SHOSHICHI KOBAYASHI

# Differential Geometry of Complex Vector Bundles

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# DIFFERENTIAL GEOMETRY OF COMPLEX VECTOR BUNDLES

*by*
Shoshichi Kobayashi

KANÔ MEMORIAL LECTURES 5

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## Kanô Memorial Lectures

In 1969, the Mathematical Society of Japan received an anonymous donation to encourage the publication of lectures in mathematics of distinguished quality in commemoration of the late Kôkichi Kanô (1865–1942).

K. Kanô was a remarkable scholar who lived through an era when Western mathematics and philosophy were first introduced to Japan. He began his career as a scholar by studying mathematics and remained a rationalist for his entire life, but enormously enlarged the domain of his interest to include philosophy and history.

In appreciating the sincere intentions of the donor, our Society has decided to publish a series of "Kanô Memorial Lectures" as a part of our Publications. This is the fifth volume in the series.

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Dedicated to
Professor Kentaro Yano

It was some 35 years ago that I learned from him Bochner's method of proving vanishing theorems, which plays a central role in this book.

# Preface

In order to construct good moduli spaces for vector bundles over algebraic curves, Mumford introduced the concept of a stable vector bundle. This concept has been generalized to vector bundles and, more generally, coherent sheaves over algebraic manifolds by Takemoto, Bogomolov and Gieseker. As the differential geometric counterpart to the stability, I introduced the concept of an Einstein-Hermitian vector bundle. The main purpose of this book is to lay a foundation for the theory of Einstein-Hermitian vector bundles. We shall not give a detailed introduction here in this preface since the table of contents is fairly self-explanatory and, furthermore, each chapter is headed by a brief introduction.

My first serious encounter with stable vector bundles was in the summer of 1978 in Bonn, when F. Sakai and M. Reid explained to me the work of Bogomolov on stable vector bundles. This has led me to the concept of an Einstein-Hermitian vector bundle. In the summer of 1981 when I met M. Lübke at DMV Seminar in Düsseldorf, he was completing the proof of his inequality for Einstein-Hermitian vector bundles, which rekindled my interest in the subject.

With this renewed interest, I lectured on vanishing theorems and Einstein-Hermitian vector bundles at the University of Tokyo in the fall of 1981. The notes taken by I. Enoki and published as Seminar Notes 41 of the Department of Mathematics of the University of Tokyo contained good part of Chapters I, III, IV and V of this book. Without his notes which filled in many details of my lectures, this writing project would not have started. In those lectures I placed much emphasis——perhaps too much emphasis——on vanishing theorems. In retrospect, we need mostly vanishing theorems for holomorphic sections for the purpose of this book, but I decided to include cohomology vanishing theorems as well.

During the academic year 1982/83 in Berkeley and in the summer of 1984 in Tsukuba, I gave a course on holomorphic vector bundles. The notes of these lectures ("Stable Vector Bundles and Curvature" in the "Survey in Geometry" series) distributed to the audience, consisted of the better part of Chapters I through V. My lectures at the Tsukuba workshop were supplemented by talks by T. Mabuchi (on Donaldson's work) and by M. Itoh (on Yang-Mills theory). In writing Chapter VI, which is mainly on the work of Donaldson on stable bundles over algebraic surfaces, I made good use of Mabuchi's notes.

During the fall of 1985 in Berkeley, H.-J. Kim gave several seminar talks on moduli of Einstein-Hermitian vector bundles. Large part of Chapter VII is based on his Berkeley thesis as well as Itoh's work on moduli of anti-self-dual connections on Kähler surfaces. While I was revising the manuscript in this final form, I had occasions to talk with Professor C. Okonek on the subject of stable bundles, and I found discussions with him particularly enlightening.

In addition to the individuals mentioned above, I would like to express my gratitude to the National Science Foundation for many years of financial support, to Professor F. Hirzebruch and Sonderforschungsbereich in Bonn, where I started my work on Einstein-Hermitian vector bundles, and to Professors A. Hattori and T. Ochiai of the University of Tokyo and the Japan Society for Promotion of Sciences for giving me an opportunity to lecture on holomorphic vector bundles. I would like to thank also Professor S. Iyanaga for inviting me to publish this work in Publications of the Mathematical Society of Japan and Mr. H. Arai of Iwanami Shoten for his efficient co-operation in the production of this book.

February, 1986

S. Kobayashi

# Contents

# Chapter I

# Connections in vector bundles

Although our primary interest lies in holomorphic vector bundles, we begin this chapter with the study of connections in differentiable complex vector bundles. In order to discuss moduli of holomorphic vector bundles, it is essential to start with differentiable complex vector bundles. In discussing Chern classes it is also necessary to consider the category of differentiable complex vector bundles rather than the category of holomorphic vector bundles which is too small and too rigid.

Most of the results in this chapter are fairly standard and should be well known to geometers. They form a basis for the subsequent chapters. As general references on connections, we mention Kobayashi-Nomizu [1] and Chern [1].

## § 1 Connections in complex vector bundles (over real manifolds)

Let $M$ be an $n$-dimensional real $C^\infty$ manifold and $E$ a $C^\infty$ complex vector bundle of rank (=fibre dimension) $r$ over $M$. We make use of the following notations:

$A^p$ = the space of $C^\infty$ complex $p$-forms over $M$,

$A^p(E)$ = the space of $C^\infty$ complex $p$-forms over $M$ with values in $E$.

A *connection* $D$ in $E$ is a homomorphism

$$D\colon A^0(E) \longrightarrow A^1(E)$$

over $\boldsymbol{C}$ such that

$$(1.1) \qquad D(f\sigma) = \sigma df + f \cdot D\sigma \qquad \text{for} \quad f \in A^0, \quad \sigma \in A^0(E),$$

Let $s = (s_1, \cdots, s_r)$ be a local frame field of $E$ over an open set $U \subset M$, i.e.,

i) $s_i \in A^0(E|_U) \qquad i = 1, \cdots, r,$

ii) $(s_1(x), \cdots, s_r(x))$ is a basis of $E_x$ for each $x \in U$.

Then given a connection $D$, we can write

$$(1.2) \qquad Ds_i = \sum s_j \omega_i^j, \qquad \text{where} \quad \omega_i^j \in A^1|_U$$

We call the matrix 1-form $\omega = (\omega_i^j)$ the *connection form* of $D$ with respect to the frame field $s$. Considering $s = (s_1, \cdots, s_r)$ as a row vector, we can rewrite (1.2) in matrix notations as follows:

$$(1.2)' \qquad Ds = s \cdot \omega\,.$$

If $\xi = \sum \xi^i s_i$, $\xi^i \in A^0|_U$, is an arbitrary section of $E$ over $U$, then (1.1) and (1.2) imply

$$(1.3) \qquad D\xi = \sum s_i (d\xi^i + \sum \omega^i_j \xi^j)\,.$$

Considering $\xi = {}^t(\xi^1, \cdots, \xi^r)$ as a column vector, we may rewrite (1.3) as follows:

$$(1.3)' \qquad D\xi = d\xi + \omega\xi\,.$$

We call $D\xi$ the *covariant derivative of* $\xi$.

Evaluating $D$ on a tangent vector $X$ of $M$ at $x$, we obtain an element of the fibre $E_x$ denoted by

$$(1.4) \qquad D_X\xi = (D\xi)(X) \in E_x\,.$$

We call $D_X\xi$ the *covariant derivative of* $\xi$ *in the direction of* $X$.

A section $\xi$ is said to be *parallel* if $D\xi = 0$. If $c = c(t)$, $0 \leqq t \leqq a$, is a curve in $M$, a section $\xi$ defined along $c$ is said to be *parallel along* $c$ if

$$(1.5) \qquad D_{c'(t)}\xi = 0 \qquad \text{for} \quad 0 \leqq t \leqq a\,,$$

where $c'(t)$ denotes the velocity vector of $c$ at $c(t)$. In terms of the local frame field $s$, (1.5) can be written as a system of ordinary differential equations

$$(1.5)' \qquad \frac{d\xi^i}{dt} + \sum \omega^i_j(c'(t))\xi^j = 0\,.$$

If $\xi_0$ is an element of the initial fibre $E_{c(0)}$, it extends uniquely to a parallel section $\xi$ along $c$, called the *parallel displacement of* $\xi_0$ *along* $c$. This is a matter of solving the system of ordinary differential equations (1.5)′ with initial condition $\xi_0$. If the initial point and the end point of $c$ coincide so that $x_0 = c(0) = c(a)$, then the parallel displacement along $c$ induces a linear transformation of the fibre $E_{x_0}$. The set of endomorphisms of $E_{x_0}$ thus obtained from all closed curves $c$ starting at $x_0$ forms a group, called the *holonomy group* of the connection $D$ with reference point $x_0$.

We shall now study how the connection form $\omega$ changes when we change the local frame field $s$. Let $s' = (s'_1, \cdots, s'_r)$ be another local frame field over $U$. It is related to $s$ by

$$(1.6) \qquad s = s' \cdot a\,,$$

where $a\colon U \to GL(r; C)$ is a matrix-valued function on $U$. Let $\omega' = (\omega_j'^i)$ be the connection form of $D$ with respect to $s'$. Then

$$(1.7) \qquad \omega = a^{-1}\omega' a + a^{-1} da .$$

In fact,

$$s\omega = Ds = D(s'a) = (Ds')a + s'da = s'\omega' a + s'da = s(a^{-1}\omega' a + a^{-1} da) .$$

We extend a connection $D\colon A^0(E) \to A^1(E)$ to a $C$-linear map

$$(1.8) \qquad D\colon A^p(E) \longrightarrow A^{p+1}(E) , \qquad p \geqq 0 ,$$

by setting

$$(1.9) \qquad D(\sigma \cdot \varphi) = (D\sigma) \wedge \varphi + \sigma \cdot d\varphi \qquad \text{for} \quad \sigma \in A^0(E) , \quad \varphi \in A^p .$$

Using this extended $D$, we define the *curvature* $R$ of $D$ to be

$$(1.10) \qquad R = D \circ D\colon A^0(E) \longrightarrow A^2(E) .$$

Then $R$ is $A^0$-linear. In fact, if $f \in A^0$ and $\sigma \in A^0(E)$, then

$$D^2(f\sigma) = D(\sigma df + f \cdot D\sigma) = D\sigma \wedge df + df \wedge D\sigma + fD^2\sigma = fD^2\sigma .$$

Hence, $R$ is a 2-form on $M$ with values in End($E$). Using matrix notations of (1.2)′, the *curvature form* $\Omega$ of $D$ with respect to the frame field $s$ is defined by

$$(1.11) \qquad s\Omega = D^2 s .$$

Then

$$(1.12) \qquad \Omega = d\omega + \omega \wedge \omega .$$

In fact,

$$s\Omega = D(s\omega) = Ds \wedge \omega + sd\omega = s(\omega \wedge \omega + d\omega) .$$

Exterior differentiation of (1.12) gives the *Bianchi identity*:

$$(1.13) \qquad d\Omega = \Omega \wedge \omega - \omega \wedge \Omega .$$

If $\omega'$ is the connection form of $D$ relative to another frame field $s' = sa^{-1}$ as in (1.6) and (1.7), the corresponding curvature form $\Omega'$ is related to $\Omega$ by

$$(1.14) \qquad \Omega = a^{-1}\Omega' a .$$

In fact,

$$s\Omega = D^2 s = D^2(s'a) = D(Ds'a + s'da) = D^2s'a - Ds' \wedge da + Ds' \wedge da$$
$$= s'\Omega' a = sa^{-1}\Omega' a .$$

Let $\{U, V, \cdots\}$ be an open cover of $M$ with a local frame field $s_U$ on each $U$. If $U \cap V \neq \emptyset$, then

$$s_U = s_V g_{VU} \quad \text{on} \quad U \cap V , \tag{1.15}$$

where $g_{VU}\colon U \cap V \to GL(r; C)$ is a $C^\infty$ mapping, called a transition function. Given a connection $D$ in $E$, let $\omega_U$ be the connection form on $U$ with respect to $s_U$. Then (1.7) means

$$\omega_U = g_{VU}^{-1}\omega_V g_{VU} + g_{VU}^{-1} dg_{VU} \quad \text{on} \quad U \cap V . \tag{1.16}$$

Conversely, given a system of $\mathfrak{gl}(r; C)$-valued 1-forms $\omega_U$ on $U$ satisfying (1.16), we obtain a connection $D$ in $E$ having $\{\omega_U\}$ as connection forms.

If $\Omega_U$ is the curvature form of $D$ relative to $s_U$, then (1.14) means

$$\Omega_U = g_{VU}^{-1}\Omega_V g_{VU} \quad \text{on} \quad U \cap V . \tag{1.17}$$

It is sometimes more convenient to consider a connection in $E$ as a connection in the associated principal $GL(r; C)$-bundle $P$. In general, let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra identified with the tangent space $T_eG$ at the identity element $e$ of $G$. Let $P$ be a principal $G$-bundle over $M$. Let $\{U\}$ be an open cover of $M$ with local sections $\{s_U\}$ of $P$. Let $\{g_{VU}\}$ be the family of transition functions defined by $\{(U, s_U)\}$; $g_{VU}\colon U \cap V \to G$ is defined by

$$s_U(x) = s_V(x) g_{VU}(x) \qquad x \in U \cap V .$$

A *connection* in $P$ is given by a family of $\mathfrak{g}$-valued 1-forms $\omega_U$ on $U$ satisfying (1.16).

A connection in $P$ induces a connection in every bundle associated to $P$. In particular, a connection in a principal $GL(r; C)$-bundle $P$ induces a connection in the vector bundle $E^\rho = P \times_\rho C^N$ associated to $P$ by any representation $\rho\colon GL(r; C) \to GL(N; C)$.

For further details on connections in principal bundles, see Kobayashi-Nomizu [1].

## § 2 Flat bundles and flat connections

Let $E$ be a $C^\infty$ complex vector bundle over a real manifold $M$ as in the preceding section. A *flat structure* in $E$ is given by an open cover $\{U, s_U\}$ with local frame fields such that the transition functions $\{g_{VU}\}$ (see (1.15)) are all constant matrices in $GL(r; C)$. A vector bundle with a flat structure is said to be *flat*. On the other hand, a connection $D$ in a vector bundle $E$ is said to be *flat* if its curvature $R$ vanishes.

A flat vector bundle $E$ admits a natural flat connection $D$; namely, if the flat structure is given by $\{U, s_U\}$, then $D$ is defined by

$$(2.1) \qquad Ds_U = 0 .$$

Since the transition functions $\{g_{VU}\}$ are all constants, the condition $Ds_U = 0$ and $Ds_V = 0$ are compatible on $U \cap V$ and the connection $D$ is well defined. We note that if $\omega_U$ is the connection form of $D$ relative to $s_U$, then (2.1) is equivalent to

$$(2.1)' \qquad \omega_U = 0 .$$

From (2.1) (or (2.1)′), it follows that the curvature of $D$ vanishes.

Conversely, a vector bundle $E$ with a flat connection $D$ admits a natural flat structure $\{U, s_U\}$. To construct a local frame field $s_U$ satisfying (2.1), we start with an arbitrary local frame field $s'$ on $U$ and try to find a function $a\colon U \to GL(r; C)$ such that $s_U = s'a$ satisfies (2.1)′. Let $\omega'$ be the connection form of $D$ relative to $s'$. By (1.7), the condition $\omega_U = 0$ is equivalent to

$$(2.2) \qquad a^{-1}\omega' a + a^{-1} da = \omega_U = 0 .$$

This is a system of differential equations where $\omega'$ is given and $a$ is the unknown. Multiplying (2.2) by $a$ and differentiating the resulting equation

$$\omega' a + da = 0 ,$$

we obtain the integrability condition

$$0 = (d\omega')a - \omega' \wedge da = (d\omega')a + (\omega' \wedge \omega')a = \Omega' a .$$

So the integrability condition is precisely vanishing of the curvature $\Omega' = 0$.

In general, if $s$ and $s'$ are two local parallel frame fields, i.e., if $Ds = Ds' = 0$, then $s = s'a$ for some constant matrix $a \in GL(r; C)$ since the connection forms $\omega$ and $\omega'$ vanish in (1.7). Hence, if $Ds_U = Ds_V = 0$, the $g_{VU}$ is a constant matrix. This proves that a flat connection $D$ gives rise to a flat structure $\{U, s_U\}$.

Let $E$ be a vector bundle with a flat connection $D$. Let $x_0$ be a point of $M$ and $\pi_1$ the fundamental group of $M$ with reference point $x_0$. Since the connection is flat, the parallel displacement along a closed curve $c$ starting at $x_0$ depends only on the homotopy class of $c$. So the parallel displacement gives rise to a representation

(2.3) $$\rho\colon \pi_1 \longrightarrow GL(r; \boldsymbol{C}).$$

The image of $\rho$ is the holonomy group of $D$ defined in the preceding section.

Conversely, given a representation (2.3), we can construct a flat vector bundle $E$ by setting

(2.4) $$E = \tilde{M} \times_\rho \boldsymbol{C}^r,$$

where $\tilde{M}$ is the universal covering of $M$ and $\tilde{M} \times_\rho \boldsymbol{C}^r$ denotes the quotient of $\tilde{M} \times \boldsymbol{C}^r$ by the action of $\pi_1$ given by

$$\gamma\colon (x, v) \in \tilde{M} \times \boldsymbol{C}^r \longmapsto (\gamma(x), \rho(\gamma)v) \in \tilde{M} \times \boldsymbol{C}^r, \qquad \gamma \in \pi_1$$

(We are considering $\pi_1$ as the covering transformation group acting on $\tilde{M}$). It is easy to see that $E$ carries a natural flat structure coming from the product structure of $\tilde{M} \times \boldsymbol{C}^r$. The vector bundle defined by (2.4) is said to be *defined by the representation* $\rho$. In summary, we have established the following

(2.5) **Proposition** *For a complex vector bundle $E$ of rank $r$ over $M$, the following three conditions are equivalent:*

(1) *$E$ is a flat vector bundle,*

(2) *$E$ admits a flat connection $D$,*

(3) *$E$ is defined by a representation $\rho\colon \pi_1 \to GL(r; \boldsymbol{C})$.*

A connection in a vector bundle $E$ may be considered as a connection in the associated principal $GL(r; \boldsymbol{C})$-bundle. More generally, let $G$ be a Lie group and $P$ a principal $G$-bundle over $M$. Let $\{U\}$ be an open cover of $M$ with local sections $\{s_U\}$ of $P$. Let $\{g_{VU}\}$ be the family of transition functions defined by $\{(U, s_U)\}$; each $g_{VU}$ is a $C^\infty$ maps from $U \cap V$ into $G$. A *flat structure* in $P$ is given by $\{(U, s_U)\}$ such that $\{g_{VU}\}$ are all constant maps. A connection in $P$ is said to be *flat* if its curvature vanishes identically.

Let $\tilde{M}$ be the universal covering space of $M$; it is considered as a principal $\pi_1$-bundle over $M$, where $\pi_1$ is the fundamental group of $M$ acting on $\tilde{M}$ as the group of covering transformations. Given a homomorphism $\rho\colon \pi_1 \to G$, we obtain a principal $G$-bundle $P = \tilde{M} \times_\rho G$ by "enlarging" the structure group from $\pi_1$ to $G$. Then $P$ inherits a flat structure from the natural flat structure of the product bundle $\tilde{M} \times G$ over $\tilde{M}$. The following proposition is a straightforward generalization of (2.5).

(2.6) **Proposition** *For a principal $G$-bundle $P$ over $M$, the following three* *conditions are equivalent:*

(1) *$P$ admits a flat structure,*

(2) *$P$ admits a flat connection,*

(3) *$P$ is defined by a representation $\rho: \pi_1 \to G$.*

Applied to the principal $GL(r; C)$-bundle associated to a vector bundle $E$, (2.6) yields (2.5). We shall now consider the case where $G$ is the projective linear group $PGL(r; C) = GL(r; C)/C^*I_r$, (where $C^*I_r$ denotes the center of $GL(r; C)$ consisting of scalar multiples of the identity matrix $I_r$). Given a vector bundle $E$, let $P$ be the associated principal $GL(r; C)$-bundle. Then $\hat{P} = P/C^*I_r$ is a principal $PGL(r; C)$-bundle. We say that $E$ is *projectively flat* when $\hat{P}$ is provided with a flat structure. A connection $D$ in $E$ (i.e., a connection in $P$) is said to be *projectively flat* if the induced connection in $\hat{P}$ is flat. As a special case of (2.6), we have

(2.7) **Corollary** *For a complex vector bundle $E$ of rank $r$ over $M$ with the associated principal $GL(r; C)$-bundle $P$, the following three conditions are equivalent:*

(1) *$E$ is projectively flat,*

(2) *$E$ admits a projectively flat connection,*

(3) *The $PGL(r; C)$-bundle $\hat{P} = P/C^*I_r$ is defined by a representation $\rho: \pi_1 \to PGL(r; C)$.*

Let $\mu: GL(r; C) \to PGL(r; C)$ be the natural homomorphism and $\mu': \mathfrak{gl}(r; C) \to \mathfrak{pgl}(r; C)$ the corresponding Lie algebra homomorphism. If $R$ denotes the curvature of a connection $D$ in $E$, then the curvature of the induced connection in $\hat{P}$ is given by $\mu'(R)$. Hence,

(2.8) **Proposition** *A connection $D$ in a complex vector bundle $E$ over $M$ is projectively flat if and only if its curvature $R$ takes values in scalar multiples of the identity endomorphism of $E$, i.e., if and only if there exists a complex 2-form $\alpha$ on $M$ such that*

$$R = \alpha I_E .$$

Let $V = C^r$. If $A: V \to V$ is a linear transformation of the form $aI_r$, $a \in C^*$, then $A \otimes {}^tA^{-1}$ is the identity transformation of $\mathrm{End}(V) = V \otimes V^*$. Hence,

(2.9) **Proposition** *If a complex vector bundle $E$ is projectively flat, then the bundle* $\mathrm{End}(E) = E \otimes E^*$ *is flat in a natural manner.*

## § 3 Connections in complex vector bundles (over complex manifolds)

Let $M$ be an $n$-dimensional complex manifold and $E$ a $C^\infty$ complex vector bundle of rank $r$ over $M$. In addition to the notations $A^p$ and $A^p(E)$ introduced in Section 1, we use the following:

$A^{p,q}$ = the space of $(p,q)$-forms over $M$,

$A^{p,q}(E)$ = the space of $(p,q)$-forms over $M$ with values in $E$,

so that

$$A^r = \sum_{p+q=r} A^{p,q}, \qquad A^r(E) = \sum_{p+q=r} A^{p,q}(E).$$

$$d = d' + d'', \quad \text{where} \quad d' : A^{p,q} \longrightarrow A^{p+1,q} \quad \text{and} \quad d'' : A^{p,q} \longrightarrow A^{p,q+1}.$$

Let $D$ be a connection in $E$ as defined in Section 1. We can write

$$D = D' + D'', \quad \text{where}$$

$$D' : A^{p,q}(E) \longrightarrow A^{p+1,q}(E) \quad \text{and} \quad D'' : A^{p,q}(E) \longrightarrow A^{p,q+1}(E).$$

Decomposing (1.1) and (1.9) according to the bidegree, we obtain

$$\text{(3.1)} \qquad \begin{aligned} D'(\sigma\varphi) &= D'\sigma \wedge \varphi + \sigma d'\varphi \\ D''(\sigma\varphi) &= D''\sigma \wedge \varphi + \sigma d''\varphi \end{aligned} \qquad \sigma \in A^0(E), \quad \varphi \in A^{p,q}.$$

Let $R$ be the curvature of $D$, i.e., $R = D \circ D \in A^2(\mathrm{End}(E))$. Then

$$\text{(3.2)} \qquad R = D' \circ D' + (D' \circ D'' + D'' \circ D') + D'' \circ D'',$$

where

$$D' \circ D' \in A^{2,0}(\mathrm{End}(E)), \qquad D'' \circ D'' \in A^{0,2}(\mathrm{End}(E)),$$
$$D' \circ D'' + D'' \circ D' \in A^{1,1}(\mathrm{End}(E)).$$

Let $s$ be a local frame field of $E$ and let $\omega$ and $\Omega$ be the connection and the curvature forms of $D$ with respect to $s$. We can write

$$\text{(3.3)} \qquad \omega = \omega^{1,0} + \omega^{0,1},$$

$$\text{(3.4)} \qquad \Omega = \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}.$$

We shall now characterize in terms of connections those complex vector bundles which admit holomorphic structures.

(3.5) **Proposition** *Let $E$ be a holomorphic vector bundle over a complex manifold $M$. Then there is a connection $D$ such that*

$$D'' = d'' . \tag{3.6}$$

*For such a connection, the* $(0,2)$*-component* $D'' \circ D''$ *of the curvature $R$ vanishes.*

*Proof* Let $\{U\}$ be a locally finite open cover of $M$ and $\{p_U\}$ a partition of 1 subordinate to $\{U\}$. Let $s_U$ be a holomorphic frame field of $E$ on $U$. Let $D_U$ be the flat connection in $E|_U$ defined by $D_U(s_U)=0$. Then $D=\sum p_U D_U$ is a connection in $E$ with the property that $D''=d''$. The second assertion is obvious. Q.E.D.

Conversely, we have

(3.7) **Proposition** *Let $E$ be a $C^\infty$ complex vector bundle over a complex manifold $M$. If $D$ is a connection in $E$ such that* $D'' \circ D''=0$, *then there is a unique holomorphic vector bundle structure in $E$ such that* $D''=d''$.

*Proof* We define an almost complex structure on $E$ by specifying a splitting of the complex cotangent spaces into their $(1,0)$- and $(0,1)$-components and verify the integrability condition. Intrinsically, such a splitting is obtained by identifying the horizontal subspace of each tangent space of $E$ with the corresponding tangent space of $M$. But we shall express this construction more explicitly in terms of local coordinates. We fix a local trivialization $E|_U = U \times \boldsymbol{C}^r$. Let $(z^1, \cdots, z^n)$ be a coordinate system in $U$, and $(w^1, \cdots, w^r)$ the natural coordinate system in $\boldsymbol{C}^r$. Let $\omega = (\omega^i_j)$ be the connection form in this trivialization. Let

$$\omega^i_j = \omega'^i_j + \omega''^i_j$$

be the decomposition into $(1,0)$- and $(0,1)$-components. Now we define an almost complex structure on $E$ by taking

$$\{dz^\alpha, dw^i + \textstyle\sum \omega''^i_j w^j\} \tag{3.8}$$

as a basis for the space of $(1,0)$-forms on $E$. Since

$$d(dw^i + \textstyle\sum \omega''^i_j w^j) \equiv \sum (d\omega''^i_j + \sum \omega''^i_k \wedge \omega''^k_j) w^j \equiv \sum d'\omega''^i_j w^j \equiv 0$$

modulo the ideal generated by (3.8), this almost complex structure is integrable. In order to show that this holomorphic structure of $E$ has the desired property, it suffices to verify that if a local section $s$ of $E$ satisfies the equation $D''s=0$, then it pulls back every $(1,0)$-form of $E$ to a $(1,0)$-form of the base manifold $M$. Let $s$ be given locally by

$$s: U \longrightarrow U \times \mathbf{C}^r, \qquad s(z) = (z, \xi(z)).$$

Then the condition $D''s = 0$ is given by

$$d''\xi^i + \sum \omega''^i_j \xi^j = 0.$$

Pulling back the (1,0)-forms in (3.8) we obtain (1,0)-forms:

$$\begin{aligned} &s^*(dz^\alpha) = dz^\alpha, \\ &s^*(dw^i + \sum \omega''^i_j w^j) = d\xi^i + \sum \omega''^i_j \xi^j = d'\xi^i \end{aligned}$$

The uniqueness is obvious. Q.E.D.

(3.9) **Proposition** *For a connection $D$ in a holomorphic vector bundle $E$, the following conditions are equivalent*:

(a) $D'' = d''$;

(b) *For every local holomorphic section $s$, $Ds$ is of degree* (1,0);

(c) *With respect to a local holomorphic frame field, the connection form is of degree* (1,0).

The proof is trivial.

## § 4 Connections in Hermitian vector bundles

Let $E$ be a $C^\infty$ complex vector bundle over a (real or complex) manifold $M$. An *Hermitian structure* or *Hermitian metric* $h$ in $E$ is a $C^\infty$ field of Hermitian inner products in the fibers of $E$. Thus,

$$\begin{aligned} &h(\xi, \eta) \text{ is linear in } \xi, \qquad \text{where} \quad \xi, \eta \in E_x, \\ &h(\xi, \eta) = \overline{h(\eta, \xi)} \\ &h(\xi, \xi) > 0 \qquad \text{for} \quad \xi \neq 0 \\ &h(\xi, \eta) \text{ is a } C^\infty \text{ function if } \xi \text{ and } \eta \text{ are } C^\infty \text{ sections}. \end{aligned} \tag{4.1}$$

We call $(E, h)$ an *Hermitian vector bundle*.

Given a local frame field $s_U = (s_1, \cdots, s_r)$ of $E$ over $U$, we set

$$h_{ij} = h(s_i, s_j), \qquad i, j = 1, \cdots, r, \tag{4.2}$$

and

(4.3) $$H_U = (h_{i\bar{j}}) .$$

Then $H_U$ is a positive definite Hermitian matrix at every point of $U$. When we are working with a single frame field, we often drop the subscript $U$. We say that $s_U$ is a *unitary frame field* or *orthonormal frame field* if $H_U$ is the identity matrix. Under a change of frame field (1.15) given by $s_U = s_V g_{VU}$, we have

(4.4) $$H_U = {}^t g_{VU} H_V \bar{g}_{VU} \quad \text{on} \quad U \cap V .$$

A connection $D$ in $(E, h)$ is called an *h-connection* if it preserves $h$ or makes $h$ parallel in the following sense:

(4.5) $$d(h(\xi, \eta)) = h(D\xi, \eta) + h(\xi, D\eta) \qquad \text{for} \quad \xi, \eta \in A^0(E) .$$

Let $\omega = (\omega^i_j)$ be the connection form relative to $s_U$ defined by (1.2). Then setting $\xi = s_i$ and $\eta = s_j$ in (4.5), we obtain

(4.6) $$dh_{i\bar{j}} = h(Ds_i, s_j) + h(s_i, Ds_j) = \sum \omega^a_i h_{a\bar{j}} + h_{i\bar{b}} \bar{\omega}^b_j .$$

In matrix notation,

(4.6)′ $$dH = {}^t\omega H + H\bar{\omega} .$$

Applying $d$ to (4.6)′ we obtain

(4.7) $${}^t\Omega H + H\bar{\Omega} = 0 .$$

Let $H$ be the identity matrix in (4.6)′ and (4.7). Thus,

(4.8) $${}^t\omega + \bar{\omega} = 0 , \qquad {}^t\Omega + \bar{\Omega} = 0 \qquad \text{if} \quad s_U \quad \text{is unitary} ,$$

that is, $\omega$ and $\Omega$ are skew-Hermitian with respect to a unitary frame. This means that $\omega$ and $\Omega$ take values in the Lie algebra $\mathfrak{u}(r)$ of the unitary group $U(r)$.

We shall now study holomorphic vector bundles. If $E$ is a holomorphic vector bundle (over a complex manifold $M$), then an Hermitian structure $h$ determines a natural $h$-connection satisfying (3.6). Namely, we have

(4.9) **Proposition** *Given an Hermitian structure $h$ in a holomorphic vector bundle $E$, there is a unique $h$-connection $D$ such that $D'' = d''$.*

*Proof* Let $s_U = (s_1, \cdots, s_r)$ be a local holomorphic frame field on $U$. Since $Ds_i = D's_i$, the connection form $\omega_U = (\omega^i_j)$ is of degree (1,0). From (4.6) we obtain

(4.10) $$d'h_{i\bar{j}} = \sum \omega^a_i h_{a\bar{j}} \qquad \text{or} \qquad d'H_U = {}^t\omega_U H_U .$$

This determines the connection form $\omega_U$, i.e.,

$$(4.11)\qquad {}^t\omega_U = d'H_U H_U^{-1}\,,$$

proving the uniqueness part. To prove the existence, we compare ${}^t\omega_U$ with ${}^t\omega_V = d'H_V H_V^{-1}$. Using (4.4) we verify by a straightforward calculation that $\omega_U$ and $\omega_V$ satisfy (1.16). Then the collection $\{\omega_U\}$ defines the desired connection. Q.E.D.

We call the connection given by (4.9) the *Hermitian connection* of a holomorphic Hermitian vector bundle $(E, h)$. Its connection form is given explicitly by (4.11).

Its curvature $R = D \circ D$ has no $(0,2)$-component since $D'' \circ D'' = d'' \circ d'' = 0$. By (4.7) it has no $(2,0)$-component either. So the curvature

$$(4.12)\qquad R = D' \circ D'' + D'' \circ D'$$

is a $(1,1)$-form with values in End$(E)$.

With respect to a local holomorphic frame field, the connection form $\omega = (\omega^i_j)$ is of degree $(1,0)$, see (3.9). Since the curvature form $\Omega$ is equal to the $(1,1)$-component of $d\omega + \omega \wedge \omega$, we obtain

$$(4.13)\qquad \Omega = d''\omega\,.$$

From (4.11) we obtain

$$(4.14)\qquad {}^t\Omega = d''d'H\cdot H^{-1} + dH'\cdot H^{-1} \wedge d''H\cdot H^{-1}\,.$$

We write

$$(4.15)\qquad \Omega^i_j = \sum R^i_{j\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta$$

so that

$$(4.16)\qquad R_{j\bar k\alpha\bar\beta} = \sum h_{i\bar k} R^i_{j\alpha\bar\beta} = -\partial_{\bar\beta}\partial_\alpha h_{j\bar k} + \sum h^{a\bar b}\partial_\alpha h_{j\bar b}\partial_{\bar\beta} h_{a\bar k}\,,$$

where $\partial_\alpha = \partial/\partial z^\alpha$ and $\partial_{\bar\beta} = \partial/\partial\bar z^\beta$.

The second part of the following proposition follows from (3.7).

(4.17) **Proposition** *The curvature of the Hermitian connection in a holomorphic Hermitian vector bundle is of degree* $(1,1)$.

*If* $(E, h)$ *is a* $C^\infty$ *complex vector bundle over a complex manifold with an Hermitian structure h and if D is an h-connection whose curvature is of degree* $(1,1)$, *then there is a unique holomorphic structure in E which makes D the Hermitian connection of the Hermitian vector bundle* $(E, h)$.

(4.18) **Proposition** *Let $(E, h)$ be a holomorphic Hermitian vector bundle and $D$ the Hermitian connection. Let $E' \subset E$ be a $C^\infty$ complex vector subbundle invariant under $D$. Let $E''$ be the orthogonal complement of $E'$ in $E$. Then both $E'$ and $E''$ are holomorphic subbundles of $E$ invariant by $D$, and they give a holomorphic orthogonal decomposition*:

$$E = E' \oplus E'' .$$

*Proof.* Since $E'$ is invariant under $D$, so is its orthogonal complement $E''$. Let $s$ be a local holomorphic section of $E$ and let $s = s' + s''$ be its decomposition according to the decomposition $E = E' \oplus E''$. We have to show that $s'$ and $s''$ are holomorphic sections of $E$. Since $D = D' + d''$ by (4.9), we have $Ds = D's$. Comparing $Ds = Ds' + Ds''$ with $D's = D's' + D's''$, we obtain $Ds' = D's'$ and $Ds'' = D's''$. Hence, $d''s' = 0$ and $d''s'' = 0$. Q.E.D.

In Riemannian geometry, normal coordinate systems are useful in simplifying explicit local calculations. We introduce analogous holomorphic local frame fields $s = (s_1, \cdots, s_r)$ for an Hermitian vector bundle $(E, h)$. We say that a holomorphic local frame field $s$ is *normal* at $x_0 \in M$ if

$$(4.19) \qquad \begin{aligned} &h_{i\bar j} = \delta_{ij} \quad \text{at } x_0 \\ &\omega^i_j = \sum h^{i\bar k} d' h_{j\bar k} = 0 \quad \text{at } x_0 . \end{aligned}$$

(4.20) **Proposition** *Given a holomorphic Hermitian vector bundle $(E, h)$ over $M$ and a point $x_0$ of $M$, there exists a normal local frame field $s$ at $x_0$.*

*Proof* We start with an arbitrary holomorphic local frame field $s$ around $x_0$. Applying a linear transformation to $s$, we may assume that the first condition of (4.19) is already fulfilled. Now we apply a holomorphic transformation

$$s \longrightarrow sa$$

such that $a = (a^i_j)$ is a holomorphic matrix which reduces to the identity matrix at $x_0$. Then the Hermitian matrix $H$ for $h$ changes as follows:

$$H \longrightarrow {}^t\bar{a} H a .$$

We have to choose $a$ in such a way that $d'({}^t\bar{a}Ha) = 0$ at $x_0$. Since $d''\bar{a} = 0$ and $a^i_j = \delta^i_j$ at $x_0$, we have

$$d'({}^t\bar{a}Ha) = d'H + d'a \qquad \text{at } x_0 .$$

It is clear that there is a solution of the form

$$a^i_j = \delta^i_j + \sum c^i_{jk} z^k, \qquad \text{where} \quad c^i_{jk} \in C. \qquad \text{Q.E.D}$$

We shall now combine flat structures discussed in Section 2 with Hermitian structures. A *flat Hermitian structure* in a $C^\infty$ complex vector bundle $E$ is given by an open cover $\{U\}$ of $M$ and a system of local frame fields $\{s_U\}$ such that the transition functions $\{g_{VU}\}$ are all constant unitary matrices in $U(r)$. The following statement can be verified in the same manner as (2.5).

(4.21) **Proposition** *For a $C^\infty$ complex vector bundle $E$ of rank $r$ over $M$, the following conditions are equivalent.*

(1) *$E$ admits a flat Hermitian structure;*

(2) *$E$ admits an Hermitian structure $h$ and a flat $h$-connection $D$;*

(3) *$E$ is defined by a representation $\rho: \pi_1(M) \to U(r)$ of $\pi_1(M)$.*

Let $P$ be the principal $GL(r; C)$-bundle associated to $E$. Then $E$ admits a flat Hermitian structure if and only if $P$ can be reduced to a principal $U(r)$-bundle admitting a flat structure in the sense of Section 2.

Let $\hat{P}$ be the principal $PGL(r; C)$-bundle associated to $E$ as in Section 2, i.e., $\hat{P} = P/C^* I_r$. Let $PU(r)$ be the projective unitary group defined by

$$PU(r) = U(r)/U(1)I_r,$$

where $U(1)I_r$ denotes the center of $U(r)$ consisting of matrices of the form $cI_r$, $|c| = 1$. Given an Hermitian structure $h$ in $E$, we obtained a reduction of $P$ to a principal $U(r)$-bundle $P'$, which in turn gives rise to a principal $PU(r)$-subbundle $\hat{P}' = P'/U(1)I_r$ of $\hat{P}$. Conversely, a reduction $P' \subset P$ (or $\hat{P}' \subset \hat{P}$) gives rise to an Hermitian structure $h$ in $E$. If $\hat{P}'$ is flat, or equivalently, if $\hat{P}'$ admits a flat connection, we say that the Hermitian structure $h$ is *projectively flat*. From (2.6) we obtain

(4.22) **Proposition** *For a complex vector bundle $E$ of rank $r$ over $M$, the following conditions are equivalent:*

(1) *$E$ admits a projectively flat Hermitian structure $h$;*

(2) *The $PGL(r; C)$-bundle $\hat{P} = P/C^* I_r$ is defined by a representation $\rho: \pi_1(M) \to PU(r)$ of $\pi_1(M)$.*

Letting $A = aI$, $|a| = 1$, in the proof of (2.9), we obtain

(4.23) **Proposition** *If a complex vector bundle $E$ admits a projectively flat Hermitian structure $h$, then the bundle* $\mathrm{End}(E)=E\otimes E^*$ *admits a flat Hermitian structure in a natural manner.*

## §5 Connections in associated vector bundles

Let $E$ be a $C^\infty$ complex vector bundle over a real manifold $M$. Let $E^*$ be the dual vector bundle of $E$. The dual pairing

$$\langle\ ,\ \rangle: E_x^* \times E_x \longrightarrow C$$

induces a dual pairing

$$\langle\ ,\ \rangle: A^0(E^*)\times A^0(E) \longrightarrow A^0 .$$

Given a connection $D$ in $E$, we define a connection, also denoted by $D$, in $E^*$ by the following formula:

$$(5.1)\qquad d\langle \xi, \eta\rangle = \langle D\xi, \eta\rangle + \langle \xi, D\eta\rangle \qquad \text{for}\quad \xi\in A^0(E)\,,\quad \eta\in A^0(E^*)\,.$$

Given a local frame field $s=(s_1, \cdots, s_r)$ of $E$ over an open set $U$, let $t=(t^1, \cdots, t^r)$ be the local frame field of $E^*$ dual to $s$ so that

$$\langle t^i, s_j\rangle = \delta^i_j \qquad \text{or} \qquad \langle t, s\rangle = I_r\,,$$

where $s$ is considered as a row vector and $t$ as a column vector. If $\omega=(\omega^i_j)$ denotes the connection form of $D$ with respect to $s$ so that (see (1.2))

$$(5.2)\qquad Ds_i = \sum s_j \omega^j_i \qquad \text{or} \qquad Ds = s\omega\,,$$

then

$$(5.3)\qquad Dt^i = -\sum \omega^i_j t^j \qquad \text{or} \qquad Dt = -\omega t\,.$$

This follows from

$$0 = d\delta^i_j = \langle Dt^i, s_j\rangle + \langle t^i, Ds_j\rangle = \langle Dt^i, s_j\rangle + \omega^i_j\,.$$

If $\eta = \sum \eta_i t^i$ is an arbitrary section of $E^*$ over $U$, then (1.1) and (5.3) imply

$$(5.4)\qquad D\eta = \sum (d\eta_i - \sum \omega^j_i \eta_j) t^i$$

Considering $\eta=(\eta_1, \cdots, \eta_r)$ as a row vector, we may rewrite (5.4) as

$$(5.4)'\qquad D\eta = d\eta - \eta\omega\,.$$

If $\Omega$ is the curvature form of $D$ with respect to the frame field $s$ so that (see

(1.11))

$$D^2 s = s\Omega\,, \tag{5.5}$$

then with respect to the dual frame field $t$ we have

$$D^2 t = -\Omega t\,. \tag{5.6}$$

This follows from

$$D^2 t = -D(\omega t) = -(d\omega t - \omega D t) = -(d\omega + \omega \wedge \omega) t = -\Omega t\,.$$

Now, let $\bar{E}$ denote the (complex) conjugate bundle of $E$. There is a natural conjugation map

$$\bar{\ } : E \longrightarrow \bar{E} \tag{5.7}$$

such that $\overline{\lambda\xi} = \bar{\lambda}\bar{\xi}$ for $\xi \in E$ and $\lambda \in \boldsymbol{C}$. The transition functions of $\bar{E}$ are the complex conjugates of those for $E$ in a natural manner.

Given a connection $D$ in $E$ we can define a connection $D$ in $\bar{E}$ by

$$D\bar{\xi} = \overline{D\xi} \qquad \text{for} \quad \xi \in A^0(E)\,. \tag{5.8}$$

If $s = (s_1, \cdots, s_r)$ is a local frame field for $E$, then $\bar{s} = (\bar{s}_1, \cdots, \bar{s}_r)$ is a local frame field for $\bar{E}$. If $\omega = (\omega^i_j)$ and $\Omega$ denote the connection and curvature forms of $D$ with respect to $s$, then we have clearly

$$D\bar{s}_i = \sum \bar{s}_j \bar{\omega}^j_i \qquad \text{or} \quad D\bar{s} = \bar{s}\bar{\omega}\,, \tag{5.9}$$

$$D^2 \bar{s} = \bar{s}\bar{\Omega}\,. \tag{5.10}$$

We shall now consider two complex vector bundles $E$ and $F$ over the same base manifold $M$. Let $D_E$ and $D_F$ be connections in $E$ and $F$, respectively. Then we can define a connection $D_E \oplus D_F$ in the direct (or Whitney) sum $E \oplus F$ and a connection $D_{E\otimes F}$ in the tensor product $E \otimes F$ in a natural manner. The latter is given by

$$D_{E\otimes F} = D_E \otimes I_F + I_E \otimes D_F\,, \tag{5.11}$$

where $I_E$ and $I_F$ denote the identity transformations of $E$ and $F$, respectively. If we denote the curvatures of $D_E$ and $D_F$ by $R_E$ and $R_F$, then

$$R_E \oplus R_F = \text{the curvature of } D_E \oplus D_F\,, \tag{5.12}$$

$$R_E \otimes I_F + I_E \otimes R_F = \text{the curvature of } D_{E\otimes F}\,. \tag{5.13}$$

If $s = (s_1, \cdots, s_r)$ is a local frame field of $E$ and $t = (t_1, \cdots, t_p)$ is a local frame field of $F$ and if $\omega_E$, $\omega_F$, $\Omega_E$, $\Omega_F$ are the connection and curvature forms with respect to these frame fields, then in a natural manner the connection and curvature forms of $D_E + D_F$ are given by

$$(5.14) \qquad \begin{pmatrix} \omega_E & 0 \\ 0 & \omega_F \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Omega_E & 0 \\ 0 & \Omega_F \end{pmatrix},$$

and those of $D_{E\otimes F}$ are given by

$$(5.15) \qquad \omega_E \otimes I_p + I_r \otimes \omega_F \quad \text{and} \quad \Omega_E \otimes I_p + I_r \otimes \Omega_F\,,$$

where $I_p$ and $I_r$ denote the identity matrices of rank $p$ and $r$.

All these formulas ((5.11)–(5.15)) extend in an obvious way to the direct sum and the tensor product of any number of vector bundles. Combined with formulas ((5.1)–(5.6)), they give formulas for the connection and curvature in

$$E^{\otimes p} \otimes E^{*\otimes q} = E \otimes \cdots \otimes E^* \otimes \cdots \otimes E^*$$

induced from a connection $D$ in $E$.

As a special case, we consider $\mathrm{End}(E) = E \otimes E^*$. With respect to a local frame field $s = (s_1, \cdots, s_r)$ of $E$ and the dual frame field $t = (t^1, \cdots, t^r)$, let

$$\xi = \sum \xi^i_j s_i \otimes t^j\,.$$

be a $p$-form with values in $\mathrm{End}(E)$, where $\xi^i_j$ are differential $p$-forms. Then

$$(5.16) \qquad \begin{aligned} D\xi &= \sum (d\xi^i_j s_i \otimes t^j + (-1)^p \xi^i_j Ds_i \otimes t^j + (-1)^p \xi^i_j s_i \otimes Dt^j) \\ &= \sum (d\xi^i_j + (-1)^p \sum \xi^k_j \wedge \omega^i_k - (-1)^p \sum \xi^i_k \wedge \omega^k_j) s_i \otimes t^j\,. \end{aligned}$$

The curvature $R$ of $D$ in $E$ is a 2-form with values in $\mathrm{End}(E)$. Write

$$R = \sum \Omega^i_j s_i \otimes t^j\,.$$

Then

$$DR = \sum (d\Omega^i_j + \sum \omega^i_k \wedge \Omega^k_j - \sum \Omega^i_k \wedge \omega^k_j) s_i \otimes t^j\,.$$

Hence, by (1.13)

$$(5.17) \qquad DR = 0\,,$$

which is nothing but the Bianchi identity.

The $p$-th exterior power $\bigwedge^p E$ of $E$ is a direct summand of the $p$-th tensor power $E^{\otimes p}$. It is easy to verify that the connection $D$ in $E^{\otimes p}$ induced by a connection $D$ of $E$ leaves $\bigwedge^p E$ invariant, i.e.,

$$D(A^0(\textstyle\bigwedge^p E)) \subset A^1(\textstyle\bigwedge^p E)\,.$$

For example,

$$D(\sum \xi^{ijk} s_i \wedge s_j \wedge s_k) = \sum \nabla \xi^{ijk} s_i \wedge s_j \wedge s_k ,$$

where

$$\nabla \xi^{ijk} = d\xi^{ijk} + \sum \omega^i_a \xi^{ajk} + \sum \omega^j_a \xi^{iak} + \sum \omega^k_a \xi^{ija} .$$

In particular, for the line bundle $\det(E) = \bigwedge^r E$, called *the determinant bundle* of $E$, we have

$$D(s_1 \wedge \cdots \wedge s_r) = (\sum \omega^i_i) s_1 \wedge \cdots \wedge s_r ,$$

i.e., the connection form for $D$ in $\det(E)$ is given by the trace of $\omega$:

$$\operatorname{tr} \omega = \sum \omega^i_i . \tag{5.18}$$

Similarly, its curvature is given by

$$\operatorname{tr} \Omega = \sum \Omega^i_i . \tag{5.19}$$

Let $h$ be an Hermitian structure in $E$. Since it defines an Hermitian map $E_x \times E_x \to C$ at each point $x$ of $M$, it may be considered as a section of $E^* \otimes \bar{E}^*$, i.e.,

$$h = \sum h_{ij} t^i \otimes \bar{t}^j .$$

If $D$ is any connection in $E$, then

$$Dh = \sum (dh_{ij} - \sum h_{kj} \omega^k_i - \sum h_{i\bar{k}} \bar{\omega}^k_j) t^i \otimes \bar{t}^j . \tag{5.20}$$

Let $E$ be a complex vector bundle over $M$ and let $N$ be a another manifold. Given a mapping $f$: $N \to M$, we obtain an induced vector bundle $f^*E$ over $N$ with the commutative diagram:

$$\begin{tikzcd} f^*E \arrow[r, "f"] \arrow[d] & E \arrow[d] \\ N \arrow[r, "f"] & M \end{tikzcd} \tag{5.21}$$

Then a connection $D$ in $E$ induces a connection in $f^*E$ in a natural manner; this induced connection will be denoted by $f^*D$. If $s = (s_1, \cdots, s_r)$ is a local frame field defined on an open set $U$ of $M$, then we obtain a local frame field $f^*s$ of $f^*E$ over $f^{-1}U$ in a natural manner. If $\omega$ is the connection form of $D$ with respect to $s$, then $f^*\omega$ is the connection form of $f^*D$ with respect to $f^*s$. Similarly, if $\Omega$ is the curvature form with respect to $s$, $f^*\Omega$ is the curvature form with respect to $f^*s$.

Let $h$ be an Hermitian structure in a complex vector bundle $E$. Then the dual bundle $E^*$ has a naturally induced Hermitian structure $h^*$. If $s = (s_1, \cdots, s_r)$ is a local frame field and $t=(t^1, \cdots, t^r)$ is the dual frame field, then $h$ and $h^*$ are related by

$$(5.22)\qquad h=\sum h_{ij}t^i\otimes\bar{t}^j, \qquad h^*=\sum h^{ij}s_i\otimes\bar{s}_j,$$

where $(h^{ij})$ is the inverse matrix of $(h_{ij})$ so that

$$\sum h^{ik}h_{j\bar{k}}=\delta^i_j .$$

This relationship is compatible with that between a connection in $E$ and the corresponding connection in $E^*$. Namely, if a connection $D$ in $E$ preserves $h$, then the corresponding connection $D$ in $E^*$ preserves $h^*$. In particular, if $E$ is holomorphic and $D$ is the Hermitian connection defined by $h$, then the corresponding connection $D$ in $E^*$ is exactly the Hermitian connection defined by $h^*$.

Similarly, for the conjugate bundle $\bar{E}$, we have an Hermitian structure $\bar{h}$ given by

$$(5.23)\qquad \bar{h}=\sum \bar{h}_{ij}\bar{t}^i\otimes t^j=\sum h_{ji}\bar{t}^i\otimes t^j .$$

Given Hermitian structures $h_E$ and $h_F$ in vector bundles $E$ and $F$ over $M$, we can define Hermitian structure $h_E\oplus h_F$ and $h_E\otimes h_F$ in $E\oplus F$ and $E\otimes F$ in a well known manner. In the determinant bundle $\det(E)=\bigwedge^r E$, we have a naturally induced Hermitian structure $\det(h)$. If $H=(h_{ij})$ is defined by (4.2), then

$$(5.24)\qquad \det(h)(s_1\wedge\cdots\wedge s_r, s_1\wedge\cdots\wedge s_r)=\det H .$$

If $(E, h)$ is an Hermitian vector bundle with Hermitian connection $D$ and curvature form $\Omega=(\Omega^i_j)$ with respect to $s=(s_1, \cdots, s_r)$, then $(\det(E), \det(h))$ is an Hermitian line bundle with curvature form $\operatorname{tr}\Omega=\sum\Omega^i_i$, (see (5.19)). By (4.15) we have the *Ricci form*

$$(5.25)\qquad \operatorname{tr}\Omega=\sum R_{\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta$$

where

$$R_{\alpha\bar{\beta}}=\sum R^i_{i\alpha\bar{\beta}}=-\partial_\alpha\partial_{\bar{\beta}}\det(h_{ij}) .$$

Finally, given $(E, h)$ over $M$, a mapping $f\colon N\to M$ induces an Hermitian structure $f^*h$ in the induced bundle $f^*E$ over $N$.

All these constructions of Hermitian structures in various vector bundles are compatible with the corresponding constructions of connections.

What we have done in this section may be best described in terms of principal bundles and representations of structure groups. Thus, if $P$ is the principal $GL(r, C)$-bundle associated to $E$ and if $\rho: GL(r; C)\to GL(k; C)$ is a representation, we obtain a vector bundle $E^\rho=P\otimes_\rho C^k$ of rank $k$. A connection in $E$ is really a connection in $P$, and the latter defines a connection in every associated bundle, in particular in $E^\rho$. An Hermitian structure $h$ in $E$ corresponds to a principal $U(r)$-subbundle $P'$. Considering $E^\rho$ as a bundle associated to the subbundle $P'$ by restricting $\rho$ to $U(r)$, we obtain an Hermitian structure $h^\rho$ in $E^\rho$. A connection in $E$ preserving $h$ is a connection in $P'$ and induces a connection in $E^\rho$ preserving $h^\rho$.

## § 6 Subbundles and quotient bundles

Let $E$ be a holomorphic vector bundle of rank $r$ over an $n$-dimensional complex manifold $M$. Let $S$ be a holomorphic subbundle of rank $p$ of $E$. Then the quotient bundle $Q=E/S$ is a holomorphic vector bundle of rank $r-p$. We can express this situation as an exact sequence

$$(6.1) \qquad 0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0 .$$

Let $h$ be an Hermitian structure in $E$. Restricting $h$ to $S$, we obtain an Hermitian structure $h_S$ in $S$. Taking the orthogonal complement of $S$ in $E$ with respect to $h$, we obtain a complex subbundle $S^\perp$ of $E$. We note that $S^\perp$ may not be a holomorphic subbundle of $E$ in general. Thus

$$(6.2) \qquad E = S \oplus S^\perp$$

is merely a $C^\infty$ orthogonal decomposition of $E$. As a $C^\infty$ complex vector bundle, $Q$ is naturally isomorphic to $S^\perp$. Hence, we obtain also an Hermitian structure $h_Q$ in a natural way.

Let $D$ denote the Hermitian connection in $(E, h)$. We define $D_S$ and $A$ by

$$(6.3) \qquad D\xi = D_S\xi + A\xi \qquad \xi \in A^0(S),$$

where $D_S\xi \in A^1(S)$ and $A\xi \in A^1(S^\perp)$. Then

(6.4) **Proposition** (1) *$D_S$ is the Hermitian connection of* $(S, h_S)$;
(2) *$A$ is a* $(1,0)$*-form with values in* $\mathrm{Hom}(S, S^\perp)$, *i.e.*, $A \in A^{1,0}(\mathrm{Hom}(S, S^\perp))$.

*Proof* Let $f$ be a function on $M$. Replacing $\xi$ by $f\xi$ in (6.3), we obtain

$$D(f\xi) = D_S(f\xi) + A(f\xi) .$$

On the other hand,

$$D(f\xi) = df \cdot \xi + fD\xi = df \cdot \xi + fD_S\xi + fA\xi .$$

Comparing the $S$- and $S^\perp$-components of the two decompositions of $D(f\xi)$, we