathrm{span}\{\mathrm{Id},J\} \cong \mathbb{C}$. (This algebra is called a "commuting subalgebra" for ρ.)

Strictly analogous remarks apply to quaternionic representations of $C\ell_{r,s}$. Here the real vector space W carries three real linear transformations I, J and K such that

$$I^2 = J^2 = K^2 = -\mathrm{Id}$$

$$IJ = -JI = K, \qquad JK = -KJ = I, \qquad KI = -IK = J.$$

This makes W into an $\mathbb{H}$-module. A representation $\rho : C\ell_{r,s} \to \mathrm{Hom}_{\mathbb{R}}(W,W)$ is quaternionic if

$$\rho(\varphi) \circ I = I \circ \rho(\varphi), \qquad \rho(\varphi) \circ J = J \circ \rho(\varphi), \qquad \rho(\varphi) \circ K = K \circ \rho(\varphi) \tag{5.3}$$

for all $\varphi \in C\ell_{r,s}$. That is, ρ has a commuting subalgebra $\mathrm{span}_{\mathbb{R}}\{\mathrm{Id},I,J,K\}$ isomorphic to $\mathbb{H}$.

REMARK 5.2. Any complex representation of $C\ell_{r,s}$ automatically extends to a representation of $C\ell_{r,s} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\ell_{r+s}$. Any quaternionic representation of $C\ell_{r,s}$ is automatically complex (by restricting to $\mathbb{C} \subset \mathbb{H}$). Of course the complex dimension of any $\mathbb{H}$-module is even.

The above remarks will prove useful when we carry these constructions over to vector bundles in Chapter II.

We now come to the notion of irreducibility.

DEFINITION 5.3. Let $V, q, k \subseteq K$, be as in definition 5.1. A K-representation $\rho: C\ell(V,q) \to \mathrm{Hom}_K(W,W)$ will be said to be **reducible** if the vector space W can be written as a non-trivial direct sum (over K).

$$W = W_1 \oplus W_2$$

such that $\rho(\varphi)(W_j) \subseteq W_j$ for $j = 1,2$ and for all $\varphi \in C\ell(V,q)$. Note that in this case we can write

$$\rho = \rho_1 \oplus \rho_2$$

where $\rho_j(\varphi) \equiv \rho(\varphi)|_{W_j}$ for $j = 1,2$. A representation is called **irreducible** if it is not reducible.

It is more conventional to call a representation "irreducible" if it has the property that there are no proper invariant subspaces. However, since $C\ell_n$ is the algebra of a finite group (see the discussion following Proposition 5.15), the two concepts are easily seen to agree in this case.

Proposition 5.4. *Every K-representation ρ of a Clifford algebra $C\ell(V,q)$ can be decomposed into a direct sum $\rho = \rho_1 \oplus \cdots \oplus \rho_m$ of irreducible representations.*

Proof. If ρ is reducible, it can be decomposed as a direct sum $\rho = \rho_1 \oplus \rho_2$. If either ρ_1 or ρ_2 is reducible, ρ can be further decomposed. This process must stop because of the finite dimensionality of the module. ■

We shall be interested here, of course, only in equivalence classes of representations.

DEFINITION 5.5. Two representations $\rho_j: C\ell(V,q) \to \mathrm{Hom}_K(W_j,W_j)$ for $j = 1,2$ are said to be **equivalent** if there exists a K-linear isomorphism $F: W_1 \to W_2$ such that $F \circ \rho_1(\varphi) \circ F^{-1} = \rho_2(\varphi)$ for all $\varphi \in C\ell(V,q)$.

From §4 we know that every algebra $C\ell_{r,s}$ is of the form $K(2^m)$ or $K(2^m) \oplus K(2^m)$ for $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The representation theory of such algebras is particularly simple.

Theorem 5.6. Let $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, *and consider the ring $K(n)$ of $n \times n$ K-matrices as an algebra over $\mathbb{R}$. Then the natural representation ρ of $K(n)$ on the vector space K^n is, up to equivalence, the only irreducible real representation of $K(n)$.*

The algebra $K(n) \oplus K(n)$ has exactly two equivalence classes of irreducible real representations. They are given by

$$\rho_1(\varphi_1,\varphi_2) \equiv \rho(\varphi_1) \qquad \text{and} \qquad \rho_2(\varphi_1,\varphi_2) = \rho(\varphi_2)$$

acting on K^n.

Proof. This follows from the classical fact that the algebras $K(n)$ are simple and that simple algebras have only one irreducible representation up to equivalence. See Lang [1]. ■

From the classification of §4 (see Table II) we immediately conclude the following:

Theorem 5.7. *Let $v_{r,s}$ denote the number of inequivalent irreducible real representations of $C\ell_{r,s}$, and let $v_n^{\mathbb{C}}$ denote the number of inequivalent irreducible complex representatons of $\mathbb{C}\ell_n$. Then*

$$v_{r,s} = \begin{cases} 2 & \text{if } r+1-s \equiv 0 \pmod 4 \\ 1 & \text{otherwise} \end{cases}$$

and

$$v_n^{\mathbb{C}} = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

This is a good time to recall (cf. Theorem 3.7) that there are isomorphisms

$$C\ell_{r,s} \cong C\ell^0_{r+1,s} \tag{5.4}$$

for all r,s, and consequently

$$\mathbb{C}\ell_n \cong \mathbb{C}\ell^0_{n+1} \tag{5.5}$$

for all n. Since

$$\mathrm{Spin}_{r,s} \subset C\ell^0_{r,s} \subset \mathbb{C}\ell^0_{r+s} \tag{5.6}$$

we see that it is the irreducible representations of $C\ell_{r-1,s}$ and $\mathbb{C}\ell_{r+s-1}$ that are relevant to constructing irreducible real and complex representations of $\mathrm{Spin}_{r,s}$.

From this point on we shall restrict our attention to the algebras $C\ell_n = C\ell_{n,0}$ (and $\mathbb{C}\ell_n = C\ell_n \otimes_{\mathbb{R}} \mathbb{C}$) in order to simplify the exposition. Corresponding facts for the general case $C\ell_{r,s}$ are easy to deduce if the reader is interested. We shall begin with a summary of information easily deduced from the classification theorem 4.3.

We begin with some definitions. For each n, let $d_n = \dim_{\mathbb{R}}(W)$ where W is an irreducible $\mathbb{R}$-module for $C\ell_n$. Similarly, let $d_n^{\mathbb{C}} = \dim_{\mathbb{C}}(W')$ where W' is an irreducible complex module for $C\ell_n$ (and therefore for $\mathbb{C}\ell_n =$