J.J. SAKURAI

Invariance Principles and Elementary Particles

PRINCETON LEGACY LIBRARY

INVARIANCE PRINCIPLES AND ELEMENTARY PARTICLES

INVESTIGATIONS IN PHYSICS

Edited by EUGENE P. WIGNER and ROBERT HOFSTADTER

1. Ferroelectricity by E. T. JAYNES
2. Mathematical Foundations of Quantum Mechanics by JOHN VON NEUMANN
3. Shell Theory of the Nucleus by EUGENE FEENBERG
4. Angular Momentum in Quantum Mechanics by A. R. EDMONDS
5. The Neutrino by JAMES S. ALLEN
6. Photonuclear Reactions by L. KATZ, R. SKINNER, and A. S. PENFOLD (to be published)
7. Radiation Damage in Solids by D. BILLINGTON and J. CRAWFORD
8. Nuclear Structure by LEONARD EISENBUD and EUGENE P. WIGNER
9. The Physics of Elementary Particles by J. D. JACKSON
10. Invariance Principles and Elementary Particles by J. J. SAKURAI

Invariance Principles and Elementary Particles

BY

J. J. SAKURAI

PRINCETON UNIVERSITY PRESS
PRINCETON, NEW JERSEY
1964

L. C. Card 62-11964

Printed in the United States of America

Nature seems to take advantage of the simple mathematical representations of the symmetry laws. When one pauses to consider the elegance and the beautiful perfection of the mathematical reasoning involved and contrast it with the complex and far-reaching physical consequences, a deep sense of respect for the power of the symmetry laws never fails to develop.

FROM C. N. YANG's NOBEL LECTURE

PREFACE

The purpose of the present monograph is to discuss various elementary particle phenomena that can be understood from a few general principles based on invariance or symmetry considerations. I have tried to strike a reasonable balance among fundamental concepts, applications to well-known problems, and implications to problems in the twilight zone. A purely formal and axiomatic approach is avoided as much as possible, and emphasis is placed on physical and empirical aspects of the various invariance principles. Since elementary particle physics today is in a confused state, it is not surprising that certain parts of the monograph seem somewhat disorganized; it is written in such a way that there will still be room for new ideas.

The monograph is based on a graduate course given at the University of Chicago in the Spring quarter of 1959 (needless to say, numerous developments since that time in this rapidly growing field have been incorporated). The students in the original course were assumed to be familiar with *elementary* field theory including the quantization of free fields and covariant calculational techniques. However, the monograph is written in such a way that the reader who is unfamiliar with field theory may gain a great deal even if he skips certain difficult sections that require a firm understanding of field theory. For instance, it may be read profitably even if all but the earlier sections of Chapter 5 and the entire Chapter 6 are omitted. I should like to emphasize that many symmetry arguments can be understood on the phenomenological level; it is unfortunate that a number of experimentalists as well as theoreticians hold the erroneous view that one must know field theory to understand simple symmetry arguments.

In general, I have tried to make this work readable for people with different backgrounds—serious graduate students of elementary particle theory, low energy physicists who want to become acquainted with high energy physics, active experimentalists in particle physics who wish to gain insight into theoretical problems, etc. In my past contacts with various people, I have found that certain concepts in elementary particle physics are tacitly assumed in the current journals by the "experts," but are not adequately treated in existing textbooks. It is hoped that the present monograph will serve to fill in this gap which is becoming increasingly serious as it widens.

Sections of the monograph are based on lecture notes taken by Frank Chilton and P. Schlein. The help of S. G. Eckstein and S. F. Tuan is also appreciated. I would like to take this opportunity to emphasize that, without their untiring work and numerous suggestions, the present monograph would never have materialized.

J. J. Sakurai

TABLE OF CONTENTS

INVARIANCE PRINCIPLES
AND ELEMENTARY PARTICLES

CHAPTER 1

Introduction

The fundamental interactions in nature can be classified in historical order into the following four groups:

(1) Gravitational interactions
(2) Electromagnetic interactions
(3) Weak interactions
(4) Strong interactions

The investigations of the gravitational interactions were initiated in 1666 by Newton, then a twenty-four-year-old student. Coulomb's pioneer work on the electromagnetic interactions appeared in 1776. Although there are numerous phenomena that can be understood as manifestations of the first two classes of interactions—the motions of planets and artificial satellites, the propagation of radio waves, molecular forces, atomic spectra, superconductivity, etc.—it is an empirically well established fact, though not clearly recognized until the 1930's, that there are forces in nature that are neither gravitational nor electromagnetic. Shortly after Chadwick's discovery of the neutron, Fermi (1934) wrote down a beta decay Hamiltonian, which, with slight modifications, is still believed to be the correct weak interaction Hamiltonian in the low energy limit. At about the same time, in order to explain nuclear forces, Yukawa (1935) spelled out what has since been recognized as the starting point of the strong interactions.

The various classes of fundamental interactions are characterized by coupling constants that differ in many orders of magnitude. The electromagnetic coupling is characterized by the well-known dimensionless constant 1/137. The analogous dimensionless constant that characterizes the gravitational repulsion between two protons is $G^2_{\text{grav.}} M_p^2/\hbar c = 2 \times 10^{-39}$, which shows that we can essentially ignore gravity in discussing elementary particle phenomena. The weak interaction constant in dimensionless units turns out to be of the order of 10^{-14} or 10^{-7} depending on whether we regard a weak decay process as a one-stage process or a two-stage process (we shall say more about this in Chapter 7), and the various strong interaction constants seem to be of the order of unity. One of the deepest mysteries in elementary particle physics is that there is such a wide gap in strength between the strong and electromagnetic couplings on the one hand and the weak couplings on the other.

The major reason why the strong and weak interactions had not been discovered before the twentieth century is that they are too short-ranged to manifest themselves in daily life. For instance, the range of the force between a proton and a neutron is expected to be of the order of the pion Compton wavelength, $\hbar/\mu_\pi c = 1.41 \times 10^{-13}$ cm. So, just from the de Broglie wave relation $\lambda = \hbar/p$, we realize that we need high-energy machines to probe the nature of such short-ranged forces. It is for this reason that the term "elementary particle physics" is synonymous with the term "high energy physics."

To be more specific, let us consider proton-proton scattering. Below 200 kev., the scattering cross section follows the Mott formula, which is computed under the assumption that the only force between the two protons is Coulomb's repulsive force. We start learning something about the tail end of the two nucleon potential by performing scatterings at 500 kev. to 1 Mev. In order to study the nature of the forces between two protons at distances of the order 0.5×10^{-13} cm., we need a beam of protons with laboratory kinetic energy ≈ 300 Mev. Similar considerations are applicable to other phenomena. For instance, if we want to study the "structure" of weak interactions at short distances, we need a high energy beam of neutrinos.

The fact that the strong and weak interactions are short-ranged implies that those interactions do not possess any macroscopic analogs whatsoever in the domain of classical physics. This situation differs drastically from the electromagnetic case. Coulomb's law that is valid between the electron and the proton in the hydrogen atom is essentially the same Coulomb's law that also holds between two charged macroscopic balls separated by macroscopic distances. In constructing quantum electrodynamics we have been guided by classical electrodynamics; we may require that quantum electrodynamics reduce to classical electrodynamics in the limit where the number of photons per volume λ^3 is much greater than unity (as in the case of radio waves), or we may write down the Lagrangian for the photon field in quantum electrodynamics in analogy with the Lagrangian for the classical Maxwell field. In contrast, in constructing theories of strong and weak interactions of elementary particles, there are no such guiding principles from macroscopic physics. There are no classical, macroscopic analogs of nuclear forces, nor of the beta decay couplings. In elementary particle physics we must start from the very beginning. This is why elementary particle physics is so fantastically difficult and, hence, so much more challenging.

It is true that elementary particle physics today is in a very unsatisfactory state. It is expected that many of our present-day concepts have to be either revised or demolished. However, there are reasons to believe that certain arguments in elementary particle physics today, namely those arguments which are firmly based on invariance or symmetry considerations, will be of more permanent value. In reviewing the history of elementary particle physics, which is less than thirty years old, we cannot underestimate the power of symmetry considerations, which have led to a number of non-trivial predictions and have provided us with remarkably orderly understandings of elementary particle phenomena. True, this is not the first time that invariance or symmetry considerations played such an important role; the whole theory of relativity rests on the idea of Lorentz invariance; in atomic physics the regularities revealed in the periodic table are a direct consequence of invariance under rotations. But in elementary particle physics the guidance from invariance considerations is particularly helpful mainly because we lack any guidance from macroscopic analogies and correspondence with classical theories. Perhaps in the future theory of elementary particles, invariance principles may play even more significant roles; in the future, as Wigner (1949) put it, we may well "derive the laws of nature and try to test their validity by means of the laws of invariance rather than to try to derive the laws of invariance from what we believe to be the laws of nature."

One of the most striking features of invariance principles is that many of the so-called invariance or symmetry *laws* are only *approximate*. At first sight, one may feel that, because of this approximate nature, invariance considerations might not be very fruitful after all. We believe that just the opposite is the case. For, when a certain invariance principle is violated, it is violated in a very definite, orderly manner and never in a chaotic manner. The very breakdown as well as the very existence of the invariance principle is likely to provide clues to the mysterious dynamics of elementary particle interactions. For instance, the law of parity conservation looks like an absolute invariance principle in the realms of strong and electromagnetic interactions, but it does not hold at all for weak interactions, and moreover, there definitely exist orderly patterns in the ways parity conservation is violated in weak processes. Similar situations hold for other invariance principles as summarized in Table 1.1. The precise meanings of "strangeness," "charge conjugation invariance," etc. will be clarified in the appropriate chapters. Invariance principles or conservation laws not mentioned in Table 1.1 such as G conjugation invariance and lepton conservation will also be discussed as we go along.

Before we proceed further, we must define what is meant by an "elementary particle." When we consider the motion of molecules in a room, for all practical purposes we can regard each molecule as "elementary," as tacitly assumed in the kinetic theory of gases. This is because the mean energy of the molecule per degree of freedom is of the order of 0.03 ev. and is too low for the non-elementary nature of the molecule to reveal itself. However, in analyzing the vibrational spectrum of diatomic molecules in the infra-red region, it is more convenient to have a picture in which the molecule is composed of two "elementary" atoms with a spring in between. In the realm of atomic physics, atoms are no longer elementary; instead, we talk

TABLE 1.1

The Validity of Invariance Principles

Symmetry operations or conserved quantities	Strong	Electromagnetic	Weak
Parity (space inversion)	yes	yes	no
Charge conjugation	yes	yes	no
Time reversal	yes	yes	yes?
Electric charge	yes	yes	yes
Baryon number	yes	yes	yes
Isospin	yes	no	no
Strangeness	yes	yes	no

about electrons and various kinds of nuclei. In atomic physics, all nuclei look elementary; by counting isotopes we note that there are more than two hundred "elementary particles." This is highly unsatisfactory. It is indeed gratifying that the series of milestone discoveries and experiments in nuclear physics—Becquerels' discovery of radioactivity in 1896, Rutherford's first laboratory transmutation of elements in 1919, Chadwick's discovery of the neutron in 1932, etc.—finally led us to the idea that what are "elementary" are not various nuclei but the proton and the neutron (Heisenberg, 1932).

We may naturally ask: Is the proton elementary? We know today that the physical proton itself is made up of the "core" and the "cloud" of virtual mesons surrounding the core. In photon-proton collisions with photon energies greater than 150 Mev., a pion in the cloud can be knocked out from the proton ($\gamma + p \rightarrow p + \pi^0$) just as an electron in the atom can be knocked out in a photoelectric process. In photon-proton collisions at even higher energies ($E_\gamma > 910$ Mev.) the proton can dissociate itself into a Λ particle and a K meson ($\gamma + p \rightarrow \Lambda^0 + K^+$), which reminds us of the photodisintegration of the deuteron ($\gamma + d \rightarrow p + n$). Is the proton a bound system of a Λ and a K? Some physicists say "yes," other physicists say "no;" still others regard such a question as meaningless. (One may argue that we should regard stable particles as elementary as much as possible, but this argument gets into difficulty since the deuteron is stable while the neutron is unstable.)

For the purpose of the present book we take, from Lee and Yang, a negative definition of elementary particles (Lee and Yang 1957b). "We believe we understand what is meant by an atom, a molecule, and a nucleus. Any small particle that is not an atom, not a molecule, not a nucleus (except the hydrogen nucleus) is called an elementary particle." With this definition, there are, without counting antiparticles, seventeen elementary particles, as shown in Table 1.2. It is not known whether or not there are more elementary particles as yet undiscovered.

TABLE 1.2

Table of Elementary Particles (Compiled by W. H. Barkas and A. H. Rosenfeld). Errors Are Not Shown

Family	Particle	Spin	Mass (Mev.)	Mean life (second)
Photon	γ	1	0	stable
Leptons	$\nu(\bar{\nu})$	$\frac{1}{2}$	$<2 \times 10^{-4}$	stable
	$e^-(e^+)$	$\frac{1}{2}$	0.511	stable
	$\mu^-(\mu^+)$	$\frac{1}{2}$	105.7	2.26×10^{-6}
Mesons	$\pi^+(\pi^-)$	0	139.6	2.6×10^{-8}
	π°	0	135.0	2×10^{-16}
	$K^+(K^-)$	0	494	1.2×10^{-8}
	$K^\circ(\bar{K}^\circ)$	0	498 $\left\{K_1^\circ:\right.$	1.0×10^{-10}
	$(\lvert m(K_1^\circ) - m(K_2^\circ)\rvert \sim 5 \times 10^{-12})$		$\left.K_2^\circ:\right.$	$\sim 7 \times 10^{-8}$
Baryons	$p(\bar{p})$	$\frac{1}{2}$	938.2	stable
	$n(\bar{n})$	$\frac{1}{2}$	939.5	1.0×10^3
	$\Lambda^\circ(\overline{\Lambda^\circ})$	$\frac{1}{2}$	1115.5	2.5×10^{-10}
	$\Sigma^+(\overline{\Sigma^+})$	$\frac{1}{2}$	1189	0.8×10^{-10}
	$\Sigma^-(\overline{\Sigma^-})$	$\frac{1}{2}$	1197	1.6×10^{-10}
	$\Sigma^\circ(\overline{\Sigma^\circ})$	$\frac{1}{2}$	1193	theory $\sim 10^{-19}$
	$\Xi^-(\overline{\Xi^-})$	?	1318	1.3×10^{-10}
	$\Xi^\circ(\overline{\Xi^\circ})$	?	~ 1312	$\sim 2 \times 10^{-10}$

The first thing one must learn about elementary particles is that they can be classified into four groups. First of all, there is the photon, which is a group by itself. The photon can interact only electromagnetically. Then there is the "lepton" family. It consists of neutrinos (ν), electrons (e), and muons (μ). (It is unfortunate that due to an historic accident, a muon is sometimes referred to as a *μ-meson.*) They are all fermions with masses considerably smaller than the proton mass, and they cannot interact strongly. In contrast, particles belonging to the "meson" family and the "baryon" family do interact strongly as well as weakly (or electromagnetically). Pions (π mesons) and K particles are bosons, and they belong to the meson family. A Λ particle, a Σ particle and a Ξ particle (cascade particle) are often referred to as "hyperons," and they are like the nucleon (proton or neutron) in the sense that they are heavy fermions capable of strong interactions. The baryon family is made up of nucleons and hyperons.

Note that some of the particle masses are almost degenerate. Take the triplet Σ^+, Σ^- and Σ^0, for instance. We are reminded here of a "hyperfine structure" splitting. It is generally believed that the three levels collapse, to a single level in the absence of the electromagnetic coupling. However we are far from being able to calculate such mass differences within the multiplet.

Of all the particles in Table 1.2, only γ, e, ν, and p are stable. Other particles in the table have fairly well-defined masses because their lifetimes are long. Note in this connection that a one per cent uncertainty in the mass value corresponds to a lifetime of the order of 10^{-22} sec. for a particle of a protonic mass. We have not included in Table 1.2 resonant states that are too short-lived to be regarded as "particles," although we believe that there is no sharp distinction between a "particle" and a resonant state with well-defined quantum numbers. We emphasize here that the longevity of the strongly interacting particles is due to the mass relation and/or the "strangeness" selection rule that forbid them to "decay strongly" (cf. Chapter 10). If the Σ mass were 60 Mev. higher, the Σ particle would appear as a resonant state of the $\pi\Lambda$ system with lifetime $\sim 10^{-23}$ to 10^{-24} sec., which is the characteristic time scale of the strong interactions. All unstable particles (with the exception of Σ° and π°) decay via weak couplings, whose characteristic time scale is $\sim 10^{14}$ times longer than the "nuclear" time scale.

We may close this introductory chapter by listing some examples of the various interactions.

(1) Strong interactions:

$$p + n \rightarrow p + n$$
$$\pi^- + p \rightarrow \pi^\circ + n$$
$$p + n \rightarrow \Lambda^\circ + n + K^+ \quad \text{(associated production)}$$
$$K^- + p \rightarrow \Sigma^+ + \pi^-$$
$$p + \bar{p} \rightarrow \Lambda^\circ + \overline{\Lambda}^\circ$$

(2) Electromagnetic interactions combined with strong interactions:

$$\gamma + p \rightarrow \pi^\circ + p \quad \text{(photopion production)}$$
$$\pi^- + p \rightarrow n + \gamma \quad \text{(radiative capture of } \pi^-\text{)}$$
$$\gamma + p \rightarrow \Lambda^\circ + K^+ \quad \text{(associated photoproduction)}$$
$$\pi^\circ \rightarrow 2\gamma$$
$$\Sigma^\circ \rightarrow \Lambda^\circ + \gamma$$

(3) Weak interactions:

$$\mu^+ \rightarrow e^+ + \nu + \bar{\nu}$$
$$n \rightarrow p + e^- + \bar{\nu} \quad \text{(beta decay)}$$
$$\mu^- + p \rightarrow n + \nu \quad \text{(muon capture)}$$
$$K^+ \rightarrow \pi^+ + \pi^\circ$$
$$\Sigma^+ \rightarrow p + \pi^\circ$$

CHAPTER 2

Continuous Space-Time Transformations

2.1. General Considerations

The connection between a conservation law and symmetry is well known in classical mechanics (see e.g. Goldstein, 1953). If L is the Lagrangian, then Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_j}\right) - \frac{\partial L(q_j, \dot{q}_j, t)}{\partial q_j} = 0 \tag{2.1}$$

Clearly, if L does not depend on q_j, then $\dfrac{\partial L}{\partial \dot{q}_j} = p_j$, the canonical momentum, is constant in time, i.e. is conserved.

In the Hamiltonian formalism, one has

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = [p_j, H]_{\text{classical}} \tag{2.2}$$

where the bracket is the classical or Poisson bracket. Under an infinitesimal transformation $q_j \to q_j + \delta q_j$, we have $\delta H = \delta q_j \dfrac{\partial H}{\partial q_j}$. If the Hamiltonian is invariant under such an infinitesimal transformation, i.e. if an infinitesimal displacement δq_j does not change H, the corresponding canonical momentum p_j is a constant of the motion. For example, the conservation of linear momentum is a result of the *homogeneity* of space while the conservation of angular momentum is due to the *isotropy* of space.

If invariance under translations and rotations holds, the laws of nature do not depend on "absolute space." The questions of which point we choose as the origin of the coordinate system and of which orientation axis we choose as the z-axis should play no essential role in the formulation of physical laws. Of course, on the cosmological scale, we may legitimately ask whether the laws of nature on the "fringes" of the universe are the same as the laws of nature in the "center" of the universe. For instance, we may examine whether the universal constants of nature such as the fine structure constant and the proton gyromagnetic ratio are truly universal in the sense that they are identical in various parts of the universe. Savedoff (1956) has shown that radio astronomical observations of the radio source Cygnus A, which is estimated to be $\sim 3 \times 10^8$ light years away, yield

$$\frac{(\alpha^2)_{\text{cyg. A}}}{(\alpha^2)_{\text{local}}} = 1.0036 \pm 0.0032$$

where α is the fine structure constant. A similar result has been obtained for the proton g-factor.

Even if the laws are the same everywhere, we might still expect that the results of measurements carried out in our solar system are influenced by the fact that the matter in our galaxy is not distributed isotropically with respect to the solar system. Cocconi and Salpeter (1958) have speculated on the possible dependence of the inertial mass M on the direction of its acceleration; because our solar system is not at the center of the galaxy, M may depend on whether its acceleration is in the direction towards the center of the galaxy or in a direction perpendicular to it. It can be estimated from nuclear resonance experiments that the fractional anisotropy of inertia is $\Delta M/M \lesssim 10^{-20}$ (Hughes, Robinson, and Beltram-Lopez, 1960).

Energy conservation is related to invariance under displacement of the time coordinate in the same way as momentum conservation is related to invariance under displacement of a space coordinate. We may ask whether the laws of nature in an early epoch of the universe are the same as the laws of nature today. It appears that energy conservation is compatible only with a static, non-expanding universe.

2.2. Translation Operator and Linear Momentum

In a quantum mechanical formalism one starts with state vectors which are eigenvectors of the Hamiltonian and of other commuting observables. A quantity is conserved if the generator of the corresponding infinitesimal transformation commutes with the Hamiltonian. We have

$$DH\Psi = HD\Psi \tag{2.3}$$

which is the mathematical expression of the quantum mechanical statement that the Hamiltonian H is unchanged by a transformation D.

In the coordinate representation an infinitesimal translation operator D is given by

$$D = 1 + \delta l_j \frac{\partial}{\partial x_j} \tag{2.4}$$

Equation (2.3) gives

$$[D, H] = 0 \text{ or } [\vec{\nabla}. H] = 0 \tag{2.5}$$

where the bracket is now a commutator bracket. The analogy to the classical equation $[p_j, H]_{\text{classical}} = 0$ is evident and leads one to identify $\vec{p}$ with $\vec{\nabla}$ except for a constant factor. There is the formal requirement for the infinitesimal transformation to be unitary because the norm of the state vector must be preserved. This leads one to choose the constant as

$$\vec{p} = -i\hbar\vec{\nabla} \tag{2.6}$$

where $\hbar$ must have the dimensions of action, e.g., erg-sec. Thus

$$D = 1 + \frac{i}{\hbar}\vec{p}\cdot\delta\vec{l} \tag{2.7}$$

That D is unitary (to first order in $\delta\vec{l}$) is evident since

$$D^\dagger D = \left(1 - \frac{i}{\hbar}\vec{p}\cdot\delta\vec{l}\right)\left(1 + \frac{i}{\hbar}\vec{p}\cdot\delta\vec{l}\right) = 1 + O((\delta\vec{l})^2)$$

where the Hermiticity of $\vec{p}$ has been assumed.

A finite transformation can be obtained by compounding successive infinitesimal transformations. (According to a mathematical theorem in the theory of Lie groups, the properties of the finite differentiable group transformations are completely determined by those of the corresponding infinitesimal transformations.) For a finite displacement $\vec{l}$, the translation operator D, is given by

$$D = \lim_{n\to\infty}\left(1 + \frac{i\vec{l}}{n\hbar}\cdot\vec{p}\right)^n = \exp\left(\frac{i\vec{p}\cdot\vec{l}}{\hbar}\right) \tag{2.8}$$

As an example, the plane wave solution to the Schrödinger equation $\exp\left[\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - Et)\right]$, goes into $\exp\left(\frac{i}{\hbar}\vec{p}\cdot\vec{l}\right)\exp\left[\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - Et)\right]$ when $\vec{x}$ goes into $\vec{x} + \vec{l}$.

There are various elementary particle reactions that are forbidden because of momentum and energy conservation. Trivial as they may seem, we list here some typical examples.

$e^- + e^+ \not\rightarrow \gamma$	in free space
$e^- \not\rightarrow e^- + e^+ + e^-$	” ”
$p \not\rightarrow p + \gamma, p + \pi^\circ$	” ”

2.3. Rotation Operator and Angular Momentum

In a way analogous to translations, to an infinitesimal rotation about some axis along $\delta\vec{\omega}$ by an amount $|\,\delta\vec{\omega}\,|$, we associate the rotation operator

$$R = 1 + i\,\delta\vec{\omega}\cdot\vec{J}, \tag{2.9}$$

where $\hbar = c = 1$ from now on. Just as $\vec{p}$ is called momentum, $\vec{J}$ is called the angular momentum and has analogous properties. The properties of $\vec{J}$ can be studied by examining the infinitesimal rotations which it generates. For the specific set of axes shown in Fig. 2.1, consider the following sequence of rotations and their effect upon the point (0, 1, 0). A rotation by ε about the x-axis gives $(0, 1, \varepsilon)$. A further rotation by η about the y-axis gives $(\varepsilon\eta, 1, \varepsilon)$. Now rotate by $-\varepsilon$ about the x-axis, this will give $(\varepsilon\eta, 1, 0)$.

A final rotation by $-\eta$ about the y-axis gives $(\varepsilon\eta, 1, 0)$. Thus the net rotation is $\varepsilon\eta$ about the z-axis. Symbolically

$$(1 - i\eta J_y)(1 - i\varepsilon J_x)(1 + i\eta J_y)(1 + i\varepsilon J_x) = (1 + i\varepsilon\eta J_z) \qquad (2.10)$$

Hence we have the commutation relations

$$[J_x, J_y] = iJ_z \qquad (2.11)$$

and the cyclic permutations thereof. From (2.11) all well-known properties of angular momentum — $J^2 = j(j+1)$, $\langle j, m|J_x \pm iJ_y|j, m \mp 1\rangle = \sqrt{(j \pm m)(j \mp m + 1)}$, etc.—can be derived. Just as in the case of trans-

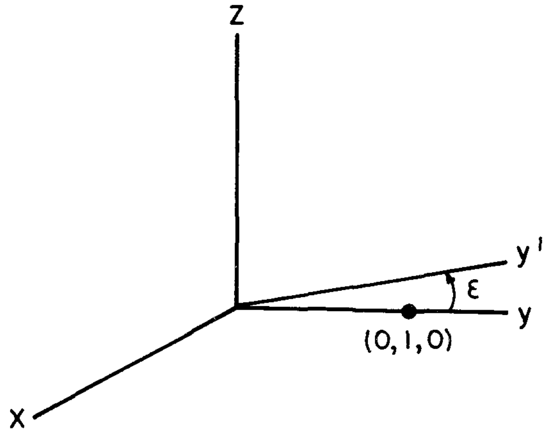

Figure 2.1. Infinitesimal rotation

lations, the infinitesimal rotations are compounded to give finite rotation operators of the form $\exp(iJ_z\omega)$ etc.

As an example, consider the plane wave expansion with the z-axis in the beam direction

$$\exp(ik|\vec{x}|\cos\theta) = \sum_{l=0}^{\infty} (2l+1)i^l j_l(k|\vec{x}|)P_l(\cos\theta) \qquad (2.12)$$

Note that the expansion contains no terms of the form $P_l^m(\cos\theta)\exp(im\varphi)$ with $m \neq 0$. The application of $1 + i\delta\omega L_z$ with $L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = i\frac{\partial}{\partial\varphi}$ just results in the identity operation, which expresses the fact that, although the plane wave contains all possible l values, the plane wave is cylindrically symmetric in the sense that it is invariant under rotations about the z-axis; hence $l_z = m = 0$. That the component of $\vec{L}$ along the beam direction must necessarily vanish is expected classically $(\vec{x} \times \vec{p})\cdot\vec{x} = 0$.

A scalar field $\varphi(\vec{x})$, by definition, transforms as $\varphi'(\vec{x}') = \varphi(\vec{x})$ under a rotation $\vec{x} \to \vec{x}'$. The value of the scalar field at a given point in space is the same in all coordinate frames. For the scalar field the angular momentum operator $\vec{J}$ is identical to the orbital angular momentum operator $\vec{L}$ that just acts on the coordinates. Thus the scalar field carries no intrinsic angular momentum, and describes a spinless particle.

In contrast, for a vector field $\vec{A}(\vec{x}) \to \vec{A}'(\vec{x}') \neq \vec{A}(\vec{x})$. Let us write the vector field, $\vec{A}$, at a given point as a product of a unit vector $\vec{\varepsilon}$ and a spatial function $f(\vec{x})$, and consider a rotation about the z-axis with the rotation operator $\exp[i(S_z + L_z)\omega]$. The operator L_z acts on $f(\vec{x})$. But the spin operator S_z acts on the polarization vector $\vec{\varepsilon}$. In the rotated frame

$$\begin{aligned} \varepsilon_{x'} &= \varepsilon_x \cos\omega - \varepsilon_y \sin\omega \\ \varepsilon_{y'} &= \varepsilon_x \sin\omega + \varepsilon_y \cos\omega \\ \varepsilon_{z'} &= \varepsilon_z \end{aligned} \tag{2.13}$$

A more suggestive way of writing these relations is obtained by rearrangement

$$\begin{aligned} \varepsilon_{x'} + i\varepsilon_{y'} &= \exp(i\omega)(\varepsilon_x + i\varepsilon_y) \\ \varepsilon_{x'} - i\varepsilon_{y'} &= \exp(-i\omega)(\varepsilon_x - i\varepsilon_y) \\ \varepsilon_{z'} &= \varepsilon_z \end{aligned} \tag{2.14}$$

It is evident that these combinations which correspond to states of definite circular polarization are eigenfunctions of S_z with the eigenvalues $+1$, -1, 0 respectively. The real transverse photon with momentum $\vec{k}$ requires $\vec{k}\cdot\vec{\varepsilon} = 0$, and taking the z-axis along $\vec{k}$, we see that there are only two spin states $+1$ and -1.

For two component spinors which are appropriate for describing spin $\frac{1}{2}$ particles, the rotation operator around the unit vector $\hat{n}$ can be written as $\exp\left(i\,\frac{\vec{\sigma}\cdot\hat{n}}{2}\,\omega\right)$. By expanding the exponential in a Taylor series, using the relation $(\vec{\sigma}\cdot\hat{n})^2 = \hat{n}\cdot\hat{n} + i\vec{\sigma}\cdot(\hat{n}\times\hat{n}) = 1$, and combining terms, one obtains

$$\exp\left(i\,\frac{\vec{\sigma}\cdot\hat{n}}{2}\,\omega\right) = \cos\frac{\omega}{2} + i\vec{\sigma}\cdot\hat{n}\sin\frac{\omega}{2} \tag{2.15}$$

Suppose the two component spinor u^P is given by

$$u^P = \begin{pmatrix}1\\0\end{pmatrix} = |s = \tfrac{1}{2}, s_z = \tfrac{1}{2}\rangle$$

in the usual representation in which σ_z is diagonal.

$$\begin{aligned} \sigma_x &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \\ \sigma_y &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \\ \sigma_z &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \end{aligned} \tag{2.16}$$

Then $\sigma_z u^P = u^P$ so the z component of the spin would be definitely $\frac{1}{2}$. In another coordinate system, say, rotated about the y-axis by ω

$$u^{P'} = \exp\left(i\,\frac{\sigma_y\omega}{2}\right)u^P = \begin{pmatrix} \cos\dfrac{\omega}{2} \\ \\ -\sin\dfrac{\omega}{2} \end{pmatrix} \tag{2.17}$$

Hence the probability for $s_{z'} = \frac{1}{2}$ is $\cos^2\dfrac{\omega}{2}$, and for $s_{z'} = -\frac{1}{2}$ is $\sin^2\dfrac{\omega}{2}$. In particular for $\omega = 90°$, the observation of spin up z' (along the old x-axis) or spin down are equally probable. In other words, if we have a beam of particles with $s_z = \frac{1}{2}$, then s_x and s_y do not have sharp values. This is not surprising because s_z and s_y (or s_x) do not commute, which means that s_z and s_x cannot be simultaneously diagonalized.

Experimentally this would be found in a Stern-Gerlach type experiment. From a beam of spin $\frac{1}{2}$ atoms moving along the y-axis in an inhomogeneous field along the z-axis, select the $s_z = \frac{1}{2}$ fraction. If this part of the beam enters a region with an inhomogeneous field in the x direction, it will be split into two equal parts corresponding to $s_x = \frac{1}{2}$ and $s_x = -\frac{1}{2}$. Now select the $s_x = \frac{1}{2}$ part, and again subject it to an inhomogeneous field along the z-axis. Both $s_z = \frac{1}{2}$ and $s_z = -\frac{1}{2}$ appear, i.e. we no longer have a pure spin $s_z = \frac{1}{2}$ beam. In Chapter 10 we shall see that the behavior of neutral K particles can be discussed along similar lines.

Given *any* two component spinor, there always exists a direction $\hat{n}$ such that $\vec{\sigma}\cdot\hat{n}$ is sharp, i.e.,

$$\vec{\sigma}\cdot\hat{n}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\theta & , & \sin\theta\exp(-i\varphi) \\ \sin\theta\exp(i\varphi), & & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.18}$$

$$|a|^2 + |b|^2 = 1$$

where θ and φ describe the orientation of $\hat{n}$.

The reader may readily verify that within an undetermined phase factor

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\dfrac{\theta}{2} \\ \\ \sin\dfrac{\theta}{2}\exp(i\varphi) \end{pmatrix} \tag{2.19}$$

will do. This means that any linear combination of the two independent pure states

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is also a pure state with a spin pointing out to *some* direction in space.

If we want to describe an *ensemble* of spin $\frac{1}{2}$ particles whose spin orientations are partially (if not completely) random, we must form an incoherent mixture of pure states. For this purpose it is convenient to use the density matrix formalism which is fully discussed in Chapter 17 of Bethe and Morrison (1956) (see also Tolhoek, 1956). Here we simply remark that a beam of spin $\frac{1}{2}$ particles is completely characterized by the 4×4 density matrix,

$$\rho = \frac{I_0}{2}(1 + \langle\vec{\sigma}\rangle\cdot\vec{\sigma}) \tag{2.20}$$

The intensity, I_0, is

$$I_0 = \mathrm{Tr}(\rho) \tag{2.21}$$

while the polarization vector $\langle\vec{\sigma}\rangle$ is given by

$$\langle\vec{\sigma}\rangle = \frac{\mathrm{Tr}(\rho\vec{\sigma})}{\mathrm{Tr}(\rho)} \tag{2.22}$$

Since $\frac{1}{2}(1 + \vec{\sigma}\cdot\hat{n})$ is the projection operator for spin along $\hat{n}$, the component of the polarization vector in the direction $\hat{n}$ is

$$\begin{aligned}\langle\vec{\sigma}\rangle\cdot\hat{n} &= \frac{\mathrm{Tr}(\rho\frac{1}{2}(1 + \vec{\sigma}\cdot\hat{n}) - \mathrm{Tr}(\rho\frac{1}{2}(1 - \vec{\sigma}\cdot\hat{n}))}{\mathrm{Tr}(\rho)} \\ &= \frac{|a_\uparrow|^2 - |a_\downarrow|^2}{|a_\uparrow|^2 + |a_\downarrow|^2}\end{aligned} \tag{2.23}$$

$|a_\uparrow|^2$ and $|a_\downarrow|^2$ refer to the probabilities of observing spin along $\hat{n}$ and opposite to $\hat{n}$. The direction of $\langle\vec{\sigma}\rangle$ is that quantization direction which makes

$$\frac{|a_\uparrow|^2 - |a_\downarrow|^2}{|a_\uparrow|^2 + |a_\downarrow|^2}$$

maximal. The magnitude of $\langle\vec{\sigma}\rangle$ is the average spin angular momentum in units of $\hbar/2$ in the polarization direction. If the beam is unpolarized, $\langle\vec{\sigma}\rangle = 0$; if the magnitude of $\langle\vec{\sigma}\rangle$ is unity, the beam is in a pure state of polarization.

2.4. Angular Momentum Selection Rules and Spin Tests

There are many applications of angular momentum considerations to elementary particle physics. Some of them are rather far-reaching.

We first show that from the two-quantum decay of the π^0, $\pi^0 \to 2\gamma$, one can conclude that its spin cannot be one (Landau, 1948; Yang, 1950). A final state wave function in momentum space for the two photons is to be constructed from the polarization vectors, $\vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$ and the relative momentum vector $\vec{k}$. Also it must clearly be linear in $\vec{\varepsilon}_1$ and in $\vec{\varepsilon}_2$ and transform like a vector under rotations if the initial π^0 has spin one. Further, Bose-Einstein statistics must be obeyed because photons are bosons. There are three independent combinations of $\vec{\varepsilon}_1$, $\vec{\varepsilon}_2$, and $\vec{k}$ that are linear in $\vec{\varepsilon}_1$ and in $\vec{\varepsilon}_2$ and transform like vectors:

(a) $\vec{\varepsilon}_1 \times \vec{\varepsilon}_2$, which is not satisfactory because it is antisymmetric under interchange of the two photons

(b) $(\vec{\varepsilon}_1 \cdot \vec{\varepsilon}_2)\vec{k}$, which is not satisfactory either for the same reason ($\vec{k} \to -\vec{k}$ under interchange).

(c) $\vec{k} \times (\vec{\varepsilon}_1 \times \vec{\varepsilon}_2) = \vec{\varepsilon}_1(\vec{k} \cdot \vec{\varepsilon}_2) - \vec{\varepsilon}_2(\vec{k} \cdot \vec{\varepsilon}_1)$ which satisfies Bose-Einstein statistics but is identically zero since $\vec{k} \cdot \vec{\varepsilon} = 0$ by the transversality condition.

Thus the π^0 spin is not one if angular momentum is conserved in $\pi^0 \to 2\gamma$. This type of argument is also applicable to positronium in a ${}^3S_1$ state, which is forbidden to decay into two photons.

When we discuss detailed balancing in Chapter 4, we will present an argument for the pion spin $= 0$. In the following we assume that the pion is spinless.

From angular momentum conservation and Bose statistics one can also conclude that if the K_1^0 meson had odd spin, then $K_1^0 \to 2\pi^0$ would be forbidden. Since the pion is spinless, the K_1^0 spin must be equal to the orbital angular momentum L of the two pion system. But for odd L, $Y_L \to -Y_L$, for the orbital wave function of the two-pion system, when the two identical bosons are interchanged. This would violate Bose-Einstein statistics. Experimentally the decay process $K_1^0 \to 2\pi^0$ does occur; hence the K_1^0 spin must be even.

Also it can be shown that $K^+ \to \pi^+ + \gamma$ is forbidden for spin-zero K. In the center-of-mass system of the K^+ one chooses the axis of quantization along the π-γ decay line. Initially $J = 0, J_z = 0$. In the final state $L_z = 0$ necessarily, from the choice of the axis of quantization, and $J_z = S_z(\text{photon}) = \pm 1$. Hence this decay process is not possible if the K is spinless. Experimentally the radiative reaction in question has *not* been observed. Dalitz (1955) has estimated that if the K spin were two (or greater), the radiative reaction $K^+ \to \pi^+ + \gamma$ would compete favorably with $K \to 2\pi, 3\pi$. In the following we assume that the K particle is spinless. (Moreover, for the K spin $\neq 0$, decay products can be emitted anisotropically in the K rest system if the parent K particles are polarized. Experimentally there is no evidence for anisotropy.)

In the same way $\gamma + He^4 \to \pi^0 + He^4$ is forbidden for an s-wave final state. (The He^4 and π^0 spins are both zero.) Quite generally a zero-zero transition via the emission or absorption of a real photon is strictly forbidden.

Perhaps it may be appropriate to quote here an experiment of Sunyar (unpublished) from low energy nuclear physics to detect possible γ transitions between two nuclear states of spin zero. The 700 kev $0^+ \to 0^+$ transition of Ge^{22} is known to be an internal conversion electron transition.

If angular momentum conservation were violated, the transition could occur with the emission of a real photon. No evidence for γ rays has been found, and it is estimated that the angular momentum violating amplitude is $> 3 \times 10^6$ weaker (Feinberg and Goldhaber, 1959).

Elementary angular momentum considerations also enable one to say something about the spins of the hyperons. For instance, the Λ particle spin can be determined as first pointed out by Adair (1955). Λ particles are produced in the reaction

$$\pi^- + p \rightarrow K^0 + \Lambda^0 \tag{2.24}$$

and the Λ subsequently decays by the reaction

$$\Lambda^0 \rightarrow \pi^- + p \tag{2.25}$$

For the initial π^--p system, $J_z = S_z^{(p)} = \pm\frac{1}{2}$ (with equal probability for unpolarized p) if the beam direction is the axis of quantization. And for Λ and K produced in the *incident beam direction* $L_z = 0$, hence $J_z = S_z^{(\Lambda)}$ is still $\pm\frac{1}{2}$. This means that for Λ spin $> \frac{1}{2}$, the Λ particles are aligned in the sense that all but $S_z^{(\Lambda)} = \pm\frac{1}{2}$ states have zero population.

Now let us consider the angular distribution of the decay products (in the Λ rest system) of those Λ's which are produced in the incident beam direction. If the Λ spin is $\frac{1}{2}$ (unpolarized Λ), the angular distribution in the decay reaction (2.25) is isotropic. This is true of either s-wave or p-wave decay, as the reader may easily verify. If both s-wave and p-wave are present, parity is not conserved in (2.25) as we shall show in Chapter 3; the distribution is also isotropic for the reason presented below. Let us consider in detail the slightly more non-trivial case, $S_\Lambda = \frac{3}{2}$. The orbital π–p state must be p or d. Consider the p state, and assume that the original proton spin had $S_z^{(p)} = \frac{1}{2}$. Using standard techniques for adding angular momenta (see Appendix A for a table of Clebsh-Gordan coefficients), we obtain for the angular momentum wave function of the decay products

$$|J = \tfrac{3}{2}, J_z = \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|L_z = 0, S_z = \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|L_z = 1, S_z = -\tfrac{1}{2}\rangle$$

But

$$|L_z = 0, S_z = \tfrac{1}{2}\rangle = -\sqrt{\frac{3}{4\pi}}\cos\theta\alpha$$

$$|L_z = 1, S_z = -\tfrac{1}{2}\rangle = \sqrt{\frac{3}{8\pi}}\sin\theta\exp(i\varphi)\beta$$

where $\alpha = |s = \frac{1}{2}, s_z = +\frac{1}{2}\rangle$ and $\beta = |s = \frac{1}{2}, s_z = -\frac{1}{2}\rangle$ represent the proton spin up and down respectively. Thus the angular distribution of the decay products is $\langle\frac{3}{2}, \frac{1}{2}|\frac{3}{2}, \frac{1}{2}\rangle \propto \frac{1}{2}(1 + 3\cos^2\theta)$. Naturally $J_z = -\frac{1}{2}$ would give the same result. It can be shown that the $d_{3/2}$-state angular distribution is also the same using essentially the same technique. If both $p_{3/2}$ and $d_{3/2}$ states are present, parity is not conserved. Recall that for given J the possible L values are $L = J \pm \frac{1}{2}$. The angular distribution would contain odd powers of $\cos\theta$ from the interference term between the two L values. But if both J_z values are present in equal numbers, which is the case experimentally for unpolarized proton targets in (2.24), the two sets of odd powers of $\cos\theta$ terms will cancel exactly, and the angular distribution will look the same as for pure $L = J + \frac{1}{2}$, or for pure $L = J - \frac{1}{2}$. To sum up, for those Λ's produced in the beam direction the decay angular distribution is a unique function of the Λ spin as shown in Table 2.1.

TABLE 2.1

Angular Distribution in Λ Decay for those Λ's produced in the Incident Beam Direction

Spin	Angular distribution
$\frac{1}{2}$	isotropic
$\frac{3}{2}$	$\frac{1}{2} + \frac{3}{2}\cos^2\theta$
$\frac{5}{2}$	$\frac{3}{4} - \frac{3}{2}\cos^2\theta + \frac{15}{4}\cos^4\theta$

Experimentally the reactions (2.24) and (2.25) were extensively studied by Eisler *et al.* (1958). They obtained decay distributions for $|\cos\theta_{\text{production}}| \gtrsim 0.6$ shown in Fig. 2.2. Also shown are the distributions expected for spins $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$. It is evident that the spin of the Λ is most likely $\frac{1}{2}$. The same group finds that the Σ spin is also likely to be $\frac{1}{2}$ using the same method.

It is easy to see that essentially the same argument holds for the reaction

$$K^- + p \rightarrow \Sigma^+ + \pi^-$$

$$\Sigma^+ \rightarrow p + \pi^0$$

if K^- is captured at rest from an s state of the K mesic atom (Treiman, 1956). Day, Snow, and Sucher (1959) have advanced an argument (based on the rapid rates for $np \rightarrow ns$ Stark transitions) to show that in a hydrogen bubble chamber K^-'s are indeed captured from s states. The experimental distribution indicates that the decay products are emitted isotropically in the rest system of Σ^+'s, which shows that the Σ spin is $\frac{1}{2}$. $S_\Sigma = \frac{3}{2}$ seems to be ruled out by about 20 standard deviations (Leitner *et al.*, 1959).

2.5. Lorentz Invariance

In contrast to the notions of translational and rotational invariance, the notion of Lorentz invariance is not immediately obvious from our daily life. Historically, the fact that the equations of electrodynamics are invariant under what can now be recognized as relativistic transformations

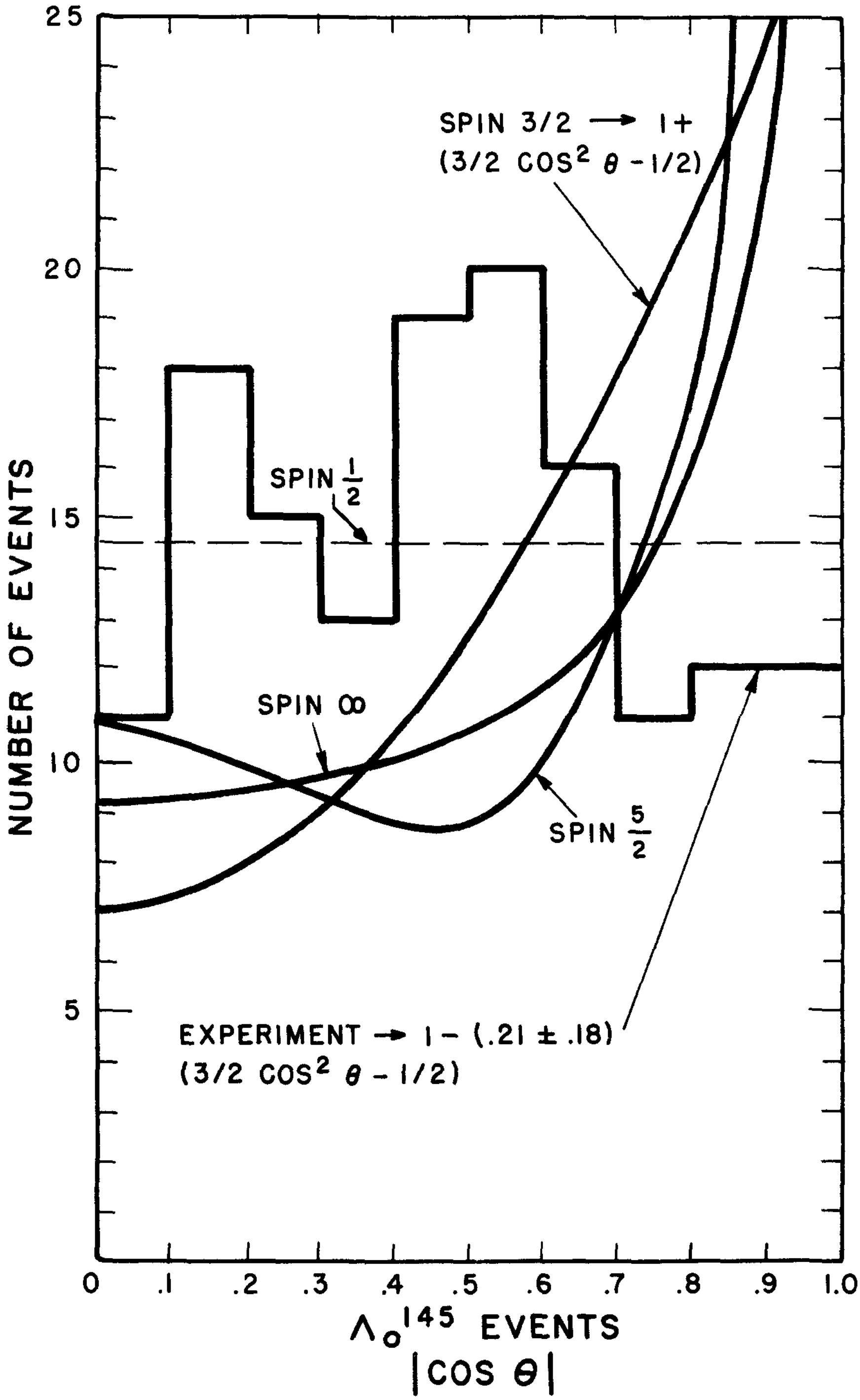

Figure 2.2. Angular distribution in Λ decay for Λ's produced in the beam direction

was first noted by H. Poincaré, who associated with it the name of Lorentz. The full significance of this new invariance, however, was not recognized until the advent of Einstein's special relativity.

The transformations that we have considered in the previous sections, translations and rotations, are Galilean, i.e., they leave $\sum_{i=1}^{3} (x_i^{(1)} - x_i^{(2)})^2$ invariant. Rotations look like

$$
\begin{aligned}
x_1' &= x_1 \cos \omega - x_2 \sin \omega \\
x_2' &= x_1 \sin \omega + x_2 \cos \omega
\end{aligned}
\tag{2.26}
$$

Pure Lorentz transformations have a similar form, e.g.

$$
\begin{aligned}
x_1' &= \frac{1}{\sqrt{1-\beta^2}}\, x_1 - \frac{i\beta}{\sqrt{1-\beta^2}}\,(ix_0) \\
ix_0' &= \frac{i\beta}{\sqrt{1-\beta^2}}\, x_1 + \frac{1}{\sqrt{1-\beta^2}}\,(ix_0)
\end{aligned}
\tag{2.27}
$$

or

$$
\begin{aligned}
x_1' &= x_1 \cosh \chi - ix_4 \sinh \chi \\
x_4' &= ix_1 \sinh \chi + x_4 \cosh \chi
\end{aligned}
\tag{2.27'}
$$

with

$$
\cosh \chi = \frac{1}{\sqrt{1-\beta^2}}
$$

$$
\sinh \chi = \frac{\beta}{\sqrt{1-\beta^2}}
$$

$$
x_4 = ix_0
$$

The only thing different is that the angle of rotations is now purely imaginary. In general, both for pure rotations and pure Lorentz transformations

$$
x_\mu' = a_{\mu\nu} x_\nu \qquad \mu = 1, 2, 3, 4
$$

$$
a_{\mu\nu} a_{\lambda\nu} = \delta_{\mu\lambda}
$$

$a_{ij} (i, j = 1, 2, 3)$ are purely real while a_{4j} are purely imaginary. We are dealing with four-space $x_\mu = (\vec{x}, x_4) = (\vec{x}, ix_0)$, and since $x_\mu x_\mu = \vec{x}^2 + x_4^2 = \vec{x}^2 - x_0^2$ is left invariant, this transformation can be looked upon as a rotation in four-space. Using this viewpoint, we can discuss the Lorentz group just as we did for the three-dimensional group.

The generators of the infinitesimal transformations will have the commutation relations

$$
[J_{\lambda\mu}, J_{\rho\sigma}] = -i(\delta_{\lambda\sigma} J_{\mu\rho} + \delta_{\mu\rho} J_{\lambda\sigma} - \delta_{\lambda\rho} J_{\mu\sigma} - \delta_{\mu\sigma} J_{\lambda\rho}) \tag{2.28}
$$

Geometrically these J's have the same significance as the angular momenta J's. J_{12} is the generator for an infinitesimal rotation in the 1-2 plane and around the 3-4 plane. These commutation relations can easily be verified in the special case $L_{\lambda\mu} = -L_{\mu\lambda} = -i\left(x_\lambda \frac{\partial}{\partial x_\mu} - x_\mu \frac{\partial}{\partial x_\lambda}\right)$ but, of course, the commutation relations hold quite generally. If we define two three-vectors

$$\begin{aligned} \vec{M} &= (J_{23}, J_{31}, J_{12}) \\ \vec{N} &= -i(J_{41}, J_{42}, J_{43}) \end{aligned} \tag{2.29}$$

then the commutation relations between their components are

$$\begin{aligned} [M_i, M_j] &= iM_k \\ [N_i, M_j] &= iN_k \\ [N_i, N_j] &= -iM_k \end{aligned} \tag{2.30}$$

$(ijk) = (123)$ (meaning i, j, k, are cyclic permutations of 1, 2, 3)

The quantities $F = \frac{1}{2}(\vec{M}^2 - \vec{N}^2)$ and $G = \vec{M}\cdot\vec{N}$ are invariant under Lorentz transformations.

The particular utility of $\vec{M}$ and $\vec{N}$ lies in their relation to the two subgroups of the orthochronous proper Lorentz groups $\mathscr{L}_+^\uparrow$ where $\uparrow$ stands for $a_{44} \geq 1$, and $+$ for $\det(a_{\mu\nu}) = +1$. Representations of the infinitesimal three-dimensional rotation group (which is a subgroup of $\mathscr{L}_+^\uparrow$) have the form

$$S_{\text{rot}} = 1 + i\delta\vec{\omega}\cdot\vec{M} \tag{2.31}$$

and representations of the infinitesimal "pure" Lorentz transformation group have the form

$$S_{\text{Lor}} = 1 + i\delta\vec{\chi}\cdot\vec{N} \tag{2.32}$$

Just for completeness, let us recall that infinitesimal translations in three-space and time have the representations

$$S_{\text{trans}} = \begin{cases} 1 + i\delta\vec{l}\cdot\vec{p} \\ 1 + i\delta l_0 p_0 \end{cases} \tag{2.33}$$

From this it is clear that ten quantities, $\vec{\omega}$, $\vec{\chi}$, l_μ are needed to specify the *inhomogeneous* Lorentz transformation $x'_\mu = a_{\mu\nu}x_\nu + l_\mu$ that can be continuously generated from the identity.

As an example consider

$$J_{\mu\nu} = \tfrac{1}{2}\sigma_{\mu\nu} \tag{2.34}$$

where

$$\sigma_{\mu\nu} = \frac{1}{2i}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

The γ_μ's are 4×4 matrices satisfying

$$\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}, \quad \gamma_\mu^\dagger = \gamma_\mu$$

In the Dirac-Pauli representation we may write these 4×4 matrices in 2×2 block form in terms of the 2×2 identity and the Pauli matrices, σ_j,

$$\vec{\gamma} = -i\beta\vec{\alpha} = \begin{pmatrix} 0 & -i\vec{\sigma} \\ i\vec{\sigma} & 0 \end{pmatrix},$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{2.35}$$

In addition, there is

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \frac{1}{4!}\,\varepsilon_{\mu\nu\lambda\rho}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho = -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

which satisfies

$$\{\gamma_5, \gamma_\mu\} = 0, \qquad \gamma_5^2 = I, \qquad \gamma_5^\dagger = \gamma_5$$

We see that in this representation $\vec{M}$ and $\vec{N}$ are given by

$$M_k = \tfrac{1}{2}\sigma_{ij} = \frac{1}{2i}\,\gamma_i\gamma_j = \tfrac{1}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \quad (ijk) = (1, 2, 3) \text{ etc.}$$

$$N_k = -\frac{i}{2}\,\sigma_{4k} = \frac{i}{2}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

That M_k and N_k satisfy the commutation relations (2.30) is easily verifiable. For an infinitesimal rotation around the 3-axis and an infinitesimal pure Lorentz transformation along the 1-axis, we have respectively

$$S_{\text{rot}} = 1 + \tfrac{1}{2}\gamma_1\gamma_2\delta\omega$$

$$S_{\text{Lor}} = 1 + \frac{i}{2}\,\gamma_1\gamma_4\delta\chi$$

For finite transformations we merely compound infinitesimal transformations as we did before to obtain

$$S_{\text{rot}} = \cos\frac{\omega}{2} + \gamma_1\gamma_2\sin\frac{\omega}{2} \tag{2.36}$$

$$S_{\text{Lor}} = \cosh\frac{\chi}{2} + i\gamma_1\gamma_4\sinh\frac{\chi}{2} \tag{2.37}$$

which correspond to (2.26) and (2.27). A little examination shows that $S_{\text{rot}}^\dagger = S_{\text{rot}}^{-1}$ while $S_{\text{Lor}}^\dagger = S_{\text{Lor}}$. This "difficulty" with S_{Lor} results from the four dimensional space-time being Lorentzian and not Euclidean. However, it is clear that in both cases

$$\gamma_4 S^\dagger \gamma_4 = S^{-1} \tag{2.38}$$

which is a very important relation.

PROBLEM 1. Find a 2×2 representation (using the Pauli matrices) of $\vec{M}$ and $\vec{N}$. Show that $S_\mu = (u^P \vec{\sigma} u^P, \ \pm i u^P u^P)$ transforms like a four-vector (the u^P's are Pauli spinors) under finite, proper, orthochronous Lorentz transformations (three dimensional rotations and pure Lorentz transformations).

2.6. The Dirac Equation and Bilinear Covariants

In this section we briefly discuss some of the properties of the Dirac equation, its solutions and the bilinear covariants. For a more complete discussion, the reader may refer to various standard treatises such as Pauli's *Handbuch* article (Pauli, 1933).

The Dirac equation is

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi = 0 \tag{2.39}$$

It must be remembered that ψ is a spinor and usually represents a column of four elements. If (2.39) is multiplied by $\left(\gamma_\mu \dfrac{\partial}{\partial x_\mu} - m\right)$ from the left, then it is found that ψ satisfies

$$\left(\frac{\partial^2}{\partial x_\mu^2} - m^2\right)\psi = 0$$

which means that each of the four components satisfies the Klein-Gordon equation.

What is meant by the invariance of the Dirac equation under a Lorentz transformation? It certainly does not mean that a solution to the Dirac equation $\psi(x)$ in the original frame is identical to the corresponding solution $\psi'(x')$ in some other frame any more than the Lorentz invariance of electrodynamics implies that the electric field $\vec{E}(x)$ is the same in all frames. We assume that $\psi'(x')$ and $\psi(x)$ are related linearly as follows

$$\psi'(x') = S\psi(x), \qquad x'_\mu = a_{\mu\nu}x_\nu$$

where S is a 4×4 matrix independent of space-time. In the primed system the Dirac equation reads

$$\gamma_\mu \frac{\partial}{\partial x'_\mu} \psi' + m\psi' = 0 \tag{2.40}$$

We must show that this equation is equivalent to the Dirac equation in the original system. For this reason, we express (2.40) in terms of the original (unprimed) quantities

$$\gamma_\mu S a_{\mu\nu} \frac{\partial \psi}{\partial x_\nu} + mS\psi = 0$$