SINGULAR POINTS OF COMPLEX HYPERSURFACES

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TO MY MOTHER

Preface

The topology associated with a singular point of a complex curve has fascinated a number of geometers, ever since K. BRAUNER* showed in 1928 that each such singular point can be described in terms of an associated knotted curve in the 3-sphere. Recently E. BRIESKORN has brought new interest to the subject by discovering similar examples in higher dimensions, thus relating algebraic geometry to higher dimensional knot theory and the study of exotic spheres.

This manuscript will study singular points of complex hypersurfaces by introducing a fibration which is associated with each singular point.

As prerequisites the reader should have some knowledge of basic algebra and topology, as presented for example in LANG, *Algebra* or VAN DER WAERDEN, *Modern Algebra*, and in SPANIER, *Algebraic Topology*.

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* See the Bibliography. Proper names in capital letters will always indicate a reference to the Bibliography.

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§1. INTRODUCTION

Let $f(z_1, \ldots, z_{n+1})$ be a non-constant polynomial in $n + 1$ complex variables, and let V be the algebraic set consisting of all $(n+1)$-tuples

$$\mathbf{z} = (z_1, \ldots, z_{n+1})$$

of complex numbers with $f(\mathbf{z}) = 0$. (Such a set is called a *complex hypersurface.*) We want to study the topology of V in the neighborhood of some point $\mathbf{z}^0$.

We will use the following construction, due to BRAUNER. Intersect the hypersurface V with a small sphere S_ε centered at the given point $\mathbf{z}^0$. Then the topology of V within the disk bounded by S_ε is closely related to the topology of the set

$$K = V \cap S_\varepsilon \ .$$

(Compare §2.10 and §2.11.)

As an example, if $\mathbf{z}^0$ is a *regular point* of f (that is if some partial derivative $\partial f/\partial z_j$ does not vanish at $\mathbf{z}^0$) then V is a smooth manifold of real dimension $2n$ near $\mathbf{z}^0$. The intersection K is then a smooth $(2n-1)$-dimensional manifold, diffeomorphic to the $(2n-1)$-sphere, and K is embedded in an unknotted manner in the $(2n+1)$-sphere S_ε. (See §2.12.)

By way of contrast, consider the polynomial

$$f(z_1, z_2) = z_1^p + z_2^q$$

in two variables, with a *critical point* $(\partial f/\partial z_1 = \partial f/\partial z_2 = 0)$ at the origin. Assume that the integers p, q are relatively prime and ≥ 2.