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On Uniformization of Complex Manifolds

The Role of Connections (MN-22)

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ON UNIFORMIZATION OF COMPLEX MANIFOLDS:

THE ROLE OF CONNECTIONS

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PREFACE

These are notes based on a course of lectures given at Princeton University during the Fall term of 1976, incorporating some material from lecture courses given during the year 1963-64 as well. The topic of the lectures is the study of complex analytic pseudogroup structures on complex manifolds, viewed as an extension of the theory of uniformization of Riemann surfaces. The particular pseudogroup structures considered, and the questions asked about them, are determined by this point of view; and this point of view also lies behind the choice of the role of connections as a unifying and limiting principal theme. A more detailed overview of the topics covered and the point of view taken is given in the introductory chapter. There remain many fascinating open questions and likely avenues to explore; and I hope these notes will provide a background for further investigations.

I should like to express my thanks here to the students and colleagues who attended these lectures, for their interest and their many helpful comments and suggestions, and to Mary Smith, for the splendid typing of these notes.

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§1. Introduction

The general uniformization theorem for Riemann surfaces is one of the most remarkable results in complex analysis, and is at the center of a circle of problems which are still very actively being investigated. An interest in extending this theorem to complex manifolds of higher dimensions has long been manifest, and indeed there have been several extensions of one or another aspect of the general uniformization theorem. As has been observed in other cases, some theorems in classical complex analysis appear as the accidental concurrence in the one-dimensional special case of rather separate phenomena in the general case; so a major difficulty is deciding just what to attempt to extend. For compact Riemann surfaces perhaps the principal use of the general uniformization theorem lies in the possibility of representing these surfaces as quotients of the unit disc or the complex plane modulo a properly discontinuous group of complex analytic automorphisms. Recent works (surveyed in [2]) have demonstrated the existence and importance of a considerable array of different representations of compact Riemann surfaces as quotients of various subdomains of the sphere modulo appropriate groups of automorphisms; but the detailed results seem to rest very heavily on purely one-dimensional tools. On the other hand any such representation has a local form, in the sense that the representation can be viewed as inducing a complex projective structure on the Riemann surface, a rather finer structure than the complex analytic structure [20]. The set of all projective structures on a compact Riemann surface, being somewhat more local in nature, can be handled much more readily than the set of uniformizations of the surface and with tools that are less restricted to the one-dimensional case; and these structures include, in addition to those induced by the classical and contemporary uniformizations, those associated to the more exotic representations investigated by Thurston [41], in which the groups of automorphisms are not discontinuous. It is the extension to manifolds of higher dimensions of this somewhat local additional structure on Riemann surfaces that I propose to discuss here; if the phrase did not already have a different generally accepted meaning, this could perhaps be called the local uniformization of complex manifolds.

There are many papers in the literature in which such structures on manifolds have been investigated, although not often have complex analytic manifolds been of primary interest; for this is really just a special case of the general problem of the investigation of pseudogroup structures on manifolds, an active area of research in differential geometry. However the model presented by the uniformization of Riemann surfaces suggests restricting attention to a very special class of pseudogroup structures, those defined by families of partial differential equations having constant coefficients; for the defining differential equations can play the role in the general case that the Schwarzian derivative plays in the one-dimensional case, and that suggests the tenor of the treatment of the general case on the model of the one-dimensional case. The principal difference between the one-dimensional case and the higher-dimensional cases is then merely the presence of nontrivial integrability conditions in the higher-dimensional cases. That in turn suggests considering the connections associated to the structures rather than the structures themselves; and the formal treatment in the general case is then precisely parallel to that in the one-dimensional case. Considering the connections rather than the structures really has the effect of linearizing the entire problem, and thus trivializing the questions of deformation of structures and of moduli of structures. The nonlinearity does appear in the investigation of integrability conditions, although even there it is frequently possible to avoid the apparent nonlinearities; and the moduli can be introduced at this stage in a rather simpler and more explicit manner. Actually for some purposes it appears that the connections are all that is really needed of the structures, as will be evident during the course of the discussion; so the emphasis here will be primarily on the connections.

Even among the restricted class of pseudogroups mentioned above there is a great variety of possible pseudogroups; and any analysis detailed enough to be nontrivial seems to require somewhat separate treatment of basically different pseudogroups. Therefore to limit the present discussion as much as reasonably possible only those pseudogroups defined by partial differential equations with constant coefficients and having unrestricted Jacobian matrices will be considered here; the latter condition can be rephrased as the condition that the pseudogroup be transitive on tangent directions. This subclass of pseudogroups is still broad enough to include all the one-dimensional pseudogroups and some of the classical pseudogroups of differential geometry, the affine and projective pseudogroups; so this is perhaps the restriction leaving the general discussion closest to that of the one-dimensional case. There are enough complex manifolds admitting pseudogroup structures of this subclass to lead to an interesting discussion. However this restriction does leave out a great many interesting and important pseudogroup structures, such as general G-structures, contact structures, and foliated structures, which must eventually be included in any complete treatment of uniformization of complex manifolds. Some of these structures are well treated in other places though [8], [14] ; and the subject is anyway not sufficiently developed to warrant any attempt at a complete treatment.

In a discussion such as this it is a matter of choice whether merely to list the pseudogroups being considered, together with their defining equations and relevant properties, or rather to derive the defining equations and their properties from a classification of the possible pseudogroups of the limited class under consideration. I have chosen the second alternative, but to avoid requiring an unwilling reader to wade through the classification it has been included in a separate first part, from which the remainder of the discussion is essentially independent; so the unwilling reader need only glance at the list of pseudogroups contained in Theorem 1 at the end of §5, and refer to the properties of the defining equations as needed. The general study of pseudogroups of transformations was begun and carried very far indeed by E. Cartan in a series of fundamental papers, [7] ; and the extension and completion of the classification of pseudogroups has been taken up recently by several differential geometers in a number of major papers, of which it may suffice here merely to mention [16], [29], and [39]. However the classification of the restricted set of pseudogroups being considered here can be carried out quite simply and completely, without use of the extensive machinery required in the general case; indeed the classification can be reduced to an algebraic investigation of the subgroups or subalgebras of an easy and quite explicit finite Lie group or algebra, and some very classical analysis. The advantage of carrying out the classification in detail in this case is that it clarifies the relevant notion of equivalence and exhibits the possible alternative forms for these pseudogroups, while it also demonstrates the role of the defining equations and the parts played by their properties. It may also appeal to others, as it does to me, to see why such peculiar operators as the Schwarzian derivative must have the forms and properties that they do.

The second part contains a general discussion of pseudogroup structures on complex manifolds for the special class of pseudogroups being considered here, with particular attention to the role played by connections. The purely formal aspects, which hold for all these pseudogroups simultaneously, are treated in §6, while the remaining three sections discuss some more detailed properties of connections for the individual pseudogroups. The properties treated are: integrability conditions, alternative characterizations of the pseudogroups (except for the projective pseudogroup, where this seems less interesting), the differentiation operators associated to the connections, and the topological restrictions imposed by the existence of complex analytic connections. To provide some illustrative examples the third part contains a discussion of some aspects of these pseudogroup structures on two-dimensional compact complex manifolds, and is devoted primarily to the topics: which compact surfaces satisfy the topological restrictions the existence of complex analytic connections imposes; and then which of these surfaces actually admit complex analytic connections; and finally briefly which of these connections are integrable.

§2. The group of k-jets and its Lie algebra.

Consider the set of all germs of complex analytic mappings from the origin to the origin in the space $\mathbb{C}^n$ of n complex variables. The <u>k-jet</u> of such a germ f, denoted by $j_k f$, is defined to consist of the terms of order $\leq k$ in the Taylor expansion of the germ f ; but since all these germs are assumed to take the origin to the origin the conventional usage will be slightly modified in that the constant terms in the Taylor expansion, the terms of order $= 0$, will not be considered as part of the k-jet. Upon identifying a k-jet with its Taylor coefficients the set $M_k = M_k(n,\mathbb{C})$ of all such k-jets can be viewed as a finite-dimensional complex vector space; indeed M_k can be viewed as the direct sum

$$M_k = T_1 \oplus \ldots \oplus T_k \ , \tag{1}$$

where $T_p = T_p(n,\mathbb{C})$ is the complex vector space of dimension $= n\binom{n+p-1}{p}$ consisting of the Taylor coefficients of order $= p$. If $\xi_i \in M_k$ then choosing any germs of complex analytic mappings f_i such that $\xi_i = j_k f_i$ define

$$\xi_1 \cdot \xi_2 = j_k(f_1 \circ f_2) \ , \tag{2}$$

noting that the k-jet of the composite mapping $f_1 \circ f_2$ depends only on the k-jets of the individual mappings f_i. It is readily verified that under the operation (2) the set M_k has the structure of a semigroup with an identity element, though not generally an abelian semigroup; the identity is the germ of the identity mapping. The subset $G_k \subseteq M_k$ of germs of local homeomorphisms form the group of invertible elements in M_k ; this group $G_k = G_k(n,\mathbb{C})$ will be called the <u>general k-fold group</u> or the <u>group of k-jets</u>, the special case $G_1 = G_1(n,\mathbb{C})$ being of course the general linear group. The group G_k consists of all the jets $\xi \in M_k$ such that the terms of order $= 1$ form a nonsingular $n \times n$ matrix; thus G_k is a dense open subset of the vector space M_k, and with the natural manifold structure inherited from that vector space it is evident that G_k is a complex Lie group.

It is a quite simple matter to write the group operation in G_k explicitly in terms of the natural global coordinates provided by the encompassing vector space M_k , or at least explicitly enough for the present purposes. To do so it is necessary to be a bit more precise about the coordinatization of the space M_k , since there are various possibilities. It seems most convenient for the present purposes to view $T_p = T_p(n,\mathbb{C})$ as the subspace of the (p+1)-fold tensor product $\mathbb{C}^n \otimes \ldots \otimes \mathbb{C}^n$ consisting of those tensors which are fully symmetric in the last p indices; the first index will be written as a superscript and the last p indices as subscripts, so an element $\xi_p \in T_p$ is a tensor

$$(3) \qquad \xi_p = \{\xi^i_{j_1 \ldots j_p}\}$$

which is symmetric in the p lower indices. An element $\xi \in M_k$ is then the set of k tensors

$$(4) \qquad \xi = \{\xi_1, \ldots, \xi_k\} \quad , \quad \text{where} \quad \xi_p \in T_p \ .$$

If f is the germ of a complex analytic mapping from the origin to the origin in $\mathbb{C}^n$ and is given by the n coordinate functions $f_i(z_1, \ldots, z_n)$ then the k-jet $\xi = j_k f$ will be taken to be the element (4) with components (3) given by

$$(5) \qquad \xi^i_{j_1\cdots j_p} = \left.\frac{\partial^p f_i(z)}{\partial z_{j_1}\cdots\partial z_{j_p}}\right|_{z=0} .$$

This means that the k-jet is actually viewed as a set of derivatives of the coordinate functions rather than as a set of Taylor coefficients, just a difference of some combinatorial numerical coefficients; but the group operation (2) can then be obtained merely by repeated applications of the chain rule for differentiation. In particular if $\xi = j_k f$, $\eta = j_k g$, and $\zeta = j_k(f \circ g) = \xi \cdot \eta$ it follows readily that

$$(6) \qquad \zeta^i_j = \Sigma_k\, \xi^i_k\, \eta^k_j ,$$

$$(7) \qquad \zeta^i_{j_1 j_2} = \Sigma_k\, \xi^i_{k_1 k_2}\, \eta^{k_1}_{j_1}\, \eta^{k_2}_{j_2} + \Sigma_k\, \xi^i_k\, \eta^k_{j_1 j_2} ,$$

$$(8) \qquad \zeta^i_{j_1 j_2 j_3} = \Sigma_k\, \xi^i_{k_1 k_2 k_3}\, \eta^{k_1}_{j_1}\, \eta^{k_2}_{j_2}\, \eta^{k_3}_{j_3} + \Sigma_k\, \xi^i_{k_1 k_2}(\eta^{k_1}_{j_1 j_2}\eta^{k_2}_{j_3} + \eta^{k_1}_{j_1 j_3}\eta^{k_2}_{j_2} + \eta^{k_1}_{j_2 j_3}\eta^{k_2}_{j_1}) + \Sigma_k\, \xi^i_k\, \eta^k_{j_1 j_2 j_3} ,$$

and so on. Formula (6) is just the usual matrix product; and while the ensuing formulas are somewhat more complicated, their general pattern is quite transparent. Indeed $\zeta^i_{j_1\cdots j_p}$ is a sum of p terms, the q-th of which is of the form $\Sigma_k\, \xi^i_{k_1\ldots k_q}\, P_{k_1\ldots k_q}(\eta)$ where $P_{k_1\ldots k_q}(\eta)$ denotes some polynomial function of the components of the tensors η. That polynomial is in turn a sum of terms of the form $\eta_{J_1}^{k_1} \cdots \eta_{J_q}^{k_q}$ where J_r are various subsets of the indices $j_1, \ldots, j_p$ such that $(J_1, \ldots, J_q)$ is a permutation of the indices $(j_1, \ldots, j_p)$; all possible sizes of subsets J_r appear, since all such differentiations appear upon iterating the chain rule, and the sum must be formally symmetric in the indices $j_1, \ldots, j_p$. Thus in general

$$(9) \qquad \zeta^i_{j_1 \ldots j_p} = \Sigma^p_{q=1} \Sigma_k \xi^i_{k_1 \ldots k_q} \Sigma_\nu \; S_J \, \eta_{J_1}^{k_1} \cdots \eta_{J_q}^{k_q} .$$

Here Σ_ν denotes a sum over all sets of integers $\nu_1, \ldots, \nu_q$ such that $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_q \geq 1$ and $\nu_1 + \cdots + \nu_q = p$; $J_1 = (j_1, \ldots, j_{\nu_1})$ and so on, J_r consisting of ν_r of the indices $j_1, \ldots, j_p$; and S_J denotes a sum over some set of permutations of the indices $j_1, \ldots, j_p$. Actually S_J consists of the minimal sum needed to ensure the formal symmetry of $\zeta^i_{j_1 \ldots j_q}$ in the lower indices, taking into account the symmetries of the tensors ξ and η ; but that is a finer point than is really needed here, so will not be proved. Indeed the general formula is not really needed, and it is an easy matter to verify any particular case of the formula. For example in the case $p = 4$, the next case after (8), the formula is

$$(10) \qquad \begin{aligned} \zeta^i_{j_1 j_2 j_3 j_4} &= \Sigma_k \xi^i_{k_1 k_2 k_3 k_4} \eta^{k_1}_{j_1} \eta^{k_2}_{j_2} \eta^{k_3}_{j_3} \eta^{k_4}_{j_4} \\ &+ \Sigma_k \xi^i_{k_1 k_2 k_3} S_1 \eta^{k_1}_{j_1 j_2} \eta^{k_2}_{j_3} \eta^{k_3}_{j_4} \\ &+ \Sigma_k \xi^i_{k_1 k_2} (S_2 \eta^{k_1}_{j_1 j_2 j_3} \eta^{k_2}_{j_4} + S_3 \eta^{k_1}_{j_1 j_2} \eta^{k_2}_{j_3 j_4}) \\ &+ \Sigma_k \xi^i_k \eta^k_{j_1 j_2 j_3 j_4} \quad , \end{aligned}$$

where S_1 is a sum over 6 permutations, S_2 is a sum over 4 permutations, and S_3 is a sum over 3 permutations; for S_1 the expression is already symmetric in the indices j_1 and j_2, and is also symmetric in the indices j_3 and j_4 (since ξ is symmetric), so the summation is only extended over a set of permutations in the symmetric group on 4 letters which represent cosets of the subgroup describing this symmetry, and similarly in the other cases.

The structure of the Lie group G_k can be described in general terms rather easily, without making much use of the preceding detailed form of the product operation; but more details will be needed later in describing subgroups of G_k. Note that for any integers $1 \le \ell \le k$ it is possible to consider the ℓ-jet $j_\ell \xi$ of a k-jet $\xi \in G_k$; this defines a mapping

$$j_\ell : G_k(n,\mathbb{C}) \to G_\ell(n,\mathbb{C}) ,$$

which is evidently a surjective group homomorphism. In terms of the representation (4) of course

$$j_\ell\{\xi_1, \ldots, \xi_k\} = \{\xi_1, \ldots, \xi_\ell\} .$$

For the special case $\ell = k-1$ the kernel of this group homomorphism can be identified with the vector space T_k; indeed the kernel of this homomorphism is clearly the point set $\delta_1 \oplus 0 \oplus \ldots \oplus 0 \oplus T_k$ in the decomposition (1), where $\delta_1 \in T_1$ is the identity matrix (the identity element in G_k being $\delta_1 \oplus 0 \oplus \ldots \oplus 0$), and it follows easily from (9) that in this subgroup the group operation amounts to addition in the vector space T_k. There thus arises