Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds

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ANNALS OF MATHEMATICS STUDIES

PRINCETON UNIVERSITY PRESS

Annals of Mathematics Studies

Number 134

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PRINCETON UNIVERSITY PRESS

PRINCETON, NEW JERSEY

1994

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Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

Library of Congress Cataloging-in-Publication Data

Kauffman, Louis, H., 1945–
Temperley-Lieb recoupling theory and invariants of 3-manifolds / by Louis H. Kauffman and Sóstenes L. Lins.
p. cm. — (Annals of mathematics studies ; no. 134)
Includes bibliographical references and index.
ISBN 0-691-03641-1 ISBN 0-691-03640-3 (pbk.)
1. Knot theory. 2. Three-manifolds (Topology).
3. Invariants (Mathematics) I. Sóstenes L. Lins. II.Title.
III. Series.
QA612.2.K39 1994
514'.224—dc20 94-15512

The publisher would like to acknowledge the authors of this volume for providing the camera-ready copy from which this book was printed

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Chapter 1

Introduction

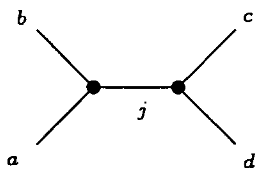
This monograph develops, in a self-contained manner, a recoupling theory for colored knots and links with trivalent graphical vertices. The theory is based on underlying properties of the bracket polynomial and the tangle-theoretic Temperley-Lieb algebra ([Kau87b],[Kau90b]). This recoupling theory is the direct analog, in this context, of the corresponding theory for q-deformed angular momentum recoupling using the quantum group $SL(2)_q$ (See [KR88].).

By working directly with the knot theory associated with the bracket polynomial we obtain a direct and combinatorial approach to this subject. In particular, we obtain a direct analysis of the recoupling theory at roots of unity that is useful for applications to topological invariants of 3-manifolds. In particular, we obtain an essentially axiomatic reconstruction of the invariant of Turaev and Viro (Chapter 10) and a corresponding treatment of the Witten-Reshetikhin-Turaev invariant in Chapter 12.

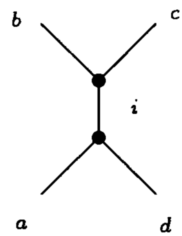
An important motivation for this work was the realization that the Temperley-Lieb context can be viewed as a q-deformed version of the spin-network theory of Roger Penrose [Pen69]. This point of view is not detailed here, but can be found in other papers by the first author ([Kau90c][Kau91] [Kau90a] [Kau92]).

In Chapter 8 we use a combination of the chromatic technique of Penrose and Mousourris [Mou79] and q analogs to obtain the basic formulas for recoupling coefficients needed in this theory. Further connections with spin networks and with mathematical physics will be given in a separate paper.

The book is organized as follows. Chapter 2 discusses the bracket polynomial model of the original Jones polynomial and its relation with the Temperley-Lieb algebra. Chapter 3 discusses a definition of Jones-Wenzl projectors in the Temperley-Lieb algebra as q-deformed symmetrizers. This construction is conceptually interesting, and it is computationally useful for the chromatic method explained in Chapter 8. Chapter 4 goes on to define the 3-vertex in terms of these projectors and ends with a discussion of examples and of n-fold cabled bracket invariants. Chapter 5 and Chapter 6 take up the computation of θ-networks at roots of unity and for generic values of q. ($\sqrt{q} = A$ where A is the variable in the bracket polynomial.) In the course of this discussion a number of basic formulas for network evaluation are derived. In Chapter 7 we do the recoupling theory both for generic q and at roots of unity. This can be summarized via formulas of the form

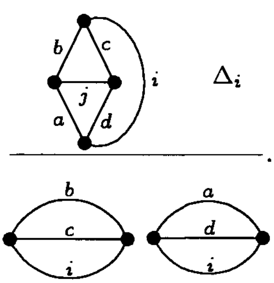


$$= \sum_i \begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix}$$



where the indices are subject to certain admissibility conditions, equality refers to global network evaluation and the symbol $\begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix}$ is a (generalized) $6j$ symbol receiving its definition from the above formula (see Chapter 7). Propositions 9 and 10 prove the orthogonality and the Biedenharn-Elliot identities for these $6j$ symbols. Proposition 11 gives our network explicit formula for $6j$:

$$\begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix} =$$



Analysis of the θ-nets in Chapter 6 guarantees that, for q-admissible labellings when q is a root of unity, these formulas are well defined and correspond to labels in the set $\{0, 1, 2, \ldots, r-2\}$ for $q = e^{i\pi/r}$. Chapter 8 contains the computations of the theta and tetrahedral coefficients via counting formulas in the classical case and recursions in the q-deformed case. Chapter 10 details our construction of a 3-manifold invariant by state summation, using this recoupling theory. This invariant is identical with the invariant of Turaev and Viro [TV92] (See [Piu92]), and our construction of it shows how the invariance follows from axiomatic features of the network recoupling. The state sum (Definition 7) that defines the invariant takes on a particularly transparent form when given in terms of a labelling of the decomposition of special spine for a 3-manifold M^3.

Chapter 9 contains a succinct summary of the facts of the recoupling theory. The reader may find it useful to go directly to it and beyond after reading Chapters 2,3 and 4. After discussing the Turaev-Viro invariant in Chapter 10, we turn to the Shadow World of Kirillov and Reshetikhin in Chapter 11. Here we show how to translate colored link invariants into partition functions of very similar type to the Turaev-Viro invariant. This is accomplished by a careful reformulation of the Kirillov-Reshetikhin work in terms of our recoupling theory. In Chapter 12 we construct the Witten-Reshetikhin-Turaev invariant, give a normalization for it at $A = e^{i\pi/2r}$ and then use the results of Chapter 11 to reformulate this invariant as a partition function on a 2-complex. The Chapter ends with a sketch of the proof of the basic Theorem of Turaev and Walker relating these invariants.

An appendix to Chapter 12 sketches the work of Justin Roberts on the Turaev-Viro invariant and its application to the Crane-Yetter invariant of 4-manifolds.

A *blink* is a finite plane graph with a bipartition on the edges. Blinks are in $1-1$ correspondence with projections of links and so, via the blackboard framing, they are very concise data structure which yield connected orientable 3-manifolds. This presentation of 3-manifolds is entirely adequate for the computation of the Witten-Reshetikhin-Turaev invariants, as we show in Chapters $12-14$. One disadvantage of the presentation by blinks is that the recognition of the 3-manifolds from them is rather difficult. This is due to the lack of a useful simplification theory. In Chapter 13 we provide a simple way to go from a blink to a particular kind of ball complex inducing the same 3-manifold. These ball complexes are special 4-valent edge colored graphs named *3-gems*. These objects do have a useful simplification theory which we survey in Chapter 13. In particular, 3-manifolds induced by 3-gems with less than 30 vertices have been topologically classified. With the simplification theory of 3-gems we recognize many of the 3-manifolds for which we give the Witten-Reshetikhin-Turaev invariants in Chapter 14. Various pairs of 3-manifolds in these tables are suspected to be homeomorphic from these invariants. They are proven to be so because they have the *same TS-code:* a canonical form computed in 3-gems (see Chapter 13). We stress the point that the implementation of the algorithm to get, as in Chapter 14, the quantum invariants for 3-manifolds from blinks is based solely on well known basic results on framed links and, otherwise, on results developed from scratch, in this monograph. Fully detailed recipes on how to reproduce these computations are discussed and crystallized.

It gives us pleasure to thank Vladimir Turaev, Ray Lickorish, Oleg Viro and Said Sidki for helpful conversations in the course of this research. The implementation of the TS_ρ-algorithm, from which we recognize various manifolds, in the tables of Chapter 14 is due to Cassiano Durand. We also thank Oscar Pereira S. Neto for the latex typing and elaboration of most of the hundreds of figures, some of them rather intricate.

The first author wishes to thank the National Science Foundation for support under NSF Grant Number DMS-9205277 and the Program for Mathematics and Molecular Biology at the University of California at Berkeley for partial support during the preparation of this research. He also thanks the Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan; the Isaac Newton Institute, Cambridge, England and the Universidade Federal de Pernambuco in Recife, Brazil for their kind hospitality during the course of this work.

The second author wants to acknowledge the partial support of FINEP and CNPq/Brazil (process number 30.1103/80). He also thanks the University of Illinois at Chigago and the Geometry Center at Minneapolis, Institutions where part of this work were produced.

Chapter 2

Bracket Polynomial, Temperley-Lieb Algebra

2.1 Bracket Polynomial

We start by recalling the definition and elementary properties of the (topological) bracket polynomial: to each crossing in an unoriented link diagram K there are associated two smoothings, labelled A and A^{-1} as shown in Figure 1.

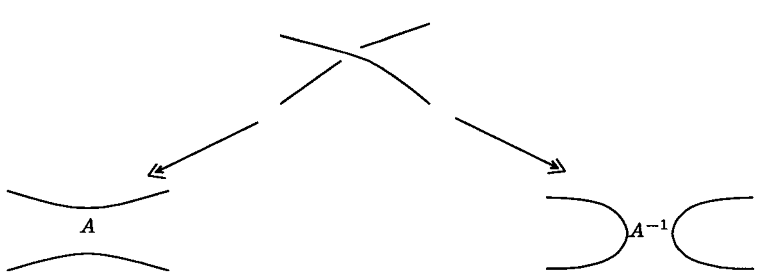
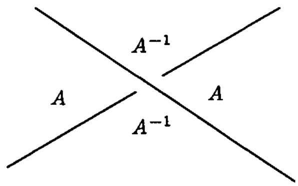


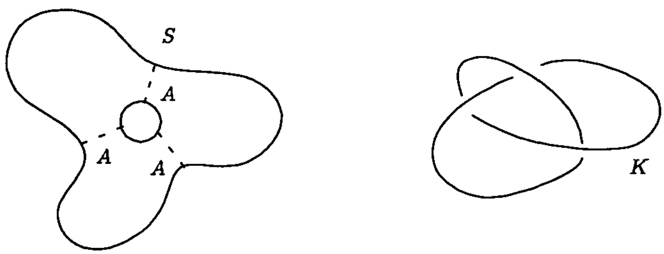
Figure 1

Labellings for these two types of smoothing are obtained by labelling the four regions at the crossing by A and A^{-1} so that the two regions swept out by turning the over-crossing line counter-clockwise are labelled A. This configuration is shown below.



The smoothing that joins the two A-labelled regions is the A-smoothing. The smoothing that joins the two A^{-1}-labelled regions is the A^{-1}-smoothing.

A *state S of the diagram K* is a choice of a smoothing for each crossing in K. Thus S appears as a disjoint set of Jordan curves in the plane that is decorated with labels at the site of each smoothing. For example, the diagram below is a state of the trefoil diagram K



Given a state S of a diagram K, let $||S||$ denote the number of disjoint Jordan curves in S, and let $< K|S >$ denote the product of the state labels (called vertex weights in the statistical mechanics context [Kau89]) of S. (In the example above, $||S|| = 2$ and $< K|S >= A^3$.)

Define the *bracket polynomial* $< K > \in \mathbb{Z}[A, A^{-1}]$ by the state summation formula,

$$< K >= \sum_S < K|S > d^{||S||}$$

where S runs all states of the diagram K, and $d = -A^2 - A^{-2}$. The following results are proved in [Kau85], [Kau87b], [Kau87a], (see also [Kau83].).

Theorem 1 *The bracket polynomial is an invariant of regular isotopy of link diagrams. If K is an oriented link diagram with $w(K)$ the writhe of K, then*

$$f_K(A) = (-A^3)^{-w(K)} < K > / < 0 >$$

is an invariant of ambient isotopy of link diagrams. The bracket polynomial has the following properties.

(i) 

where the small diagrams stand for parts of larger ones that differ only as indicated by them.

(ii) $< 0 \sqcup K >= d < K >$
where $0\sqcup$ denotes disjoint union of the diagram K with a Jordan curve in the plane, and $d = -A^2 - A^{-2}$.

(iii) If $V_K(t)$ is the original Jones polynomial [Jon86], then

$$V_K(t) = f_K(t^{-1/4}).$$

These results are all elementary consequences of the definition of the bracket polynomial. Recall, however the definitions of regular isotopy, ambient isotopy and writhe. Two link diagrams are said to be ambient isotopic if one can be obtained from the other by a sequence of Reidemeister moves of the type I, II, III (plus underlying graphical changes induced by homeomorphisms of the plane). Two link diagrams are regularly isotopic if one can be obtained from the other by a sequence of Reidemeister moves of type II and III. The Reidemeister moves are shown in Figure 2.

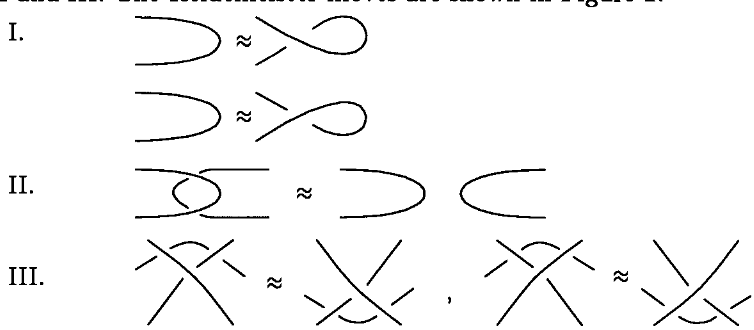
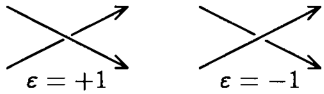


Figure 2

The *writhe*, $w(K)$, of an oriented link diagram K is the sum of the signs of the crossings in the diagrams, where these signs are given by the convention shown below.

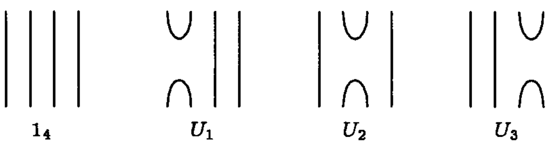


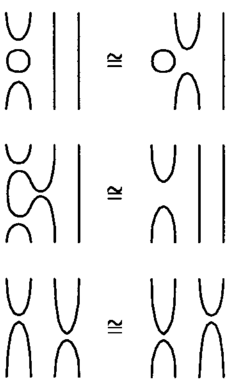
$$w(K) = \sum_p \varepsilon(p).$$

2.2 Temperley-Lieb Algebra

The Jones polynomial [Jon86] was originally defined via a representation of the Artin braid group [Art25] into the Temperley-Lieb algebra ([Jon87], [Jon83]). The bracket model for Jones polynomial allows a tangle-theoretic interpretation of the Temperley-Lieb algebra (see [Kau85], [Kau87b]) that is fundamental to the rest of the work in this paper.

In order to elucidate this version of Temperley-Lieb algebra, we define the *elementary tangles* $U_1, U_2, \ldots, U_{n-1} \in T_n$ where T_n denotes the n-strand Temperley-Lieb algebra. That is, each U_i is a tangle with n input strands and n output strands. In U_i, the k^{th} input is connected to the k^{th} output for $k \neq i, i+1$, while the i^{th} input is connected to the $(i+1)th$ input and the i^{th} output is connected to the $(i+1)th$ output. See Figure 3 for an illustration of T_4.





$U_i^2 = dU_i$

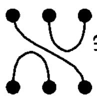
$U_i U_{i\pm1} U_i = U_i$

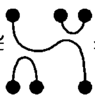
$U_i U_j = U_j U_i, \; |i-j| > 1$

Figure 3

Two tangles of the same number of strands are multiplied by attaching the output strands of the first tangle to the input strands of the second tangle. Two tangles are *equivalent* if they are regularly isotopic relative to their end points. This means that the regular isotopy of a given tangle is restricted to a box; the input and output strands emanate from this box; the regular isotopy leaves the intersections of the input and output strands with the boundary of the box fixed.

It is not hard to see that the equivalence class of a product of the $U_i's$ is determined entirely by the pattern of connections of inputs and outputs and that these products include all possible such connections that can be drawn in a planar box without self intersections. For example, in T_3,



$\cong$  $= U_2U_1.$

The $U_i's$ enjoy the following basic relations (Compare Figure 3):

(i) $U_i^2 = dU_i$ where d denotes a (commuting) value assigned to a closed loop.

(ii) $U_iU_{i\pm 1}U_i = U_i$

(iii) $U_iU_j = U_jU_i$ for $|i-j| > 1$.

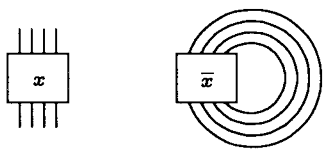
Call a tangle *planar-non-intersecting* if it can be represented in the plane with no intersections and no crossings. In [Kau90b] it is shown that every planar non-intersecting n-tangle (n inputs and n outputs) is equivalent to a product of $1_n, U_1, U_2, \ldots, U_{n-1}$ (1_n is the tangle that connects the i^{th} input with the i^{th} output) and that two such products represent equivalent tangles if and only if one product can be obtained from the other by the relations (i), (ii), (iii) above.

The *Temperley-Lieb algebra* T_n is the free additive algebra over $\tilde{\mathbb{Z}}[A, A^{-1}]$ with multiplicative generators $1_n, U_1, U_2, \ldots, U_{n-1}$ and relations (i), (ii) and (iii) as given above. We take $d = -A^2 - A^{-2}$, and it is given that A and A^{-1} commute with all elements of T_n. [Here $\tilde{\mathbb{Z}}[A, A^{-1}]$ denotes the set of rational functions P/Q with $P, Q \in \mathbb{Z}[A, A^{-1}]$].

Remark 1 *For our purposes it is convenient to parametrize the loop d as* $d = -A^2 - A^{-2}$. *Other treatments of Temperley-Lieb algebra use different loop parametrizations.*

It is useful to generalize the Temperley-Lieb algebra to an algebra T_n that is generated multiplicatively by the regular isotopy equivalence classes of all n-strand tangles. Thus a generator of T_n is any link diagram with $2n$ free ends such that n of them are designated as inputs and the remaining n of them are designated as outputs. The diagram is regarded as enclosed in a box from which emanate the inputs and the outputs. The *tangle algebra* T_n is the free additive algebra over $\tilde{\mathbb{Z}}[A, A^{-1}]$ generated multiplicatively by the n-strand tangles (n-tangles).

If x is an n-tangle, let $\bar{x}$ denote the *standard closure of* x obtained by attaching the k^{th} input to the k^{th} output as shown below.



Now define the *trace*

$$tr : T_r \longrightarrow \mathbb{Z}[A, A^{-1}]$$

by the formulas

(i) If x is an n-tangle then $tr(x) =< \bar{x} >$ where $< >$ denotes the bracket polynomial.

(ii) $tr(x + y) = tr(x) + tr(y)$.

Note that $tr(ab) = tr(ba)$ is an immediate consequence of the properties of the bracket polynomial and of the form of closure for the tangles.

Since the Temperley-Lieb algebra is a sub-algebra of the tangle algebra, this trace function restricts to a trace on T_n. In this case, if $x \in T_n$ is a product of the $U_i's$, then $\bar{x}$ is a disjoint union of Jordan curves in the plane, and $tr(x) =< \bar{x} >= d^{||\bar{x}||}$ where $||\bar{x}||$ denotes the number of Jordan curves. Each product of $U_i's$ corresponds to a single bracket state.

Remark 2 *This trace on the Temperley-Lieb algebra corresponds directly to the Jones trace* [Jon83]. *In* [Jon83] *this trace is defined in terms of normal forms for words in the algebra.*

Elements of the Artin braid group B_n are a special case of n-tangles. A *braid* $b \in B_n$ is a n-tangle that is regularly isotopic to a product of *elementary braids* $1_n, \sigma_1, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}$. The elementary braid $\sigma_i^{\pm}$ takes input i to output $i+1$ and input $i+1$ to output i. The braids σ_i and σ_i^{-1} have opposite crossing type, with σ_i such that the smoothing that joins input to input and output to output is A-type. (See Figure 4.) Call this the *horizontal smoothing* of σ_i and denote it by $H(\sigma_i)$. Similarly, let $V(\sigma_i)$ denote the input-output (vertical) smoothing of σ_i. Thus we have

$$H(\sigma_i^{\pm}) = U_i \ , \qquad V(\sigma_i^{\pm}) = 1_n.$$

Since a state (bracket state) of the closure $\bar{b}$ of a braid b is obtained by choosing a smoothing for each $\sigma_i^{\pm}$ in b, it follows that *each state of $\bar{b}$ corresponds to the strand closure of an element in the Temperley-Lieb algebra.*

For example, the states of $\bar{\sigma_1}$ in B_2 are $\overline{H(\sigma_1)} = \overline{U_1}$ and $\overline{V(\sigma_1)} = \overline{1_2}$. We have

$$< \overline{\sigma_1} >= Ad^{||\overline{U_1}||} + A^{-1}d^{||\overline{1_2}||} = Ad + A^{-1}d^2 \quad (d = -A^2 - A^{-2}).$$

This gives an algebraic algorithm for computing $< \bar{b} >$ for any braid b via a sum of trace evaluations of elements of the Temperley-Lieb algebra. In fact the method applies with slight generalization to any tangle, but in the case of the braid group there is an underlying representation $\rho : B_n \longrightarrow T_n$ of the braid group to the Temperley-Lieb algebra. This representation is determined by the formulas

$$\rho(\sigma_i) = AU_i + A^{-1}1_n$$

$$\rho(\sigma_i^{-1}) = A^{-1}U_i + A1_n.$$

It is easy to check [Kau87b],[Kau90b] that with loop value $d = -A^2 - A^{-2}$ that ρ is a representation is equivalent to the representation constructed by Jones [Jon83], and it follows from our definitions that

$$tr\rho(b) =< \bar{b} >,$$

giving the bracket as a trace on the representation of the braid group into the Temperley-Lieb algebra.

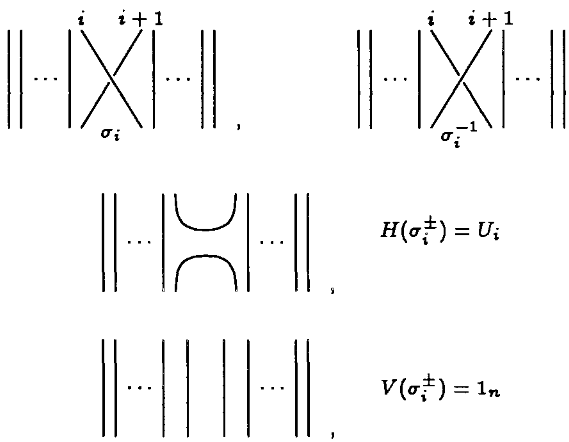


Figure 4

Chapter 3

Jones-Wenzl Projectors

The next construction is motivated by a number of considerations, ranging for the need for graph invariants to the representation theory of the Temperley-Lieb algebra and an analogue of the theory of angular momentum.

3.1 A Standard Projector in T_n

Definition 1 *Let $f_i \in T_n$ be defined inductively for $i = 0, 1, 2, \ldots, n-1$ by the following formulas:*

$$\begin{aligned} f_0 &= 1_n \\ f_{k+1} &= f_k - \mu_{k+1} f_k U_{k+1} f_k \end{aligned}$$

where $\mu_1 = d^{-1}, \mu_{k+1} = (d - \mu_k)^{-1}$. Here d is the loop value in T_n, $d = -A^2 - A^{-2}$, and $U_i^2 = dU_i$ for each i.

Lemma 1 *The elements $f_i \in T_n$ enjoy the following properties.*

(i) $f_i^2 = f_i$ *for* $i = 0, 1, \ldots, n-1$.

(ii) $f_i U_j = U_j f_i = 0$ *for* $j \geq i$.

(iii) $tr(f_{n-1}) = \Delta_n = \Delta_n(-A^2)$ *and* $\mu_{k+1} = \Delta_k / \Delta_{k+1}$ *with* $\Delta_0 = 1$.

where $\Delta_n(x) = \left(\dfrac{x^{n+1} - x^{-n-1}}{x - x^{-1}}\right)$ *is the* $n^{\underline{th}}$ *Chebyschev polynomial.*

Proof: See [Jon83], [Lic91], or [Kau91]. ■

Remark 3 *By adding one more input and one more output, and a strand between them, we have inclusions $T_1 \subset T_2 \subset T_3 \subset \cdots \subset T_n \subset T_{n+1} \subset \cdots$. In this sense, each $f_{i-1} \in T_i$ and we can take this as the usual placement of f_{i-1}. In this case, we can state that $tr(f_{i-1}) = \Delta_i$ (meaning the trace with respect to i-tangles).*

Proposition 1 *There is a unique non-zero element $f \in T_n$ such that*
(i) $f^2 = f$
(ii) $fU_i = U_i f = 0,\ i = 1, 2, \ldots, n-1$.

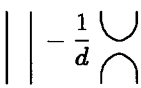
Proof: Lemma 1 asserts the existence of the element. Suppose that g also satisfies (i) and (ii). Since $g \in T_n, g$ is a linear combination of products of the elements $U_1, U_2, \ldots, U_{n-1}$ and 1_n. Since $g^2 = g, g = 1_n + U$ where $g\ U = 0$, U being a sum of such products, each including at least one U_i for some i. Similarly $f = 1 + U'$. Hence

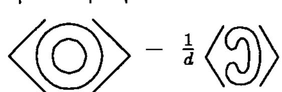
$$\begin{aligned} f &= f + fU = f(1+U) = fg \\ &= (1+U')g = g + U'g = g \end{aligned}$$

■

Example: In T_2,

$$\begin{aligned} f_1 &= f_0 - \mu_1 f_0 U_1 f_0 \\ &= 1_2 - d^{-1} U_1. \end{aligned}$$

$f_1 =$ 

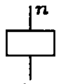
$Tr(f_1) =$ 

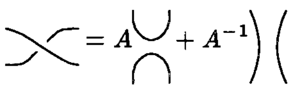
$$\begin{aligned} &= d^2 - (\tfrac{1}{d})d \\ &= d^2 - 1 \\ &= (-A^2 - A^{-2})^2 - 1 \\ &= A^4 + 1 + A^{-4}. \end{aligned}$$

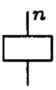
$$\begin{aligned} \Delta_2(x) &= (x^3 - x^{-3})/(x - x^{-1}) \\ &= x^2 + 1 + x^{-2} \\ \therefore \quad tr(f_1) &= \Delta_2(-A^2). \end{aligned}$$

Proposition 1 is useful because it allows us to verify other realizations of these projectors. In particular, we now can give a global formula for f_{n-1}.

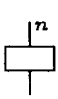
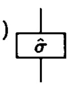
3.2 The Projectors as Sums of Tangles

For a given positive integer n, let  denote the sum (over rational functions in A and A^{-1} as coefficients) of n-tangles described below. Note that any n-tangle can be expended via the bracket identity



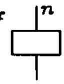
to (a sum of) elements in the Temperley-Lieb algebra. Thus, we can regard  as in T_n. The definition is:

Definition 2

 $= \frac{1}{\{n\}!} \sum_{\sigma \in S_n} (A^{-3})^{t(\sigma)}$ 

where $\{n\}! = \sum_{\sigma \in S_n} (A^{-4})^{t(\sigma)} = \Pi_{k=1}^{n} \left(\frac{1 - A^{-4k}}{1 - A^{-4}} \right)$. *Here* S_n *denotes the symmetric group on* n *letters, so that* $\sigma \in S_n$ *may be thought of as a permutation of* $1, 2, \ldots, n$, *and* $\hat{\sigma}$ *denotes the* n*-tangle obtained from any minimal representation of* σ *as a product of transpositions, so that each transposition is replaced by a braid in the form* σ_i *(note the positivity) for* $i = 1, 2, \ldots, n-1$. *For example, the permutation* $a = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ *corresponds to the braid* $\hat{\sigma} =$  .

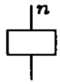
The integer $t(\sigma)$ *is the number of transpositions in the minimal representation of the permutation* σ. *(In the example above,* $t(\sigma) = 2$.*) The term* $\{n\}!$ *is a version of* q*-deformed factorial with* $\sqrt{q} = A$. *When* $A = 1$ *or* -1, $\{n\}!$ *is the usual factorial function.*

Proposition 2 *If $g_n \in T_n$ denotes the image of*  *in the Temperley-Lieb algebra T_n, then*

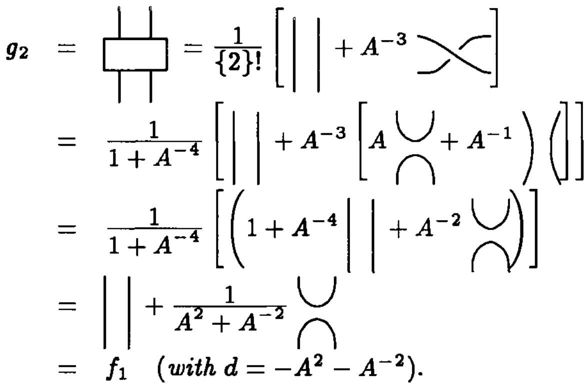
(i) $g_n^2 = g_n$

(ii) $g_n U_i = U_i g_n = 0$ *for* $i = 1, 2, \ldots, n-1$.

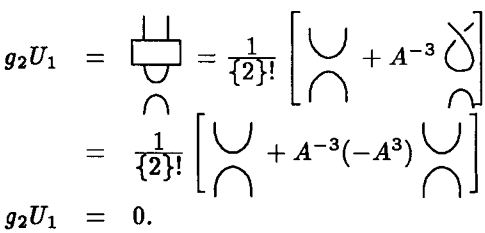
Therefore, by Proposition 1 we conclude that $f_{n-1} = g_n$ in T_n. That is,

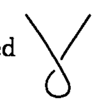
$f_{n-1} =$  .

Remark 4 *Before proving the Proposition 2, an example is in order. Here is the explicit expansion of g_2 in T_2.*

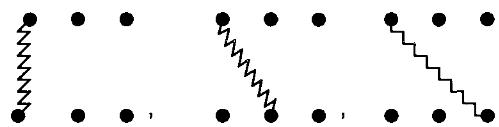


Note also that $gU_1 = 0$ via the following calculation

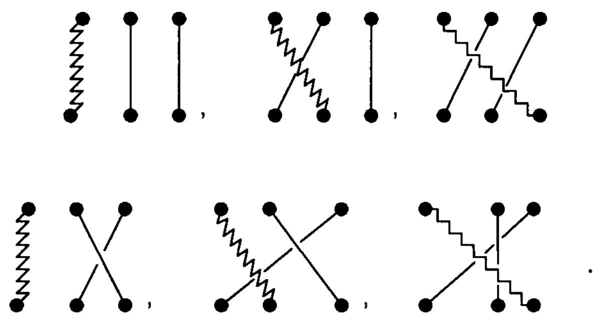


Here we used  $= (-A^3)$ , a fact about the bracket expansion.

As background for the proof of Proposition 2 it is useful to have a canonical inductive construction for the braids $\hat{\sigma}$ with $\sigma \in S_n$. To this end, note that by the convention of lifting to positive braids, the strand from input 1 to output i will *overcross* all the other strands that it meets. Hence the set $\{\hat{\sigma}|\sigma \in S_n\}$ is constructed from the set $\{\hat{\tau}|\tau \in S_{n-1}\}$ by inserting the braids $\hat{\tau}$ as inputs $2, 3, \ldots, n$ and outputs $1, 2, \ldots, i-1, i+1, \ldots, n$ with an overcrossing strand from input 1 to output i for each $i = 1, 2, \ldots, n$. For example, $\{\hat{\sigma}|\sigma \in S_3\}$ is constructed from $\{\hat{\tau}|\tau \in S_2\}$ by filling in the templates



as shown below.



Call this the *canonical inductive construction* of the braid set $\{\hat{\sigma}|\sigma \in S_n\}$.

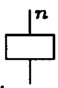
Proof: [of Proposition 2] It is easy to see from the canonical inductive construction of the braid set that $\{\hat{\sigma}|\sigma \in S_n\}$ can be written, for any given $i \in \{1, 2, \ldots, n-1\}$, as a disjoint union of a set of braids W and the set $W' = \{w\sigma_i | w \in W\}$. (Note that this entails the assumption that no braid word in W ends in σ_i or σ_i^{-1} due to minimality.) As a result, we have

$$\{n\}!g_n = \sum_{w \in W} (A^{-3})^{t(w)} w + (A^{-3})^{t(w)+1} w\sigma_i.$$

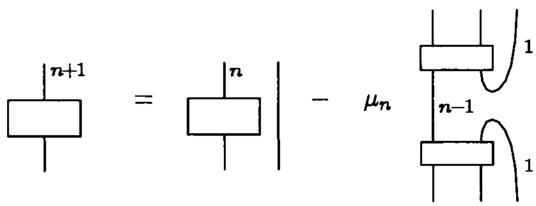
Since $w\sigma_i U_i = (-A^3) w U_i$ in T_n, it follows that $g_n U_i = 0$ for $i = 1, 2, \ldots, n-1$. This completes the proof of property (ii).

Next it is claimed that the coefficient of 1_n (the identify braid) in $\hat{g}_n = \sum_{\sigma \in S_n} (A^{-3})^{t(\sigma)} \hat{\sigma}$ is precisely $\{n\}!$, so that the coefficient of 1_n in

g_n is just equal to 1. To see this claim, note that for each $\hat{\sigma}$, its individual expansion in T_n contains one copy of 1_n. Hence each $\hat{\sigma}$ contributes $(A^{-3})^{t(\sigma)}(A^{-1})^{t(\sigma)}$ to the coefficient of 1_n. The factor $(A^{-1})^{t(\sigma)}$ is the product of vertex weights in the bracket expansion that yields the state 1_n. Hence the coefficient of 1_n in $\tilde{g}_n$ is the sum $\sum_{\sigma \in S_n} (A^{-4})^{t(\sigma)} = \{n\}!$, as claimed. Thus we know that g_n can be written in the form $g_n = 1_n + \mathcal{U}$ where $\mathcal{U}$ is a sum of products of $U_i^! s$. Therefore, $g_n^2 = g_n(1_n + \mathcal{U}) = g_n + g_u\mathcal{U} = g_n + 0 = g_n$ since $g_n U_i = 0$ for $i = 1, \ldots, n-1$. Thus $g_n^2 = g_n$, and the proof is complete. ■

The above result is useful since it shows that $\{n\}! f_{n-1}$ is an element of T_n all of whose coefficients are in $\mathbb{Z}[A, A^{-1}]$, hence all possibilities of poles in the coefficients of f_{n-1} are referred to questions about the zeroes of $\{n\}!$ (for specific complex values of A). It is also useful to use the tangle notation $f_{n-1} =$ 

. Thus the recursion relation of definition 1 becomes the diagrammatic equation

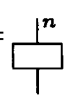


with

$$\mu_1 = 1/d \quad (d = -A^2 - A^{-2})$$
$$\mu_{k+1} = (d - \mu_k)^{-1}.$$

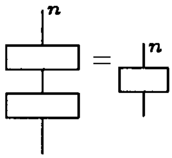
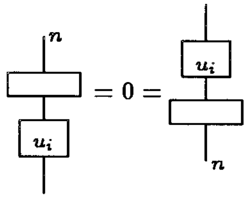
3.3 Diagrams and Structural Recursion

The above diagrammatic recursion will be of fundamental use in proving our results on recoupling theory at roots of unity. That it is true follows from the uniqueness given in Proposition 1. A direct proof from the tangle definition of g_n is also possible.

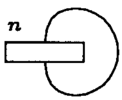
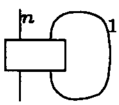
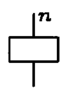
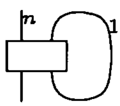
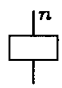
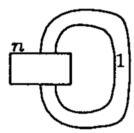
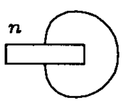
The diagrammatic notation makes it easy to see how this recursive relation is structurally determined. Assume that $g_n =$ 

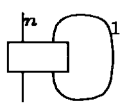
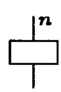
$\in T_n$ is given

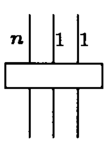
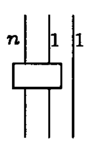
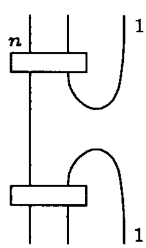
and that

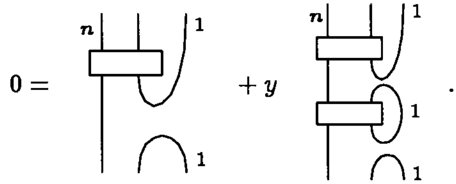
 $(g_n^2 = g_n)$ and 

for $1 \le i \le n-1$.

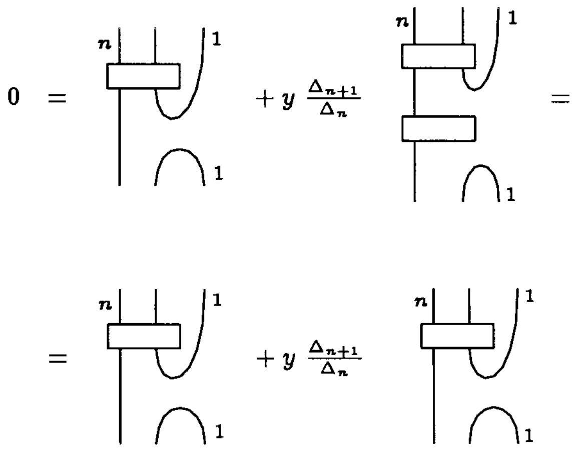
Let Δ_n =  denote the bracket evaluation of this closure. Then  must be a multiple of  by the argument of Proposition 1. But  $= x$  imples that  $=$ x , hence $\Delta_{n+1} = x\Delta_n$ and $x = \Delta_{n+1}/\Delta_n$:

 $= \dfrac{\Delta_{n+1}}{\Delta_n}$  .

Now, if  $=$  $+\, y$  it follows that (by composing with U_{n+1})



Hence



since it is easy to see that