TRANSCENDENTAL NUMBERS

By Carl Ludwig Siegel

ANNALS OF MATHEMATICS STUDIES

NUMBER 16

ANNALS OF MATHEMATICS STUDIES

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PRINCETON
PRINCETON UNIVERSITY PRESS
1949

LONDON: GEOFFREY CUMBERLEGE, OXFORD UNIVERSITY PRESS

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

This booklet reproduces with slight changes a course of lectures delivered in Princeton during the Spring term 1946. It would be misleading to call it a theory of transcendental numbers, our knowledge concerning transcendental numbers being narrowly restricted. The text deals with a few special transcendency problems of some interest, but it is more than a mere collection of scattered examples, since it involves a method which might be useful in the search of more general results.

Carl Ludwig Siegel.

April, 1949
Princeton, New Jersey.

CONTENTS

CHAPTER I

THE EXPONENTIAL FUNCTION

The most widely known result on transcendental numbers is the transcendency of π proved by Lindemann in 1882. His method is based on Hermite's previous work who discovered the transcendency of e in 1873. Both results are contained in the general Lindemann-Weierstrass theorem which will be proved in §12. We shall start with some simpler problems, namely the irrationality of e and π and related questions.

§1. The irrationality of e

The usual proof of the irrationality of e runs as follows. From the series

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

we get the decomposition

$$e = s_n + r_n, \quad s_n = \sum_{k=0}^{n} \frac{1}{k!}, \quad r_n = \sum_{k=n+1}^{\infty} \frac{1}{k!}$$

$$(n=1,2,\dots).$$

Since

$$r_n = \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots\right) < \frac{e-1}{(n+1)!},$$

we find that

$$e = s_1 + r_1 < 2 + \frac{e-1}{2}, \quad e < 3;$$

therefore

$$0 < r_n < \frac{2}{(n+1)!}.$$

Put

$$n!\, s_n = a_n, \qquad n!\, r_n = b_n,$$

then the number a_n is integral and

$$0 < b_n < \frac{2}{n+1} \leq 1$$

for $n = 1, 2, \ldots.$ This proves that $n!\, e = a_n + b_n$ and, a fortiori, $n\, e$ is never an integer. In other words, e is irrational.

The proof is still simpler, if we use the series for e^{-1} instead of e. Then

$$e^{-1} = \sigma_n + \rho_n, \; \sigma_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \quad \rho_n = \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}$$

$$(n=1, 2, \ldots)$$

and

$$0 < (-1)^{n+1} \rho_n = \frac{1}{(n+1)!} - \frac{1}{(n+2)!} + \cdots < \frac{1}{(n+1)!} \; .$$

Defining

$$n!\, \sigma_n = \alpha_n, \qquad n!\, \rho_n = \beta_n$$

we see that α_n is integral and

$$0 < (-1)^{n+1}\beta_n < \frac{1}{n+1} < 1.$$

Therefore $n!e^{-1} = \alpha_n + \beta_n$ and, a fortiori, $n.e^{-1}$ is never an integer.

We can prove a little more, namely that e is not the root of a quadratic equation $ax^2 + bx + c = 0$ with integral a, b, c, not all 0. Consider the expression

$$E_n = n!\ (ae + ce^{-1})$$

with integral a and c, not both 0. Then

$$E_n = S_n + R_n, \quad S_n = aa_n + c\alpha_n, \quad R_n = ab_n + c\beta_n,$$

where S_n is integral and the absolute value

$$|\ R_n\ | \leq |\ ab_n\ | + |c\beta_n| < \frac{2|a|+|c|}{n+1},$$

so that

$$|\ R_n\ | < 1$$

for all $n \geq 2|a|+|c|$. On the other hand we have the recursion formula

$$nR_{n-1} - R_n = a(nb_{n-1} - b_n) + c(n\beta_{n-1} - \beta_n)$$
$$= a + (-1)^n c.$$

It follows that at least one of the three numbers R_{n-1}, R_n, R_{n+1} is different from 0, since otherwise $a+c=0$, $a-c=0$ and $a=0$, $c=0$. This shows the existence of a positive integer ν such that E_ν is not integral, and therefore the number