The Action Principle and Partial Differential Equations

BY

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Contents

The Action Principle and Partial Differential Equations

General Introduction

The principle of stationary action, in the form in which it arose in classical mechanics in the work of Lagrange, led to the discovery of symplectic geometry and the development of Hamiltonian methods which are indispensable in the study of the equations of motion.

In classical mechanics the domain of the unknowns is the real line of time, the action is a single integral and the Euler-Lagrange equations are ordinary differential equations. The principle of stationary action was subsequently generalized to the case that the domain of the unknowns, the manifold of independent variables, is multi-dimensional, the action is a multiple integral, and the Euler-Lagrange equations are partial differential equations.

This generalization originates in Lagrange's derivation of the partial differential equation which a function of two variables must satisfy so that its graph is a surface of least area in Euclidean space. Dirichlet's principle provided a simpler example of an action as a multiple integral, which stimulated the development of direct methods of the calculus of variations for several independent variables, beginning with the work of Hilbert [Hil]. This development led to the solution of the problem of surfaces of least area by Douglas [Dou] and Radó [R].

In the mean time Hilbert had extended the principle of stationary action to the case that the domain manifold is the four dimensional manifold of space-time and derived from a particular form of this principle Maxwell's equations for the electromagnetic field. Hilbert's work in this direction culminated with his discovery of the action principle which governs the geometry of the space-time manifold itself [Hi2], leading to Einstein's equations of general relativity. The general form of the stationary action principle as envisioned by Hilbert, is at the present time a central unifying theme in theoretical physics.

Now, a great deal of deep work has been done in the calculus of variations in the case that the Euler-Lagrange equations are of elliptic type, in developing direct methods and in working out the regularity theory of solutions. Among the milestones in this development are Morrey's solution of the harmonic map problem from the unit disc to a Riemannian manifold [M1], the first variational problem to be solved where the Euler-Lagrange equations constitute a non-linear system of partial differential equations, leading him directly to the solution of the problem of surfaces of least area in a Riemannian manifold. Also, the breakthrough in the regularity theory of more than two independent variables by De Giorgi [DeG] and Nash [Na], in the case of a single unknown function, which led, in particular, to the solution of the problem of hypersurfaces of minimal volume in higher dimensional spaces.

However, despite the remarkable progress, just outlined, in the elliptic case, the aspects of the action principle which are relevant to the case where the Euler-Lagrange equations are of hyperbolic type, the case occuring in physics, have been left largely undeveloped, with the notable exception of the principle connecting symmetries to conserved quantities propounded by Noether [No]. The purpose of the present book is to contribute toward remedying this situation. Thus our aim is to introduce concepts and prove theorems which will be found useful in developing the theory of non-linear systems of hyperbolic type. For this reason we have completely left out all subject matter which pertains exclusively to the elliptic case. Nevertheless, since there is a number of concepts and theorems of a general nature, which apply equally well to the elliptic and hyperbolic cases in particular, we expound these in the first four chapters of the book.

After the introductory first chapter, the main developments expounded in this book are the following. First, in Chapter 2, symplectic geometry in the case of a multi-dimensional domain manifold is explored. In analogy with classical hydrodynamics, a theory of flows is developed. A flow corresponds to an n-parameter family of solutions of the Euler-Lagrange equations (n is the number of dependent variables), and the theory is an extension of the Hamilton-Jacobi theory of classical mechanics. In Chapter 3, a general theory of integral identities is developed, the theory of *compatible currents*, which extends the work of Noether. Whereas other methods, such as the maximum principle, are avaivalble for the treatment of elliptic equations, integral identities provide the only known general basis for approaching hyperbolic equations. In the development of the theory of compatible currents the great gulf between the case of two independent variables and the case of more than two independent variables becomes apparent. Chapter 4 deals with the case that the unknown is a section of a vector bundle over the domain manifold, rather than a mapping of the domain manifold into a target manifold. This is necessary for the developments of the last two chapters. Chapter 5 begins with our notion of hyperbolicity, which represents a significant departure from notions in the existing literature and suggests new methods for the solution of problems. We show how the new notion overcomes the difficulties associated with singularities of the characteristic variety. The causal structure on the domain manifold defined by a hyperbolic Lagrangian is then studied and the domain of dependence property of solutions is established. As is usual in this subject, the methods of the domain of dependence theorem lead readily to a local existence theorem, for given initial data. The results apply in particular to the theory of non-linear elasticity. We should note here that Leray's pioneering theory of strictly hyperbolic systems [L] is applicable to non-linear elasticity only under certain restrictions. In fact, Fritz John has found a physical example [J] to which Leray's theory does not apply. Moreover, Tahvildar-Zadeh has recently shown [T-Z] that Fritz John's example is stable within the framework of non-linear elasticity, thereby reinforcing its importance. In contrast to Leray's theory, our results apply without restrictions to the general framework of the theory of elasticity.

Finally, the last chapter deals with electromagnetic theory, the electrodynamics of a general non-linear continuous medium. Although electromagnetic theory may be considered to be a theory of sections of the cotangent bundle of the domain manifold, it cannot be reduced to the general theory of sections of vector bundles, for then the Legendre transformation which takes us from the Lagrangian to the Hamiltonian picture would be singular. This is a direct consequence of the requirement of gauge invariance and requires a reworking of all constructions. We have thus devoted a separate chapter to this theory, in view also of its physical significance. We establish results analogous to those of Chapters 3 and 5 (including the domain of dependence theorem) in the framework of non-linear electrodynamics. The results in this chapter are the first general results in this framework going beyond the linear approximation.

Chapter 1

1.0 Introduction

In this chapter we begin with the basic concepts of the calculus of variations in several independent variables, the central theme being the principle of stationary action. The independent variables constitute an m dimensional manifold $\mathcal{M}$, the domain of the dependent variables. The latter take values in a n dimensional target manifold $\mathcal{N}$. The action is a m fold integral over a domain $\mathcal{D}$ with compact closure in $\mathcal{M}$. The subject matter of the first section is thus entirely classical. Our presentation is however from the global perspective, with the intention of making explicit the geometric structures which need to be introduced at each step and distinguishing those constructs which turn out to be independent of specific choices of such structures. The second section presents the transition from the Lagrangian to the Hamiltonian picture from the same perspective. The Hamiltonian picture in the case of several independent variables originates in the work of De Donder [DeD], and is much less widely known than the Lagrangian picture, in contrast to the case of a single independent variable. Lastly, the third section is devoted to examples. The first two are from classical differential geometry: harmonic maps and minimal surfaces. We then present a novel formulation of relativistic continuum mechanics, to which the last two examples belong: fluid dynamics and the dynamics of crystalline solids. The latter is treated in the framework of our recently developed continuum theory of dislocations ([Ch]).

1.1 The Lagrangian Picture

Let $\mathcal{M}$ and $\mathcal{N}$ be differentiable manifolds, $\dim \mathcal{M} = m$, $\dim \mathcal{N} = n$, and let $\mathcal{M}$ be oriented. We shall study Lagrangian theories of maps $u : \mathcal{M} \to \mathcal{N}$. The *configuration space* is the product

$$\mathcal{C} = \mathcal{M} \times \mathcal{N} \ \ (\dim \mathcal{C} = m + n) \tag{1.1}$$

The *velocity space* is the bundle:

$$\mathcal{V} = \bigcup_{(x,q)\in\mathcal{C}} \mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N}) \ \ (\dim \mathcal{V} = m + n + mn) \tag{1.2}$$

In general, if U and V are vector spaces we denote by $\mathcal{L}(U, V)$ the space of linear maps of U into V. An element v of $\mathcal{V}$ is called *canonical velocity.* Letting $\pi_{\mathcal{V},\mathcal{C}}$ be the projection $\mathcal{V}$ onto $\mathcal{C}$,

$$\begin{array}{l} \pi_{\mathcal{V},\mathcal{C}} : \mathcal{V} \to \mathcal{C} \\ v \in \mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N}) \mapsto (x, q) \in \mathcal{C} \end{array}$$

$(\mathcal{V}, \pi_{\mathcal{V},\mathcal{C}})$ is a vector bundle over $\mathcal{C}$. Also, letting $\pi_{\mathcal{V},\mathcal{M}}$ and $\pi_{\mathcal{V},\mathcal{N}}$ be the projections of $\mathcal{V}$ onto $\mathcal{M}$ and $\mathcal{N}$, $\pi_{\mathcal{V},\mathcal{M}}(v)$ and $\pi_{\mathcal{V},\mathcal{N}}(v)$ being respectively the 1st and 2nd components of $\pi_{\mathcal{V},\mathcal{C}}(v)$, $(\mathcal{V}, \pi_{\mathcal{V},\mathcal{M}})$ is a bundle over $\mathcal{M}$ and $(\mathcal{V}, \pi_{\mathcal{V},\mathcal{N}})$ is a bundle over $\mathcal{N}$.

In general, if $\mathcal{M}$ and $\mathcal{N}$ are differentiable manifolds, $(\mathcal{B}, \pi_{\mathcal{B},\mathcal{N}})$ is a vector bundle over $\mathcal{N}$, and f is a map of $\mathcal{M}$ into $\mathcal{N}$, we denote by $f^*\mathcal{B}$ the *pullback bundle*:

$$f^*\mathcal{B} = \bigcup_{x\in\mathcal{M}} \{x\} \times \pi^{-1}_{\mathcal{B},\mathcal{N}}(f(x)) \tag{1.3}$$

a vector bundle over $\mathcal{M}$. We have:

$$\pi_{f^*\mathcal{B},\mathcal{M}}(x, b) = x \ \ ; b \in \pi^{-1}_{\mathcal{B},\mathcal{N}}(f(x)) \tag{1.4}$$

Denoting by $\wedge_k\mathcal{M}$ the vector bundle

$$\wedge_k\mathcal{M} = \bigcup_{x\in\mathcal{M}} \wedge_k(T_x\mathcal{M}) \tag{1.5}$$

where $\wedge_k(T_x\mathcal{M})$ is the space of totally antisymmetric k-linear forms on $T_x\mathcal{M}$, we consider the pullback bundle $\pi^*_{\mathcal{V},\mathcal{M}} \wedge_k \mathcal{M}$, a vector bundle over $\mathcal{V}$.

An element of this bundle belonging to the fiber over $v \in \mathcal{V}$ is an element of $\wedge_k(T_x\mathcal{M})$, where $x = \pi_{\mathcal{V},\mathcal{M}}(v)$ is the corresponding base point in $\mathcal{M}$.

The *Lagrangian* L is a differentiable section of the pullback bundle $\pi^*_{\mathcal{V},\mathcal{M}} \wedge_m \mathcal{M}$. If $u : x \mapsto u(x)$ is a map of $\mathcal{M}$ into $\mathcal{N}$ then $\sigma : x \mapsto v(x) = du(x)$ is a section of $(\mathcal{V}, \pi_{\mathcal{V},\mathcal{M}})$ and $L \circ \sigma$ is an exterior differential form of the top degree, m, on $\mathcal{M}$. Thus $L \circ \sigma$ can be integrated on any open set $\mathcal{D}$ with compact closure in $\mathcal{M}$ and the integral

$$\mathcal{S}\,[u; \mathcal{D}] = \int_{\mathcal{D}} L \circ \sigma \tag{1.6}$$

is the *action of* u *in* $\mathcal{D}$.

The $\mathcal{C}$-vertical derivative of L at $v \in \mathcal{V}$, $\pi_{\mathcal{V},\mathcal{C}}(v) = (x, q)$, is the element $(\partial L/\partial v)(v)$ of

$$\mathcal{L}(\mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N}), \wedge_m(T_x\mathcal{M}))$$

defined by:

$$\begin{aligned}\left(\frac{\partial L}{\partial v}\right)(v) \cdot \dot{v} &= \lim_{t\to 0} \frac{1}{t}\{L(v + t\dot{v}) - L(v)\} \\ &: \ \forall \dot{v} \in \pi^{-1}_{\mathcal{V},\mathcal{C}}(x, q) = \mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N})\end{aligned} \tag{1.7}$$

Now $\mathcal{L}(T_q\mathcal{N}, \wedge_{m-1}(T_x\mathcal{M}))$ is canonically isomorphic to $\mathcal{L}(\mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N}), \wedge_m(T_x\mathcal{M}))$; the isomorphism i takes α to $i\alpha$, where

$$(i\alpha) \cdot \dot{v} = \dot{v} \wedge \alpha \ : \forall \dot{v} \in \mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N}) \tag{1.8}$$

with $\dot{v} \wedge \alpha \in \wedge_m(T_x\mathcal{M})$ given by:

$$(\dot{v} \wedge \alpha)(X_1, \ldots, X_m) = \sum_{i=1}^{m} (-1)^{i-1} \alpha(\dot{v}(X_i))(X_1, \ldots < X_i > \ldots, X_m)$$
$$\forall X_1, \ldots, X_m \in T_x\mathcal{M} \tag{1.9}$$

We can thus identify $(\partial L/\partial v)(v)$ with an element of $\mathcal{L}(T_q\mathcal{N}, \wedge_{m-1}(T_x\mathcal{M}))$. Let us define the vector bundles over $\mathcal{C}$.

$$\begin{aligned}\wedge_{k,\ell}(\mathcal{M}, \mathcal{N}) &= \bigcup_{(x,q)\in\mathcal{C}} \mathcal{L}(\wedge^k(T_x\mathcal{M}), \wedge_\ell(T_q\mathcal{N})) \\ &= \bigcup_{(x,q)\in\mathcal{C}} \mathcal{L}(\wedge^\ell(T_q\mathcal{N}), \wedge_k(T_x\mathcal{M}))\end{aligned} \tag{1.10}$$

(In general, if V is a vector space we denote by $\wedge^k(V)$ the totally antisymmetric k-fold tensor product of V with itself). Then $\partial L/\partial v$ can be identified with a section of the pullback bundle $\pi^*_{\mathcal{V},\mathcal{C}} \wedge_{m-1,1} (\mathcal{M}, \mathcal{N})$. We define the *canonical momentum* to be the section:

$$p = \frac{\partial L}{\partial v} \tag{1.11}$$

Next, we shall define the $\mathcal{N}$-horizontal derivative of L. To do this requires the choice of a connection A in $T\,\mathcal{N}$. We require A to be *symmetric*. Thus, if D is the associated covariant derivative and Y, Z are vectorfields on $\mathcal{N}$ then:

$$D_Y Z - D_Z Y = [Y, Z] \tag{1.12}$$

Given a curve $\gamma : (-1,1) \to \mathcal{N}$ in $\mathcal{N}$ through $q, \gamma(0) = q$, with tangent vector Q at $q, \dot{\gamma}(0) = Q$, we can then define a curve $\gamma^{\#Q'}_{T\mathcal{N}} : (-1,1) \to T\mathcal{N}$

in $T\mathcal{N}$ through $Q' \in T_q\mathcal{N}$, $\gamma_{T\mathcal{N}}^{\sharp Q'}(0) = Q'$, the *horizontal lift* of γ to $T\mathcal{N}$ through Q'. We have: $\pi_{T\mathcal{N},\mathcal{N}} \circ \gamma_{T\mathcal{N}}^{\sharp Q'} = \gamma$. The tangent vector $\dot{\gamma}_{T\mathcal{N}}^{\sharp Q'}(0)$ of this curve at Q' is the horizontal lift to $T\mathcal{N}$ through $Q' \in T_q\mathcal{N}$ of the vector $Q \in T_q\mathcal{N}$ and is denoted by $Q_{T\mathcal{N}}^{\sharp Q'}$.

If $M_a, a \in \Re$, is the transformation of $T\mathcal{N}$ given by:

$$M_a(Q) = aQ \; : \forall Q \in T\mathcal{N} \tag{1.13}$$

then we have:

$$Q_{T\mathcal{N}}^{\sharp aQ'} = dM_a \cdot Q_{T\mathcal{N}}^{\sharp Q'} \tag{1.14}$$

where dM_a is the differential of the map $M_a \; : T\mathcal{N} \to T\mathcal{N}$.

If $\tilde{A}$ is another symmetric connection in $T\mathcal{N}$, then the difference $\tilde{A} - A$ corresponds to a tensorfield B on $\mathcal{N}$, a section of the vector bundle

$$S_2^1\mathcal{N} = \bigcup_{q \in \mathcal{N}} S_2^1(T_q\mathcal{N}) \tag{1.15}$$

over $\mathcal{N}$, where $S_2^1(T_q\mathcal{N})$ is the space of symmetric bilinear maps of $T_q\mathcal{N}$ into $T_q\mathcal{N}$. Then $Q_{T\mathcal{N}}^{\tilde{\sharp} Q'} - Q_{T\mathcal{N}}^{\sharp Q'}$, the difference of the horizontal lifts of $Q \in T_q\mathcal{N}$ to $T\mathcal{N}$ through $Q' \in T_q\mathcal{N}$ defined by the two connections is given by:

$$Q_{T\mathcal{N}}^{\tilde{\sharp} Q'} - Q_{T\mathcal{N}}^{\sharp Q'} = -B(Q, Q') \in T_q\mathcal{N} \tag{1.16}$$

Let again $\gamma \; : (-1,1) \to \mathcal{N}$ be a curve in $\mathcal{N}$ through $q, \gamma(0) = q$, with tangent vector Q at $q, \dot{\gamma}(0) = Q$. Given any $v \in \mathcal{V}$ such that $\pi_{\mathcal{V},\mathcal{N}}(v) = q$, we can define a curve $\gamma_{\mathcal{V}}^{\sharp v} \; : (-1,1) \to \mathcal{V}$ in $\mathcal{V}$ through $v, \gamma_{\mathcal{V}}^{\sharp v}(0) = v$, by requiring that $\pi_{\mathcal{V},\mathcal{N}} \circ \gamma_{\mathcal{V}}^{\sharp v} = \gamma$ while $\pi_{\mathcal{V},\mathcal{M}} \circ \gamma_{\mathcal{V}}^{\sharp v}$ maps $(-1,1)$ to the single point $x = \pi_{\mathcal{V},\mathcal{M}}(v) \in \mathcal{M}$, and for each $X \in T_x\mathcal{M}$ the curve

$$\gamma_{\mathcal{V}}^{\sharp v} \cdot X \; : t \in (-1,1) \mapsto \gamma_{\mathcal{V}}^{\sharp v}(t) \cdot X \in T_{\gamma(t)}\mathcal{N}$$

in $T\mathcal{N}$ coincides with $\gamma^{\sharp v\cdot X}_{T\mathcal{N}}$, the horizontal lift of γ to $T\mathcal{N}$ through $v \cdot X \in T_q\mathcal{N}$:

$$\gamma^{\sharp v}_{\mathcal{V}} \cdot X = \gamma^{\sharp v\cdot X}_{T\mathcal{N}} \ : \forall X \in T_x\mathcal{M} \tag{1.17}$$

If we denote by $Q^{\sharp v}_{\mathcal{V}}$ the tangent vector at $v \in \mathcal{V}$ of the curve $\gamma^{\sharp v}_{\mathcal{V}}$ in $\mathcal{V}$, $Q^{\sharp v}_{\mathcal{V}} = \dot\gamma^{\sharp v}_{\mathcal{V}}(0)$, we have, according to the above,

$$Q^{\sharp v}_{\mathcal{V}} \cdot X = Q^{\sharp v\cdot X}_{T\mathcal{N}} \ : \forall X \in T_x\mathcal{M} \tag{1.18}$$

The vector $Q^{\sharp v}_{\mathcal{V}} \in T_v\mathcal{V}$ is the horizontal lift to $\mathcal{V}$ through v of the vector $Q \in T_q\mathcal{N}$. If $\tilde{A}$ is another symmetric connection in $T\mathcal{N}$ then by 1.16 the difference of the horizontal lifts of Q to $\mathcal{V}$ defined by the two connections is:

$$Q^{\tilde{\sharp} v}_{\mathcal{V}} - Q^{\sharp v}_{\mathcal{V}} = -B(Q, v) \tag{1.19}$$

a $\mathcal{C}$-vertical vector at v. Here $B(Q, v) \in \mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N})$ is defined by:

$$B(Q, v) \cdot X = B(Q, v \cdot X) \ : \forall X \in T_x\mathcal{M}$$

We define the $\mathcal{N}$-horizontal derivative of L at v relative to the connection A to be the element $DL(v)$ of $\mathcal{L}(T_q\mathcal{N}, \wedge_m(T_x\mathcal{M}))$, $(x, q) = \pi_{\mathcal{V},\mathcal{C}}(v)$, defined by:

$$(DL)(v) \cdot Q = \left(\frac{d}{dt} L(\gamma^{\sharp v}_{\mathcal{V}}(t))\right)_{t=0} \ : \forall Q \in T_q\mathcal{N} \tag{1.20}$$

where γ is any curve in $\mathcal{N}$ through q with tangent vector Q at q, $\dot\gamma(0) = Q$. Denoting by δL the $\mathcal{M}$-vertical differential of L, that is the restriction of the differential of L to vectors tangent to the fibers $\{\pi^{-1}_{\mathcal{V},\mathcal{M}}(x) \ : x \in \mathcal{M}\}$, we can write:

$$\left(\frac{d}{dt}L(\gamma_{\mathcal{V}}^{\sharp v}(t))\right)_{t=0} = \delta L \cdot Q_{\mathcal{V}}^{\sharp v} \tag{1.21}$$

Thus,

$$(DL)(v) \cdot Q = \delta L \cdot Q_{\mathcal{V}}^{\sharp v} \ : \forall Q \in T_q\mathcal{N}, q = \pi_{\mathcal{V},\mathcal{N}}(v) \tag{1.22}$$

The $\mathcal{N}$-horizontal derivative DL of L, defined by a given connection A in $T\mathcal{N}$, is a section of the pullback bundle $\pi^*_{\mathcal{V},\mathcal{C}} \wedge_{m,1} (\mathcal{M},\mathcal{N})$. We define the *canonical force* to be the section:

$$f = DL \tag{1.23}$$

By 1.19, the difference of the $\mathcal{N}$-horizontal derivatives of L defined by the connections $\tilde{A}$ and A is:

$$(\tilde{D}L - DL)(v) \cdot Q = -\left(\frac{\partial L}{\partial v}\right)(v) \cdot B(Q,v)$$

Thus, in view of 1.11 the difference of the canonical forces is given by:

$$(\tilde{f} - f)(v) \cdot Q = -p(v) \cdot B(Q,v) \tag{1.24}$$

Given a mapping $u : \mathcal{M} \to \mathcal{N}$, the connection A in $T\mathcal{N}$ induces a connection u^*A in the pullback bundle

$$u^*T\mathcal{N} = \dot{\mathcal{C}}_u \tag{1.25}$$

a vector bundle over $\mathcal{M}$, as follows. A curve k^* in $u^*T\mathcal{N}$ is a mapping of the form:

$$\begin{aligned} k^* \ : \ \ & (-1,1) \to u^*T\mathcal{N}, \\ & t \mapsto (c(t), k(t)) \end{aligned} \tag{1.26}$$

where $c:(-1,1)\to\mathcal{M}$ is a curve in $\mathcal{M}$ and $k:(-1,1)\to T\mathcal{N}$ is a curve in $T\mathcal{N}$ projecting to the curve $u\circ c:(-1,1)\to\mathcal{N}$ in $\mathcal{N}$. We say that k^* is a horizontal curve in $u^*T\mathcal{N}$ if and only if k is a horizontal curve in $T\mathcal{N}$. The tangent vector $\dot{k}^*(0)$ to the curve k^* at the point $k^*(0)$ is then the horizontal lift to $u^*T\mathcal{N}$ through $k^*(0)$ of the tangent vector $\dot{c}(0)$ to the curve c at $c(0)$:

$$\dot{k}^*(0)=\dot{c}(0)^{\sharp k^*(0)}_{u^*T\mathcal{N}} \tag{1.27}$$

and we have:

$$\dot{k}^*(0)=(\dot{c}(0),\ \dot{k}(0)) \tag{1.28}$$

with $\dot{k}(0)$ the horizontal lift to $T\mathcal{N}$ through $k(0)$ of the tangent vector to the curve $u\circ c$ at $(u\circ c)(0)$. Setting $c(0)=x\in\mathcal{M},\dot{c}(0)=X\in T_x\mathcal{M}$, $k(0)=Q\in T_{u(x)}\mathcal{N}$, we can write:

$$X^{\sharp(x,Q)}_{u^*T\mathcal{N}}=(X,(du(x)\cdot X)^{\sharp Q}_{T\mathcal{N}}) \tag{1.29}$$

On $u^*T\mathcal{N}$ we can define for each $a\in\Re$, a transformation M_a of $u^*T\mathcal{N}$, analogous to the transformation 1.13 of $T\mathcal{N}$, by:

$$M_a(x,Q)=(x,aQ)\ :\forall x\in\mathcal{M},\forall Q\in T_{u(x)}\mathcal{N} \tag{1.30}$$

and by 1.14 we have:

$$X^{\sharp(x,aQ)}_{u^*T\mathcal{N}}=dM_a\cdot X^{\sharp(x,Q)}_{u^*T\mathcal{N}} \tag{1.31}$$

The connection u^*A defines a covariant derivative D^* of sections of $u^*T\mathcal{N}=\dot{\mathcal{C}}_u$. If s is a section of $\dot{\mathcal{C}}_u$ and $c:(-1,1)\to\mathcal{M}$ is a curve in $\mathcal{M}$ through x with tangent vector X at x, then $s\circ c$ is a curve in $u^*T\mathcal{N}$. The tangent vector to this curve at $s(x)$ is $ds(x)\cdot X$ where ds is the differential of s as a map of $\mathcal{M}$ into $\dot{\mathcal{C}}_u$. The covariant derivative $D^*s(x) \cdot X$ of s at x evaluated on $X \in T_x\mathcal{M}$ is then defined to be the vertical part of the vector $ds(x) \cdot X$:

$$D^*s(x) \cdot X = ds(x) \cdot X - X^{\sharp s(x)}_{u^*T\mathcal{N}} \tag{1.32}$$

D^*s itself is a section of the vector bundle:

$$\bigcup_{x \in \mathcal{M}} \mathcal{L}(T_x\mathcal{M}, T_{u(x)}\mathcal{N}) = \dot{\mathcal{V}}_u \subset \mathcal{V} \tag{1.33}$$

If f is a function on $\mathcal{M}$ and s is a section of $\dot{\mathcal{C}}_u$, the section fs of $\dot{\mathcal{C}}_u$ is defined by:

$$(fs)(x) = f(x)s(x) \ : \forall x \in \mathcal{M} \tag{1.34}$$

From 1.31, 1.32 it follows that

$$D^*(fs)(x) \cdot X = dM_{f(x)} \cdot (D^*s(x) \cdot X) + (df(x) \cdot X)s(x) \ : \forall x \in \mathcal{M}, \ \forall X \in T_x\mathcal{M} \tag{1.35}$$

The first term on the right can be written simply as $f(x)(D^*s(x) \cdot X)$ in view of the fact that the vector $D^*s(x) \cdot X$ is vertical and the fibers of $u^*T\mathcal{N}$ have linear structure.

If $\tilde{A}$ is another symmetric connection in $T\mathcal{N}$ with the difference $\tilde{A} - A$ corresponding to a tensor field B on $\mathcal{N}$ then from 1.29 and 1.16 we have, in reference to 1.32,

$$\begin{aligned} X^{\tilde{\sharp} s(x)}_{u^*T\mathcal{N}} - X^{\sharp s(x)}_{u^*T\mathcal{N}} &= (du(x) \cdot X)^{\tilde{\sharp} s(x)}_{T\mathcal{N}} - (du(x) \cdot X)^{\sharp s(x)}_{T\mathcal{N}} \\ &= -B(du(x) \cdot X, s(x)) = -B(s(x), du(x)) \cdot X, \end{aligned}$$

the last step by virtue of the symmetry of B. Hence, at each $x \in \mathcal{M}$, the difference at x of the covariant derivatives of s defined by the two connections is given by:

$$\tilde{D}^* s(x) - D^* s(x) = B(s(x), du(x)) \tag{1.36}$$

The Euler-Lagrange equations for a mapping $u : \mathcal{M} \to \mathcal{N}$ express the condition that for any domain $\mathcal{D}$ with compact closure in $\mathcal{M}$ the action S is stationary at u with respect to variations supported in any sub-domain $\mathcal{D}'$ with compact closure in $\mathcal{D}$. Let $\{u_t : t \in (-1, 1)\}$ be a differentiable family of maps $\mathcal{M}$ into $\mathcal{N}, u_0 = u$, agreeing in $\mathcal{M} \backslash \mathcal{D}'$. Then for each $x \in \mathcal{M}$, $\gamma_x : t \to u_t(x)$ is a curve in $\mathcal{N}$ through $u(x)$. Let $\dot{u}(x)$ be the tangent vector of this curve at $u(x), \dot{u}(x) = \dot{\gamma}_x(0)$. Then $\dot{u} : x \to (x, \dot{u}(x))$, the *variation of the map* u, is a section of the pullback bundle $u^* T\mathcal{N} = \dot{\mathcal{C}}_u$.

Consider, for each $x \in \mathcal{M}$, the curve $\delta_x : t \to du_t(x)$. It is a curve in $\mathcal{V}$ through $\sigma(x) = du(x)$ which projects to the curve γ_x in $\mathcal{N}$ and to the single point x in $\mathcal{M}$. Its tangent vector $\dot{\delta}_x(0)$ at $\sigma(x)$ can be expressed as the sum of a $\mathcal{N}$-horizontal vector $\dot{u}(x)^{\sharp\sigma(x)}_{\mathcal{V}}$, the horizontal lift of $\dot{\gamma}_x(0)$ to $\mathcal{V}$ through $\sigma(x)$, and a $\mathcal{C}$-vertical vector which we denote $\dot{\sigma}(x)$:

$$\dot{\delta}_x(0) = \dot{u}(x)^{\sharp\sigma(x)}_{\mathcal{V}} + \dot{\sigma}(x) \tag{1.37}$$

Thus $\dot{\sigma}$ is a section of the vector bundle $\dot{\mathcal{V}}_u$ (see 1.33).

Let c be a curve in $\mathcal{M}$ through x with tangent vector X at x:

$$c : (-1, 1) \to \mathcal{M}, \;\; c(0) = x, \;\; \dot{c}(0) = X$$

Consider the map:

$$h : (-1, 1)^2 \to \mathcal{N}$$

by:

$$h(t,s) = u_t(c(s)) \tag{1.38}$$

Let Y, Z be the vectorfields:

$$Y = h_*\left(\frac{\partial}{\partial t}\right), \quad Z = h_*\left(\frac{\partial}{\partial s}\right) \tag{1.39}$$

defined along $h((-1,1)^2) \subset \mathcal{N}$. We have:

$$[Y,Z] = h_*\left(\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right]\right) = 0 \tag{1.40}$$

and:

$$Y_{h(t,s)} = \dot{u}_t(c(s)), \quad Z_{h(t,s)} = du_t(c(s)) \cdot \dot{c}(s) \tag{1.41}$$

Thus, considering the curves $t \mapsto Z_{h(t,0)}$, $s \mapsto Y_{h(0,s)}$ in $T\mathcal{N}$, we have:

$$(D_Y Z)_{h(0,0)} = \text{vertical part of } \left(\frac{\partial}{\partial t} Z_{h(t,0)}\right)_{t=0} = \dot{\sigma}(x) \cdot X$$

$$(D_Z Y)_{h(0,0)} = \text{vertical part of } \left(\frac{\partial}{\partial s} Y_{h(t,0)}\right)_{s=0} = D^*\dot{u}(x) \cdot X$$

Therefore, by the symmetry of the connection A, 1.12, and 1.40,

$$(\dot{\sigma}(x) - D^*\dot{u}(x)) \cdot X = (D_Y Z - D_Z Y)_{h(0,0)} = 0$$

We conclude that:

$$\dot{\sigma} = D^*\dot{u} \tag{1.42}$$

In view of 1.37, 1.42 and the definitions 1.11, 1.23 we can express:

$$\begin{aligned}\left(\frac{d\mathcal{S}[u_t,\mathcal{D}]}{dt}\right)_{t=0} &= \int_{\mathcal{D}}\left\{(DL\circ\sigma)\cdot\dot{u}+\left(\frac{\partial L}{\partial v}\circ\sigma\right)\cdot\dot{\sigma}\right\} \\ &= \int_{\mathcal{D}}\{(f\circ\sigma)\cdot\dot{u}+D^*\dot{u}\wedge(p\circ\sigma)\} \qquad (1.43)\end{aligned}$$

Let τ be a section of the vector bundle $\wedge_{k,\ell}(\mathcal{M},\mathcal{N})$ (see 1.10) considered as a bundle over $\mathcal{M}$. Let $u:\mathcal{M}\to\mathcal{N}$ be the composition of τ with the projection of $\wedge_{k,\ell}(\mathcal{M},\mathcal{N})$ to $\mathcal{N}$ on the left. We can then define $d\tau$, the *covariant exterior derivative* of τ, a section of the bundle $\wedge_{k+1,\ell}(\mathcal{M},\mathcal{N})$ over $\mathcal{M}$, as follows. Given vectors $Q_1,\ldots,Q_\ell\in T_{u(x)}\mathcal{N}$ we extend them to local sections $s_1,\ldots,s_\ell$ of $u^*T\mathcal{N}$. Then $\tau(s_1,\ldots,s_\ell)$ is a locally defined exterior differential form of degree k on $\mathcal{M}$ and its exterior derivative is defined in the usual manner. We then set:

$$(d\tau)(Q_1,\ldots,Q_\ell)=d(\tau(s_1,\ldots,s_\ell))-\sum_{j=1}^{\ell}(D^*s_j\wedge\tau)(s_1,\ldots<s_j>\ldots,s_\ell) \qquad (1.44)$$

Now 1.35 implies that the right hand side is a ℓ-linear form on the space of sections of $u^*T\mathcal{N}$ with linearity defined with respect to multiplication by the ring of functions on $\mathcal{M}$ as in 1.34. It follows that the above definition is meaningful for it does not depend on the manner in which the vectors $Q_1,\ldots,Q_\ell$ are extended to local sections $s_1,\ldots,s_\ell$. In the case $k=m-1$, $\ell=1$ the formula 1.44 reduces to:

$$(d\tau)(Q)=d(\tau(s))\;-\;D^*s\wedge\tau \qquad (1.45)$$

a formula which in particular applies to the section $p\circ\sigma$ of $\wedge_{m-1,1}(\mathcal{M},\mathcal{N})$.

We can thus write:

$$D^*\dot{u}\wedge(p\circ\sigma)=d((p\circ\sigma)(\dot{u}))\;-\;(d(p\circ\sigma))(\dot{u})$$

and since by Stokes' theorem

$$\int_{\mathcal{D}} d((p \circ \sigma)(\dot{u})) = \int_{\partial\mathcal{D}} (p \circ \sigma)(\dot{u}) = 0,$$

in view of the fact that $\dot{u}$ vanishes in neighborhood of $\partial\mathcal{D}$, 1.43 takes the form:

$$\left(\frac{d\mathcal{S}[u_t, \mathcal{D}]}{dt}\right)_{t=0} = \int_{\mathcal{D}} \{f \circ \sigma - d(p \circ \sigma)\} \cdot \dot{u} \tag{1.46}$$

The requirement that, for any domain $\mathcal{D}$ with compact closure in $\mathcal{M}, \mathcal{S}$ be stationary with respect to arbitrary variations $\dot{u}$ vanishing in a neighborhood of $\partial\mathcal{D}$ then yields the *Euler-Lagrange equation*:

$$d(p \circ \sigma) = f \circ \sigma \tag{1.47}$$

It follows from 1.36 and 1.45 that the difference at $(x, Q) \in u^*T\mathcal{N}$ of the covariant derivatives of $p \circ \sigma$ defined by the connections $\tilde{A}$ and A is given by:

$$(\tilde{d}(p \circ \sigma) - d(p \circ \sigma)) \cdot Q = -B(Q, v) \wedge (p \circ \sigma) \tag{1.48}$$

on the other hand, according to 1.24, the difference of the corresponding canonical forces is also given by:

$$(\tilde{f} \circ \sigma - f \circ \sigma) \cdot Q = -B(Q, v) \wedge (p \circ \sigma) \tag{1.49}$$

We conclude that *the Euler-Lagrange equation is independent of the particular choice of a symmetric connection in $T\mathcal{N}$.*

1.2 The Hamiltonian Picture

We now consider the relationship between the *canonical velocity* v and the *canonical momentum* p. As we have seen, p is a section of the pullback bundle $\pi^*_{\mathcal{V},\mathcal{C}}\mathcal{P}$ where $\mathcal{P}$ is the bundle

$$\mathcal{P} = \wedge_{m-1,1}(\mathcal{M}, \mathcal{N}) \tag{2.1}$$

which we call *phase space*. According to our definition 1.11, p is the restriction of the differential of L to the fibers of $\mathcal{V}$ over $\mathcal{C}$. At each $(x,q) \in \mathcal{C}$ this relation constitutes a nonlinear mapping of

$$\pi_{\mathcal{V},\mathcal{C}}^{-1}(x,q) = \mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N})$$

into

$$\pi_{\mathcal{P},\mathcal{C}}^{-1}(x,q) = \mathcal{L}(\wedge^{m-1}(T_x\mathcal{M}), T_q^*\mathcal{N}) = \mathcal{L}(T_q\mathcal{N}, \wedge_{m-1}(T_x\mathcal{M}))$$

These two vector spaces are isomorphic. For, upon choosing a volume form ϵ on $\mathcal{M}$, to each $p \in \mathcal{L}(\wedge^{m-1}(T_x\mathcal{M}), T_q^*\mathcal{N})$ we can associate

$$p^* \in \mathcal{L}(T_x^*\mathcal{M}, T_q^*\mathcal{N}) = \mathcal{L}(T_q\mathcal{N}, T_x\mathcal{M}) = (\mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N}))^*$$

by:

$$\begin{aligned} p(X_1, \ldots, X_{m-1}) \cdot Q &= \epsilon(p^*(Q), X_1, \ldots, X_{m-1}) \\ &: \forall X_1, \ldots, X_{m-1} \in T_x\mathcal{M}, \forall Q \in T_q\mathcal{N} \end{aligned} \tag{2.2}$$

The linear mapping $p^* \mapsto p$ given by 2.2 is an isomorphism of $(\mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N}))^*$ onto $\mathcal{L}(\wedge^{m-1}(T_x\mathcal{M}), T_q^*\mathcal{N})$. Since $\mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N})$ is isomorphic with its dual, it follows that the vector spaces $\pi_{\mathcal{V},\mathcal{C}}^{-1}(x,q)$ and $\pi_{\mathcal{P},\mathcal{C}}^{-1}(x,q)$ are likewise isomorphic. In particular,

$$\dim \mathcal{P} = \dim \mathcal{V} = m + n + mn$$

We now introduce the *hypothesis* that for each $(x,q) \in \mathcal{C}$ the non-linear mapping of $\pi_{\mathcal{V},\mathcal{C}}^{-1}(x,q)$ into $\pi_{\mathcal{P},\mathcal{C}}^{-1}(x,q)$ defined by 1.11 is a diffeomorphism of $\pi^{-1}_{\mathcal{V},\mathcal{C}}(x,q)$ onto $\pi^{-1}_{\mathcal{P},\mathcal{C}}(x,q)$. This hypothesis allows us to pass from the Lagrangian to the Hamiltonian theory. Denoting by $\pi_{\mathcal{P},\mathcal{M}}$ the projection of $\mathcal{P}$ to $\mathcal{M}$, we define the *Hamiltonian* H to be the differentiable section of the pullback bundle $\pi^*_{\mathcal{P},\mathcal{M}} \wedge_m \mathcal{M}$ given by:

$$H(p) = v \wedge p - L(v) \tag{2.3}$$

with p and v related according to 1.11:

$$p = \frac{\partial L}{\partial v}(v) \tag{2.4}$$

We shall refer to 2.3, 2.4 as the *Legendre transformation*, for, in the case $m = 1$ it reduces to the Legendre transformation of classical mechanics.

The $\mathcal{C}$-vertical derivative of H at $p \in \mathcal{P}, \pi_{\mathcal{P},\mathcal{C}}(p) = (x,q)$ is the element $(\partial H/\partial p)(p)$ of

$$\mathcal{L}(\mathcal{L}(\wedge^{m-1}(T_x\mathcal{M}), T^*_q\mathcal{N}), \wedge_m(T_x\mathcal{M}))$$

defined by:

$$\begin{aligned} \left(\frac{\partial H}{\partial p}\right)(p)\cdot\dot{p} &= \lim_{t\to 0}\frac{1}{t}\{H(p+t\dot{p}) - H(p)\} \\ &: \quad \forall \dot{p} \in \mathcal{L}(\wedge^{m-1}(T_x\mathcal{M}), T^*_q\mathcal{N}) \end{aligned} \tag{2.5}$$

Now $\mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N})$ is canonically isomorphic to $\mathcal{L}(\mathcal{L}(\wedge^{m-1}(T_x\mathcal{M}), T^*_q\mathcal{N}), \wedge_m(T_x\mathcal{M}))$; the isomorphism j takes β to $j\cdot\beta$, where

$$\begin{aligned} (j\cdot\beta)\cdot\dot{p} &= \beta\wedge\dot{p} \\ &: \quad \forall \dot{p} \in \mathcal{L}(\wedge^{m-1}(T_x\mathcal{M}), T^*_q\mathcal{N}) \end{aligned} \tag{2.6}$$

(see 1.9). We can thus identify $(\partial H/\partial p)(p)$ with an element of $\mathcal{L}(T_x\mathcal{M}, T_q\mathcal{N})$. This element is in fact v. For, according to 2.3 we have:

$$\frac{\partial H}{\partial p} \cdot \dot{p} = \left(\frac{\partial v}{\partial p} \cdot \dot{p}\right) \wedge p + v \wedge \dot{p} - \frac{\partial L}{\partial p} \cdot \dot{p}$$

and by 2.4 and the isomorphism i of 1.8,

$$\frac{\partial L}{\partial p} \cdot \dot{p} = \frac{\partial L}{\partial v} \cdot \left(\frac{\partial v}{\partial p} \cdot \dot{p}\right) = \left(\frac{\partial v}{\partial p} \cdot \dot{p}\right) \wedge p$$

Hence:

$$\frac{\partial H}{\partial p} \cdot \dot{p} = v \wedge \dot{p} \tag{2.7}$$

which according to the isomorphism j of 2.6 means:

$$\frac{\partial H}{\partial p} = v \tag{2.8}$$

We note that in the Lagrangian picture v is an element of $\mathcal{V}$ and p is a section of the pullback bundle $\pi^*_{\mathcal{V},\mathcal{C}}\mathcal{P}$, while in the Hamiltonian picture p is an element of $\mathcal{P}$ and v is a section of the pullback bundle $\pi^*_{\mathcal{P},\mathcal{C}}\mathcal{V}$.

To a section of σ of $(\mathcal{V}, \pi_{\mathcal{V},\mathcal{M}})$ which is a solution of the Euler-Langrange equation 1.47 there corresponds, through the Legendre transformation 2.3, 2.4 a section τ of $(\mathcal{P}, \pi_{\mathcal{P},\mathcal{M}})$ which is a solution of the *canonical equations*. The first canonical equation corresponds to the condition that if

$$u = \pi_{\mathcal{V},\mathcal{N}} \circ \sigma = \pi_{\mathcal{P},\mathcal{M}} \circ \tau,$$

then $\sigma = du$. That is:

$$du = v \circ \tau \tag{2.9}$$

where now v is the section of the pullback bundle $\pi^*_{\mathcal{P},\mathcal{C}}\mathcal{V}$, defined by the Hamiltonian according to 2.8. The second equation corresponds to the Euler-Lagrange equation. Since $p \circ \sigma = \tau$, the left hand side of 1.47

is simply $d\tau$, the covariant derivative of the section τ of $(\mathcal{P}, \pi_{\mathcal{P},\mathcal{M}})$ with respect to a symmetric connection A in $T\mathcal{N}$ as defined by 1.45. To express the right hand side of 1.47 in terms of the Hamiltonian, we consider the $\mathcal{N}$-horizontal derivative of H.

Given a curve $\gamma : (-1,1) \to \mathcal{N}$ in $\mathcal{N}$ through $q, \gamma(0) = q$, with tangent vector Q at $q, \dot{\gamma}(0) = Q$, and given any $p \in \mathcal{P}$ such that $\pi_{\mathcal{P},\mathcal{N}}(p) = q$, we define a curve $\gamma^{\sharp p}_{\mathcal{P}} : (-1,1) \to \mathcal{P}$ in $\mathcal{P}$ through $p, \gamma^{\sharp p}_{\mathcal{P}}(0) = p$, the *horizontal lift* of γ to $\mathcal{P}$ through p, by requiring that $\pi_{\mathcal{P},\mathcal{N}} \circ \gamma^{\sharp p}_{\mathcal{P}} = \gamma$ while $\pi_{\mathcal{P},\mathcal{M}} \circ \gamma^{\sharp p}_{\mathcal{P}}$ maps $(-1,1)$ to the single point $x = \pi_{\mathcal{P},\mathcal{M}}(p) \in \mathcal{M}$, and for each $X_1, \ldots, X_{m-1} \in T_x\mathcal{M}$ the curve

$$\gamma^{\sharp p}_{\mathcal{P}} \cdot (X_1, \ldots, X_{m-1}) : t \in (-1,1) \mapsto \gamma^{\sharp p}_{\mathcal{P}}(t) \cdot (X_1, \ldots, X_{m-1}) \in T^*_{\gamma(t)}\mathcal{N}$$

coincides with $\gamma^{\sharp p \cdot (X_1, \ldots, X_{m-1})}_{T^*\mathcal{N}}$, the horizontal lift of γ to $T^*\mathcal{N}$ through

$$p \cdot (X_1, \ldots, X_{m-1}) \in T^*_q\mathcal{N}$$

We recall here that $\gamma^{\sharp\alpha}_{T^*\mathcal{N}}$, the horizontal lift of γ to $T^*\mathcal{N}$ through $\alpha \in T^*_q\mathcal{N}$ is defined by the condition that $\forall Q \in T_q\mathcal{N}$,

$$\gamma^{\sharp\alpha}_{T^*\mathcal{N}}(t) \cdot \gamma^{\sharp Q}_{T\mathcal{N}}(t) = \alpha \cdot Q \;:\; \forall t \in (-1,1)$$

where $\gamma^{\sharp Q}_{T\mathcal{N}}$ is the horizontal lift of γ to $T\mathcal{N}$ through $Q \in T_q\mathcal{N}$. If we denote by $Q^{\sharp p}_{\mathcal{P}}$ the tangent vector at $p \in \mathcal{P}$ of the curve $\gamma^{\sharp p}_{\mathcal{P}}$ in $\mathcal{P}$, $\dot{\gamma}^{\sharp p}_{\mathcal{P}}(0)$, we then have:

$$\begin{aligned} Q^{\sharp p}_{\mathcal{P}} \cdot (X_1, \ldots, X_{m-1}) &= Q^{\sharp p \cdot (X_1, \ldots, X_{m-1})}_{T^*\mathcal{N}} \\ &: \forall (X_1, \ldots, X_{m-1}) \in T_x\mathcal{M} \end{aligned} \tag{2.10}$$

where $Q^{\sharp\alpha}_{T^*\mathcal{N}} = \dot{\gamma}^{\sharp\alpha}_{T^*\mathcal{N}}(0)$ is the horizontal lift to $T^*\mathcal{N}$ through $\alpha \in T^*_q\mathcal{N}$ of the vector $Q = \dot{\gamma}(0) \in T_q\mathcal{N}$.

We define the $\mathcal{N}$-horizontal derivative of H at p relative to the connection A to be the element $(DH)(p)$ of $\mathcal{L}(T_q\mathcal{N}, \wedge_m(T_x\mathcal{M}))$, $(x,q) = \pi_{\mathcal{P},\mathcal{C}}(p)$, defined by:

$$(DH)(p) \cdot Q = \left(\frac{d}{dt} H(\gamma_{\mathcal{P}}^{\sharp p}(t))\right)_{t=0} : \forall Q \in T_q\mathcal{N} \tag{2.11}$$

where γ is any curve in $\mathcal{N}$ through q with tangent vector Q at $q, \dot{\gamma}(0) = Q$. Denoting by δH the $\mathcal{M}$-vertical differential of H, that is the restriction of the differential of H to vectors tangent to the fibers $\{\pi_{\mathcal{P},\mathcal{M}}^{-1}(x) : x \in \mathcal{M}\}$, we can write:

$$\left(\frac{d}{dt} H(\gamma_{\mathcal{P}}^{\sharp p}(t))\right)_{t=0} = \delta H \cdot Q_{\mathcal{P}}^{\sharp p} \tag{2.12}$$

Thus, DH is the section of the pullback bundle $\pi_{\mathcal{P},\mathcal{M}}^{*} \wedge_{m,1} (\mathcal{M}, \mathcal{N})$ given by:

$$(DH)(p) \cdot Q = \delta H \cdot Q_{\mathcal{P}}^{\sharp p} : \forall Q \in T_q\mathcal{N}, q = \pi_{\mathcal{P},\mathcal{N}}(p) \tag{2.13}$$

If $\tilde{A}$ is another symmetric connection in $T\mathcal{N}$, with the difference $\tilde{A} - A$ corresponding to a tensorfield B on $\mathcal{N}$, a section of $S_2^1\mathcal{N}$, then $Q_{T^*\mathcal{N}}^{\tilde{\sharp}\alpha} - Q_{T^*\mathcal{N}}^{\sharp\alpha}$, the difference of the horizontal lifts of $Q \in T_q\mathcal{N}$ to $T^*\mathcal{N}$ through $\alpha \in T_q^*\mathcal{N}$ defined by the two connections is given by:

$$Q_{T^*\mathcal{N}}^{\tilde{\sharp}\alpha} - Q_{T^*\mathcal{N}}^{\sharp\alpha} = \alpha \cdot B(Q, \cdot) \tag{2.14}$$

The difference of the corresponding horizontal lifts of Q to $\mathcal{P}$ through p is, therefore,

$$Q_{\mathcal{P}}^{\tilde{\sharp}p} - Q_{\mathcal{P}}^{\sharp p} = p \cdot B(Q, \cdot) \tag{2.15}$$

where

$$p \cdot B(Q, \cdot) \in \mathcal{L}(\wedge^{m-1}(T_x\mathcal{M}), T_q^*\mathcal{N})$$