RADICALLY ELEMENTARY PROBABILITY THEORY

BY

EDWARD NELSON

ANNALS OF MATHEMATICS STUDIES
PRINCETON UNIVERSITY PRESS

Annals of Mathematics Studies

Number 117

RADICALLY ELEMENTARY PROBABILITY THEORY

BY

EDWARD NELSON

PRINCETON UNIVERSITY PRESS

PRINCETON, NEW JERSEY

1987

Library of Congress Cataloging-in-Publication Data

Nelson, Edward, 1932–
Radically elementary probability theory.
(Annals of mathematics studies ; no. 117)
Includes index.
1. Martingales. 2. Stochastic processes. 3. Probabilities.
I. Title. II. Series
QA274.5.N45 1987 519.2 87-3160

ISBN 0-691-08473-4
ISBN 0-691-08474-2(pbk.)

Princeton University Press books are printed on acid-free paper and meet the guidelines for permanence and durability of the Committee on Production Guidelines for Book Longevity of the Council on Library Resources

http://pup.princeton.edu

Printed in the United States of America

3 5 7 9 10 8 6 4

Table of contents

Preface

More than any other branch of mathematics, probability theory has developed in conjunction with its applications. This was true in the beginning, when Pascal and Fermat tackled the problem of finding a fair way to divide the stakes in a game of chance, and it continues to be true today, when the most exciting work in probability theory is being done by physicists working on statistical mechanics.

The foundations of probability theory were laid just over fifty years ago, by Kolmogorov. I am sure that many other probabilists teaching a beginning graduate course have also had the feeling that these measure-theoretic foundations serve more to salve our mathematical consciences than to provide an incisive tool for the scientist who wishes to apply probability theory.

This work is an attempt to lay new foundations for probability theory, using a tiny bit of nonstandard analysis. The mathematical background required is little more than that which is taught in high school, and it is my hope that it will make deep results from the modern theory of stochastic processes readily available to anyone who can add, multiply, and reason.

What makes this possible is the decision to leave the results in nonstandard form. Nonstandard analysts have a new way of thinking about mathematics, and if it is not translated back into conventional terms then it is seen to be remarkably elementary.

Mathematicians are quite rightly conservative and suspicious of new ideas. They will ask whether the results developed here are as powerful as the conventional results, and whether it is worth their while to learn nonstandard methods. These questions are addressed in an appendix, which assumes a much greater level of mathematical knowledge than does the main text. But I want to emphasize that the main text stands on its own.

Acknowledgments

I am grateful to Eric Carlen, Mai Gehrke, Klaus Kaiser, and Brian White for helpful comments, and to Pierre Cartier for a critical reading of the manuscript and the correction of many errors, together with helpful suggestions. This work was partially supported by the National Science Foundation. I thank Morgan Phillips for the six illustrations that LaTeX did not draw.