TOPICS IN TOPOLOGY

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BY

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INTRODUCTION

The present monograph has been planned in such a way as to form a natural companion to the author's volume Algebraic Topology appearing at the same time in the Colloquium Series and hereafter referred to as AT. The topics dealt with have for common denominator the relations between polytopes and general topology. The first chapter takes up the relations between polytopes in general and the topologies which they may receive and in these questions we lean particularly heavily upon J. Tukey. The second chapter completes in certain important points the treatment of singular elements of AT. The third chapter deals with mappings of spaces on polytopes and certain related imbedding questions; it contains also a modern treatment of retraction for separable metric spaces. The last chapter is devoted to the group of questions centering around the general concept of local connectedness. Comparisons with retracts are considered at length, there is a full treatment of the homology and fixed point properties. The chapter concludes with an outline of the relations with "homology" local connectedness (the so-called HLC properties).

The general notations are those of AT. In addition to a short reference bibliography, a mere supplement to that of AT, there is also given a fairly comprehensive bibliography on locally connected spaces and retraction.

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Chapter I.

POLYTOPES

§1. AFFINE SIMPLEXES AND COMPLEXES

1. Affine Simplexes. In spite of the evident analogy with the treatment of Euclidean simplexes of (AT, III, VIII). it will be more convenient and also clearer to repeat the necessary introductory definitions and properties.

Our simplexes are considered here also as subsets of a real vector space $\mathfrak{B}$ whose elements are to be called points.

(1.1) DEFINITION. Let $\sigma^p = a_o \dots a_p$ be a p-simplex whose vertices are independent points of a real vector space $\mathfrak{B}$. By the affine p-simplex associated with σ^p is meant the set, written σ^p_v given by

(1.2) $$x = x^i a_i$$

(1.3) $$p = 0 : x^o = 1,$$

(1.4) $$p > 0 : 0 < x^i < 1, \quad \sum x^i = 1.$$

The x^i's are the barycentric coordinates of x. To the face $\sigma^q = a_{i_o} \dots a_{i_q}$ of σ^p there corresponds the set of points obtained by replacing $0 < x_{i_h}$ by $0 = x_{i_h}$ in (1.4); it is the σ^q_v associated with σ^q and is called a q-face of σ^p_v. We transfer to σ^p_v and to its faces the terminology previously adopted for σ^p. In particular we speak of the open or closed affine simplex, the boundary $\mathfrak{B}\sigma_v^p$ etc. The set of all points in an element of $\mathfrak{B}\sigma_v^p$ or of $\mathrm{Cl}\,\sigma_v^p$ is denoted by $|\mathfrak{B}\,\sigma_v^p|$ or $|\mathrm{Cl}\,\sigma_v^p|$.

(1.5) The open and the closed affine simplexes are convex.

Let $x', x'' \in \mathrm{Cl}\,\sigma_v^p$. The segment $l = \overline{x'\,x''}$ joining them consists of the points

(1.6) $x = t'x' + t''x'',\ 0 \leq t',\ t'' \leq 1, t' + t'' = 1.$

Hence if $x' = x'^i a_i,\ x'' = x''^i a_i$ we have

$$x = x^i a_i,\ y^i = t'\,x'^i + t''\,x''^i$$

and we verify readily that $x \in |\mathrm{Cl}\,\sigma_v^p|$. Similarly for σ_v^p.

(1.7) If $\sigma_v^p = \sigma_v'\sigma_v''$ (complementary faces) there passes through each point x a unique segment $\overline{x'x''}$ with $x' \in \sigma_\alpha',\ x'' \in \sigma_\alpha''$.

(Same proof as for (AT,VIII,2.1).

2.(2.1) DEFINITION. Let $S = \{\sigma_{vi}\}$, $S' = \{\sigma'_{vi}\}$ be two sets of affine simplexes, where the simplexes in each set are disjoint. We shall say that S' is a simplicial partition of S whenever each σ'_{vi} is in some σ_{vj} and each σ_{vj} is a union of a finite number of σ'_{vi}. Thus S' is a partition of S in the sense of (AT,IV,29).

(2.2) Let $S = \{\sigma_{vi}\}$ be a simplicial partition of $\mathfrak{B}\sigma_v^p$ and $\hat{\sigma}^p$ any point of $\bar{\sigma}_v^p$. Then: (a) if $\hat{\sigma}^p \subset \sigma_v^p$, $S' = \{\hat{\sigma}^p,\ \hat{\sigma}^p\sigma_{vi}\}$ is a simplicial partition of σ_v^p; (b) if $\hat{\sigma}^p \in \sigma_{vi}$, $S' = \{\hat{\sigma}^p\sigma_{vj} \mid j \neq i\}$ has the same property.

Since (2.2) is trivial for $p = 0$ we assume $p > 0$. Suppose first $\hat{\sigma}^p \subset \sigma_v^p$ and let $x \neq \hat{\sigma}^p$. By (1.5) the segment $\hat{\sigma}^p x$ extended meets $|\mathfrak{B}\sigma_v^p|$ in a point x' in some σ_{vi} and so $x \in \hat{\sigma}^p \sigma_{vi}$. Thus σ_v^p is the union of the elements of S'. Since $\hat{\sigma}^p$ is in no $\hat{\sigma}^p\sigma_{vi}$ we only have to prove the disjunction property for a pair $\hat{\sigma}^p\sigma_{vi}, \hat{\sigma}^p\sigma_{vh}$, $i \neq h$. Now if x is a point common to both, $\hat{\sigma}^p x$ extended will meet $\mathfrak{B}\sigma_v^p$ in a point common to σ_{vi}, σ_{vh} and this is ruled out since S is a simplicial partition of $\mathfrak{B}\sigma_v^p$. The treatment of (b) is essentially similar.

(2.3) Let $\{\sigma_{vi}\}$ be the set of all the proper faces of σ_v^p and $\hat{\sigma}_i, \hat{\sigma}^p$ points on σ_{vi}, σ_v^p. Then the affine simplexes

(2.4) $\zeta = \hat{\sigma}_i \ldots \hat{\sigma}_j \hat{\sigma}^p,\ \sigma_{vi} \prec \cdots \prec \sigma_{vj}$ make up a simplicial partition of σ_v^p.

This is trivial for $p = 0$ so we assume it for dimensions $< p$ and prove it for p. Under the hypothesis of the induction the collection of all the $\zeta_0' = \hat{\sigma}_i \ldots \hat{\sigma}_j$ $\sigma_{vi} \prec \cdots \prec \sigma_{vj}$ terminating with $\hat{\sigma}_j$ is a simplicial partition of σ_{vj}. Since the σ_{vj} are disjoint $\{\zeta'\}$ is a simplicial partition of $\mathfrak{B}\sigma_v^p$, so that (2.3) follows now from (2.2).

The decomposition of $(\mathrm{Cl}\,\sigma_v^p)$ by the simplexes (2.11) is its first derived $(\mathrm{Cl}\,\sigma_v^p)'$. Usually the <u>centroid</u> $(\frac{1}{p+1}, \ldots, \frac{1}{p+1})$ is chosen as $\hat{\sigma}^p$ and similarly for the faces. The corresponding $(\mathrm{Cl}\,\sigma_v^p)'$ is known as the <u>barycentric first derived</u>. We can treat similarly the simplexes of $(\mathrm{Cl}\,\sigma_v^p)'$, and obtain the successive derived or barycentric derived as the case may be. In general, unless otherwise stated, "derived" shall stand for "barycentric derived".

(24.5) The following designations will be found very convenient. The simplexes of $(\mathrm{Cl}\,\sigma_v^p)^{(n)}$ will be designated by σ_n (we omit the subscript v). Since the σ_n

make up a dissection of $\mathrm{Cl}\,\sigma_v^p$ every point x of the latter belongs to one and only one σ_n which will be denoted by $\sigma_n(x)$.

3. The vector space $\mathfrak{V}$ or its subspaces may be metrized in various ways. For our purpose it is sufficient to consider an <u>Euclidean metric</u> relative to a base $B = \{b_i\}$. If $x = x^i b_i$, $y = y^i b_i$ (finite sums) such a metric is defined by

$$(3.1) \qquad d(x,y) = \left(\sum (x^i - y^i)^2 \right)^{1/2}$$

and it has a meaning for all (x,y). The simplexes of $\mathfrak{V}$ are then Euclidean and may be written σ_e^p as in AT. The simplexes of the nth derived $(\mathrm{Cl}\sigma_e^p)^{(n)}$ have a maximum diameter: the <u>mesh</u> of the derived.

As a special case one may utilize the metric (3.1) attached to the subspace spanned by the vertices a_i of σ^p in $\mathfrak{V}$ relative to the base $\{a_i\}$ for the subspace. We thus obtain a metric for σ^p, and in fact for $|\mathrm{Cl}\sigma_v^p|$, given by (3.1) where x^i, y^i are now the barycentric coordinates of x, y. This particular metric will be called the <u>natural</u> metric of σ_v^p. Notice that if $\sigma_v^q \prec \sigma_v^p$, the induced metric in σ_v^q, is likewise its natural metric.

(3.1a) <u>Remark.</u> If no topology is specified for σ_v^p it will be understood that the set has been topologized by means of its natural metric. In point of fact the various topologies that may be specified in the sequel for σ_v^p will always be equivalent to the one induced by its general metric. This property is readily verified in each case and no further reference will be made to it later.

(3.2) The Euclidean p-simplex σ_e^p is a p-cell; its boundary $\mathfrak{B}\sigma_e^p$ is a $(p-1)$-sphere and $\bar{\sigma}_e^p$ is a p-dimensional parallelotope.

This is a consequence of (AT, I, 12.9) and the fact that $|\mathrm{Cl}\,\sigma_v^p|$ is a bounded convex subset of a Euclidean space $\mathfrak{E}^p$, the metrized subspace of the vertices.

(3.3) Let $\sigma_e^p = a_o \ldots a_p$, be a simplex in an Euclidean space $\mathfrak{E}^n$ and x any point of $\mathfrak{E}^n$. Then $d(x,y)$, $y \in \sigma_e^p$, does not exceed the maximum distance ρ from x to the vertices. (AT, VIII, 2.2).

(3.4) The diameter of σ_e^p is the length of its longest edge (AT, VIII, 2.3).

(3.5) Mesh $(\mathrm{Cl}\,\sigma_e^p)' \leqq \frac{p}{p+1}$ diam σ_e^p. (AT, VIII, 2.4).

(3.6) If σ_v^p is assigned the natural metric then mesh $(\mathrm{Cl}\,\sigma_v^p)^{(n)} \leqq \sqrt{2}\,(\frac{p}{p+1})^n$, which $\to 0$ as $n \to \infty$.

For in the natural metric the edges of σ_v^p all have the length $\sqrt{2}$ and so (3.6) is a consequence of (3.5).

(3.7) If x,x' are distinct points of σ_v^p there is an n such that the simplexes $\sigma_n(x)$, $\sigma_n(x')$ containing x,x' have no common vertices. (3.6).

(3.8) Let $\{\sigma_n\}$ be such that $\bar{\sigma}_{n+1} \subset \sigma_n$ (notations of 2.5). Then $\bigcap \bar{\sigma}_n = x$ a point of $\bar{\sigma}_v^p$.

In the natural metric $\bar{\sigma}_v^p$ is a compactum and $\{\bar{\sigma}_n\}$ a collection of closed subsets with the finite intersection property. Hence $\bigcap \bar{\sigma}_n \neq \phi$ and since diam $\bar{\sigma}_n \to 0$ the intersection is a point.

(3.9) Let $x \in \sigma_v^q \prec \sigma_v^p$. Then there exists an n such that $\sigma_n(x)$ has all its vertices in St σ_v^q (star in Cl σ_v^p).

For diam $\sigma_n(x) \longrightarrow 0$ and the distance from x to the set of simplexes not in St σ_v^q is positive.

4. Affine complexes. Just as for simplexes it is convenient as well as clearer to separate the affine and other complexes. The affine complex serves to specify the point-set which under suitable topologies becomes a geometric or an Euclidean complex.

(4.1) DEFINITION. Let $K = \{\sigma\}$ be a simplicial complex and let $\{A_i\}$ be its vertices where $\{i\}$ is any set whatever. Let $\{a_i\}$ be vectors of a real vector space with the following properties:

(4.2) $a_i \longleftrightarrow A_i$ is one-one;

(4.3) if $\sigma = A_i \dots A_j \in K$ then $a_i, \dots, a_j$ are independent, and so they are the vertices of an affine simplex denoted by σ_v;

(4.4) $\sigma \not> \sigma' \Longrightarrow \sigma_v \cap \sigma'_v = \phi$.

If we transfer to $\{\sigma_v\}$ the incidences "is a face of" prevailing in K, likewise the same incidence-numbers, it becomes a complex $K_v \cong K$, known as an affine simplicial complex. Its relation to K is also described by the statement: K_v is an affine realization of K. We also refer sometimes to K as an antecedent of K_v.

We transfer to K_v the full terminology attached to K.

Example. $\mathrm{Cl}\sigma_v^p$, $\mathfrak{B}\sigma_v^p$ are affine realizations of $\mathrm{Cl}\sigma^p$, $\mathfrak{B}\sigma^p$ and σ_v^p is an open subcomplex of $\mathrm{Cl}\,\sigma_v^p$.

The set of all the points of the simplexes of K_v is denoted by $|K_v|$.

It follows from the definition of K_v that every point $x \in |K_v|$ satisfies a relation

(4.5) $$x = x^i a_i$$

where if $x \in \sigma_v$ considered in (4.3), the coordinates $x^i, \dots, x^j$ are the barycentric coordinates of x in σ_v.

It follows that the x^i are unique and satisfy (1.3), (1.4). The x^i are known here also as the barycentric coordinates of x.

(4.6) Barycentric mapping. The definition is the same as for Euclidean complexes (AT,VIII,6.1) and need not be repeated.

(4.7) A noteworthy special case is when K_v, K_{1v} are both realizations of the same complex K. Let $\{A_i\}$, $\{a_i\}$, $\{a_i'\}$ be the vertices of K, K_v, K_{1v} where a_i, a_i' are the images of A_i. Then $a_i \longrightarrow a_i'$ is a one-one transformation which induces a one-one barycentric mapping τ, referred to as the natural barycentric mapping $K_v \longrightarrow K_{1v}$.

We notice the following properties:

(4.8) Every simplicial complex K has an affine realization K_v.

For if $\{A_i\}$ is chosen as a base for a real vector-space $\mathfrak{V}$ (its elements being all the finite forms t^iA_i with the t_i real) the three conditions (4.2), (4.3), (4.4) are naturally satisfied and so K_v may be constructed with $a_i = A_i$ throughout.

It is important to observe that this special choice of the a_i is not unique. Thus consider the two-complex K^2 consisting of a $\mathfrak{B}\sigma^3$ with one two-face removed. K^2 has the following affine realization: take a plane triangle ABC and let D be its centroid; K^2 consists of the triangles DAB, DBC, DCA with all their sides and vertices. This is a realization as a subset of a plane, whereas the above construction would require a four-space.

(4.9) Let $\hat{\sigma}$ be some point on $\sigma_v \in K_v$. Then:

(a) $\zeta = \hat{\sigma}_1 \ldots \hat{\sigma}_j, \sigma_{v1} \prec \cdots \prec \sigma_{vj}$
is an affine simplex and

$$\zeta \subset \sigma_{vj};$$

(b) $K_v' = \{\zeta\}$ is an affine realization of K', and is known as a first derived of K_v;

(c) $|K_v| = |K_v'|$.

This is an immediate consequence of (1.10) together with (AT,IV,26).

When the new vertices $\hat{\sigma}$ are the centroids of the corresponding σ, the affine complex K_v' is called the barycentric first derived. The definition of the nth derived, barycentric or otherwise is now obvious. It is written $K_v^{(n)}$ and is an affine realization of $K^{(n)}$ which coincides with K_v as a point set.

(4.10) Notations. Extending the notations introduced in (2.5) we designate by σ_n the simplexes of $K^{(n)}$ (also σ for σ_o) and by $\sigma_n(x)$ the $\sigma_n \ni x$.

The following property is needed later.

(4.11) Let $\hat{\sigma}$ be a point of σ_v and let K_v undergo the set-transformation S (in the sense of AT,IV,7): S is the identity outside of St σ_v; $S\sigma_v = \hat{\sigma}\mathfrak{B}\sigma_v$; if $\sigma_v' \in \text{St}\,\sigma_v - \sigma_v$; $S\sigma_v' = \hat{\sigma}(\mathfrak{B}\sigma_v' - \sigma_v)$. Then S is a simplicial partition of K_v into a new complex K_{1v}, and K_{1v} is a subdivision of K_v.

The partition property is an immediate consequence of (1.9). It is also clear that S fulfills the conditions of (AT,IV,24.8) and so it is a subdivision.

(4.12) Consider the function d(x,y) defined on K_v by the expression (3.1). If K_{1v} is the affine realization of (4.8) with $|A_1|$ as the base for the vector space $\mathfrak{V}$, then d(x,y) is a metric for $\mathfrak{V}$ and hence for K_{1v}. Since the natural barycentric transformation $K_{1v} \rightarrow K_v$ is one-one and preserves the barycentric co-