Braids, Links and Mapping Class Groups

BY

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based on lecture notes by James Cannon

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To the memory of Ralph H. Fox

PREFACE

This manuscript is based upon lectures given at Princeton University during the fall semester of 1971-72. The central theme is Artin's braid group, and the many ways that the notion of a braid has proved to be important in low dimensional topology.

Chapter 1 is concerned with the concept of a braid as a group of motions of points in a manifold. Structural and algebraic properties of the braid groups of two manifolds are studied, and systems of defining relations are derived for the braid groups of the plane and sphere. Chapter 2 focuses on the connections between the classical braid group and the classical knot problem. This is an area of research which has not progressed rapidly, yet there seem to be many interesting questions. The basic results are reviewed, and we then go on to prove an important theorem which was announced by Markov in 1935 but never proved in detail. This is followed by a discussion of a much newer result, Garside's solution to the conjugacy problem in the braid group. The last section of Chapter 2 explores some of the possible implications of the Garside and Markov theorems.

In Chapter 3 we discuss matrix representations of the free group and of subgroups of the automorphism group of the free group. These ideas come to a focus in the difficult open question of whether Burau's matrix representation of the braid group is faithful. In Chapter 4, we give an overview of recent results on the connections between braid groups and mapping class groups of surfaces. Finally, in Chapter 5, we discuss briefly the theory of "plats." The Appendix contains a list of problems. All are of a research nature, many of unknown difficulty.

It will be assumed that the reader is familiar with the basic ideas of elementary homotopy theory, such as the notions of covering spaces and fiber spaces, and of exact sequences of homotopy groups of pairs of spaces (a good reference is Hu's "Homotopy Theory," 1959); also, with elementary concepts in infinite group theory, such as the Schreier-Reidemeister rewriting process (see, for example, Magnus, Karass and Soliar, "Combinatorial Group Theory," 1966). With this qualification, we have attempted to make the manuscript self-contained.

On the matter of notation: Theorems are labeled consecutively within each chapter, e.g., Theorem 3.2 means the second theorem in Chapter 3; Corollary 3.2.2 means the second corollary to Theorem 3.2; Lemma 3.2.1 means the first lemma used in the proof of Theorem 3.2. Equations are numbered consecutively within each chapter, e.g., equation (3-33) means the thirty-third equation of Chapter 3. A double bar $\|$ is used to signify the end of a proof.

The suggestion that the lecture notes be the basis for a monograph originated with Ralph H. Fox. His lively interest and continuing encouragement, and his willingness to share completely the wealth of his knowledge and experience, did much to make this manuscript a reality.

I am deeply indebted to Charles F. Miller III, whose careful reading of the manuscript and many questions, criticisms and suggestions helped to make it both more readable and more accurate. The monograph was also reviewed by José María Montesinos; there is no adequate way to thank him for the time and effort and expertise which he brought to the task. The original lecture notes were taken by James Cannon, and I wish to thank him for his interest in the topics presented, and for the large amount of time and energy which he expended in the preparation of the notes. However, any errors which exist are certainly mine, because the manuscript has undergone extensive revisions from the original notes. I would also like to thank K. Murasugi, for numerous discussions about the possibility of applying braid theory to knots, which helped to clarify for me many of my own ideas. I am also grateful to all who attended the lectures, for their interest, questions and insights.

Finally, my thanks to J. H. Roberts, for communicating his unpublished proof of Theorem 4.4; to an unknown seminar speaker at Princeton University, circa 1954, for his notes on Theorem 2.3; and to the National Science Foundation of the United States for partial support.

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CHAPTER 1
BRAID GROUPS

The central theme in this manuscript is the concept of a braid group, and the many ways that the notion of a braid has been important in low dimensional topology. In particular, we will be interested in the largely unexplored possibility of applying braid theory to the study of knots and links, and also to the study of surface mappings.

Our object in this first chapter will be to develop the main structural and algebraic properties of braid groups on manifolds. (Our braid groups will be limited to groups of motions of points; we will not treat generalizations to motions of a sub-manifold of dimension > 0 in a manifold.) In this setting the "classical" braid group B_n of Artin appears as the "full braid group of the Euclidean plane E^2."

Section 1.1 is concerned with definitions. The problem of how to properly define a braid, in order to capture the essential significant properties of "weaving patterns" and so study them mathematically, is a very basic one, for if the definition is too narrow the range of application will be severely limited, while if it is too broad there will not be an interesting theory. It is a tribute to Artin's extraordinary insight as a Mathematician that the definition he proposed in 1925 [see Artin, 1925] for equivalence of geometric braids could ultimately be broadened and generalized in many different directions without destroying the essential features of the theory. For a discussion of several such generalizations, see Section 1.1; for generalizations to higher dimensions see [D. Dahm, 1962] and [D. Goldsmith, 1972]; for other generalizations, see [Brieskorn and Saito, 1972; Arnold, 1968b; Gorin and Lin, 1969].

Section 1.2 contains a development of the main properties of "configuration spaces," introduced by E. Fadell and L. Neuwirth in 1962. Configuration spaces will be our tool for finding defining relations in the braid groups of surfaces. We chose this method because we felt that it gave particular geometric insight into the algebraic structure of the classical braid group as a sequence of semi-direct products of free groups. This same structure is exhibited by other methods in [Magnus 1934; Markoff 1945; Chow 1948].

In Section 1.3 we review the chief properties of braid groups on manifolds other than E^2 and S^2. Theorem 1.5 shows that braid groups of manifolds M of dimension $n > 2$ are really not of much interest, since they are finite extensions (by the full symmetric group) of the nth cartesian product of $\pi_1 M$. Theorems 1.6 and 1.7 are concerned with the relationships between Artin's classical braid group on E^2 and the braid groups of other closed 2-manifolds.

In Section 1.4 we study the braid group of E^2. In Theorem 1.8 we find generators and defining relations for the full braid group B_n of E^2. Corollaries 1.8.1 and 1.8.2 relate to the algebraic structure of B_n as a sequence of semi-direct products of free groups, and lead to solutions to the "word problem" in B_n. In Corollary 1.8.3 we establish that B_n has a faithful representation as a subgroup of the automorphism group of a free group. This subgroup is characterized in Theorem 1.9, by giving necessary and sufficient conditions for an automorphism of a free group of rank n to be in B_n. Corollary 1.8.4 identifies the center of B_n. Finally, in Theorem 1.10 we establish another interpretation of B_n as the group of topologically-induced automorphisms of the fundamental group of an n-punctured disc, where admissible maps are required to keep the boundary of the disc fixed pointwise.

Section 1.5 discusses the braid group of the sphere, which will play an important role later in this book, in relation to the theory of surface mappings. In Section 1.6 we give a list of references for further results on braid groups of closed 2-manifolds.

1.1. *Definitions*

We begin not with the classical braid group, but with a somewhat more general concept of a braid as a motion of points in a manifold. Our definition will be shown to reduce to the classical case when the manifold is taken to be the Euclidean plane.

Let M be a manifold of dimension ≥ 2, let $\prod_{i=1}^{n} M$ denote the n-fold product space, and let $F_{0,n}M$ denote the subspace

$$F_{0,n}M = \left\{ (z_1,\cdots,z_n) \in \prod_{i=1}^{n} M / \ z_i \neq z_j \ \text{ if } \ i \neq j \right\}.$$

(The meaning of the subscript "0" in the symbol $F_{0,n}$ will become clear later.) The fundamental group $\pi_1 F_{0,n}M$ of the space $F_{0,n}M$ is the *pure* (or *unpermuted*) *braid group* with n strings of the manifold M.

Two points z and z' of $F_{0,n}M$ are said to be equivalent if the coordinates $(z_1,\cdots,z_n)$ of z differ from the coordinates $(z'_1,\cdots,z'_n)$ of z' by a permutation. Let $B_{0,n}M$ denote the identification space of $F_{0,n}M$ under this equivalence relation. The fundamental group $\pi_1 B_{0,n}M$ of the space $B_{0,n}M$ is called the *full braid group* of M, or more simply, the *braid group* of M. Note that the natural projection $p : F_{0,n}M \to B_{0,n}M$ is a regular covering projection.

The classical braid group of Artin [cf. Artin 1925 and 1947a] is the braid group $\pi_1 B_{0,n}E^2$, where E^2 denotes the Euclidean plane. Artin's geometric definition of $\pi_1 B_{0,n}E^2$ can be recovered from the definition above as follows:

Choose a base point $z^0 = (z_1^0,\cdots,z_n^0) \in F_{0,n}E^2$ for $\pi_1 F_{0,n}E^2$ and a point $\tilde{z}^0 \in B_{0,n}E^2$ such that $p(z^0) = \tilde{z}^0$. Any element in $\pi_1 B_{0,n}E^2 = \pi_1(B_{0,n}E^2, \tilde{z}^0)$ is represented by a loop

$$\ell : I, \{0,1\} \to B_{0,n}E^2, \tilde{z}^0$$

which lifts uniquely to a path

$$\ell : I, \{0\} \to F_{0,n}E^2, z^0 .$$

If $\ell(t) = (\ell_1(t),\cdots,\ell_n(t))$, $t \in I$, then each of the coordinate functions ℓ_i defines (via its graph) an arc $\mathfrak{A}_i = (\ell_i(t), t)$ in $E^2 \times I$. Since $\ell(t) \in F_{0,n}E^2$ the arcs $\mathfrak{A}_1,\cdots,\mathfrak{A}_n$ are disjoint. Their union $\mathfrak{A} = \mathfrak{A}_1 \cup \cdots \cup \mathfrak{A}_n$ is called a *geometric braid* (see Figure 1). The arc $\mathfrak{A}_i$ is called the ith *braid string*.

A geometric braid is a representative of a path class in the fundamental group $\pi_1 B_{0,n}E^2$. Thus if $\mathfrak{A}$ and $\mathfrak{A}'$ are geometric braids, then $\mathfrak{A} \sim \mathfrak{A}'$ (that is, they represent the same element of $\pi_1 B_{0,n}$) if the paths ℓ and ℓ' which define these braids are homotopic relative to the base point $(z_1^0,\cdots,z_n^0)$ in the space $F_{0,n}E^2$. Thus we require the existence of a continuous mapping $\mathcal{F} : I \times I \to F_{0,n}E^2$ with

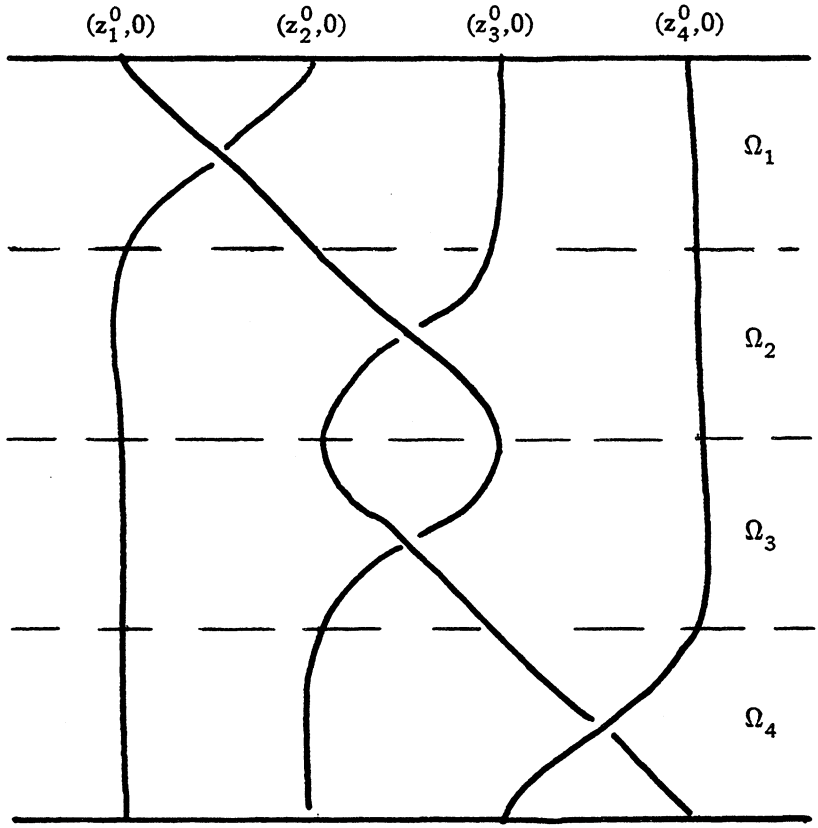


Fig. 1.

$$\mathcal{F}(t,0) = (\mathcal{F}_1(t,0),\cdots,\mathcal{F}_n(t,0)) = (\ell_1(t),\cdots,\ell_n(t))$$

$$\mathcal{F}(t,1) = (\mathcal{F}_1(t,1),\cdots,\mathcal{F}_n(t,1)) = (\ell'_1(t),\cdots,\ell'_n(t))$$

$$\mathcal{F}(0,s) = (\mathcal{F}_1(0,s),\cdots,\mathcal{F}_n(0,s)) = (z_1^0,\cdots,z_n^0)$$

$$\mathcal{F}(1,s) = (\mathcal{F}_1(1,s),\cdots,\mathcal{F}_n(1,s)) = (z_{\mu_1}^0,\cdots,z_{\mu_n}^0)$$

where $(\mu_1,\cdots,\mu_n)$ is a permutation of the array $(1,\cdots,n)$. The homotopy $\mathcal{F}$ defines a continuous sequence of geometric braids $\mathcal{A}(s) = \mathcal{A}_1(s)\cup\cdots\cup\mathcal{A}_n(s)$, $s \in I$, where $\mathcal{A}_i(s) = (\mathcal{F}_i(t,s),t)$, such that $\mathcal{A}(0) = \mathcal{A}$ and $\mathcal{A}(1) = \mathcal{A}'$. The reader is referred to Figures 2(b) and 2(c) for pictures of geometric braids which are equivalent to the "trivial" braid.

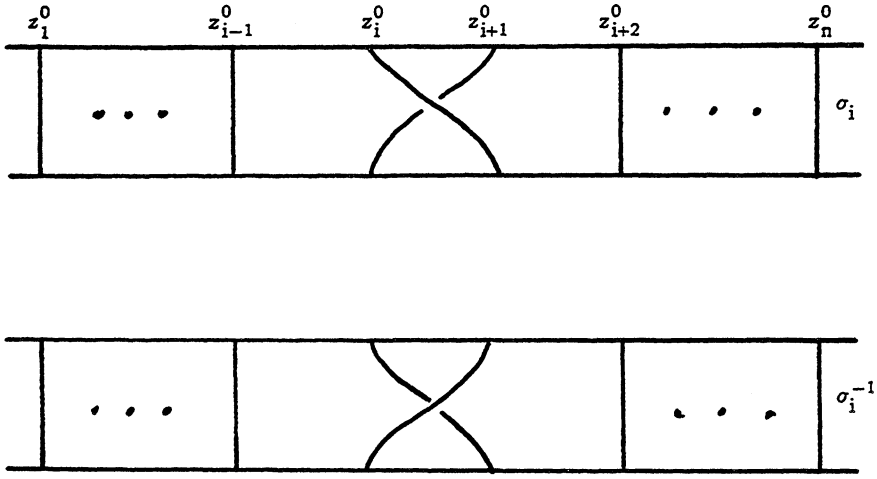
One may also define various stronger and weaker forms of equivalence between geometric braids, and we mention several of these briefly:

i). Let $\mathcal{A}$ and $\mathcal{A}'$ be geometric braids. Note that $\mathcal{A}$ and $\mathcal{A}'$ are subsets of $E^2 \times I$. Then, we write $\mathcal{A} \approx \mathcal{A}'$ if there is an isotopic deformation $\mathcal{G}_s$ of $E^2 \times I$ which is the identity on $E^2 \times \{0\}$ and on $E^2 \times \{1\}$ for each $s \in [0,1]$ and which has the property:

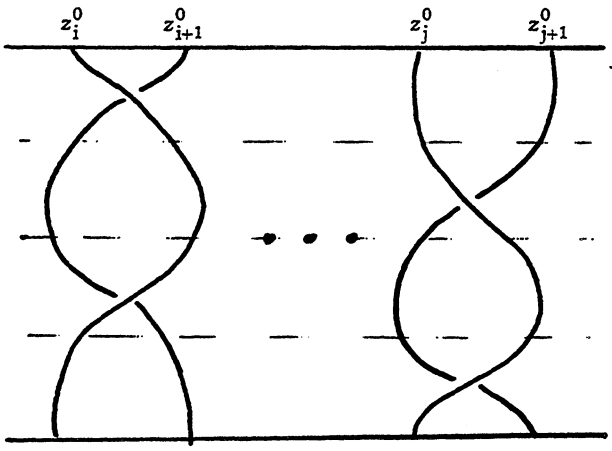
> *For each $s \in [0,1]$ the image set $\mathcal{A}(s)$ of $\mathcal{A}$ under $\mathcal{G}_s$ is a geometric braid, that is, $\mathcal{A}(s)$ meets each plane $E^2 \times \{t_0\}$, $t_0 \in I$, in precisely n points, and moreover $\mathcal{A}(0) = \mathcal{A}$, $\mathcal{A}(1) = \mathcal{A}'$.

It was proved by Artin [see 1947a] that $\mathcal{A} \approx \mathcal{A}'$ if and only if $\mathcal{A} \sim \mathcal{A}'$. Thus a braid homotopy may always be "extended" to $E^2 \times I$, in the sense defined above.

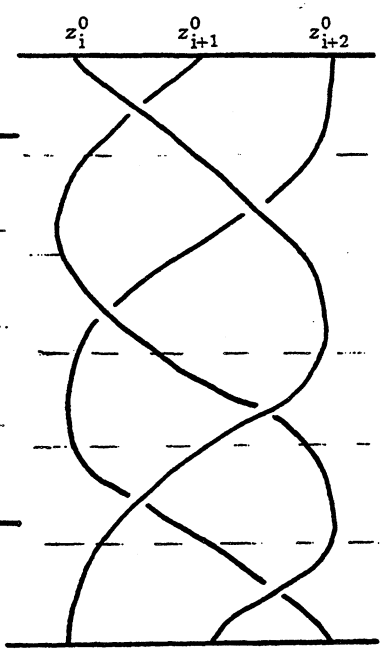
ii). If we think of our braid strings $\mathcal{A}_1,\cdots,\mathcal{A}_n$ as being made of elastic, one might imagine a more general type of equivalence in which the strings could be stretched or deformed in the region $E^2 \times I$ *without* requiring that $\mathcal{A}(s)$ meet each plane $E^2 \times \{t_0\}$, $t_0 \in I$, in precisely n points. In this situation, it might happen, for example, that some intermediate set $\mathcal{A}_1(s_0)\cup\cdots\cup\mathcal{A}_n(s_0)$ is as illustrated in Figure 3. (This intermediate set is not a geometric braid.) More precisely, under this



(a) Geometric braids representing σ_i and σ_i^{-1}



(b) $\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} \approx$ identity if $|i-j| > 2$



(c) $\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}$ $\approx$ identity

Fig. 2.

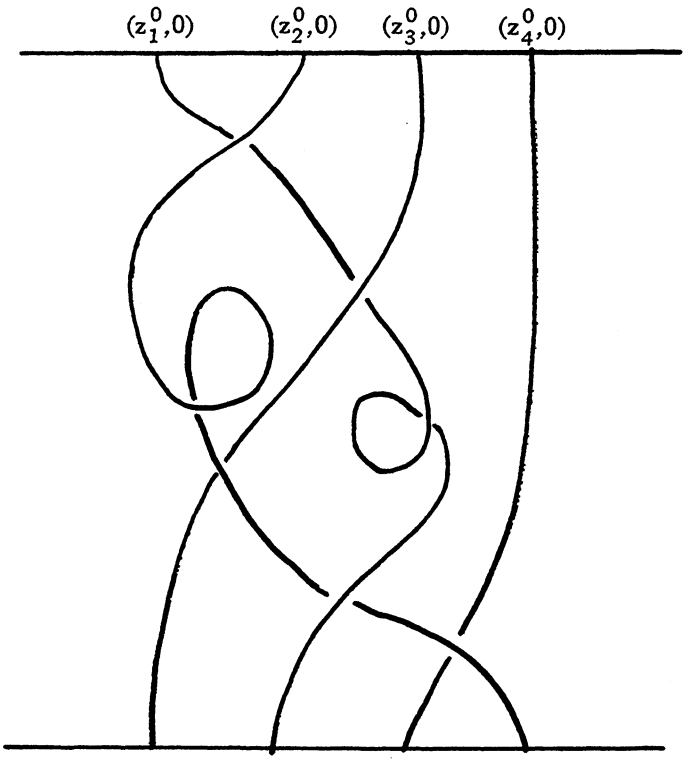


Fig. 3

more general notion, $\mathfrak{A} \approx \mathfrak{A}'$ if there is an isotopy $\mathcal{G}_s$ which is exactly like that defined in i) above except that $\mathcal{G}_s$ need not satisfy the property $*$. Again, Artin established [1947a] that $\mathfrak{A} \approx \mathfrak{A}'$ if and only if $\mathfrak{A} \sim \mathfrak{A}'$. (This is the first hint of a relationship between the concepts of equivalence of braids and equivalence of links, a relationship which will be studied in detail in Chapter 2.)

iii). D. Goldsmith [1974] has defined a concept of "homotopy" of braids by defining two geometric braids to be homotopic if one can be deformed to the other by simultaneous homotopies of the individual paths $(\varphi_i(t),t)$ in $E^2 \times I$, fixing the end points, and subject to the restriction that a string may intersect *itself*, but not any other string. Note that if $\mathfrak{A} \sim \mathfrak{A}'$ then $\mathfrak{A}$ and $\mathfrak{A}'$ are also equivalent under Goldsmith's rule, but

the converse need not be true. In fact, Goldsmith has exhibited non-trivial elements of the group $\pi_1 B_{0,3}E^2$ which are homotopic to the identity element of $\pi_1 B_{0,3}E^2$. She goes on to define a "homotopy braid group," $\hat{B}_n$, and finds a group presentation for $\hat{B}_n$ which exhibits $\hat{B}_n$ as a quotient group of the group $\pi_1 B_{0,n}E^2$. We note that Goldsmith's results were suggested by J. Milnor's work on homotopy of links and isotopy of links [see J. Milnor, 1954].

iv). The concept of a braid group has been generalized by D. Dahm [1962] and by D. Goldsmith [1972] to a group of motions of a submanifold in a manifold. We now give Goldsmith's definition of that group. Let N be a subspace contained in the interior of a manifold M. Denote by $\mathcal{H}(M)$ the group of autohomeomorphisms of M with the compact open topology, where if M has boundary ∂M, all homeomorphisms are required to fix ∂M pointwise. Denote the identity map of M by $1_M : M \to M$. A *motion of* N *in* M is a path ℓ_t in $\mathcal{H}(M)$ beginning at $\ell_0 = 1_M$ and ending at ℓ_1, where $\ell_1(N) = N$. The motion is said to be a *stationary motion of* N *in* M if $\ell_t(N) = N$ for all $t \in [0,1]$. To compose two motions, translate the second by multiplication in the group $\mathcal{H}(M)$ so that its initial point coincides with the endpoint of the first, and multiply as in the groupoid of paths. Define the inverse f^{-1} of a motion f to be the inverse of the path f in $\mathcal{H}(M)$, translated so that its initial point is 1_M.

Finally, let motions f and g be equivalent if the path $f^{-1}g$ is homotopic modulo its endpoint to a stationary motion. The *group of motions of* N *in* M is the set of equivalence classes of motions of N in M, with multiplication induced by composition of motions. From this point of view, the group of motions of an interior point in a manifold M is the group $\pi_1 M$, and the group of motions of n distinct points is the pure braid group of M (cf. Chapter 4 of this text, also Theorem 1.10).

Dahm [1962] studies the group of motions of n disjoint circles in S^3, and Goldsmith [1972, studies the group of motions of torus links in S^3.

1.2. *Configuration spaces*

PROPOSITION 1.1. *The natural projection map* $p : F_{0,n}M \to B_{0,n}M$ *is a regular covering space projection. The group of covering transformations is the full symmetric group* Σ_n *on* n *letters. Therefore there is a canonical isomorphism*

$$\pi_1 B_{0,n}M/\pi_1 F_{0,n}M \approx \Sigma_n . \tag{1-3}$$

The geometric interpretation of the product of two braids is immediate; suffice it to say that, in Figure 1, the configuration of arcs between any two consecutive dotted horizontal levels can be considered to be geometric braids $(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ of which the entire braid $\Omega = \Omega_1\Omega_2\Omega_3\Omega_4$ is the product.

Geometric intuition suggests that an arbitrary braid is equivalent to a braid that is a product of simple braids of the types illustrated in Figure 2. The equivalence classes of these elementary braids will be denoted by the symbols σ_i and σ_i^{-1}. In the example of Figure 1,

$$\Omega = \sigma_1 \sigma_2^2 \sigma_3^{-1} .$$

Geometric intuition thus suggests that $\sigma_1, \cdots, \sigma_{n-1}$ generate the group $\pi_1 B_{0,n}E^2$, a fact which will be proved later.

The following relations in $\pi_1 B_{0,n}E^2$ are obvious from Figure 2:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2,\ 1 \leq i, j \leq n-1 \tag{1-1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad 1 \leq i \leq n-2 . \tag{1-2}$$

It will be proved below that (1-1) and (1-2) comprise a set of defining relations in $\pi_1 B_{0,n}E^2$. Our proof, which allows us at the same time to compute defining relations for the braid groups of arbitrary 2-manifolds, will make use of the concept of the "configuration space" of a manifold. Other proofs (for the special case $M = E^2$) can be found in [Artin 1925; 1947a; Magnus 1934; Bohnenblust 1947; Fox and Neuwirth 1962].

Proof. Clear. ||

Since the map p is known explicitly, it follows from Proposition 1.1 that it is not difficult to analyze $\pi_1 B_{0,n}M$ once $\pi_1 F_{0,n}M$ is known. Therefore, the remainder of this section will be devoted to the group $\pi_1 F_{0,n}M$.

Let $Q_m = \{q_1, \cdots, q_m\}$ be a set of fixed distinguished points of M. Following Fadell and Neuwirth [1962] and Fadell and Van Buskirk [1962] we define the *configuration space* $F_{m,n}M$ of M to be the space $F_{0,n}(M - Q_m)$. Note that the topological type of $F_{m,n}M$ does not depend on the choice of the particular points Q_m, since one may always find an isotopy of M which deforms any one such point set Q_m into any other Q'_m. Note that $F_{m,1}M = M - Q_m$. (One may, similarly, define spaces $B_{m,n}M = B_{0,n}(M - Q_m)$, however we will only be interested in $B_{0,n}M$.)

We are interested in the relationship between the configuration spaces $F_{n,m}M$ and $F_{0,n}M$. The key observation is the following theorem:

THEOREM 1.2 [Fadell and Neuwirth, 1962]. *Let* $\pi : F_{m,n}M \to F_{m,r}M$ *be defined by*

$$\pi(z_1, \cdots, z_n) = (z_1, \cdots, z_r), \quad 1 \le r < n . \tag{1-4}$$

Then π *exhibits* $F_{m,n}M$ *as a locally trivial fibre space over the base space* $F_{m,r}M$, *with fibre* $F_{m+r,n-r}M$.

Proof. First consider, for some base point $(z_1^0, \cdots, z_r^0)$ in $F_{m,r}M$, the fibre $\pi^{-1}(z_1^0, \cdots, z_r^0)$:

$$\pi^{-1}(z_1^0, \cdots, z_r^0) = \{(z_1^0, \cdots, z_r^0, y_{r+1}, \cdots, y_n), \text{ where } z_1^0, \cdots, z_r^0, y_{r+1}, \cdots, y_n \text{ are distinct and in } M - Q_m\}.$$

If we select Q_{m+r} equal to $Q_m \cup \{z_1^0, \cdots, z_r^0\}$, then

$$F_{m+r,n-r}M = \{(y_{r+1}, \cdots, y_n), \text{ where } y_{r+1}, \cdots, y_n \text{ are distinct and in } M - Q_{m+r}\},$$

and there is an obvious homeomorphism

$$h : F_{m+r,n-r}M \to \pi^{-1}(z_1^0,\cdots,z_r^0)$$

defined by

$$h(y_{r+1},\cdots,y_n) = (z_1^0,\cdots,z_r^0, y_{r+1},\cdots,y_n) .$$

The proof of the local triviality of π will be carried out, for notational and descriptive convenience, only in the case of $r = 1$. The other cases will be left to the reader as exercises. Fix for consideration, therefore, a point $x_0 \in M - Q_m = F_{m,1}M = F_{m,r}M$. Add another point q_{m+1} to the set Q_m to form Q_{m+1} and pick a homeomorphism $\alpha : M \to M$, fixed on Q_m, such that $\alpha(q_{m+1}) = x_0$. Let U denote a neighborhood of x_0 in $M - Q_m$ which is homeomorphic to an open ball, and let $\bar{U}$ denote the closure of U. Define a map $\theta : U \times \bar{U} \to \bar{U}$ with the following properties. Setting $\theta_z(y) = \theta(z,y)$ we require:

(i) $\theta_z : \bar{U} \to \bar{U}$ is a homeomorphism which fixes $\partial\bar{U}$.

(ii) $\theta_z(z) = x_0$.

By (i), θ can be extended to $\theta : U \times M \to M$ by defining $\theta(z,y) = y$ for $y \notin U$. The required local product representation

$$U \times F_{m+1,n-1}M \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} \pi^{-1}(U)$$

is given by

$$\phi(z,z_2,\cdots,z_n) = (z,\theta_z^{-1}\alpha(z_2),\cdots,\theta_z^{-1}\alpha(z_n)) .$$

$$\phi^{-1}(z,z_2,\cdots,z_n) = (z,\alpha^{-1}\theta_z(z_2),\cdots,\alpha^{-1}\theta_z(z_n)) . \|$$

Two important consequences of Theorem 1.2 now follow:

PROPOSITION 1.3. *If* $\pi_2(M - Q_m) = \pi_3(M - Q_m) = 0$ *for each* $m \geq 0$, *then* $\pi_2 F_{0,n}M = 0$.

Proof. The exact homotopy sequence of the fibration $\pi : F_{m,n}M \to F_{m,1}M = M - Q_m$ of Theorem 1.2 gives an exact sequence

$$\cdots \to \pi_3(M-Q_m) \to \pi_2 F_{m+1,n-1}M \to \pi_2 F_{m,n}M \to \pi_2(M-Q_m) \to \cdots .$$

Since $\pi_2(M-Q_m) = \pi_3(M-Q_m) = 0$, it follows that $\pi_2 F_{m+1,n-1}M$ and $\pi_2 F_{m,n}M$ are isomorphic. An inductive argument shows that

$$\pi_2 F_{0,n}M \approx \pi_2 F_{n-1,1}M = \pi_2(M-Q_{n-1}) = 0 . \tag{1-5}$$

This completes the proof. ||

Let π be the projection map from $F_{0,n}M$ to $F_{0,n-1}M$ defined by (1-4). Let $(z_1^0, \cdots, z_n^0)$ be base point for $\pi_1 F_{0,n}M$. Let $F_{n-1,1}M = M - Q_{n-1} = M - \{z_1^0, \cdots, z_{n-1}^0\}$. Let j be the inclusion map from $F_{n-1,1}M$ to $F_{0,n}M$, defined by

$$j(z_n) = (z_1^0, \cdots, z_{n-1}^0, z_n) \qquad z_n \in M - \{z_1^0, \cdots, z_{n-1}^0\} . \tag{1-6}$$

THEOREM 1.4. *If* $\pi_2(M-Q_m) = \pi_3(M-Q_m) = \pi_0(M-Q_m) = 1$ *for every* $m \geq 0$, *then the following sequence of groups and homomorphism is exact*:

$$1 \longrightarrow \pi_1(F_{n-1,1}M, z^0) \xrightarrow{j_*} \pi_1(F_{0,n}M, (z_1^0, \cdots, z_n^0)) \xrightarrow{\pi_*} \pi_1(F_{0,n-1}M, (z_1^0, \cdots, z_{n-1}^0)) \longrightarrow 1 \tag{1-7}$$

where π_* *and* j_* *are the homomorphism induced by the mappings* π *and* j.

Proof. The sequence (1-7) is part of the exact homotopy sequence of the fibration of Theorem 1.2. The identity terms reflect the equalities $\pi_2 F_{0,n-1} = 1$, established in Proposition 1.3, and $\pi_0 F_{n-1,1}M = \pi_0(M-Q_{n-1}) = 1$. ||